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Sharp Hardy–Sobolev–Maz'ya, Adams and Hardy–Adams inequalities on quaternionic hyperbolic spaces and on the Cayley hyperbolic plane

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Abstract. The main purpose of this paper is to establish the higher order Poincaré–Sobolev and Hardy–Sobolev–Maz'ya inequalities on quaternionic hyperbolic spaces and on the Cayley hyperbolic plane using the Helgason–Fourier analysis on symmetric spaces. A crucial part of our work is to establish appropriate factorization theorems on these spaces, which can be of independent interest. To this end, we need to identify and introduce the "quaternionic Geller operators" and the "octonionic Geller operators", which have been absent on these spaces. Combining the factorization theorems and the Geller type operators with the Helgason–Fourier analysis on symmetric spaces, some precise estimates for the heat and the Bessel–Green–Riesz kernels, and the Kunze–Stein phenomenon for connected real simple groups of real rank one with finite center, we succeed to establish the higher order Poincaré–Sobolev and Hardy–Sobolev–Maz'ya inequalities on quaternionic hyperbolic spaces and on the Cayley hyperbolic plane. The kernel estimates required to prove these inequalities are also sufficient to establish the Adams and Hardy–Adams inequalities on these spaces. This paper, together with our earlier works on real and complex hyperbolic spaces, completes our study of the factorization theorems, higher order Poincaré–Sobolev, Hardy–Sobolev–Maz'ya, Adams and Hardy–Adams inequalities on all rank one symmetric spaces of noncompact type.

1. Introduction

Let G be a simple Lie group of real rank one. That is, G is one of the four groups $\mathrm{SO}(n,1)$, $\mathrm{SU}(n,1)$, $\mathrm{Sp}(n,1)$ and F_4 (see, e.g., [43,44]). Let K be a maximal compact subgroup of G, and set $\mathbb{X} = G/K$. Then $\mathbb{X}$ is a rank one symmetric space of non-compact type, which is known as the real, complex and quaternionic hyperbolic spaces, and the Cayley hyperbolic plane, which we denote by $H^n_{\mathbb{R}}$, $H^n_{\mathbb{C}}$, $H^n_{\mathbb{Q}}$ and $H^2_{\mathbb{O}}$, respectively. Throughout this paper,

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we let $\Delta_{\mathbb{X}}$ be the Laplace–Beltrami operator of X, and $\rho_{\mathbb{X}}$ will be the half-sum of the positive roots of $\mathbb{X}$. We note that

$$\rho_{\mathbb{X}} = \begin{cases} (n-1)/2, & \text{if } \mathbb{X} = H^n_{\mathbb{R}}, \\ n, & \text{if } \mathbb{X} = H^n_{\mathbb{C}}, \\ 2n+1, & \text{if } \mathbb{X} = H^n_{\mathbb{Q}}, \\ 11, & \text{if } \mathbb{X} = H^2_{\mathbb{O}}, \end{cases}$$

and that $\rho^2_{\mathbb{X}}$ is the spectral gap of $-\Delta_{\mathbb{X}}$.

Our main object of study is the sharp higher order Poincaré–Sobolev and Hardy–Sobolev–Maz'ya inequalities and their borderline cases, the Adams and Hardy–Adams inequalities, on $\mathbb{X}$. The Hardy–Sobolev–Maz'ya inequalities, studied firstly by Maz'ya in [65], combine the Hardy and Sobolev inequalities into a single inequality, that can be stated as follows:

$$\int_{\mathbb{R}^n_+} |\nabla u|^2 \, dx - \frac{1}{4}\int_{\mathbb{R}^n_+} \frac{u^2}{x_1^2} \, dx \geq C_n \Big(\int_{\mathbb{R}^n_+} x_1^{\gamma} |u|^p \, dx \Big)^{2/p}, \quad \text{for } u \in C_0^{\infty}(\mathbb{R}^n_+), n \geq 3,$$

where $2 < p \leq 2n/(n-2)$, $\gamma = (n-2)p/2 - n$, $\mathbb{R}^n_+ = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_1 > 0\}$ and C_n is a positive constant which is independent of u. (See also [24], [73], and [74] and many references therein for Hardy type inequalities in the non-Euclidean setting.) In terms of the half-space model of real hyperbolic spaces, one can see that such an inequality is equivalent to the Poincaré–Sobolev inequality on $H^n_{\mathbb{R}}$. The borderline case of the Hardy–Trudinger–Moser inequality when $n = 2$ has been studied in [59, 72], and when $n > 2$, in [56]. (See also the case when $n = 1$ in [13].) The higher order inequalities of such type, namely the so-called Hardy–Adams inequalities, have been established in [53, 54, 60, 64].

1.1. The case $\mathbb{X} = H^n_{\mathbb{R}}$

We firstly recall the Poincaré half space model and the ball model of $H^n_{\mathbb{R}}$. The Poincaré half space model is given by $\mathbb{R}_+ \times \mathbb{R}^{n-1} = \{(x_1, \dots, x_n) : x_1 > 0\}$ equipped with the Riemannian metric $ds^2 = (dx_1^2 + \cdots + dx_n^2)/x_1^2$. The induced Riemannian measure can be written as $dV = dx/x_1^n$, where dx is the Lebesgue measure on $\mathbb{R}^n$. The ball model is given by the unit ball

$$\mathbb{B}^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : |x| < 1\}$$

equipped with the usual Poincaré metric

$$ds^2 = \frac{4(dx_1^2 + \cdots + dx_n^2)}{(1-|x|^2)^2}.$$

The factorization theorem on $H^n_{\mathbb{R}}$ is given,

- in the ball model (see [57]), by

$$\Big(\frac{1-|x|^2}{2}\Big)^{k+n/2} (-\Delta)^k \Big[\Big(\frac{1-|x|^2}{2}\Big)^{k-n/2} f \Big] = P_k f,$$

- and in the half space model (see [61]), by

$$x_1^{n/2+k}(-\Delta)^k(x_1^{k-n/2}f) = P_k f,$$

where $f \in C^\infty(H^n_{\mathbb{R}})$, Δ is the Laplacian on Euclidean space, $P_1 = -\Delta_{\mathbb{X}} - n(n-2)/4$ and

$$P_k = P_1(P_1+2)\cdot\cdots\cdot(P_1+k(k-1))$$

is the GJMS operator of order $2k$ on $H^n_{\mathbb{R}}$ (see [6, 22, 23, 32, 45]). On the other hand, the Poincaré–Sobolev inequality reads as

$$\int_{H^n_{\mathbb{R}}}(\zeta^2-\rho^2_{\mathbb{X}}-\Delta_{\mathbb{X}})^s(-\rho^2_{\mathbb{X}}-\Delta_{\mathbb{X}})^{\alpha/2}u\cdot u\,dV \geq C\|u\|^2_{L^p(H^n_{\mathbb{R}})},$$

where $0<\alpha<3$, $\zeta>0$ and $u\in C_0^\infty(H^n_{\mathbb{R}})$. Therefore, in terms of the Poincaré half space model and the ball model of $H^n_{\mathbb{R}}$, we have the following Hardy–Sobolev–Maz'ya inequalities of higher order (see [61]).

Theorem A. *Let $2\leq k<n/2$ and $2<p\leq 2n/(n-2k)$. There exists a positive constant $C=C(n,k,p)$ such that, for each $u\in C_0^\infty(H^n_{\mathbb{R}})$,*

$$\int_{H^n_{\mathbb{R}}}(P_k u)u\,dV-\prod_{i=1}^{k}\frac{(2i-1)^2}{4}\int_{H^n_{\mathbb{R}}}u^2\,dV \geq C\Big(\int_{H^n_{\mathbb{R}}}|u|^p\,dV\Big)^{2/p}. \tag{1.1}$$

We mention in passing that the best constant C in the above Hardy–Sobolev–Maz'ya inequalities when $k=1$ and $n=3$ is the same as the Sobolev constant (see [11]), and is otherwise strictly smaller than the Sobolev constant when $k=1$ and $n>3$ (see [33, 34]). In the higher order derivative cases (i.e., for $k\geq 2$), it was proved in all the cases of $n=2k+1$, the best constants are the same as the Sobolev constants [62] (see also [37]), and are strictly less than the Sobolev constant for $n\geq 2k+2$.

In the borderline case, there holds the Hardy–Adams inequality. We state it as follows (see [53, 54, 62]).

Theorem B. *Let $n\geq 3$, $\zeta>0$ and $0<s<3/2$. Then there exists a constant $C_{\zeta,n}>0$ such that for all $u\in C_0^\infty(H^n_{\mathbb{R}})$ with*

$$\int_{H^n_{\mathbb{R}}}(\zeta^2-\rho^2_{\mathbb{X}}-\Delta_{\mathbb{X}})^s\,(-\rho^2_{\mathbb{X}}-\Delta_{\mathbb{X}})^{\alpha/2}u\cdot u\,dV\leq 1.$$

there holds

$$\int_{H^n_{\mathbb{R}}}\left(e^{\beta_0(n/2,n)u^2}-1-\beta_0\,(n/2,n)\,u^2\right)dV\leq C_{\zeta,n},$$

where

$$\beta_0(\alpha,n)=\frac{n}{\omega_{n-1}}\Big[\frac{\pi^{n/2}\,2^\alpha\,\Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}\Big]^{p'},\quad 0<\alpha<n,$$

is the best Adams constant on $\mathbb{R}^n$, and ω_{n-1} is the area of the surface of the unit n-ball.

In terms of the ball model, we have the following Hardy–Adams inequalities on $\mathbb{B}^n$ (see [53, 60, 72].)

