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Contractive inequalities between Dirichlet and Hardy spaces

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Abstract. We prove a conjecture of Brevig, Ortega-Cerdà, Seip and Zhao about contractive inequalities between Dirichlet and Hardy spaces and discuss its consequent connection with the Riesz projection.

1. Introduction

As usual, let $\mathbb{D}$ and $\mathbb{T}$ be the open unit disk in the complex plane $\mathbb{C}$ and its boundary, respectively. Let $\mathcal{H}(\mathbb{D})$ denote the set of all holomorphic functions in $\mathbb{D}$. Given $f \in \mathcal{H}(\mathbb{D})$ and an exponent $p > 0$, $M_p(r, f)$ will denote its integral p-mean on the circle centered at the origin with radius r,

$$M_p(r, f) := \Big(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p\, dt\Big)^{1/p}, \quad 0 \le r < 1.$$

We will say that f belongs to the *Hardy space* H^p if its M_p means are uniformly bounded with respect to the radius. It is well known that the radial limit $f(e^{it}) := \lim_{r\to 1^-} f(re^{it})$ exists for almost every $t \in [0, 2\pi]$ if $f \in H^p$. Moreover, these radial limits are p-integrable on $\mathbb{T}$, so we can define the H^p-norm as follows:

$$\|f\|_{H^p} := \Big(\frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^p\, dt\Big)^{1/p}.$$

Of course, $\|\cdot\|_{H^p}$ is a proper norm if $p \ge 1$. Furthermore, in this case H^p can be identified with the subspace of $L^p(\mathbb{T})$ functions with vanishing negative Fourier coefficients. That is,

$$H^p = \{f \in L^p(\mathbb{T}) \mid \widehat{f}(k) = 0, \forall k < 0\},$$

where

$$\widehat{f}(k) := \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\, e^{-ikt}\, dt, \quad k \in \mathbb{Z}.$$

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For more information regarding the elementary properties of Hardy spaces, we refer the reader to [9].

Let $dA(re^{it}) := \frac{r}{\pi}\, dr\, dt$ be the normalized area measure of $\mathbb{D}$. For $0 < p$ and $\alpha > -1$, we define the *standard weighted Bergman space* A^p_α as the set of all analytic functions whose moduli are p-integrable with respect to the measure $(1-|z|^2)^\alpha\, dA(z)$. If $\alpha = 0$, we will simply write A^p instead of A^p_0. It can be checked that, for a fixed f, the quantities

$$\|f\|_{A^p_\alpha} := \Big((\alpha+1)\int_{\mathbb{D}} (1-|z|^2)^\alpha\, |f(z)|^p\, dA(z)\Big)^{1/p}$$

converge to $\|f\|_{H^p}$ when α approaches -1. Thus, we will understand A^p_{-1} as H^p.

For $p = 2$, A^2_α is a Hilbert space and $\|f\|_{A^2_\alpha}$ can be computed in terms of the Taylor coefficients of f. Specifically, we have that

$$\|f\|_{A^2_\alpha} = \Big(\sum_{n=0}^{\infty} \frac{|a_n|^2}{c_{\alpha+2}(n)}\Big)^{1/2},$$

where $(1-z)^{-\beta} = \sum_{n\ge 0} c_\beta(n)\, z^n$ and $f(z) = \sum_{n\ge 0} a_n\, z^n$.

As a companion of A^2_α, we introduce the *weighted Dirichlet space* D_β as the set of all holomorphic functions f such that

$$\|f\|_{D_\beta} := \Big(\sum_{n=0}^{\infty} c_\beta(n)\, |a_n|^2\Big)^{1/2} \tag{1.1}$$

is finite. Throughout this paper, the space D_β should not be confused with the *weighted Besov space* B^2_γ (which is called a Dirichlet-type space in some references like [1] or [8]) consisting of all analytic functions whose derivatives belong to A^2_γ. It follows from the classical Euler–Gauss formula for Euler's Gamma function (see [10], p. 255) that

$$\lim_{n\to\infty} \frac{c_\beta(n)}{n^{\beta-1}} = \frac{1}{\Gamma(\beta)}, \quad \beta \ge 1,$$

and hence $D_\beta = B^2_{2-\beta}$ when $1 \le \beta < 3$, but we maintain the different notation in order to highlight the choice of an alternative norm for B^2_γ, which will be settled later in this section.

