Rev. Mat. Iberoam. (Online first)
DOI 10.4171/RMI/1441

On Bernstein type quantitative estimates for Ornstein non-inequalities

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Abstract. For the sequence of multi-indices $\{\alpha_j\}_{j=1}^m$ and β, we study the inequality

$$\|D^\beta f\|_{L_1(\mathbb{T}^d)} \leqslant K_N \sum_{j=1}^m \|D^{\alpha_j} f\|_{L_1(\mathbb{T}^d)},$$

where f is a trigonometric polynomial of degree at most N on the d-dimensional torus. Assuming some natural geometric property of the set $\{\alpha_j\} \cup \{\beta\}$, we show that

$$K_N \geqslant C(\ln N)^\phi,$$

where $\phi < 1$ depends only on the set $\{\alpha_j\} \cup \{\beta\}$.

1. Introduction

In his inspiring article [12], D. Ornstein showed that if $Q(D), P_1(D), \ldots, P_m(D)$ are homogeneous differential operators of the same order, and if $Q \notin \operatorname{span}\{P_j\}$, then, for any $C > 0$, the inequality

$$\|Q(D)f\|_{L_1(\mathbb{T}^d)} \leqslant C \sum_{j=1}^m \|P_j(D)f\|_{L_1(\mathbb{T}^d)} \tag{1.1}$$

does not hold. In particular, for any $C > 0$ and any multi-indices $\beta, \alpha_1, \ldots, \alpha_m$ with $|\beta| = |\alpha_1| = \cdots = |\alpha_m|$, $\beta \notin \{\alpha_j\}_{j=1}^m$, the inequality

$$\|D^\beta f\|_{L_1(\mathbb{T}^d)} \leqslant C \sum_{j=1}^m \|D^{\alpha_j} f\|_{L_1(\mathbb{T}^d)} \tag{1.2}$$

does not hold (in this paper, $L_1(\mathbb{T}^d)$ is considered with respect to the normalized Haar measure).

2020 Mathematics Subject Classification: Primary 42A55; Secondary 46E35, 42B37.
Keywords: Ornstein non-inequalities, Bernstein inequality, Riesz products.

In this paper, we deal with the quantitative version of this theorem. We are interested in the constant of Bernstein type, i.e., what is the growth of the best constant C in (1.1) when the inequality is restricted to the polynomials of degree at most n. To the best of our knowledge, no result of such type is known.

Our main idea is to use the properties of finite Riesz products [14]. In fact, we are constructing explicitly the trigonometric polynomials for which our bounds hold. For different (but qualitative) proofs of the Ornstein non-inequality in the isotropic case, check for example [3, 9].

Focusing for a moment on the simplest case of our results, we get the following.

Corollary 1.1. *For every $N \in \mathbb{N}$, there exists a trigonometric polynomial P_N on $\mathbb{T}^2$, of degree N, which satisfies*

$$\left\| \frac{\partial^2}{\partial x^2} P_N \right\|_{L_1(\mathbb{T}^d)} + \left\| \frac{\partial^2}{\partial y^2} P_N \right\|_{L_1(\mathbb{T}^d)} \leqslant 1,$$

but

$$\left\| \frac{\partial^2}{\partial x \partial y} P_N \right\|_{L_1(\mathbb{T}^d)} > C \ln^{1/2} N.$$

We do not know if the bound from Corollary 1.1 is sharp. In fact, one can establish in a rather standard way that the mixed derivative from Corollary 1.1 could not have norm greater than $c \ln N$. Indeed, the (linear and invariant) operator T which retrieves the mixed derivative from the pure ones is of a weak type $(1, 1)$ (see [5]). Hence, by the Nikolskii type inequality for Lorentz spaces (see Theorem 3 in [15] and Lemma 3.1 in [1]), for a trigonometric polynomial f of degree N,

$$\begin{aligned} \left\| \frac{\partial^2}{\partial x \partial y} f \right\|_{L_1(\mathbb{T}^2)} &= \|Tf\|_{L_{1,1}(\mathbb{T}^2)} \leqslant C \ln(1+N) \|Tf\|_{L_{1,\infty}(\mathbb{T}^2)} \\ &\leqslant C \ln(1+N) \Big(\left\| \frac{\partial^2}{\partial x^2} f \right\|_{L_1(\mathbb{T}^2)} + \left\| \frac{\partial^2}{\partial y^2} f \right\|_{L_1(\mathbb{T}^2)} \Big). \end{aligned}$$

The same comment concerns all the results obtained in this paper. All bounds from below presented here are of the form $(\ln N)^{\phi}$ for some $\phi < 1$, while the common upper bound is $\ln N$. Nevertheless, we conjecture that the optimal exponent ϕ should be equal to one (see Remark 4.2).

This paper contains final results of the study we began in [7]. It seems it is the first use in the literature of trigonometric polynomials in the context of Ornstein non-inequalities.

2. Results

In this paper, we consider a more general, anisotropic case of the inequality (1.2), i.e., such that there exists $\Lambda \in \mathbb{N}^d$ with $\langle \alpha_1, \Lambda \rangle = \langle \alpha_2, \Lambda \rangle = \cdots = \langle \beta, \Lambda \rangle$. This case was already considered in the literature (see [6]), with an additional assumption on the parity of derivatives $|\alpha_j| \equiv_2 |\beta|$. (We mention that the results obtained there were only of the qualitative nature.) In our present approach, we remove this "parity assumption". However, we still need other geometric conditions.