Theorem C. *There exists a constant $C > 0$ such that for all $u \in C_0^\infty(\mathbb{B}^n)$ with*

$$\int_{\mathbb{B}^n} |\nabla^{n/2} u|^2 \, dx - \prod_{k=1}^{n/2} (2k-1)^2 \int_{\mathbb{B}^n} \frac{u^2}{(1-|x|^2)^n} \, dx \le 1,$$

there holds

$$\int_{\mathbb{B}^n} \frac{e^{\beta_0(n/2,n) u^2} - 1 - \beta_0(n/2,n) u^2}{(1-|x|^2)^n} \, dx \le C.$$

1.2. The case $\mathbb{X} = H_{\mathbb{C}}^n$

The complex hyperbolic space is a simply connected complete Kaehler manifold of constant holomorphic sectional curvature -4. There are two models of complex hyperbolic space, the Siegel domain model $\mathcal{U}^n$ and the ball model $\mathbb{B}_{\mathbb{C}}^n$. The Siegel domain $\mathcal{U}^n \subset \mathbb{C}^n$ is defined as

$$\mathcal{U}^n := \{z \in \mathbb{C}^n : \varrho(z) > 0\},$$

where

$$\varrho(z) = \operatorname{Im} z_n - \sum_{j=1}^{n-1} |z_j|^2. \tag{1.2}$$

The Bergman metric on $\mathcal{U}^n$ is the metric with Kaehler form $\omega = \frac{i}{2} \partial\bar{\partial} \log \frac{1}{\varrho}$. Its boundary $\partial\mathcal{U}^n := \{z \in \mathbb{C}^n : \varrho(z) = 0\}$ can be identified with the Heisenberg group $\mathbb{H}^{2n-1}$, which is a nilpotent group of step two with the group law

$$(z,t) \circ (z',t') = (z + z', t + t' + 2\operatorname{Im}(z, z')),$$

where $z, z' \in \mathbb{C}^{n-1}$ and (z, z') is the Hermite inner product

$$(z, z') = \sum_{j=1}^{n} z_j \bar{z}'_j.$$

Set $z_j = x_j + i\, y_j (1 \le j \le n-1)$ and define

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t} \quad \text{for } j = 1, \ldots, n-1, \quad \text{and} \quad T = \frac{\partial}{\partial t}\cdot$$

The $2n-1$ vector fields $X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_{n-1}, T$ are left-invariant and form a basis for the Lie algebra of $\mathbb{H}^{2n-1}$. Let

$$\mathcal{L}_0 = \frac{1}{4} \sum_{j=1}^{n-1} (X_j^2 + Y_j^2)$$

be the sub-Laplacian on $\mathbb{H}^{2n-1}$. Then the Laplace–Beltrami operator is given by

$$\Delta_{\mathbb{X}} = 4\varrho \, [\varrho \, (\partial_{\varrho\varrho} + T^2) + \mathcal{L}_0 - (n-1) \, \partial_\varrho].$$

The ball model is given by the unit ball

$$\mathbb{B}_{\mathbb{C}}^{n}=\{z=(z_1,\dots,z_n)\in\mathbb{C}^n : |z|<1\}$$

equipped with the Kaehler metric

$$ds^2=-\partial\bar{\partial}\log(1-|z|^2).$$

The Laplace–Beltrami operator is given by

$$\Delta_{\mathbb{X}}=4(1-|z|^2)\sum_{j,k=1}^{n}(\delta_{jk}-z_j\bar{z}_k)\frac{\partial^2}{\partial z_j\partial\bar{z}_k},$$

where

$$\delta_{j,k}=\begin{cases}1, & \text{if } j=k,\\ 0, & \text{if } j\neq k.\end{cases}$$

The Geller operator $\Delta_{\alpha,\beta}$ is defined by (see [29])

$$\begin{aligned}\Delta_{\alpha,\beta}=4(1-|z|^2)\Big[&\sum_{j,k=1}^{n}(\delta_{jk}-z_j\bar{z}_k)\frac{\partial^2}{\partial z_j\partial\bar{z}_k}\\ &+\alpha\sum_{j=1}^{n}z_j\frac{\partial}{\partial z_j}+\beta\sum_{j=1}^{n}\bar{z}_j\frac{\partial}{\partial\bar{z}_j}-\alpha\beta\Big].\end{aligned}\tag{1.3}$$

Denote by

$$R=\sum_{j=1}^{n}z_j\frac{\partial}{\partial z_j}\quad\text{and}\quad\overline{R}=\sum_{j=1}^{n}\bar{z}_j\frac{\partial}{\partial\bar{z}_j}.$$

Then we have

$$\Delta_{\alpha,\beta}=4(1-|z|^2)\Big[\frac{1-|z|^2}{|z|^2}R\overline{R}-\frac{1}{|z|^2}\mathcal{L}_0'+\frac{n-1}{2}\cdot\frac{1}{|z|^2}(R+\overline{R})+\alpha R+\beta\overline{R}-\alpha\beta\Big],$$

where $\mathcal{L}_0'$ is the Folland–Stein operator ([25, 31]) on the CR sphere, defined as follows:

$$\mathcal{L}_0'=-\frac{1}{2}\sum_{j<k}(M_{jk}\overline{M}_{jk}+\overline{M}_{jk}M_{jk}),\quad\text{where } M_{j,k}=z_j\partial_{\bar{z}_k}-\bar{z}_k\partial_{z_j}.$$

For simplicity, we set

$$\Delta_{\alpha,\beta}'=\frac{1}{4(1-|z|^2)}\Delta_{\alpha,\beta}.$$

These Geller's operators are closely related to CR invariant operators on the Heisenberg group in the works of Jerison and Lee [40–42].

The factorization theorem involving Geller's operators on the complex hyperbolic space plays an important role in establishing both the higher order Poincaré–Sobolev and the Hardy–Sobolev–Maz'ya inequalities on the complex hyperbolic spaces, and can be stated as follows (see [63]).

Theorem D. *Let $a \in \mathbb{R}$ and $k \in \mathbb{N} \setminus \{0\}$. In terms of the Siegel domain model, we have, for $u \in C^\infty(\mathcal{U}^n)$,*

$$\prod_{j=1}^{k} \left[\varrho\,\partial_{\varrho\varrho} + a\,\partial_\varrho + \varrho T^2 + \mathcal{L}_0 - i(k+1-2j)T\right](\varrho^{(k-n-a)/2}u)$$

$$= 4^{-k}\varrho^{-(k+n+a)/2} \prod_{j=1}^{k} \left[\Delta_{\mathbb{X}} + n^2 - (a-k+2j-2)^2\right] u. \tag{1.4}$$

In terms of the ball model, we have, for $f \in C^\infty(\mathbb{B}^n_{\mathbb{C}})$,

$$\prod_{j=1}^{k} \left[\Delta'_{(1-a-n)/2,(1-a-n)/2} + \frac{(k+1-2j)^2}{4} - \frac{k+1-2j}{2}(R-\overline{R})\right][(1-|z|^2)^{(k-n-a)/2} f]$$

$$= 4^{-k}(1-|z|^2)^{-(k+n+a)/2} \prod_{j=1}^{k} \left[\Delta_{\mathbb{X}} + n^2 - (a-k+2j-2)^2\right] f. \tag{1.5}$$

We note that the left sides of (1.4) and (1.5) are closely related to the CR invariant differential operators on the Heisenberg group and CR sphere, respectively.

We also have the following Poincaré–Sobolev inequality on $H^n_{\mathbb{C}}$:

$$\int_{H^n_{\mathbb{C}}} (\zeta^2 - \rho^2_{\mathbb{X}} - \Delta_{\mathbb{X}})^s (-\rho^2_{\mathbb{X}} - \Delta_{\mathbb{X}})^{\alpha/2} u \cdot u \, dV \geq C\|u\|^2_{L^p(H^n_{\mathbb{R}})},$$

where $0 < \alpha < 3$, $\zeta > 0$ and $u \in C_0^\infty(H^n_{\mathbb{C}})$. Therefore, in terms of two models of $H^n_{\mathbb{C}}$, we have the following Hardy–Sobolev–Maz'ya inequalities:

Theorem E. *Let $a \in \mathbb{R}$, $1 \leq k < n$ and $2 < p \leq 2n/(n-k)$. In terms of the Siegel domain model, there exists a positive constant C such that for each $u \in C_0^\infty(\mathcal{U}^n)$, we have*

$$\int_{\mathbb{H}^{2n-1}} \int_0^\infty u \prod_{j=1}^{k} \left[-\varrho\,\partial_{\varrho\varrho} - a\partial_\varrho - \varrho T^2 - \mathcal{L}_0 + i(k+1-2j)T\right] u \, \frac{dz\,dt\,d\varrho}{\varrho^{1-a}}$$

$$- \prod_{j=1}^{k} \frac{(a-k+2j-2)^2}{4} \int_{\mathbb{H}^{2n-1}} \int_0^\infty \frac{u^2}{\varrho^{k+1-a}} \, dz\,dt\,d\varrho$$

$$\geq C\left(\int_{\mathbb{H}^{2n-1}} \int_0^\infty |u|^p \varrho^\gamma \, dz\,dt\,d\varrho\right)^{2/p},$$

where $\gamma = (n-k+a)p/2 - n - 1$. In terms of the ball model, we have for $f \in C_0^\infty(\mathbb{B}^n_{\mathbb{C}})$,

$$\int_{\mathbb{B}^n_{\mathbb{C}}} f \prod_{j=1}^{k} \left[\Delta'_{(1-a-n)/2,(1-a-n)/2} + \frac{(k+1-2j)^2}{4} - \frac{k+1-2j}{2}(R-\overline{R})\right] f \, \frac{dz}{(1-|z|^2)^{1-a}}$$

$$- \prod_{j=1}^{k} \frac{(a-k+2j-2)^2}{4} \int_{\mathbb{B}^n_{\mathbb{C}}} \frac{f^2}{(1-|z|^2)^{k+1-a}} \, dz$$

$$\geq C\left(\int_{\mathbb{B}^n_{\mathbb{C}}} |f|^p (1-|z|^2)^\gamma \, dz\right)^{2/p}.$$

In the borderline case, there holds the Hardy–Adams inequality; we state it as follows.