Contractive inclusions between spaces of analytic functions have attracted the attention of the experts because of their multiple applications. For this work is especially relevant the following inequality, which was conjectured by Brevig, Ortega-Cerdà, Seip and Zhao [5] and Lieb and Solovej [15], and was recently proved by Kulikov [14].

Theorem A. *Let* $0 < p < q$ *and* $-1 \le \alpha < \beta$ *such that* $(\alpha+2)/p = (\beta+2)/q$. *Then we have that*

$$\|f\|_{A^q_\beta} \le \|f\|_{A^p_\alpha}, \quad \forall f \in A^p_\alpha, \tag{1.2}$$

and equality is possible if and only if there exists $\zeta \in \mathbb{D}$ *and* $C \in \mathbb{C}$ *such that*

$$f(z) = C\, \frac{(1-|\zeta|^2)^{(\alpha+2)/p}}{(1-\bar{\zeta} z)^{2(\alpha+2)/p}}, \quad z \in \mathbb{D}. \tag{1.3}$$

It is worth mentioning that, for $\alpha = -1$ and $q = kp$ being k a positive integer, the inequality (1.2) was already known because of the works of Carleman [7] and Burbea [6].

Although it was not explicitly stated in [14], from Theorem A we can derive a complete characterization of contractive inclusions between weighted Bergman spaces.

Corollary 1.1. *Assume that $A^p_\alpha \subset A^q_\beta$ for some $p, q \in (0, \infty)$ and $\alpha, \beta \geq -1$. Then the inclusion operator $\iota\colon A^p_\alpha \to A^q_\beta$ is contractive if and only if $p < q$ or $p \geq q$ and $\alpha \leq \beta$.*

Proof. The case $q \leq p$ was already known (see for example [16] for a complete characterization for mixed norm spaces). If $p < q$, then the condition $A^p_\alpha \subset A^q_\beta$ yields that $(\alpha + 2)/p \leq (\beta + 2)/q$ (see Theorem 69 in [22]), and consequently,

$$\|f\|_{A^q_\beta} \leq \|f\|_{A^q_{\frac{q}{p}(\alpha+2)-2}} \leq \|f\|_{A^p_\alpha}, \quad \forall f \in A^p_\alpha,$$

proving that the inclusion has norm equal to 1 in this case too. ■

For a suitable choice of p, q, α and β, from (1.2) we can see that

$$\Big(\sum_{n=0}^{\infty} \frac{|a_n|^2}{c_{2/p}(n)} \Big)^{1/2} \leq \|f\|_{H^p},$$

for all $f \in \mathcal{H}(\mathbb{D})$ and $p \in (0, 2]$. For $2 < p$, Brevig, Ortega-Cerdà, Seip and Zhao [5] conjectured the following contractive inequality:

Conjecture B. *If $p > 2$, then the inequality*

$$\|f\|_{H^p} \leq \|f\|_{D_{p/2}}$$

holds for all f analytic function in $\mathbb{D}$.

The main goal of this paper is to prove Conjecture B and to show its connection with the Riesz projection. In fact, we will actually prove the following inequality, which implies the veracity of Conjecture B.