The paper contains two results which are proved in quite similar way. Each of them provides a geometric criterion for a set of symbols of partial derivatives which yields quantitative estimates of Ornstein type.

Theorem 2.1. *Assume* $\alpha_1, \dots, \alpha_n, \beta$ *are multi-indices in* $(\mathbb{N} \cup \{0\})^d$, *and that there exists a pair of vectors* $\Gamma,\ \Lambda \in \mathbb{N}^d$ *for which the following conditions are satisfied*:

$$\langle \alpha_1, \Lambda \rangle = \langle \beta, \Lambda \rangle = \langle \alpha_2, \Lambda \rangle = \cdots = \langle \alpha_m, \Lambda \rangle$$

and

$$\langle \alpha_1, \Gamma \rangle > \langle \beta, \Gamma \rangle > \langle \alpha_2, \Gamma \rangle \geqslant \cdots \geqslant \langle \alpha_m, \Gamma \rangle, \tag{2.1}$$

Let K_N *be the smallest constant such that, for every trigonometric polynomial* f *of degree at most* N, *the following estimate holds*:

$$\|D^\beta f\|_{L_1(\mathbb{T}^d)} \leqslant K_N \sum_{j=1}^{m} \|D^{\alpha_j} f\|_{L_1(\mathbb{T}^d)}. \tag{2.2}$$

Then, there exists a constant $C > 0$ *such that*

$$K_N > C(\ln N)^\phi,$$

where $\phi = \frac{1}{2}\big(1 - \frac{\langle \alpha_1 - \beta, \Gamma \rangle}{\langle \alpha_1 - \alpha_2, \Gamma \rangle}\big)$.

Remark 2.2. The inequalities (2.1) could be satisfied for different vectors Γ, and then for different permutations of the set $\{\alpha_1, \dots, \alpha_m\}$. For any fixed set of multi-indices, the choice of the optimal vector Γ is a simple optimization problem. In dimension 2, for fixed α_1, the value of ϕ does not depend on the choice of Γ.

Theorem 2.3. *Assume* $\alpha_1, \dots, \alpha_n, \beta$ *are multi-indices in* $(\mathbb{N} \cup \{0\})^d$, *and that there exists a vector*[1] $\Lambda \in \mathbb{N}^d$ *for which the following condition is satisfied*:

$$\langle \alpha_1, \Lambda \rangle = \langle \beta, \Lambda \rangle = \langle \alpha_2, \Lambda \rangle = \cdots = \langle \alpha_m, \Lambda \rangle.$$

Suppose moreover that there exists $\boldsymbol{\varepsilon} \in \{0, 1\}^d$ *such that*

$$\langle \beta, \boldsymbol{\varepsilon} \rangle \not\equiv \langle \alpha_1, \boldsymbol{\varepsilon} \rangle \mod 2 \quad \textit{and} \quad \langle \alpha_j, \boldsymbol{\varepsilon} \rangle \equiv \langle \alpha_1, \boldsymbol{\varepsilon} \rangle \mod 2 \tag{2.3}$$

for $j \in \{1, \dots, m\}$. *Let* K_N *be the smallest constant such that, for every trigonometric polynomial* f *of degree at most* N, *the following estimate holds*:

$$\|D^\beta f\|_{L_1(\mathbb{T}^d)} \leqslant K_N \sum_{j=1}^{m} \|D^{\alpha_j} f\|_{L_1(\mathbb{T}^d)}. \tag{2.4}$$

Then, there exists a constant $C > 0$ *such that*

$$K_N > C(\ln N)^{1/2}.$$

Remark 2.4. The case $\boldsymbol{\varepsilon} = (1, 1, \dots, 1)$ corresponds to the anisotropic Sobolev space, which contains an invariant, complemented, infinite dimensional subspace isomorphic to a Hilbert space (see [13] for details).

[1] In this paper, we put $\mathbb{N} = \{1, 2, 3, \dots\}$

3. Proof of Theorem 2.1

Proof. Let $\Lambda = (\lambda_1, \dots, \lambda_d)$ and $\Gamma = (\gamma_1, \dots, \gamma_d)$. We introduce an auxiliary sequence $(b_k)_{k\geqslant 1}$ depending on the parity of our multi-indices. If $|\alpha_1| - |\beta|$ is even, we put $b_k := (2 + (-1)^k)$, and if not, we put $b_k := 1$. For a fixed $n \in \mathbb{N}$, we define a sequence of vectors $(a_k)_{k\geqslant 1}$ in $\mathbb{N}^d$ by the formula $a_k = (a_k(1), \dots, a_k(d))$, where

$$a_k(j) := 3^{\lambda_j 2kn}\, b_k^{\gamma_j} \lfloor n^{\theta\gamma_j} \rfloor \tag{3.1}$$

and

$$\theta := \frac{1}{\langle \alpha_1 - \alpha_2, \Gamma \rangle}. \tag{3.2}$$

Since γ_j and λ_j are positive integers, we know that, for any $j \in \{1, 2, \dots, d\}$,