Theorem F. *Let $n \geq 3$, $\zeta > 0$ and $0 < s < 3/2$. Then there exists a constant $C_{\zeta,n} > 0$ such that for all $u \in C_0^\infty(H_{\mathbb{C}}^n)$ with*

$$\int_{H_{\mathbb{C}}^n} (\zeta^2 - \rho_{\mathbb{X}}^2 - \Delta_{\mathbb{X}})^s (-\rho_{\mathbb{X}}^2 - \Delta_{\mathbb{X}})^{\alpha/2} u \cdot u \, dV \leq 1,$$

there holds

$$\int_{H_{\mathbb{C}}^n} (e^{\beta_0(n,2n)u^2} - 1 - \beta_0(n,2n)u^2) \, dV \leq C_{\zeta,n}.$$

Furthermore, in terms of the Siegel domain model, we have that for all $u \in C_0^\infty(\mathcal{U}^n)$ with

$$4^n \int_{\mathbb{H}^{2n-1}} \int_0^\infty u \prod_{j=1}^{n} \left[-\varrho \partial_{\varrho\varrho} - a\partial_\varrho - \varrho T^2 - \mathcal{L}_0 + i(k+1-2j)T\right] u \, \frac{dz\,dt\,d\varrho}{\varrho^{1-a}}$$
$$- \prod_{j=1}^{n} (a - n + 2j - 2)^2 \int_{\mathbb{H}^{2n-1}} \int_0^\infty \frac{u^2}{\varrho^{n+1-a}} \, dz\,dt\,d\varrho \leq 1,$$

there holds

$$\int_{\mathbb{H}^{2n-1}} \int_0^\infty \frac{e^{\beta_0(n,2n)\varrho^a u^2} - 1 - \beta_0(n,2n)\varrho^a u^2}{\varrho^{n+1}} \, dz\,dt\,d\varrho \leq C.$$

In terms of the ball model, we have that for all $u \in C_0^\infty(\mathbb{B}_{\mathbb{C}}^n)$ with

$$4^n \int_{\mathbb{B}_{\mathbb{C}}^n} f \prod_{j=1}^{n} \left[\Delta'_{(1-a-n)/2,(1-a-n)/2} + \frac{(n+1-2j)^2}{4} - \frac{n+1-2j}{2}(R - \bar{R})\right] f \, \frac{dz}{(1-|z|^2)^{1-a}}$$
$$- \prod_{j=1}^{k} (a - k + 2j - 2)^2 \int_{\mathbb{B}_{\mathbb{C}}^n} \frac{u^2}{(1-|z|^2)^{n+1-a}} \, dz \leq 1,$$

there holds

$$\int_{\mathbb{B}_{\mathbb{C}}^n} \frac{e^{\beta_0(n,2n)(1-|z|^2)^a u^2} - 1 - \beta_0(n,2n)(1-|z|^2)^a u^2}{(1-|z|^2)^{n+1}} \, dz \leq C.$$

1.3. Our main results

In this paper, we will consider the higher order Poincaré–Sobolev and the Hardy–Sobolev–Maz'ya inequalities on the remaining two rank one symmetric spaces of non-compact type, i.e., the quaternionic hyperbolic spaces $H_{\mathbb{Q}}^m$ and the Cayley hyperbolic plane $H_{\mathbb{O}}^2$. The first main result is the factorization theorems. We shall use the NA group model (or Damek–Ricci space) and the ball model. We note (see [3, 19, 20]) that the Damek–Ricci space is a solvable Lie group with a left invariant Riemannian structure which includes all the rank one symmetric spaces of non-compact type.

The Damek–Ricci space NA is a semi-direct product of $A \cong \mathbb{R}$ with a group of Heisenberg type N. Let $\mathfrak{n}$ be a Lie algebra of N, let $\mathfrak{z}$ be the center of $\mathfrak{n}$, and let $\mathfrak{h}$ be its orthogonal complement. Denote by $Q = \frac{1}{2}\dim \mathfrak{h} + \dim \mathfrak{z}$ the homogeneous dimension of N. We parameterize the elements in $N = \exp \mathfrak{n}$ by (X, Z), for $X \in \mathfrak{h}$ and $Z \in \mathfrak{z}$. Then the group law is given by

$$(X, Z)(X', Z') = (X + X', Z + Z' + \tfrac{1}{2}[X, X']).$$

Thus the multiplication in $S = NA$ is given by

$$(X, Z, a)(X', Z', a') = (X + a^{1/2}X', Z + aZ' + \tfrac{1}{2}a^{1/2}[X, X'], aa'), \quad a, a' > 0.$$

Let Δ_Z denote the Euclidean Laplacian on the center of N, and let $\mathcal{L}_0$ denote the sub-Laplacian on N. Let ϱ denote the A-coordinate of a general point in S, and let ∂_ϱ denote the unit vector in the Lie algebra of A. Then the Laplace–Beltrami operator Δ_S on S is given by

$$\Delta_S = 4\varrho\left[\varrho(\partial_{\varrho\varrho} + \Delta_Z) + \mathcal{L}_0 - (Q-1)\partial_\varrho\right]$$

and the bottom of the spectrum of $-\Delta_S$ is Q^2.

Firstly, we establish the factorization theorem on a Damek–Ricci space, from which the factorization theorems on the quaternionic hyperbolic spaces and the Cayley hyperbolic plane follow naturally. We state it as follows.

Theorem 1.1. *Let $a \in \mathbb{R}$ and $f \in C^\infty(\mathcal{U})$. There holds*

$$\varrho^{(k+Q+a)/2}\prod_{j=1}^{k}\left[\varrho\,\partial_{\varrho\varrho} + a\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0 - i(k+1-2j)\sqrt{-\Delta_Z}\right]\left(\varrho^{(k-Q-a)/2}f\right)$$
$$= \prod_{j=1}^{k}\left\{\varrho\left[\varrho(\partial_{\varrho\varrho} + \Delta_Z) + \mathcal{L}_0 - (Q-1)\partial_\varrho\right] + \frac{Q^2}{4} - \frac{(a-k+2j-2)^2}{4}\right\}f.$$

To state the factorization theorem on the ball model of $H^m_{\mathbb{Q}}$, we need to introduce some conventions. First recall that the quaternionic space $\mathbb{Q}^m$ may be identified with $\mathbb{C}^{2m}$ by the correspondence

$$\mathbb{Q}^m \ni q = (q_1, \ldots, q_m) \leftrightarrow \mathbb{C}^{2m} \ni z = (z_1, \ldots, z_{2m}),$$

where $q_j = z_j + z_{m+j}\, i_2$. This allows us to write Δ in terms of the complex coordinates z:

$$\begin{aligned}\Delta_{\mathbb{X}} f(z) = 4(1-|z|^2)\Bigg\{\sum_{i,j=1}^{m}\Bigg(&(\delta_{ij} - z_i\,\bar{z}_j - \bar{z}_{m+i}\,z_{m+j})\,\frac{\partial^2 f}{\partial z_i\,\partial \bar{z}_j} f\\ &+ (\bar{z}_i\,z_{m+j} - z_{m+i}\,\bar{z}_j)\,\frac{\partial^2 f}{\partial z_{m+i}\,\partial \bar{z}_j} + (\bar{z}_{m+i}\,z_j - z_i\,\bar{z}_{m+j})\,\frac{\partial^2 f}{\partial z_i\,\partial \bar{z}_{m+j}}\\ &+ (\delta_{ij} - \bar{z}_i\,z_j - z_{m+i}\,\bar{z}_{m+j})\,\frac{\partial^2 f}{\partial z_{m+i}\,\partial \bar{z}_{m+j}}\Bigg) + R + \overline{R}\Bigg\},\end{aligned}$$

where now

$$R=\sum_{j=1}^{2m} z_j\,\frac{\partial}{\partial z_j}\qquad\text{and}\qquad \overline{R}=\sum_{j=1}^{2m}\bar z_j\,\frac{\partial}{\partial \bar z_j}\cdot$$

We introduce the following "quaternionic Geller operators": given $\alpha\in\mathbb{C}$, define the quaternionic Geller operator

$$\begin{aligned}\Delta_\alpha f(z)=4(1-|z|^2)\bigg\{\sum_{i,j=1}^{m}\Big(&(\delta_{ij}-z_i\,\bar z_j-\bar z_{m+i}\,z_{m+j})\,\frac{\partial^2 f}{\partial z_i\,\partial \bar z_j}f\\ &+(\bar z_i\,z_{m+j}-z_{m+i}\,\bar z_j)\,\frac{\partial^2 f}{\partial z_{m+i}\,\partial\bar z_j}+(\bar z_{m+i}\,z_j-z_i\,\bar z_{m+j})\,\frac{\partial^2 f}{\partial z_i\,\partial\bar z_{m+j}}\\ &+(\delta_{ij}-\bar z_i\,z_j-z_{m+i}\,\bar z_{m+j})\,\frac{\partial^2 f}{\partial z_{m+i}\,\partial \bar z_{m+j}}\Big)+(1+\alpha)(R+\overline{R})-\alpha(\alpha+1)\bigg\}.\end{aligned}$$

In particular, $\Delta_0=\Delta_{\mathbb{X}}$, and if we set

$$\Delta'_\alpha=\frac{1}{4(1-|z|^2)}\,\Delta_\alpha,$$

then

$$\Delta'_\alpha=\Delta'_0+\alpha(R+\overline{R})-\alpha(\alpha+1).$$

We emphasize the analogy between Δ_α and $D_{\alpha,\beta}$ by pointing out the following intertwining relationships: for $u\in C^\infty(\mathbb{B}^n_{\mathbb{C}})$ and $s\in\mathbb{R}$, there holds

$$\Delta_{s-n,s-n}\left[(1-|z|^2)^{s-n}\,u\right]=4^{-1}(1-|z|^2)^{s-n}\,[\Delta_{0,0}+4s(n-s)]\,u\quad\text{on }\mathbb{B}^n_{\mathbb{C}}$$

and, for $u\in C^\infty(\mathbb{B}^m_{\mathbb{Q}})$ and $s\in\mathbb{R}$, there holds

$$\Delta_{s-2m-1}\left[(1-|z|^2)^{s-2m-1}\,u\right]=(1-|z|^2)^{s-2m-1}\,[\Delta_0+4s(2m+1-s)]\quad\text{on }\mathbb{B}^m_{\mathbb{Q}}.$$

Recall that the spectral gaps of $-\Delta_{0,0}$ and $-\Delta_0$ are $(2m+1)^2$ and n^2, respectively. Similarly, we can also define the Geller's operators Δ_α on $H^2_{\mathbb{O}}$ through the intertwining relationships in terms of the ball model,

$$\Delta_\alpha\left[(1-|x|^2)^{s-11}\,u\right]=(1-|x|^2)^{s-11}\,[\Delta_{\mathbb{X}}+4s(11-s)],$$

where 11 is the spectral gap of $-\Delta_{\mathbb{X}}$ on $H^2_{\mathbb{O}}$. Now we can state the factorization theorem on the ball model of $H^m_{\mathbb{Q}}$.