Theorem 1.2. *Let $p > 2$. Then, for all $f \in \mathcal{H}(\mathbb{D})$, we have that*

$$\|f\|_{H^p} \leq \Big(\sum_{n=0}^{\infty} \frac{|a_n|^2}{c_{2/p}(n)} \Big)^{1/2} =: \|f\|_{B^2_{2/p}}. \tag{1.4}$$

2. Proof of Theorem 1.2

For the sake of clarity, $C_{p,n}$ will denote the norm of the restriction of the inclusion operator from $B^2_{2/p}$ into H^p to the subspace of polynomials of degree n. That is,

$$C_{p,n} := \sup \left\{ \Big\| \sum_{k=0}^{n} a_k z^k \Big\|_{H^p} \;\Big|\; \sum_{k=0}^{n} \frac{|a_k|^2}{c_{2/p}(k)} = 1 \right\}.$$

Clearly, (1.4) holds for any holomorphic function f if and only if $C_{p,n} = 1$ for all $n \geq 1$. Observe that $C_{p,n} \geq 1$ for all p and n, since constant functions are considered in the supremum above. For a fixed n, it is trivial that there exists a polynomial Q of degree at most n such that $\|Q\|_{B^2_{2/p}} = 1$ and $\|Q\|_{H^p} = C_{p,n}$. We are going to prove that such polynomials must be unimodular constants.

Theorem 2.1. *Let $p > 2$. Then, for every $n \geq 1$, we have that $C_{p,n} = 1$. Moreover, if Q is a polynomial, then the identity $\|Q\|_{H^p} = \|Q\|_{B^2_{2/p}}$ is possible if and only if Q is constant.*

Proof. Let $n \geq 1$ and take Q a polynomial of degree at most n satisfying that $\|Q\|_{B^2_{2/p}} = 1$ and $\|Q\|_{H^p} = C_{p,n}$. Consider $r \in (0, 1)$ and let $Q_r(z) := Q(rz)$. It is clear that Q_r is also a polynomial of degree at most n, so $\|Q_r\|_{H^p} \leq C_{p,n}\|Q_r\|_{B^2_{2/p}}$.

On the one hand,

$$\|Q_r\|^2_{B^2_{2/p}} = \sum_{k=0}^{n} \frac{r^{2k}\,|a_k|^2}{c_{2/p}(k)} = 1 - \sum_{k=1}^{n} \frac{(1 - r^{2k})\,|a_k|^2}{c_{2/p}(k)},$$

and therefore,

$$\|Q_r\|^p_{B^2_{2/p}} = 1 - \frac{p}{2}\sum_{k=1}^{n} \frac{(1 - r^{2k})\,|a_k|^2}{c_{2/p}(k)} + O((1-r)^2), \quad \text{as } r \to 1^-.$$

On the other hand, it is obvious that $\|Q_r\|^p_{H^p} = M_p^p(r, Q)$. Thus, we have that

$$M_p^p(r, Q) - C^p_{p,n} \leq -C^p_{p,n}\,\frac{p}{2}\sum_{k=1}^{n} \frac{(1 - r^{2k})\,|a_k|^2}{c_{2/p}(k)} + O((1-r)^2), \quad \text{as } r \to 1^-,$$

and then we see that

$$\frac{M_p^p(1, Q) - M_p^p(r, Q)}{1 - r} \geq C^p_{p,n}\,\frac{p}{2}\sum_{k=1}^{n} \frac{1 - r^{2k}}{1 - r}\,\frac{|a_k|^2}{c_{2/p}(k)} + \frac{O((1-r)^2)}{1-r}, \quad \text{as } r \to 1^-.$$

Taking $r \to 1^-$, the *Hardy–Stein identity* (see Theorem 2.18 in [19])

$$\frac{d}{dr}M_p^p(r, f) = \frac{p^2}{2r}\int_{r\mathbb{D}} |f'(z)|^2\,|f(z)|^{p-2}\,dA(z), \quad f \in \mathcal{H}(\mathbb{D}),$$

and the fact that polynomials are entire functions yield that

$$\frac{p}{2}\int_{\mathbb{D}} |Q'(z)|^2\,|Q(z)|^{p-2}\,dA(z) \geq C^p_{p,n}\sum_{k=1}^{n} \frac{k\,|a_k|^2}{c_{2/p}(k)}. \tag{2.1}$$