$$a_k(j) > 3^{2(n-1)}\, a_{k-1}(j) \tag{3.3}$$

and

$$\|a_k\|_2 > 3^{2(n-1)}\, \|a_{k-1}\|_2. \tag{3.4}$$

We define a modified Riesz product based on this sequence,

$$R_n(x) = -1 + \prod_{k=1}^{n} (1 + \cos\langle x, a_k \rangle), \tag{3.5}$$

and the family of sets

$$A_k = \Big\{ q : q = a_k + \sum_{j=1}^{k-1} \xi_j\, a_j,\ \xi_j \in \{-1, 0, 1\} \text{ for } j \in \{1, \dots, k-1\} \Big\}.$$

From (3.4), by standard calculations, we know that every point in A_k has a unique representation as $a_k + \sum_{j=1}^{k-1} \xi_j\, a_j$. From (3.3), there exists a constant $2 \geqslant \tau > 1$, independent of k and j, such that

$$\frac{1}{\tau}\, a_k(j) \leqslant |q(j)| \leqslant \tau\, a_k(j), \tag{3.6}$$

for all $q \in A_k$.

For $\mu \in \mathbb{Z}^d$, we denote

$$n^{\mu} = \prod_{j=1}^{d} n_j^{\mu(j)}.$$

For $q \in A_k$ of the form $q = a_k + \sum_{j=1}^{k-1} \xi_j\, a_j$, we set

$$r(q) = \#\{j : \xi_j \neq 0\} + 1 \quad \text{and} \quad r(-q) = r(q).$$

Let $W_n(x)$ be a polynomial given by the formula

$$W_n(x) = \sum_{k=1}^{n} \sum_{q \in A_k \cup -A_k} \frac{i^{-|\alpha_1|}}{q^{\alpha_1}} \frac{1}{2^{r(q)}} e^{i\langle q,x\rangle}. \tag{3.7}$$

Note that

$$D^{\alpha_1} W_n(x) = \sum_{k=1}^{n} \sum_{q \in A_k \cup -A_k} \frac{1}{2^{r(q)}} e^{i\langle q,x\rangle} = R_n(x).$$

Moreover, for $\mu \in \{\beta, \alpha_2, \dots, \alpha_m\}$,

$$D^{\mu} W_n(x) = \sum_{k=1}^{n} \sum_{q \in A_k \cup -A_k} i^{|\mu|-|\alpha|} \frac{q^{\mu}}{q^{\alpha_1}} \frac{1}{2^{r(q)}} e^{i\langle q,x\rangle},$$

which could be represented as

$$D^{\mu} W_n(x) = B_{\mu,n}(x) + G_{\mu,n}(x), \tag{3.8}$$

where

$$B_{\mu,n}(x) = \sum_{k=1}^{n} \sum_{q \in A_k} \frac{i^{|\mu|-|\alpha_1|}}{2^{r(q)}} \Big(\Big(\frac{q^{\mu}}{q^{\alpha_1}} - \frac{a_k^{\mu}}{a_k^{\alpha_1}} \Big) e^{i\langle q,x\rangle} + \Big(\frac{(-q)^{\mu}}{(-q)^{\alpha_1}} - \frac{(-a_k)^{\mu}}{(-a_k)^{\alpha_1}} \Big) e^{i\langle -q,x\rangle} \Big) \tag{3.9}$$

and

$$G_{\mu,n}(x) = \sum_{k=1}^{n} i^{|\mu|-|\alpha_1|} \frac{a_k^{\mu}}{a_k^{\alpha_1}} \sum_{q \in A_k} \frac{1}{2^{r(q)}} \big(e^{i\langle q,x\rangle} + (-1)^{|\mu|-|\alpha_1|} e^{i\langle -q,x\rangle} \big). \tag{3.10}$$

First we estimate the L_1-norm of $B_{\mu,n}$. Let $v = (q(1)/a_k(1), \dots, q(d)/a_k(d))$ for $q \in A_k$. From (3.3),

$$\|v - \mathbf{1}\|_2 \leqslant C(d)\, 3^{-2n}, \tag{3.11}$$

where $\mathbf{1} := (1, \dots, 1)$. Observe that for $q \in A_k$, by (3.6), (3.11) and by the Lipschitz continuity of functions x^{α_1}, x^{μ} on the cube $[1/\tau, \tau]^d$, we get

$$\begin{aligned} \big|q^{\mu} a_k^{\alpha_1} - a_k^{\mu} q^{\alpha_1}\big| &\leqslant \big|q^{\mu} (a_k^{\alpha_1} - q^{\alpha_1})\big| + \big|q^{\alpha_1} (q^{\mu} - a_k^{\mu})\big| \\ &\leqslant C \left(|q^{\alpha_1}||a_k^{\mu}||1 - v^{\alpha_1}| + |q^{\mu}||a_k^{\alpha_1}||1 - v^{\mu}| \right) \\ &\overset{(3.6)}{\leqslant} C |a_k^{\alpha_1}||a_k^{\mu}| \big(|1 - v^{\alpha_1}| + |1 - v^{\mu}| \big) \\ &\overset{\text{Lip.}}{\leqslant} C \|\mathbf{1} - v\|_2 |a_k^{\alpha_1}||a_k^{\mu}| \\ &\overset{(3.11)}{\leqslant} C\, 3^{-2n} |a_k^{\alpha_1}||a_k^{\mu}|. \end{aligned} \tag{3.12}$$