Theorem 1.2. *Let $a\in\mathbb{R}$ and $k\in\mathbb{N}_{>0}$. Set $\Gamma=(R-\overline{R})^2-2D_1\overline{D}_1-2\overline{D}_1D_1$, where*

$$D_1=\sum_{a=1}^{n}\left\{\bar z_a\,\frac{\partial}{\partial z_{n+a}}-\bar z_{n+a}\,\frac{\partial}{\partial z_a}\right\}\quad\textit{and}\quad \overline{D}_1=\sum_{a=1}^{n}\left\{z_a\,\frac{\partial}{\partial \bar z_{n+a}}-z_{n+a}\,\frac{\partial}{\partial \bar z_a}\right\}.$$

Then, in the ball model, for all $f \in C^\infty(\mathbb{B}^m_{\mathbb{Q}})$, *there holds*

$$4^k(1-|z|^2)^{\frac{k+a+(2m+1)}{2}}\prod_{j=1}^{k}\Big[\Delta'_{(1-a-(2m+1))/2}+\frac{(k+1-2j)^2}{4}-i\,\frac{k+1-2j}{2}\sqrt{\Gamma+1}\Big]f$$
$$=\prod_{j=1}^{k}\big[\Delta_{\mathbb{X}}+(2m+1)^2-(a-k+2j-2)^2\big]\big[(1-|z|^2)^{-(k-a-(2m+1))/2}f\big].$$

The factorization theorem on $H^2_{\mathbb{O}}$ in terms of the ball model is more complex than that in $H^m_{\mathbb{Q}}$ and $H^n_{\mathbb{C}}$, and involves rather involved computations. We shall address it in a forthcoming paper.

The second main result is the higher order Poincaré–Sobolev inequality. Using precise estimates for the Bessel–Green–Riesz and the heat kernels, we obtain the following.

Theorem 1.3. *Let* $0<\gamma<3$, $0<\gamma'$, $2<p$ *and* $0<\zeta$. *Call* $N=\dim\mathbb{X}$. *If* $0<\gamma'<N-\gamma$, *suppose further that* $2<p\le\frac{2N}{N-(\gamma+\gamma')}$. *Then there exists a constant* $C>0$ *such that, for all* $u\in C_0^\infty(\mathbb{X})$, *there holds*

$$\|u\|_p\le C\left\|(-\Delta_{\mathbb{X}}-\rho^2_{\mathbb{X}}+\zeta^2)^{\gamma'/4}(-\Delta-\rho^2_{\mathbb{X}})^{\gamma/4}u\right\|_2.$$

Using Theorem 1.3 and the factorization Theorems 1.1 and 1.2, we obtain the following Hardy–Sobolev–Maz'ya inequalities on $\mathbb{X}$. Here we state only for $H^m_{\mathbb{Q}}$.

Theorem 1.4. *Let* $a\in\mathbb{R}$, $1\le k<2m$, $2<p<\frac{4m}{2m-k}$ *and* $\lambda\le\prod_{j=1}^{k}(a-k+2j-2)^2/4$. *Then there exists a constant* $C>0$ *so that, for all* $u\in C_0^\infty(\mathcal{U}^m_{\mathbb{Q}})$, *there holds*

$$\int_{\mathbb{H}^{m-1}_{\mathbb{Q}}}\int_0^\infty u\prod_{j=1}^{k}\Big[-\varrho\,\partial_{\varrho\varrho}-a\partial_\varrho-\varrho\Delta_Z-\mathcal{L}_0-i(k+1-2j)\sqrt{-\Delta_Z}\Big]u\,\frac{dx\,dz\,d\varrho}{\varrho^{1-a}}$$
$$-\lambda\int_{\mathbb{H}^{m-1}_{\mathbb{Q}}}\int_0^\infty\frac{u^2\,dx\,dz\,d\varrho}{\varrho^{k+1-a}}\ge C\Big(\int_{\mathbb{H}^{m-1}_{\mathbb{Q}}}\int_0^\infty|u|^p\varrho^{\frac{(2m+1-k+a)p}{2}-(2m-2)}\,dx\,dz\,d\varrho\Big)^{2/p},$$

where $\mathcal{U}^m_{\mathbb{Q}}$ *is the quaternionic Siegel domain and* $\mathbb{H}^{m-1}_{\mathbb{Q}}$ *is the quaternionic Heisenberg group. In terms of the ball model, for all* $f\in C_0^\infty(\mathbb{B}^m_{\mathbb{Q}})$, *there holds*

$$\int_{\mathbb{B}^m_{\mathbb{Q}}}f\prod_{j=1}^{k}\Big[\Delta'_{(1-a-(2m+1))/2}+\frac{(k+1-2j)^2}{4}-i\,\frac{k+1-2j}{2}\sqrt{\Gamma+1}\Big]f\,\frac{dz}{(1-|z|^2)^{1-a}}$$
$$-\lambda\int_{\mathbb{B}^m_{\mathbb{Q}}}\frac{f^2}{(1-|z|^2)^{k+1-a}}\,dz\ge C\Big(\int_{\mathbb{B}^m_{\mathbb{Q}}}|f|^p(1-|z|^2)^{\frac{(2m+1-k+a)}{2}-(2m-2)}\,dz\Big)^{2/p}.$$

In the limiting case, we can establish the Adams inequality on $\mathbb{X}$.

Theorem 1.5. *Let* $0<\alpha<3$ *and* $\zeta>0$. *Then there exists a constant* $C>0$ *such that, for all* $u\in C_0^\infty(\mathbb{X})$ *with*

$$\|(-\Delta_{\mathbb{X}}-\rho^2_{\mathbb{X}}+\zeta^2)^{(2n-\alpha)/4}(-\Delta_{\mathbb{X}}-\rho^2_{\mathbb{X}})^{\alpha/4}u\|_2\le 1,$$

there holds

$$\int_{\mathbb{X}} (e^{\beta_0\,(N/2,N)\,u^2} - 1 - \beta_0(N/2,N)\,u^2)\,dV \le C.$$

As an application of Theorem 1.5 and the factorization theorem, we have the following Hardy–Adams inequalities on $\mathbb{X}$. We also state them only for $H^m_{\mathbb{Q}}$.

Theorem 1.6. *Let $a \in \mathbb{R}$. There exists a constant $C > 0$ such that, for all $u \in C_0^\infty(\mathbb{B}^m_{\mathbb{Q}})$ with*

$$4^{2m}\int_{\mathbb{B}^n_{\mathbb{Q}}} u\prod_{j=1}^{2m}\Big[\Delta'_{(1-a-(2m+1))/2}+\frac{(2m+1-2j)^2}{4}-i\,\frac{2m+1-2j}{2}\sqrt{\Gamma+1}\Big]\,\frac{u\,dz}{(1-|z|^2)^{1-a}}$$
$$-\prod_{j=1}^{2m}(a-2m+2j-2)^2\int_{\mathbb{B}^n_{\mathbb{Q}}}\frac{u^2}{(1-|z|^2)^{2m+1-a}}\,dz \le 1,$$

there holds

$$\int_{\mathbb{B}^n_{\mathbb{Q}}}\frac{e^{\beta_0(2m,4m)\,(1-|z|^2)^{(a+1)/2}\,u^2} - 1 - \beta_0(2m,4m)\,(1-|z|^2)^{(a+1)/2}\,u^2}{(1-|z|^2)^{2m+2}}\,dz \le C.$$

In terms of the Siegel domain model, we have that, for all $u \in C_0^\infty(\mathcal{U}^n_{\mathbb{Q}})$ with

$$4^{2m}\int_{\mathbb{H}^{m-1}_{\mathbb{Q}}}\int_0^\infty u\prod_{j=1}^{n}\Big[-\varrho\,\partial_{\varrho\varrho} - a\partial_\varrho - \varrho\Delta_Z - \mathcal{L}_0 + i(k+1-2j)\sqrt{-\Delta_Z}\Big]\,u\,\frac{dx\,dz\,d\varrho}{\varrho^{1-a}}$$
$$-\prod_{j=1}^{2m}(a-n+2j-2)^2\int_{\mathbb{H}^{-1}_{\mathbb{Q}}}\int_0^\infty\frac{u^2}{\varrho^{2m+1-a}}\,dx\,dz\,d\varrho \le 1,$$

there holds

$$\int_{\mathbb{H}^{m-1}_{\mathbb{Q}}}\int_0^\infty\frac{e^{\beta_0(2m,4m)\varrho^a u^2} - 1 - \beta_0\,(2m,4m)\,\varrho^a\,u^2}{\varrho^{2m+2}}\,dx\,dz\,d\varrho \le C.$$

Finally, we set up some Adams type inequalities on Sobolev spaces $W^{\alpha,N/\alpha}(\mathbb{X})$ on $\mathbb{X}$ with dimension N, for arbitrary positive fractional order $\alpha < N$. More precisely, we have the following.

Theorem 1.7. *Let $N \ge 2$, let $0 < \alpha < N$ be an arbitrary real positive number, set $p = N/\alpha$, and let ζ satisfy $\zeta > 0$ if $1 < p < 2$, and $\zeta > \rho_{\mathbb{X}}(1/2 - 1/p)$ if $p \ge 2$. Then for measurable E with finite Riemannian volume measure in $\mathbb{X}$, there exists $C = C(\zeta, \alpha, N, |E|)$ such that*

$$\frac{1}{|E|}\int_E \exp(\beta_0(\alpha,N)\,|u|^{p'})\,dV \le C$$

for any $u \in W^{\alpha,p}(\mathbb{X})$ with $\int_{\mathbb{X}} |(-\Delta_{\mathbb{X}} - \rho^2_{\mathbb{X}} + \zeta^2)^{\alpha/2}u|^p\,dV \le 1$. Here $p' = p/(p-1)$. Furthermore, this inequality is sharp in the sense that if $\beta_0(\alpha, N)$ is replaced by any $\beta > \beta_0(\alpha, N)$, the above inequality can no longer hold with some C independent of u.

Theorem 1.8. *Let $N \geq 2$, let $0 < \alpha < N$ be an arbitrary real positive number, set $p = N/\alpha$, and let ζ satisfy $\zeta > 2\rho_{\mathbb{X}} |1/2 - 1/p|$. Then there exists $C = C(\zeta, \gamma, n)$ such that the inequality*

$$\int_{\mathbb{X}} \Phi_p(\beta_0(\alpha, N)|u|^{p'})\, dV \leq C$$

holds simultaneously for any $u \in W^{\alpha,p}(\mathbb{X})$ with $\int_{\mathbb{X}} |(-\Delta_{\mathbb{X}} - \rho_{\mathbb{X}}^2 + \zeta^2)^{\alpha/2} u|^p\, dV \leq 1$. Here

$$\Phi_p(t) = e^t - \sum_{j=0}^{j_p - 2} \frac{t^j}{j!}, \quad \text{with } j_p = \min\{j \in \mathbb{N} : j \geq p\}.$$

Furthermore, this inequality is sharp in the sense that if $\beta_0(\alpha, N)$ is replaced by any $\beta > \beta(2n, \alpha)$, then the above inequality can no longer hold with some C independent of u.