We are going to show that (2.1) is possible if and only if Q is constant. To this end, assume that $a_{k_0} \neq 0$ for some $k_0 \in \{1, \ldots, n\}$. Since $1 = \frac{p+2}{2p} + \frac{p-2}{2p}$, Hölder's inequality implies that

$$\int_{\mathbb{D}} |Q'(z)|^2\,|Q(z)|^{p-2}\,dA(z) \leq \|Q'\|^2_{A^{4p/(p+2)}}\,\|Q\|^{p-2}_{A^{2p}}.$$

From Theorem A we deduce that

$$\int_{\mathbb{D}} |Q'(z)|^2 \, |Q(z)|^{p-2} \, dA(z) < \|Q'\|^2_{A^2_{2/p-1}} \, \|Q\|^{p-2}_{H^p} = \frac{2C^{p-2}_{p,n}}{p} \sum_{k=1}^{n} \frac{k\,|a_k|^2}{c_{2/p}(k)}$$

because Q is not one of the reproducing kernels (1.3). Hence, (2.1) implies that $C_{p,n} < 1$, but this is obviously not possible. Thus, we deduce that the extremal polynomial Q is constant and then $C_{p,n} = 1$. ∎

Here a couple of relevant remarks are in order. The first is that the space $B^2_{2/p}$ does not appear in this paper on a mere whim. The key property used during the proof of Theorem A in [14] is that the operators

$$T_a f(z) := (\varphi_a'(z))^{(\alpha+2)/p} f(\varphi_a(z)), \quad z \in \mathbb{D},$$

where $\varphi_a(z) := \frac{a-z}{1-\bar{a}z}$, are isometries for all $a \in \mathbb{D}$ both in A^p_α and A^q_β if $(\alpha+2)/p = (\beta+2)/q$. Following the terminology proposed by Aleman and Mas [1], this implies that these spaces have the same kind of *weighted conformal invariance*. It was proven in [1] that, if $p > 2$, the unique Hilbert space (under some reasonable conditions) which has the same weighted conformal invariance as H^p is $B^2_{2/p}$. This is the main reason to consider such weighted Besov spaces.

The second relevant comment is that the norm $\|\cdot\|_{B^2_{2/p}}$ is not the standard norm for the space $B^2_{2/p}$ that we can find in the literature. However, this is the most natural choice in this context since reproducing kernels are also extremal for this norm. In fact, the argument given in the proof above shows that if $f \in \mathcal{H}(\rho\mathbb{D})$ for some $\rho > 1$, then the identity $\|f\|_{H^p} = \|f\|_{B^2_{2/p}}$ holds if and only if f is a reproducing kernel. In addition, we will show that the inclusion of $D_{p/2}$ into $B^2_{2/p}$ is contractive, and therefore it yields that Conjecture B is true.

Lemma 2.2. *If $\beta > 0$, then*

$$\frac{1}{c_\beta(n)} \le c_{1/\beta}(n), \quad \forall n \ge 0.$$

Proof. This inequality is a direct consequence of the monotonicity of the sequence $A_n := c_\beta(n)\,c_{1/\beta}(n)$, $n \ge 0$. Indeed, $\{A_n\}_{n\ge0}$ is increasing since

$$\frac{A_{n+1}}{A_n} = \frac{(n+\beta)(n+1/\beta)}{(n+1)^2} = \frac{n^2 + (\beta+1/\beta)n + 1}{n^2+2n+1} \ge 1,$$

for all $n \ge 0$. ∎

Corollary 2.3. *Let $p > 2$ and $\beta > 0$. Then $\iota\colon D_\beta \to H^p$ is contractive if and only if $\beta \ge p/2$. In particular, Conjecture B is true.*

Proof. The necessity of the condition $\beta \ge p/2$ was already noticed in Section 4 of [5]. Its sufficiency follows from Theorem 1.2, Lemma 2.2 and the monotonicity of the weight $\{c_\beta(n)\}_{n\ge0}$ with respect to β. ∎