We calculate a_k^{μ} and $a_k^{\alpha_1}$:

$$\begin{aligned} a_k^{\mu} &= 3^{\sum_{j=1}^{d}\mu(j)\lambda_j 2kn}\, b_k^{\sum_{j=1}^{d}\mu(j)\gamma_j} \prod_{i=1}^{d} \lfloor n^{\theta\gamma_i}\rfloor^{\mu(j)} \\ &= 3^{\langle\mu,\Lambda\rangle 2kn}\, b_k^{\langle\mu,\Gamma\rangle} \prod_{j=1}^{d} \lfloor n^{\theta\gamma_j}\rfloor^{\mu(j)}, \end{aligned} \tag{3.13}$$

and similarly (replacing μ by α_1),

$$a_k^{\alpha_1} = 3^{\langle\alpha_1,\Lambda\rangle 2kn}\, b_k^{\langle\alpha_1,\Gamma\rangle} \prod_{j=1}^{d} \lfloor n^{\theta\gamma_j}\rfloor^{\alpha_1(j)}. \tag{3.14}$$

Since we only use a finite number of exponents, there is a constant $C > 1$ such that for any $\nu \in \{\beta, \alpha_1, \dots, \alpha_m\}$,

$$\frac{1}{C}\, n^{\theta\cdot\langle\nu,\Gamma\rangle} \leqslant \prod_{j=1}^{d} \lfloor n^{\theta\gamma_j}\rfloor^{\nu(j)} \leqslant C n^{\theta\cdot\langle\nu,\Gamma\rangle}. \tag{3.15}$$

For every μ as above, we have $\langle\mu,\Lambda\rangle = \langle\alpha_1,\Lambda\rangle$. By (3.6), (3.12) and (3.13), (3.14), (3.15), we get

$$\left|\left(\frac{q^{\mu}}{q^{\alpha_1}} - \frac{a_k^{\mu}}{a_k^{\alpha_1}}\right)\right| = \left|\frac{q^{\mu} a_k^{\alpha_1} - a_k^{\mu} q^{\alpha_1}}{q^{\alpha_1} a_k^{\alpha_1}}\right| \leqslant C\, 3^{-2n}\, \frac{|a_k^{\alpha_1}|\,|a_k^{\mu}|}{|a_k^{\alpha_1}|^2} \leqslant C\, 3^{-2n}\, n^{\langle\mu-\alpha_1,\Gamma\rangle} \leqslant C\, 3^{-n}.$$

Plugging the above estimates for $\mu \in \{\beta, \alpha_2, \dots, \alpha_m\}$ into the formula for $B_{\mu,n}$, we get

$$\|B_{\mu,n}\|_{L_1(\mathbb{T}^d)} \leqslant \sum_{k=1}^{n} \sum_{q\in A_k} 2C\, 3^{-n} \leqslant 2C\, 3^{-n}\, 3^{n} = 2C.$$

We pass to the estimates of the L_1 norm of $G_{\alpha_j,n}$ for $j \geqslant 2$. For $k \in \mathbb{N}$ and $1 \leqslant k \leqslant n$, we define

$$\psi_k(x) = \prod_{l=1}^{k-1} (1 + \cos\langle x, a_l\rangle).$$

A simple algebraic manipulation gives us

$$G_{\alpha_j,n}(x) = \sum_{k=1}^{n} i^{|\mu|-|\alpha_1|}\, \frac{a_k^{\alpha_j}}{a_k^{\alpha_1}}\, \frac{1}{2}\left(e^{i\langle a_k,x\rangle} + (-1)^{|\alpha_j|-|\alpha_1|}\, e^{i\langle -a_k,x\rangle}\right) \psi_k(x).$$

Since $\langle\alpha_j,\Lambda\rangle = \langle\alpha_1,\Lambda\rangle$, by (3.13) and (3.14) we get

$$\left|\frac{a_k^{\alpha_j}}{a_k^{\alpha_1}}\right| \leqslant C n^{\theta\cdot\langle\alpha_j-\alpha_1,\Gamma\rangle}.$$

Therefore,

$$\|G_{\alpha_j,n}\|_{L_1(\mathbb{T}^d)} \leqslant C n^{\theta\cdot\langle\alpha_j-\alpha_1,\Gamma\rangle} \sum_{k=1}^{n} \|\psi_k\|_{L_1(\mathbb{T}^d)}.$$

As the Riesz products ψ_l's have L_1 norms equal to 1, we deduce that

$$\|G_{\alpha_j,n}\|_{L_1(\mathbb{T}^d)} \leqslant C n^{\theta\cdot\langle\alpha_j-\alpha_1,\Gamma\rangle+1}.$$