Notice that $|1/2 - 1/p| < 1/2$ provided $p > 1$. Choosing $\zeta = \rho_{\mathbb{X}}$ in Theorem 1.8, we have the following.

Corollary 1.1. *Let $N \geq 2$, let $0 < \alpha < N$ be an arbitrary real positive number, and set $p = N/\alpha$. There exists $C = C(\alpha, n)$ such that the inequality*

$$\int_{\mathbb{X}} \Phi_p(\beta_0(\alpha, N)|u|^{p'})\, dV \leq C$$

holds simultaneously for any $u \in W^{\alpha,p}(\mathbb{X})$ with $\int_{\mathbb{X}} |(-\Delta_{\mathbb{X}})^{\alpha/2} u|^p\, dV \leq 1$.

To summarize, the following remarks are in order. In recent years, the second and third authors of this paper used the Helgason–Fourier analysis techniques on hyperbolic spaces to establish higher order Hardy–Sobolev–Maz'ya inequalities in our earlier works [61] and [62], and Hardy–Adams inequalities with Li in [53, 54, 60] on real hyperbolic spaces, and on complex hyperbolic spaces in [63]. The main purpose of this paper is to establish the higher order Poincaré–Sobolev and Hardy–Sobolev–Maz'ya inequalities on quaternionic hyperbolic spaces and on the Cayley hyperbolic plane using the Helgason–Fourier analysis on symmetric spaces. A crucial part of our work is to establish appropriate factorization theorems on these spaces, which can of independent interest. To this end, we need to identify and introduce the "quaternionic Geller operators" and the "octonionic Geller operators", which have been absent on these spaces. Combining the factorization theorems and the Geller type operators with the Helgason–Fourier analysis on symmetric spaces, some precise estimates for the heat and Bessel–Green–Riesz kernels, and the Kunze–Stein phenomenon for connected real simple groups of real rank one with finite center, we succeed to establish the higher order Poincaré–Sobolev and Hardy–Sobolev–Maz'ya inequalities on quaternionic hyperbolic spaces and on the Cayley hyperbolic plane. The kernel estimates required to prove these inequalities are also sufficient to establish the Adams and Hardy–Adams inequalities on these spaces. This paper, together with our earlier works on higher order Hardy–Sobolev–Maz'ya inequalities on real hyperbolic spaces (see [61, 62]) and Hardy–Adams inequalities on real hyperbolic spaces (see [53, 54, 60]) and on complex hyperbolic spaces [63], completes our study of the factorization theorems, higher order Poincaré–Sobolev, Hardy–Sobolev–Maz'ya, Adams and Hardy–Adams inequalities on all rank one symmetric spaces of noncompact type. The factorization theorems and higher order Poincaré–Sobolev and Hardy–Sobolev–Maz'ya inequalities on general higher rank symmetric spaces of noncompact type will be studied in a forthcoming paper.

The organization of the paper is as follows. In Section 2, we recall some necessary preliminary facts of quaternionic hyperbolic spaces and the Cayley hyperbolic plane. We shall prove the factorization theorem, namely Theorem 1.1 and 1.2, in Section 3. In Section 4, we recall some necessary facts of Funk–Hecke formulas for $\mathrm{Sp}(m) \times \mathrm{Sp}(1)$ and Spin(9), and use them to compute some integrals in term of hypergeometric function. Sharp estimates of Bessel–Green–Riesz kernels and their rearrangement estimates are given in Section 5 and Section 6, respectively. We shall prove the higher order Hardy–Sobolev–Maz'ya inequalities, namely Theorems 1.3 and 1.4, in Section 7. In Section 8, we prove the Hardy–Adams inequality, namely Theorems 1.5 and 1.6. In Appendix A, we show the Adams type inequality, namely Theorems 1.7 and 1.8.

2. Preliminaries

We begin by setting up notations and then recall proper definitions shortly after.

Let $\mathbb{Q}$ and $\mathbb{C}a$ denote, respectively, the quaternions and the Cayley algebra (i.e., octonions). Let $H^m_{\mathbb{Q}}$ denote the quaternionic hyperbolic space of real dimension $4m$, and let $H_{\mathbb{C}a}$ denote the Cayley plane of real dimension 16. In general, we will use $\mathbb{F}$ to denote any of the three normed division algebras $\{\mathbb{C}, \mathbb{Q}, \mathbb{C}a\}$, and $H^m_{\mathbb{F}}$ to denote the corresponding hyperbolic space with $\mathbb{F}$-dimension m. We recall that $H^m_{\mathbb{F}}$ is a Riemannian symmetric space and that, as homogeneous spaces, there hold $H^m_{\mathbb{Q}} = \mathrm{Sp}(m,1)/\mathrm{Sp}(m) \times \mathrm{Sp}(1)$ and $H_{\mathbb{C}a} = F_4/\,\mathrm{Spin}(9)$. Since there is only one Cayley plane, we shall often remove dimensional superscript and subscript decorations whenever specifying $\mathbb{F} = \mathbb{C}a$; for example, $H^m_{\mathbb{F}}$ with $\mathbb{F} = \mathbb{C}a$ shall be written simply as $H_{\mathbb{C}a}$.

We will also use $\mathbb{B}^m_{\mathbb{F}} \subset \mathbb{F}^m$ and $\mathcal{U}^m_{\mathbb{F}}$ to denote $H^m_{\mathbb{F}}$ when realized, respectively, in the Beltrami–Klein ball model and in the Siegel domain model. Let $S^{4m-1} = \partial \mathbb{B}^m_{\mathbb{Q}}$ and $S^{15} = \partial \mathbb{B}_{\mathbb{C}a}$ denote, respectively, the quaternionic and octonionic spheres, and let $d\sigma$ denote the round measure (i.e., the standard surface measure endowed from the ambient Euclidean space). Note that $\mathbb{B}_{\mathbb{C}a} \subset \mathbb{C}a^2 = \mathbb{R}^{16}$.

Next, let $\mathbb{H}^n_{\mathbb{F}}$ denote the Heisenberg group over $\mathbb{F} \in \{\mathbb{C}, \mathbb{Q}, \mathbb{C}a\}$, and let $Z = Z(H^n_{\mathbb{F}})$ denote the center of $\mathbb{H}^n_{\mathbb{F}}$. We make the identifications $\mathbb{H}^n_{\mathbb{C}} = \mathbb{R}^{2n} \times \mathbb{R}$, $\mathbb{H}^n_{\mathbb{Q}} = \mathbb{R}^{4n} \times \mathbb{R}^3$ and $\mathbb{H}_{\mathbb{C}a} = \mathbb{R}^8 \times \mathbb{R}^7$, and note that $Z(\mathbb{H}^n_{\mathbb{C}}) = \mathbb{R}$, $Z(\mathbb{H}^n_{\mathbb{Q}}) = \mathbb{R}^3$ and $Z(\mathbb{H}_{\mathbb{C}a}) = \mathbb{R}^7$. The homogeneous dimension of $\mathbb{H}^n_{\mathbb{F}}$ is given by $Q = \dim_{\mathbb{R}} \mathbb{H}^n_{\mathbb{F}} + \dim_{\mathbb{R}} \mathrm{Im}\,\mathbb{F}$. In particular, the homogeneous dimensions for $\mathbb{H}^n_{\mathbb{C}}$, $\mathbb{H}^n_{\mathbb{Q}}$ and $\mathbb{H}_{\mathbb{C}a}$ are, respectively, $2n + 2$, $4n + 6$ and 22.

Recalling that the boundary of $H^m_{\mathbb{F}}$ has a natural group structure given by $\mathbb{H}^{m-1}_{\mathbb{F}}$, we shall choose the normalization of the metric on $H^m_{\mathbb{F}}$ and sign convention on $\Delta_{\mathbb{X}}$ so that

$$\mathrm{spec}(-\Delta_{\mathbb{X}}) = [Q^2/4, \infty).$$

We recall that $Q/2$ also has the interpretation as $\rho_{\mathbb{X}}$, the half sum of positive roots of $H^m_{\mathbb{F}}$ counted with multiplicities. In particular, on $H^m_{\mathbb{C}}$, $H^m_{\mathbb{Q}}$ and $H_{\mathbb{C}a}$ we evaluate $Q/2$ to be, respectively, m, $2m + 1$, and 11.

For the convenience of the reader, we include a short dictionary of the Laplacians considered in this paper:

Δ	$\leftrightsquigarrow$	Laplace–Beltrami operator on $H^m_{\mathbb{F}}$ when $\mathbb{F} = \mathbb{Q}$ or $\mathbb{C}a$,
$\Delta_{H^n_{\mathbb{R}}}$	$\leftrightsquigarrow$	Laplace–Beltrami operator on $H^n_{\mathbb{R}}$ for a specified n,
Δ_Z	$\leftrightsquigarrow$	Euclidean Laplacian on the center $Z = Z(\mathbb{H}^{m-1}_{\mathbb{F}})$,
Δ_b	$\leftrightsquigarrow$	The sub-Laplacian on $\mathbb{H}^{m-1}_{\mathbb{F}}$.

In the ball model, the Riemannian volume forms on $H^m_{\mathbb{Q}}$ and $H_{\mathbb{C}a}$ are given, respectively, by

$$dV = \frac{dz}{(1-|z|^2)^{2m+2}} \quad \text{and} \quad dV = \frac{dx}{(1-|x|^2)^{12}},$$

where dz and dx denote, respectively, the Lebesgue measure on $\mathbb{C}^m$ and $\mathbb{R}^{16}$.

2.1. Automorphisms and convolution

In this section, we recall a family of automorphisms on $\mathbb{B}^m_{\mathbb{Q}}$ which are isometries and which are used to define convolution on $\mathbb{B}^m_{\mathbb{Q}}$. Analogous automorphisms are also defined for $\mathbb{B}_{\mathbb{C}a}$, but require more notation, and thus we direct the reader to [71], p. 56, for formal definitions.