It is not difficult to see that, for $p' = p/(p-1)$, $D_{2/p'}$ is included in H^p and this inclusion is critical in the sense that $D_\beta \not\subset H^p$ for any $0 < \beta < 2/p'$. However, Corollary 2.3 shows that the inclusion operator $\iota\colon D_{2/p'} \to H^p$ is not contractive. Nevertheless, a computation similar to the proof of Lemma 2.2 shows the sharp estimates

$$\|f\|_{D_{2/p'}} \leq \|f\|_{B^2_{2/p}} \leq \sqrt{\pi\Big(1-\frac{2}{p}\Big)\csc\Big(\frac{2\pi}{p}\Big)}\,\|f\|_{D_{2/p'}}, \quad p > 2,$$

and hence the set $D_{2/p'}$ *is* contractively included in H^p, but of course when equipped with a suitable equivalent (and somehow natural) norm.

3. Extension to several variables and Dirichlet series

In this section we are going to show that the inequality

$$\|f\|_{H^p} \leq \|f\|_{B^2_{2/p}}$$

naturally induces an analogous inequality in the setting of spaces of Dirichlet series, in the same way that Theorem A extended Helson's inequality [12],

$$\Big(\sum_{k=1}^{n}\frac{|a_k|^2}{d_2(k)}\Big)^{1/2} \leq \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\Big|\sum_{k=1}^{n}\frac{a_k}{k^{it}}\Big|\,dt,$$

to further Hardy spaces of Dirichlet series.

For this purpose, we will need the following notation. For $d \geq 1$ and $p > 2$, we consider the spaces

$$H^p(\mathbb{T}^d) := \{f \in L^p(\mathbb{T}^d) \mid \widehat{f}(k_1,\ldots,k_d) = 0 \text{ if } k_j < 0 \text{ for some } j\}$$

and

$$B^2_{2/p}(\mathbb{D}^d) := \Big\{f \in \mathcal{H}(\mathbb{D}^d) \,\Big|\, \sum_{k_1,\ldots,k_d \geq 0}\frac{|a_{k_1,\ldots,k_d}|^2}{c_{2/p}(k_1)\cdots c_{2/p}(k_d)} < \infty\Big\}.$$

Here of course $\mathbb{T}^d$ and $\mathbb{D}^d$ represent the Cartesian product of d copies of $\mathbb{T}$ and $\mathbb{D}$, respectively. Let $\|\cdot\|_{H^p(\mathbb{T}^d)}$ and $\|\cdot\|_{B^2_{2/p}(\mathbb{D}^d)}$ be the obvious choice of norms for these spaces. As a consequence of the contractivity of $\iota\colon B^2_{2/p} \to H^p$, we can deploy the ingenious argument of Helson [12] (also known as Bonami's lemma because of Lemma 1 in [3]) to show that the inclusion of $B^2_{2/p}(\mathbb{D}^d)$ in $H^p(\mathbb{T}^d)$ is also contractive for all $d \geq 2$ and $p > 2$. Since this technique is considered standard for the experts, we omit the details of the proof.

Theorem 3.1. *Let $d \geq 1$ and $p > 2$. Then we have that*

$$\|f\|_{H^p(\mathbb{T}^d)} \leq \|f\|_{B^2_{2/p}(\mathbb{D}^d)}$$

for all $f \in B^2_{2/p}(\mathbb{D}^d)$.

The Hardy space of Dirichlet series $\mathcal{H}^p$ can be identified with $H^p(\mathbb{T}^\infty)$ by means of the Bohr lift [2], so from Theorem 3.1 we deduce the inequality below.

Corollary 3.2. *If $p > 2$ and $n \geq 1$, we have that*

$$\Big(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \Big| \sum_{k=1}^{n} \frac{a_k}{k^{it}} \Big|^p \, dt\Big)^{1/p} \leq \Big(\sum_{k=1}^{n} \frac{|a_k|^2}{d_{2/p}(k)}\Big)^{1/2},$$

where $\{d_\alpha(n)\}_{n\geq 1}$ is the sequence of coefficients of $\zeta^\alpha(s)$ as a Dirichlet series.