By (2.1) and (3.2), for any $j \in \{2,\dots,m\}$,

$$\theta\cdot\langle\alpha_j-\alpha_1,\Gamma\rangle = \frac{\langle\alpha_j-\alpha_1,\Gamma\rangle}{\langle\alpha_1-\alpha_2,\Gamma\rangle} \leqslant -1.$$

Hence there exists $C>0$ such that

$$\|G_{\alpha_j,n}\|_{L_1(\mathbb{T}^d)} \leqslant C,$$

for any $n\in\mathbb{N}$ and any $j\in\{2,\dots,m\}$. Therefore for $j\in\{2,\dots,m\}$ and $n\in\mathbb{N}$,

$$\|D^{\alpha_j}W_n\|_{L_1(\mathbb{T}^d)} \leqslant \|B_{\alpha_j,n}\|_{L_1(\mathbb{T}^d)} + \|G_{\alpha_j,n}\|_{L_1(\mathbb{T}^d)} \leqslant C.$$

Since $D^{\alpha_1}W$ is a modified Riesz product,

$$\|D^{\alpha_1}W_n\|_{L_1(\mathbb{T}^d)} \leqslant 2.$$

Summing the above inequalities, we get

$$\sum_{j=1}^{m}\|D^{\alpha_j}W_n\|_{L_1(\mathbb{T}^d)} \leqslant C. \tag{3.16}$$

Now we estimate $\|D^{\beta}W_n\|_{L_1(\mathbb{T}^d)}$ from below. Since the norm of $B_{\beta,n}$ is uniformly bounded with respect to n, it is enough to show that the norm of $G_{\beta,n}$ is large.

Remark 3.1. In the article [11], see Remark on p. 563, Y. Meyer observes that the condition $\sum_{k=1}^{\infty}\frac{a_k(j)}{a_{k+1}(j)} < +\infty$ yields

$$\Big\| \sum_{\xi\in\{-1,0,1\}^n} b\Big(\sum_{k=1}^{n}\xi_k a_k(j)\Big)\exp\Big(i\sum_{k=1}^{n}\xi_k a_k(j)t\Big)\Big\|_{L^1(\mathbb{T})} \simeq \Big\| \sum_{\xi\in\{-1,0,1\}^n} b\Big(\sum_{k=1}^{n}\xi_k a_k(j)\Big)\exp\Big(i\sum_{k=1}^{n}\xi_k t_k\Big)\Big\|_{L^1(\mathbb{T}^n)}.$$

The constant in the above isomorphism depends only on the value of $\sum_{k=1}^{\infty}\frac{a_k(j)}{a_{k+1}(j)}$. For elementary proofs of this fact, see [8] or Proposition 4 in [2]. By a simple tensoring argument,

$$\Big\| \sum_{\xi\in\{-1,0,1\}^n} b\Big(\sum_{k=1}^{n}\xi_k a_k\Big)\exp\Big(i\sum_{k=1}^{n}\langle\xi_k a_k, t\rangle\Big)\Big\|_{L^1(\mathbb{T}^d)} \simeq \Big\| \sum_{\xi\in\{-1,0,1\}^n} b\Big(\sum_{k=1}^{n}\xi_k a_k\Big)\exp\Big(i\sum_{k=1}^{n}\langle\xi_k, t_k\rangle\Big)\Big\|_{L^1(\mathbb{T}^{nd})}.$$

In our case, there exists a constant $C > 0$ independent of n such that the finite sequence $(a_k(j))_{k=1}^n$ defined by (3.1) satisfies

$$\sum_{k=1}^{n} \frac{a_k(j)}{a_{k+1}(j)} \overset{(3.3)}{<} n\,3^{-2n} < C < +\infty$$

for any $j \in \{1, \ldots, d\}$. Hence, when calculating L_1 norm, we can treat exponents with different a_k as independent random variables.

We consider two cases separately.

Case I. $|\alpha_1| - |\beta|$ *is even.*

In this case,

$$\begin{aligned} |G_{\beta,n}(x)| &= \Big| \sum_{k=1}^{n} \frac{a_k^{\beta}}{a_k^{\alpha_1}} \cos(\langle a_k, x\rangle)\, \psi_k(x) \Big| \\ &= \Big| \frac{a_1^{\beta}}{a_1^{\alpha_1}}\, \psi_1 + \frac{a_n^{\beta}}{a_n^{\alpha_1}}\, \psi_{n+1} + \sum_{k=1}^{n-1} \Big(\frac{a_{k+1}^{\beta}}{a_{k+1}^{\alpha_1}} - \frac{a_k^{\beta}}{a_k^{\alpha_1}} \Big) \psi_{k+1} \Big|. \end{aligned}$$

Since ψ_l are Riesz products, by (3.4) and by the inequality of Latała (Theorem 1 [10], see also [2, 4]),

$$\|G_{\beta,n}\|_{L_1(\mathbb{T}^d)} \geqslant C \Big(\Big| \frac{a_1^{\beta}}{a_1^{\alpha_1}} \Big| + \Big| \frac{a_n^{\beta}}{a_n^{\alpha_1}} \Big| + \sum_{k=1}^{n-1} \Big| \frac{a_{k+1}^{\beta}}{a_{k+1}^{\alpha_1}} - \frac{a_k^{\beta}}{a_k^{\alpha_1}} \Big| \Big).$$