Following [71], we define for each $w \in \mathbb{B}^m_{\mathbb{Q}}$ the automorphism $\varphi_w : \mathbb{B}^m_{\mathbb{Q}} \to \mathbb{B}^m_{\mathbb{Q}}$ as

$$\varphi_w(z) = (1 - \langle z, w\rangle_{\mathbb{Q}})^{-1} \big(w - P_w(z) - \sqrt{1-|w|^2}\, Q_w(z)\big),$$

where

$$P_w(z) = \begin{cases} \langle z, w\rangle_{\mathbb{Q}} |w|^{-2} w & \text{if } w \neq 0, \\ 0 & \text{if } w = 0, \end{cases} \quad \text{and} \quad Q_w(z) = z - P_w(z).$$

We recall some properties of these automorphisms in the next proposition (see [71]). Note that property (iv) is not present in [71], but it is straightforward to prove.

Lemma A. *For each $w \in \mathbb{B}^m_{\mathbb{Q}}$, the automorphism φ_w satisfies the following properties*:

(i) $\varphi_w(0) = w$ *and* $\varphi_w(w) = 0$;

(ii) *for* $z \in \overline{\mathbb{B}^m_{\mathbb{Q}}}$, *there holds*

$$1 - |\varphi_w(z)|^2 = \frac{(1-|w|^2)(1-|z|^2)}{|1 - \langle z, w\rangle_{\mathbb{Q}}|^2};$$

(iii) φ_w *is an involutory isometry of* $\mathbb{B}^m_{\mathbb{Q}}$;

(iv) *for* $z \in \overline{\mathbb{B}^m_{\mathbb{Q}}}$, *there holds*

$$\sinh(\rho(\varphi_w(z))) = \frac{|\varphi_w(z)|}{\sqrt{1-|\varphi_w(z)|^2}} = \Big(\frac{|z-w|^2 + |\langle z, w\rangle_{\mathbb{Q}}|^2 - |z|^2|w|^2}{(1-|w|^2)(1-|z|^2)}\Big)^{1/2},$$

$$\cosh(\rho(\varphi_w(z))) = \frac{1}{\sqrt{1-|\varphi_w(z)|^2}} = \frac{|1 - \langle z, w\rangle_{\mathbb{Q}}|}{\sqrt{(1-|w|^2)(1-|z|^2)}}.$$

We will use φ_w to also denote the analogous automorphisms on $\mathbb{B}_{\mathbb{C}a}$. We record in the following lemma the analogues to the properties recorded in the preceding lemma. In preparation, if $z = (z_1, z_2)$, $w = (w_1, w_2) \in \mathbb{B}_{\mathbb{C}a} \subset \mathbb{C}a^2$, then let

$$\Psi_{\mathbb{C}a}(z,w) = \begin{cases} |1 - (\bar{z}_1 w_2)(w_2^{-1} w_1) - z_2 \overline{w}_2|^2 & \text{if } w_2 \neq 0, \\ |1 - \bar{z}_1 w_1|^2 & \text{if } w_2 = 0. \end{cases}$$

We also have $\Psi_{\mathbb{C}a}(z,w) = \Phi_{\mathbb{C}a}(z,w) - 2\langle z, w\rangle_{\mathbb{R}} + 1$, where

$$\Phi_{\mathbb{C}a}(z,w) = |z_1|^2 |w_1|^2 + |z_2|^2 |w_2|^2 + 2\Re((z_1 z_2)(\overline{w_1 w_2})),$$

and $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ is the Euclidean inner product on $\mathbb{R}^{16}$. We also remark that $\Phi_{\mathbb{C}a}(z,w)$ is an analogue of the form $|\langle z, w\rangle_{\mathbb{F}}|^2$, and $\Psi_{\mathbb{C}a}(z,w)$ is an analogue of the form $|1 - \langle z, w\rangle_{\mathbb{F}}|^2$, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{Q}\}$. We point out that $\Psi_{\mathbb{C}a}(z,w) \leq |z|^2 |w|^2$.

Lemma B. *For each $w \in \mathbb{B}_{\mathbb{C}a}$, the automorphism φ_w satisfies the following properties*:

(i) $\varphi_w(0) = w$ *and* $\varphi_w(w) = 0$;

(ii) *for $z \in \overline{\mathbb{B}_{\mathbb{C}a}}$, there holds*

$$1 - |\varphi_w(z)|^2 = \frac{(1-|w|^2)(1-|z|^2)}{\Psi_{\mathbb{C}a}(z,w)};$$

(iii) *φ_w is an involutory isometry of* $\mathbb{B}_{\mathbb{C}a}$;

(iv) *for $z \in \overline{\mathbb{B}_{\mathbb{C}a}}$, there holds*

$$\sinh(\rho(\varphi_w(z))) = \frac{|\varphi_w(z)|}{\sqrt{1-|\varphi_w(z)|^2}} = \left(\frac{\Psi_{\mathbb{C}a}(z,w) - (1-|z|^2)(1-|w|^2)}{(1-|w|^2)(1-|z|^2)}\right)^{1/2},$$

$$\cosh(\rho(\varphi_w(z))) = \frac{1}{\sqrt{1-|\varphi_w(z)|^2}} = \frac{\sqrt{\Psi_{\mathbb{C}a}(z,w)}}{\sqrt{(1-|w|^2)(1-|z|^2)}}.$$

With these automorphisms defined, we introduce the following convolution on $\mathbb{B}^m_{\mathbb{F}}$: for two functions f and g on $\mathbb{B}^m_{\mathbb{F}}$, let

$$(f * g)(z) = \int_{\mathbb{B}^m_{\mathbb{F}}} f(\varphi_w(z))\, g(w)\, dV(w),$$

whenever this is well defined. It is easy to see that, if f is radial, then $f * g = g * f$, when defined.

2.2. Helgason–Fourier transform on quaternionic hyperbolic spaces and on the Cayley plane

In this section, we recall the Helgason–Fourier transforms on the quaternionic hyperbolic spaces and on the Cayley plane, as well as the resulting Plancherel and inversion formulas (see [27, 35, 36, 70]). Given a function f on $\mathbb{B}^m_{\mathbb{Q}}$, the Helgason–Fourier transform $\widehat{f}$ is defined by the formula

$$\widehat{f}(\lambda, \varsigma) = \int_{\mathbb{B}^m_{\mathbb{Q}}} f(z)\, e_{-\lambda,\varsigma}(z)\, dV,$$

for $\lambda \in \mathbb{R}$ and $\varsigma \in S^{4m-1}$, provided this integral exists. Here,

$$e_{\lambda,\varsigma}(z) = \Big(\frac{1-|z|^2}{|1-\langle z,\varsigma\rangle_{\mathbb{Q}}|^2}\Big)^{((2m+1)+i\lambda)/2},$$

defined for $z \in \mathbb{B}^m_{\mathbb{Q}}$, $\lambda \in \mathbb{R}$ and $\varsigma \in S^{4m-1}$, are eigenfunctions of Δ with respective eigenvalues $-(2m+1)^2 - \lambda^2$. Note that, for $z \in \mathbb{B}^m_{\mathbb{Q}}$ and $\varsigma \in S^{4m-1}$, the function

$$\Big(\frac{1-|z|^2}{|1-\langle z,\varsigma\rangle_{\mathbb{Q}}}\Big)^{2m+1}$$

is the Poisson kernel on $\mathbb{B}^m_{\mathbb{Q}}$.

Analogously, if f is a function on $\mathbb{B}_{\mathbb{C}a}$, then its Helgason–Fourier transform $\widehat{f}$ is defined by the formula

$$\widehat{f}(\lambda,\varsigma) = \int_{\mathbb{B}^m_{\mathbb{Q}}} f(z)\, e_{-\lambda,\varsigma}(z)\, dV,$$

for $\lambda \in \mathbb{R}$ and $\varsigma \in S^{4m-1}$, provided this integral exists, where now

$$e_{\lambda,\varsigma}(z) = \Big(\frac{1-|z|^2}{\Psi_{\mathbb{C}a}(z,\varsigma)}\Big)^{(11+i\lambda)/2},$$

defined for $z \in \mathbb{B}_{\mathbb{C}a}$, with $\lambda \in \mathbb{R}$ and $\varsigma \in S^{15}$, are eigenfunctions of Δ with respective eigenvalues $-121 - \lambda^2$. Note that, for $z \in \mathbb{B}^m_{\mathbb{Q}}$ and $\varsigma \in S^{4m-1}$, the function

$$\Big(\frac{1-|z|^2}{\Psi_{\mathbb{C}a}(z,\varsigma)}\Big)^{11}$$

is the Poisson kernel on $\mathbb{B}_{\mathbb{C}a}$.

The Helgason–Fourier transform enjoys the following properties:

(i) For $f, g \in C_0^\infty(\mathbb{B}^m_{\mathbb{F}})$ and g radial, there holds

$$\widehat{f * g} = \widehat{f} \cdot \widehat{g}.$$

(ii) For $f \in C_0^\infty(\mathbb{B}^m_{\mathbb{F}})$, there holds the inversion formula

$$f(z) = C_m \int_{-\infty}^{\infty} \int_{S_{\mathbb{F}}} \widehat{f}(\lambda,\varsigma)\, e_{\lambda,\varsigma}(z)\, |\mathfrak{c}(\lambda)|^{-2}\, d\lambda\, d\sigma(\varsigma),$$

where C_m is a positive constant and $\mathfrak{c}(\lambda)$ denotes the Harish-Chandra $\mathfrak{c}$-function; see [35], p. 436, for an explicit formula.