Before concluding this section, we include a comment about the application of these inclusions to the better understanding of some properties of $\mathcal{H}^p$ functions that might result intractable when trying a more direct approach. Indeed, let $\mathcal{B}^2_{2/p}$ and $\mathcal{D}_{p/2}$ be the spaces of all Dirichlet series $f(s) = \sum_{n\geq 1} a_n n^{-s}$ such that the quantities

$$\|f\|^2_{\mathcal{B}^2_{2/p}} := \sum_{n=1}^{\infty} \frac{|a_n|^2}{d_{2/p}(n)} \quad \text{and} \quad \|f\|^2_{\mathcal{D}_{p/2}} := \sum_{n=1}^{\infty} |a_n|^2 d_{p/2}(n),$$

respectively, are finite. Olsen showed in [18] that $\mathcal{D}_{p/2}$ behaves locally like the space $B^2_{2/p}(\mathbb{C}_{1/2})$ of analytic functions in the half-plane $\mathbb{C}_{1/2} := \{s = \sigma + it \mid \sigma > 1/2, t \in \mathbb{R}\}$ with $\lim_{\sigma\to\infty} f(\sigma) = 0$ and

$$\|f\|^2_{B^2_{2/p}(\mathbb{C}_{1/2})} := \int_{\mathbb{C}_{1/2}} |f'(s)|^2 \left(\sigma - \frac{1}{2}\right)^{2/p} dA(s) < \infty.$$

Since the generalized divisor function satisfies that

$$\frac{1}{n}\sum_{k=1}^{n} d_{2/p}(k) = \frac{1}{\Gamma(2/p)} (\log n)^{2/p-1} + O\big((\log n)^{2/p-2}\big)$$

(see for instance (18) in [4]), it follows from [18] that $\mathcal{B}^2_{2/p}$ also has the same local behaviour. In a later work, Seip [21] employed Olsen's results to characterize the bounded zero sets of $\mathcal{D}_{p/2}$ by means of the zero sets of $B^2_{2/p}(\mathbb{C}_{1/2})$.

It is plain that Lemma 2.2 and Corollary 3.2 imply that

$$\|f\|_{\mathcal{H}^p} \leq \|f\|_{\mathcal{D}_{p/2}} \tag{3.1}$$

for all $f \in \mathcal{D}_{p/2}$ and, unlike in the finite dimensional case, there is no $\beta < p/2$ such that $\mathcal{D}_\beta \subset \mathcal{H}^p$. It should be pointed out that it was already known that (3.1) holds if $p = 2k$ (see Lemma 3 in [4]), and interpolation methods showed that, for all $p \in (2,\infty) \setminus 2\mathbb{N}$, there exists a $\beta > p/2$ such that the inclusion of $\mathcal{D}_\beta$ in $\mathcal{H}^p$ is contractive [21]. Thus we can use (3.1) to complete one of the concluding remarks of [21] as follows.

Corollary 3.3. *Let $p > 2$. If $\{s_n\}_{n\geq 1}$ is a bounded zero set of $B^2_{2/p}(\mathbb{C}_{1/2})$, then there exists a non zero $f \in \mathcal{H}^p$ which vanishes on $\{s_n\}_{n\geq 1}$.*

4. Contractive inequalities for the Riesz projection

In this section we are going to expose the connection of Theorem 1.2 with the classical Riesz's projection P_+. We recall that P_+ is defined as

$$P_+F(e^{it}) := \sum_{k=0}^{\infty} \widehat{F}(k)\, e^{ikt}$$

for all $F \in L^1(\mathbb{T})$.