From the definition of a_k (and the fact that $\langle \alpha_1, \Lambda\rangle = \langle \beta, \Lambda\rangle$), we get

$$\Big| \frac{a_{k+1}^{\beta}}{a_{k+1}^{\alpha_1}} - \frac{a_k^{\beta}}{a_k^{\alpha_1}} \Big| \geqslant C n^{\theta\cdot\langle \beta-\alpha_1, \Gamma\rangle}\, |b_k^{\langle \beta,\Gamma\rangle} - b_{k+1}^{\langle \beta,\Gamma\rangle}| \geqslant C n^{\theta\cdot\langle \beta-\alpha_1, \Gamma\rangle}\, |3^{\langle \beta,\Gamma\rangle} - 1|.$$

From the definition of θ, we get

$$\|G_{\beta,n}\|_{L_1(\mathbb{T}^d)} \geqslant C n^{1-\langle \beta-\alpha_1,\Gamma\rangle/\langle \alpha_2-\alpha_1,\Gamma\rangle},$$

and consequently,

$$\|D^{\beta} W_n\|_{L_1(\mathbb{T}^d)} \geqslant C n^{1-\langle \beta-\alpha_1,\Gamma\rangle/\langle \alpha_2-\alpha_1,\Gamma\rangle}.$$

There exists $C > 0$ independent of n such that $|a_k| \leqslant 3^{Cn^2}$ for $1 \leqslant k \leqslant n$. Therefore, $\deg W_n(x) \leqslant 3^{Cn^2}$ and $\ln(\deg W_n(x)) \leqslant Cn^2$. Hence, for large enough n, there exists a constant $C > 0$ such that

$$\|D^{\beta} W_n\|_{L_1(\mathbb{T}^d)} \geqslant C(\ln \deg W_n)^{\frac{1}{2}(1-\langle \beta-\alpha_1,\Gamma\rangle/\langle \alpha_2-\alpha_1,\Gamma\rangle)}.$$

From (3.16) and (2.2), we get

$$K \geqslant C(\ln \deg W_n)^{\frac{1}{2}(1-\langle \beta-\alpha_1,\Gamma\rangle/\langle \alpha_2-\alpha_1,\Gamma\rangle)}.$$

Case II. $|\alpha_1| - |\beta|$ *is odd.*

In this case,

$$|G_{\beta,n}(x)| = \Big| \sum_{k=1}^{n} \frac{a_k^{\beta}}{a_k^{\alpha_1}} \sin(\langle a_k, x\rangle)\, \psi_k(x) \Big|.$$

By the definition of the sequence a_k,

$$\begin{aligned}\Big| \frac{a_k^{\beta}}{a_k^{\alpha_1}} - n^{-\langle \beta-\alpha_1,\Gamma\rangle/\langle \alpha_2-\alpha_1,\Gamma\rangle} \Big| &= \Big| \frac{\prod_{s=1}^{d} \lfloor n^{\theta\gamma_j} \rfloor^{\beta(s)}}{\prod_{s=1}^{d} \lfloor n^{\theta\gamma_j} \rfloor^{\alpha_1(s)}} - n^{-\langle \beta-\alpha_1,\Gamma\rangle/\langle \alpha_2-\alpha_1,\Gamma\rangle} \Big| \\ &\leqslant \frac{C}{n^{\theta}} \cdot n^{-\langle \beta-\alpha_1,\Gamma\rangle/\langle \alpha_2-\alpha_1,\Gamma\rangle}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|G_{\beta,n}\|_{L_1(\mathbb{T}^d)} \geqslant n^{-\langle \beta-\alpha_1,\Gamma\rangle/\langle \alpha_2-\alpha_1,\Gamma\rangle} \Big\| \sum_{k=1}^{n} \sin(\langle a_k, x\rangle)\psi_k(x) \Big\|_{L_1(\mathbb{T}^d)} \\ - \frac{C}{n^{\theta}} \cdot n^{1-\langle \beta-\alpha_1,\Gamma\rangle/\langle \alpha_2-\alpha_1,\Gamma\rangle}.\end{aligned}$$

However,

$$\begin{aligned}\Big\| \sum_{k=1}^{n} \sin(\langle a_k, x\rangle)\, \psi_k(x) \Big\|_{L_1(\mathbb{T}^d)} &= \Big\| \sum_{k=1}^{n} \big(\cos(\langle a_k, x\rangle) - e^{i\langle a_k,x\rangle}\big)\, \psi_k(x) \Big\|_{L_1(\mathbb{T}^d)} \\ &\geqslant \Big\| \sum_{k=1}^{n} e^{i\langle a_k,x\rangle}\, \psi_k(x) \Big\|_{L_1(\mathbb{T}^d)} - \|\psi_{n+1}\|_{L_1(\mathbb{T}^d)} \\ &\geqslant \Big\| \sum_{k=1}^{n} e^{i\langle a_k,x\rangle}\, \psi_k(x) \Big\|_{L_1(\mathbb{T}^d)} - 1.\end{aligned}$$