(iii) For $f \in C_0^\infty(\mathbb{B}^m_{\mathbb{F}})$, there holds the Plancherel formula

$$\int_{\mathbb{B}^m_{\mathbb{F}}} |f(z)|^2\, dV = C_m \int_{-\infty}^{\infty} \int_{S_{\mathbb{F}}} |\widehat{f}(\lambda,\varsigma)|^2\, |\mathfrak{c}(\lambda)|^{-2}\, d\lambda\, d\sigma(\varsigma).$$

(iv) For $f \in C_0^\infty(\mathbb{B}^m_{\mathbb{F}})$, there holds

$$\widehat{\Delta f}(\lambda,\varsigma) = -\Big(\lambda^2 + \frac{Q^2}{4}\Big)\widehat{f}(\lambda,\varsigma).$$

3. Factorization theorems for the operators on $\mathbb{X}$: proof of Theorems 1.1 and 1.2

3.1. The factorization theorem on the Damek–Ricci space

Lemma 3.1. *Let $a \in \mathbb{R}$ and $f \in C^\infty(\mathcal{U})$. There holds*

$$\begin{aligned}&\left[\varrho\partial_{\varrho\varrho} + a\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\right]\left(\varrho^{(1-Q-a)/2} f\right)\\&\quad = \varrho^{-(1+Q+a)/2}\left\{\varrho\left[\varrho(\partial_{\varrho\varrho} + \Delta_Z) + \mathcal{L}_0 - (Q-1)\varrho\right] + \frac{Q^2}{4} - \frac{(a-1)^2}{4}\right\} f.\end{aligned}$$

Proof. For reference, we provide explicit computations as follows. Observing that, for any $\beta \in \mathbb{R}$, there holds

$$\begin{aligned}&\varrho^{\beta+1}\left[\varrho\partial_{\varrho\varrho} + a\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\right](\varrho^{-\beta} f)\\&\quad = \varrho\left[\varrho(\partial_{\varrho\varrho} + \Delta_Z) + \mathcal{L}_0 - (2\beta - a)\partial_\varrho\right] f + \beta(\beta + 1 - a) f,\end{aligned}$$

we may choose $\beta = (Q - 1 + a)/2$ to obtain

$$\begin{aligned}&\varrho^{(1+Q+a)/2}\left[\varrho\partial_{\varrho\varrho} + a\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\right]\left(\varrho^{(1-Q-a)/2} f\right)\\&\quad = \left\{\varrho\left[\varrho(\partial_{\varrho\varrho} + \Delta_Z) + \mathcal{L}_0 - (Q-1)\partial_\varrho\right] + \frac{Q^2}{4} - \frac{(a-1)^2}{4}\right\} f.\end{aligned}$$

The desired result follows. ■

Lemma 3.2. *Let $\beta \in \mathbb{R}$. There holds*

$$\begin{aligned}&\left[\varrho\partial_{\varrho\varrho} + (a+\beta)\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\right]\left\{\left[\varrho\partial_{\varrho\varrho} + (a-1)\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\right]^2 + (\beta-1)^2\Delta_Z\right\}\\&\quad = \left\{[\varrho\partial_\varrho + a\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0]^2 + \beta^2\Delta_Z\right\}\left[\varrho\partial_{\varrho\varrho} + (a+\beta-2)\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\right].\end{aligned}$$

Proof. Since

$$\partial_\varrho\left[\varrho\partial_{\varrho\varrho} + (a-1)\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\right] = \left[\varrho\partial_{\varrho\varrho} + a\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\right]\partial_\varrho + \Delta_Z,$$

we have

$$\begin{aligned}&\partial_\varrho\left[\varrho\partial_{\varrho\varrho} + (a-1)\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\right]^2\\&\quad = \left[\varrho\partial_{\varrho\varrho} + a\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\right]\partial_\varrho\left[\varrho\partial_{\varrho\varrho} + (a-1)\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\right]\\&\qquad + \left[\varrho\partial_{\varrho\varrho} + (a-1)\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\right]\Delta_Z\\&\quad = \left[\varrho\partial_{\varrho\varrho} + a\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\right]^2\partial_\varrho + \left[\varrho\partial_{\varrho\varrho} + a\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\right]\Delta_Z\\&\qquad + \left[\varrho\partial_{\varrho\varrho} + (a-1)\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\right]\Delta_Z\\&\quad = \left[\varrho\partial_{\varrho\varrho} + a\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\right]^2\partial_\varrho + 2\left[\varrho\partial_{\varrho\varrho} + a\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\right]\Delta_Z - \Delta_Z\partial_\varrho.\end{aligned}$$

Similarly,

$$
\begin{aligned}
&\big[\varrho\partial_{\varrho\varrho} + (a-1)\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big]^2 \\
&\quad = \big[\varrho\,\partial_{\varrho\varrho} + a\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big]\big[\varrho\partial_\varrho + (a-1)\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big] \\
&\qquad - \partial_\varrho\big[\varrho\,\partial_{\varrho\varrho} + (a-1)\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big] \\
&\quad = \big[\varrho\partial_{\varrho\varrho} + a\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big]^2 - 2\big[\varrho\partial_{\varrho\varrho} + a\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big]\partial_\varrho - \Delta_Z.
\end{aligned}
$$

Combining these two computations, we obtain

$$
\begin{aligned}
&\big[\varrho\,\partial_{\varrho\varrho} + (a+\beta)\,\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big]\big\{\big[\varrho\,\partial_{\varrho\varrho} + (a-1)\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big]^2 + (\beta-1)^2\Delta_Z\big\} \\
&= \big[\varrho\,\partial_{\varrho\varrho} + a\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big]\big\{\big[\varrho\,\partial_{\varrho\varrho} + (a-1)\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big]^2 + (\beta-1)^2\Delta_Z\big\} \\
&\quad + \beta\partial_\varrho\big\{\big[\varrho\,\partial_{\varrho\varrho} + (a-1)\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big]^2 + (\beta-1)^2\Delta_Z\big\} \\
&= \big[\varrho\,\partial_{\varrho\varrho} + a\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big] \\
&\quad \times\big\{\big[\varrho\,\partial_{\varrho\varrho} + a\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big]^2 - 2\big[\varrho\,\partial_{\varrho\varrho} + a\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big]\partial_\varrho + \beta(\beta-2)\Delta_Z\big\} \\
&\quad + \beta\big\{\big[\varrho\partial_{\varrho\varrho} + a\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big]^2\partial_\varrho + 2\big[\varrho\partial_{\varrho\varrho} + a\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big]\Delta_Z \\
&\qquad + \beta(\beta-2)\Delta_Z\partial_\varrho\big\} \\
&= \big\{\big[\varrho\,\partial_{\varrho\varrho} + a\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big]^2 + \beta^2\Delta_Z\big\}\big[\varrho\partial_{\varrho\varrho} + (a+\beta-2)\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big].
\end{aligned}
$$

This provides the desired identity. ■

Lemma 3.3. *For $k \in \mathbb{N}\setminus\{0\}$, there holds*

$$
\begin{aligned}
&\big[\varrho\partial_{\varrho\varrho} + (a+2k)\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big] \\
&\prod_{j=1}^{k}\big\{[\varrho\,\partial_{\varrho\varrho} + (a-1)\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0]^2 + (2j-1)^2\Delta_Z\big\} \\
&\quad = \big(\varrho\,\partial_{\varrho\varrho} + a\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big)\prod_{j=1}^{k}\big\{\big[\varrho\partial_{\varrho\varrho} + (a-1)\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big]^2 + 4j^2\Delta_Z\big\},
\end{aligned}
$$

and

$$
\begin{aligned}
&\big[\varrho\,\partial_{\varrho\varrho} + (a+2k)\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big]\big(\varrho\,\partial_{\varrho\varrho} + a\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big) \\
&\quad\times\prod_{j=1}^{k-1}\big[\big(\varrho\,\partial_{\varrho\varrho} + (a-1)\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big)^2 + 4j^2\Delta_Z\big] \\
&\quad = \prod_{j=1}^{k}\big\{\big[\varrho\partial_{\varrho\varrho} + (a-1)\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big]^2 + (2j-1)^2\Delta_Z\big\}.
\end{aligned}
$$

Proof. By Lemma 3.2, we have

$$\begin{aligned}
&\big[\varrho\partial_{\varrho\varrho} + (a+2k)\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big] \\
&\quad\times\prod_{j=1}^{k}\big\{\big[\varrho\partial_{\varrho\varrho} + (a-1)\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big]^2 + (2j-1)^2\Delta_Z\big\} \\
&\qquad= \big[\varrho\partial_{\varrho\varrho} + (a+2k)\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big] \\
&\qquad\quad\times\big\{\big[\varrho\partial_{\varrho\varrho} + (a-1)\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big]^2 + (2k-1)^2\Delta_Z\big\} \\
&\qquad\quad\times\prod_{j=1}^{k-1}\big\{\big[\varrho\partial_{\varrho\varrho} + (a-1)\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big]^2 + (2j-1)^2\Delta_Z\big\} \\
&\qquad= \big\{\big[\varrho\partial_{\varrho\varrho} + a\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big]^2 + 4k^2\Delta_Z\big\} \\
&\qquad\quad\times\big[\varrho\partial_{\varrho\varrho} + (a+2k-2)\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big] \\
&\qquad\quad\times\prod_{j=1}^{k-1}\big\{\big[\varrho\partial_{\varrho\varrho} + (a-1)\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0\big]^2 + (2j-1)^2\Delta_Z\big\}.
\end{aligned}$$

By repeating this process, we get the first identity in the lemma. The second identity is similarly obtained. ■

Proof of Theorem 1.1. It is sufficient to show

$$\begin{aligned}
&\prod_{j=1}^{k}\big[\varrho\partial_{\varrho\varrho} + a\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0 - i(k+1-2j)\sqrt{-\Delta_Z}\,\big]\big(\varrho^{(k-Q-a)/2}f\big) \\
&= \varrho^{-(k+Q+a)/2}\prod_{j=1}^{k}\Big\{\varrho\big[\varrho(\partial_{\varrho\varrho}+\Delta_Z)+\mathcal{L}_0-(Q-1)\partial_\varrho\big]+\frac{Q^2}{4}-\frac{(a-k+2j-2)^2}{4}\Big\}f.
\end{aligned}$$

We shall prove the theorem by induction. We have that, by Lemma 3.1, the identity above is valid for $k=1$. Now assume it is valid for $k=l$, i.e.,

$$\begin{aligned}
&\prod_{j=1}^{l}\big[\varrho\partial_{\varrho\varrho} + a\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0 - i(l+1-2j)\sqrt{-\Delta_Z}\,\big]\big(\varrho^{(l-Q-a)/2}f\big) \\
&= \varrho^{-(l+Q+a)/2}\prod_{j=1}^{l}\Big\{\varrho\big[\varrho(\partial_{\varrho\varrho}+\Delta_Z)+\mathcal{L}_0-(Q-1)\partial_\varrho\big]+\frac{Q^2}{4}-\frac{(a-l+2j-2)^2}{4}\Big\}f.
\end{aligned}$$