As a consequence of a celebrated result of Riesz [20], P_+ is bounded from $L^q(\mathbb{T})$ to itself for every finite $q > 1$, a fact that was quantified by Hollenbeck and Verbitsky [13] when they proved the following sharp inequality:

$$\|P_+G\|_{H^q} \le \csc\left(\frac{\pi}{q}\right)\|G\|_{L^q}, \quad \forall G \in L^q(\mathbb{T}).$$

Marzo and Seip [17] showed that P_+ is a contractive operator from $L^q(\mathbb{T})$ to $H^{\frac{4q}{q+2}}$ if $2 \le q \le \infty$, and using duality arguments the same can be said from $L^q(\mathbb{T})$ to $H^{\frac{2q}{4-q}}$ if $4/3 \le q \le 2$ [5]. However, the norm of $P_+\colon L^q(\mathbb{T}) \to X$ is not known in general if X is a space of analytic functions containing H^q. In the case that X is a Hardy space, we can find in the literature the conjecture below.

Conjecture C (Brevig, Ortega-Cerdà, Seip and Zhao [5]). *The Riesz projection P_+ is a contraction from $L^{p'}(\mathbb{T})$ to $H^{4/p}$ for all $1 \le p < \infty$, where $p' = p/(p-1)$. If $p = \infty$, we have that*

$$\exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|P_+F(e^{it})|\, dt\right) \le \|F\|_{L^1},$$

for all $F \in L^1(\mathbb{T})$.

It was already known that if Conjecture C holds in the interval $(1,2)$, then Conjecture B is true (Theorem 10 in [5]). Thus Theorem 1.2 can be understood as new evidence in favour of Conjecture C. In fact, we finish this paper showing that Theorem 1.2 has an actual application to the Riesz projection.

Corollary 4.1. *Let $p > 2$ and $\beta > 0$. Then, we have that $P_+\colon L^{p'}(\mathbb{T}) \to D_\beta$ is contractive if and only if $\beta \le 2/p$.*

Proof. First of all, it has to be pointed out that a classical result of Hardy and Littlewood, Theorem 13 in [11], yields that $H^{p'}$ is included in D_β if $\beta \le 2/p$, and therefore the closed graph theorem implies that Riesz projection is bounded from $L^{p'}(\mathbb{T})$ to D_β for all such β.

Consider the test functions

$$F_\varepsilon(e^{it}) := \frac{1 - \varepsilon e^{it}}{(1 - \varepsilon e^{-it})^{1-2/p'}}, \quad 0 < \varepsilon < 1. \tag{4.1}$$

It is reasonable to consider this family of functions here because it shows that, if true, the exponent $4/p$ in Conjecture C cannot be improved, see Theorem 9 in [5]. It is a straightforward computation to check that $\|P_+F_\varepsilon\|_{D_\beta} \le \|F_\varepsilon\|_{L^{p'}}$ for all $\varepsilon \in (0,1)$ implies that $\beta \le 2/p$, and hence we have proved the necessity of this condition.

Take $F \in L^{p'}(\mathbb{T})$. Since the dual of $D_{2/p}$ with respect to the H^2-pairing is $B^2_{2/p}$ and P_+ is self-adjoint, we have that

$$|\langle P_+ F, g\rangle_{H^2}| = |\langle F, g\rangle_{L^2(\mathbb{T})}| \le \|F\|_{L^{p'}} \|g\|_{H^p} \le \|F\|_{L^{p'}} \|g\|_{B^2_{2/p}},$$

for all $g \in B^2_{2/p}$ and hence P_+ is contractive from $L^{p'}(\mathbb{T})$ to $D_{2/p}$. Thus, the monotonicity of the weight $\{c_\beta(n)\}_{n\ge 0}$ with respect to β implies that

$$\|P_+ F\|_{D_\beta} \le \|P_+ F\|_{D_{2/p}} \le \|F\|_{L^{p'}}, \quad \forall F \in L^{p'}(\mathbb{T}),$$

if $\beta \le 2/p$ and then we have completed the proof. ■

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