The sequence a_k satisfies the assumptions of Meyer's theorem (see Remark 3.1). Because of that, we can use Lemma 2 from [16], which gives

$$\Big\| \sum_{j=1}^{n} e^{i\langle a_k,x\rangle}\, \psi_k(x) \Big\|_{L_1(\mathbb{T}^d)} \geqslant C n.$$

Thus

$$\|G_{\beta,n}\|_{L_1(\mathbb{T}^d)} \geqslant \Big(C_1 - \frac{C}{n^{\theta}}\Big)\, n^{1-\langle \beta-\alpha_1,\Gamma\rangle/\langle \alpha_2-\alpha_1,\Gamma\rangle}.$$

Therefore, for large enough n we get

$$\|G_{\beta,n}\|_{L_1(\mathbb{T}^d)} \geqslant C\, n^{1-\langle \beta-\alpha_1,\Gamma\rangle/\langle \alpha_2-\alpha_1,\Gamma\rangle}$$

and similarly as in Case I, we obtain

$$K \geqslant C(\ln \deg W_n)^{\frac{1}{2}\,(1-\langle \beta-\alpha_1,\Gamma\rangle/\langle \alpha_2-\alpha_1,\Gamma\rangle)}. \qquad \blacksquare$$

4. Proof of Theorem 2.3

Proof. We prove Theorem 2.3 in an analogous way. First we define the sequence a_k by the formula

$$a_k(j) = 3^{\lambda_j 2kn}\,(-1)^{\varepsilon_j k}.$$

Once again, we use the modified Riesz products of (3.5) and the corresponding polynomials W_n of (3.7). As in the proof of Theorem 2.1, we have

$$D^{\mu} W_n(x) = B_{\mu,n}(x) + G_{\mu,n}(x)$$

for any $\mu \in \{\beta, \alpha_2, \ldots, \alpha_m\}$ and $B_{\mu,n}(x)$ and $G_{\mu,n}(x)$ defined as in (3.9) and (3.10). Since the sequence a_k has super-exponential growth,

$$\frac{|a_k(j)|}{|a_{k+1}(j)|} \leqslant 3^{-2n},$$

we obtain the following bounds on $B_{\mu,n}(x)$:

$$\|B_{\mu,n}\|_{L_1(\mathbb{T}^d)} \leqslant C$$

for any $\mu \in \{\beta, \alpha_2, \ldots, \alpha_m\}$. Note that, by (2.3),

$$a_k^{\alpha_j} = 3^{\langle \Lambda, \alpha_j\rangle}\,(-1)^{k\langle \varepsilon, \alpha_j\rangle} = 3^{\langle \Lambda, \alpha_1\rangle}\,(-1)^{k\langle \varepsilon, \alpha_1\rangle} = a_k^{\alpha_1}.$$

Hence, by (3.10),

$$G_{\alpha_j,n}(x) = \sum_{k=1}^{n} i^{|\alpha_j|-|\alpha_1|} \sum_{q\in A_k} \frac{1}{2^{r(q)}}\left(e^{i\langle q,x\rangle} + (-1)^{|\alpha_j|-|\alpha_1|}\, e^{i\langle -q,x\rangle}\right).$$

Thus for $|\alpha_1| \equiv |\alpha_j| \bmod 2$, we get

$$\|G_{\alpha_j,n}\|_{L_1(\mathbb{T}^d)} = \left\|-1 + \prod_{k=1}^{n}(1+\cos\langle x, a_k\rangle)\right\|_{L_1(\mathbb{T}^d)} = \|R_n\|_{L_1(\mathbb{T}^d)} \leqslant 2,$$

and for $|\alpha_1| \equiv |\alpha_j| + 1 \bmod 2$,

$$\|G_{\alpha_j,n}\|_{L_1(\mathbb{T}^d)} = \left\|-1 + \prod_{k=1}^{n}(1+\sin\langle x, a_k\rangle)\right\|_{L_1(\mathbb{T}^d)} \leqslant 2.$$

The only thing left to do is the estimate on the norm of $G_{\beta,n}$ from below. By (2.3), we get

$$a_k^{\beta} = 3^{\langle \Lambda, \beta\rangle}\,(-1)^{k\langle \varepsilon, \beta\rangle} = 3^{\langle \Lambda, \alpha_1\rangle}\,(-1)^{k(\langle \varepsilon, \alpha_1\rangle+1)} = (-1)^k\, a_k^{\alpha_1}.$$

Therefore,

$$G_{\beta,n}(x) = \sum_{k=1}^{n}(-1)^k\, i^{|\beta|-|\alpha_1|} \sum_{q\in A_k} \frac{1}{2^{r(q)}}\left(e^{i\langle q,x\rangle} + (-1)^{|\beta|-|\alpha_1|}\, e^{i\langle -q,x\rangle}\right).$$

Let g_m be given by the formula

$$g_m(x) = \begin{cases} \prod_{k=1}^{m}(1+\cos\langle x, a_k\rangle), & \text{for } |\beta| \equiv |\alpha_1| \mod 2, \\ \prod_{k=1}^{m}(1+\sin\langle x, a_k\rangle), & \text{for } |\beta| \not\equiv |\alpha_1| \mod 2. \end{cases}$$

Then

$$\begin{aligned} |G_{\beta,n}(x)| &= \Big|\sum_{k=0}^{n-1}(-1)^k\,(g_{k+1}(x) - g_k(x))\Big| \\ &= \Big|(-1)^{n-1}g_n(x) + g_0(x) + \sum_{k=1}^{n-1} 2(-1)^{k-1}g_k(x)\Big|. \end{aligned}$$

Applying Latała's inequality (Theorem 1 in [10]), we obtain

$$\|G_{\beta,n}\|_{L_1(\mathbb{T}^d)} \geqslant Cn. \tag{4.1}$$

As in previous section, we get

$$K \geqslant C(\ln \deg W_n)^{1/2}. \qquad \blacksquare$$

Remark 4.1. Actually, to obtain estimate (4.1) we could use a weaker (random) form of Latała's inequality (see Lemma 1 in [16]). However, to do this one needs to adjust the construction to the randomness of choice of signs, which significantly complicates the redaction.

Remark 4.2. The careful study of the above proofs shows that the reason behind the sub-logarithmic growth of the constant lies in the super-exponential growth of the sequence a_k (see (3.3)). There are two reasons for the significant growth of this sequence. In order to use the Latała inequality, just geometrical growth would be enough (see [2]). However, we use a Riesz product as one of the derivatives involved in the proof. To recover the Riesz product structure for the remaining derivatives – which we need to use in the inequality of Latała –, we perturbed the actual functions. Our method to control the arising error terms requires super-exponential growth of the sequence a_k. It seems that any improvement of this method would require a more delicate study of the aforementioned error terms.

Acknowledgements. Authors would like to thank Maciej Rzeszut for helpful discussions. We would like to express our gratitude to anonymous reviewers, whose comments and suggestions helped improve and clarify this manuscript.

Funding. This research was supported by the National Science Centre Poland and Austrian Science Foundation FWF joint CEUS programme: FWF project no. I5231, and National Science Centre project no. 2020/02/Y/ST1/00072.

References

[1] Akishev, G.: Estimates for the best approximation of functions of a class of logarithmic smoothness in a Lorentz space. (Russian) *Tr. Inst. Mat. Mekh.* **23** (2017), no. 3, 3–21.

[2] Bonami, A., Latała, R., Nayar, P. and Tkocz, T.: Bounds on moments of weighted sums of finite Riesz products. *J. Fourier Anal. Appl.* **26** (2020), no. 6, article no. 84, 31 pp.

[3] Conti, S., Faraco, D. and Maggi, F.: A new approach to counterexamples to L^1 estimates: Korn's inequality, geometric rigidity, and regularity for gradients of separately convex functions. *Arch. Ration. Mech. Anal.* **175** (2004), no. 2, 287–300.

[4] Damek, E., Latała, R., Nayar, P. and Tkocz, T.: Two-sided bounds for L_p-norms of combinations of products of independent random variables. *Stochastic Process. Appl.* **125** (2015), no. 4, 1688–1713.

[5] Fabes, E. B. and Rivière, N. M.: Singular integrals with mixed homogeneity. *Studia Math.* **27** (1966), no. 1, 19–38.

[6] Kazaniecki, K., Stolyarov, D. M. and Wojciechowski, M.: Anisotropic Ornstein noninequalities. *Anal. PDE* **10** (2017), no. 2, 351–366.

[7] Kazaniecki, K. and Wojciechowski, M.: Ornstein's non-inequalities Riesz product approach. Preprint 2014, arXiv: 1406.7319.

[8] Kazaniecki, K. and Wojciechowski, M.: On the equivalence between the sets of the trigonometric polynomials. Preprint 2014, arXiv: 1502.05994.

[9] Kirchheim, B. and Kristensen, J.: On rank one convex functions that are homogeneous of degree one. *Arch. Ration. Mech. Anal.* **221** (2016), no. 1, 527–558.

[10] Latała, R.: L_1-norm of combinations of products of independent random variables. *Israel J. Math.* **203** (2014), no. 1, 295–308.

[11] Meyer, Y.: Endomorphismes des idéaux fermés de $L^1(G)$, classes de Hardy et séries de Fourier lacunaires. *Ann. Sci. École Norm. Sup. (4)* **1** (1968), no. 4, 499–580.

[12] Ornstein, D.: A non-equality for differential operators in the L_1 norm. *Arch. Rational Mech. Anal.* **11** (1962), 40–49.

[13] Pelczyński, A. and Wojciechowski, M.: Paley projections on anisotropic Sobolev spaces on tori. *Proc. London Math. Soc. (3)* **65** (1992), no. 2, 405–422.

[14] Riesz, F.: Über die Fourierkoeffizienten einer stetigen Funktion von beschränkter Schwankung. *Math. Z.* **2** (1918), no. 3-4, 312–315.

[15] Sherstneva, L. A.: Nikol'skii inequalities for trigonometric polynomials in Lorentz spaces. (Russian) *Vestnik Moskov. Univ. Ser. 1. Mat. Mekh.* **4** (1984), 75–79.

[16] Wojciechowski, M.: On the strong type multiplier norms of rational functions in several variables. *Illinois J. Math.* **42** (1998), no. 4, 582–600.

Received November 20, 2022; revised July 10, 2023. Published online July 25, 2023.

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