Making the substitution $a \to a-1$, we obtain

$$\begin{aligned}
&\prod_{j=1}^{l}\big[\varrho\partial_{\varrho\varrho} + (a-1)\partial_\varrho + \varrho\Delta_Z + \mathcal{L}_0 - i(l+1-2j)\sqrt{-\Delta_Z}\big]\big(\varrho^{(l-Q-a+1)/2}f\big) \\
&=\varrho^{-(l+Q+a-1)/2}\prod_{j=1}^{l}\Big\{\varrho\big[\varrho(\partial_{\varrho\varrho}+\Delta_Z)+\mathcal{L}_0-(Q-1)\partial_\varrho\big]+\frac{Q^2}{4}-\frac{(a-1-l+2j-2)^2}{4}\Big\}f.
\end{aligned}$$

If l is even, then Lemma 3.3 gives us

$$\begin{aligned}
&\big[\varrho\partial_\varrho + (a+l)\partial_\varrho + \varrho\Delta_Z + \mathscr{L}_0\big] \\
&\qquad \times \prod_{j=1}^{l} \big[\varrho\,\partial_{\varrho\varrho} + (a-1)\partial_\varrho + \varrho\Delta_Z + \mathscr{L}_0 - i(l+1-2j)\sqrt{-\Delta_Z}\,\big] \\
&\quad = \big[\varrho\partial_{\varrho\varrho} + (a+l)\partial_\varrho + \varrho\Delta_Z + \mathscr{L}_0\big] \\
&\qquad \times \prod_{j=1}^{l/2} \big\{\big[\varrho\partial_{\varrho\varrho} + (a-1)\partial_\varrho + \varrho\Delta_Z + \mathscr{L}_0\big]^2 + (2j-1)^2\Delta_Z\big\} \\
&\quad = \big(\varrho\partial_{\varrho\varrho} + a\partial_\varrho + \varrho\Delta_Z + \mathscr{L}_0\big) \prod_{j=1}^{l/2} \big\{\big[\varrho\partial_{\varrho\varrho} + (a-1)\partial_\varrho + \varrho\Delta_Z + \mathscr{L}_0\big]^2 + 4j^2\Delta_Z\big\} \\
&\quad = \prod_{j=1}^{l+1} \big[\varrho\,\partial_{\varrho\varrho} + (a-1)\partial_\varrho + \varrho\Delta_Z - i(l+2-2j)\sqrt{-\Delta_Z}\,\big].
\end{aligned}$$

Therefore, by Lemma 3.1, there holds

$$\begin{aligned}
&\big[\varrho\partial_{\varrho\varrho} + (a+l)\partial_\varrho + \varrho\Delta_Z + \mathscr{L}_0\big]\,\varrho^{-(l+Q+a-1)/2} \\
&\quad \times \prod_{j=1}^{l} \left\{\varrho\big[\varrho(\partial_{\varrho\varrho} + \Delta_Z) + \mathscr{L}_0 - (Q-1)\partial_\varrho\big] + \frac{Q^2}{4} - \frac{(a-l+2j-3)^2}{4}\right\} f \\
&= \varrho^{-(l+Q+a+1)/2} \prod_{j=1}^{l} \left\{\varrho\big[\varrho(\partial_{\varrho\varrho} + \Delta_Z) + \mathscr{L}_0 - (Q-1)\partial_\varrho\big] + \frac{Q^2}{4} - \frac{(a-l+2j-3)^2}{4}\right\} f.
\end{aligned}$$

The case for l is odd is obtained by the second identity in Lemma 3.3. ∎

3.2. The factorization theorem on the ball model of $H^m_{\mathbb{Q}}$

Recall that

$$\begin{aligned}
\Delta_\alpha f(z) = 4(1-|z|^2)\Bigg\{ \sum_{i,j=1}^{m} \bigg(&(\delta_{ij} - z_i\,\bar z_j - \bar z_{m+i}\,z_{m+j})\,\frac{\partial^2 f}{\partial z_i\,\partial \bar z_j} f \\
&+ (\bar z_i\,z_{m+j} - z_{m+i}\,\bar z_j)\,\frac{\partial^2 f}{\partial z_{m+i}\,\partial \bar z_j} + (\bar z_{m+i}\,z_j - z_i\,\bar z_{m+j})\,\frac{\partial^2 f}{\partial z_i\,\partial \bar z_{m+j}} \\
&+ (\delta_{ij} - \bar z_i\,z_j - z_{m+i}\,\bar z_{m+j})\,\frac{\partial^2 f}{\partial z_{m+i}\,\partial \bar z_{m+j}} \bigg) + (1+\alpha)(R+\overline{R}) - \alpha(\alpha+1) \Bigg\}
\end{aligned}$$

and

$$\Delta'_\alpha = \frac{1}{4(1-|z|^2)}\,\Delta_\alpha.$$

It is easy to check that

$$\Delta'_\alpha = \Delta'_\beta + (\alpha-\beta)(R+\overline{R}) + (\beta-\alpha)(\beta+\alpha+1). \tag{3.1}$$

Denote by $r = |z|$ and write

$$\rho = \frac{1}{2} \ln \frac{1+r}{1-r}.$$

Then

$$\cosh \rho = \frac{1}{\sqrt{1-r^2}}, \quad \sinh \rho = \frac{r}{\sqrt{1-r^2}} \quad \text{and} \quad \partial_\rho = (1-r^2)\partial_r.$$

Furthermore, if $f = f(\rho)$, then

$$\Delta f(\rho) = \partial_\rho^2 f + ((4m-1)\coth \rho + 3\tanh \rho)\,\partial_\rho f. \tag{3.2}$$

By using the identity

$$\Delta(fg) = g\Delta f + 2\langle \nabla f, \nabla g\rangle + f\Delta g$$

and (3.2), we have

$$\begin{aligned}\Delta[(\cosh\rho)^a f] &= f\Delta(\cosh\rho)^a + 2\langle\nabla(\cosh\rho)^a, \nabla f\rangle + (\cosh\rho)^a\Delta f\\ &= \left[(4m+a+2)a(\cosh\rho)^a - a(a+2)(\cosh\rho)^{a-2}\right] f\\ &\quad + 2a(\cosh\rho)^{a-1}\sinh\rho\partial_\rho f + (\cosh\rho)^a\Delta f \quad (\because \langle\nabla\rho, \nabla f\rangle = \partial_\rho f),\end{aligned}$$

i.e.,

$$\begin{aligned}[\Delta-(4m+a+2)a][(\cosh\rho)^a f] &= [\Delta-(4m+a+2)a][(1-|z|^2)^{a/2} f]\\ &= (\cosh\rho)^{a-2}\left[(\cosh\rho)^2\Delta + 2a\tanh\rho\partial_\rho - a(a+2)\right] f\\ &= (\cosh\rho)^{a-2}\left[4\Delta_0' + 2ar\partial_r - a(a+2)\right] f\\ &= (\cosh\rho)^{a-2}\left[4\Delta_0' + 2a(R+\overline{R}) - a(a+2)\right] f \quad (\because R+\overline{R} = r\partial_r)\\ &= 4(\cosh\rho)^{a-2}\Delta_{a/2}' f.\end{aligned} \tag{3.3}$$

We are now ready to give the:

Proof of Theorem 1.2. It suffices to show the following:

$$\begin{aligned}4^k(\cosh\rho)^{-k-a-(2m+1)} &\prod_{j=1}^k \left[\Delta'_{(1-a-(2m+1))/2} + \frac{(k+1-2j)^2}{4} - i\,\frac{k+1-2j}{2}\sqrt{\Gamma+1}\right] f\\ &= \prod_{j=1}^k \left[\Delta + (2m+1)^2 - (a-k+2j-2)^2\right]\left[(\cosh\rho)^{k-a-(2m+1)} f\right].\end{aligned}$$

We shall prove this by induction. For $k = 1$, we have, by (3.3),

$$\begin{aligned}&\left[\Delta + (2m+1)^2 - (a-1)^2\right]\left[(\cosh\rho)^{1-a-(2m+1)} f\right]\\ &\qquad = 4(\cosh\rho)^{-1-a-(2m+1)}\,\Delta'_{(1-a-(2m+1))/2} f.\end{aligned} \tag{3.4}$$

Assume it holds for k. Replacing a by $a-1$, we have

$$
\begin{aligned}
&4^k(\cosh\rho)^{-k+1-a-(2m+1)}\\
&\times\prod_{j=1}^{k}\Big[\Delta'_{(2-a-(2m+1))/2}+\frac{(k+1-2j)^2}{4}-i\,\frac{k+1-2j}{2}\sqrt{\Gamma+1}\Big]f \qquad (3.5)\\
&=\prod_{j=1}^{k}\big[\Delta+(2m+1)^2-(a-1-k+2j-2)^2\big]\big[(\cosh\rho)^{k+1-a-(2m+1)}f\big].
\end{aligned}
$$

Then for $k+1$, we have, by using (3.4) and (3.5),

$$
\begin{aligned}
&\prod_{j=1}^{k+1}\big[\Delta+(2m+1)^2-(a-1-k+2j-2)^2\big]\big[(\cosh\rho)^{k+1-a-(2m+1)}f\big]\\
&\quad=\big[\Delta+(2m+1)^2-(a-1+k)^2\big]4^k(\cosh\rho)^{-k+1-a-(2m+1)}\\
&\qquad\times\prod_{j=1}^{k}\Big[\Delta'_{(2-a-(2m+1))/2}+\frac{(k+1-2j)^2}{4}-i\,\frac{k+1-2j}{2}\sqrt{\Gamma+1}\Big]f\\
&\quad=4^{k+1}(\cosh\rho)^{-k-1-a-(2m+1)}\cdot\Delta'_{(1-k-a-(2m+1))/2}\\
&\qquad\times\prod_{j=1}^{k}\Big[\Delta'_{(2-a-(2m+1))/2}+\frac{(k+1-2j)^2}{4}-i\,\frac{k+1-2j}{2}\sqrt{\Gamma+1}\Big]f.
\end{aligned}
$$

The rest of the proof is similar to that given in [63] by using Lemma 3.5, and we omit it. The proof of Theorem 1.2 is thereby completed. ∎

Before the proof of Lemma 3.5, we need the following:

Lemma 3.4. *There holds*

$$[\Delta'_0,[R+\overline{R}]]=\Delta'_0-\frac{1}{2}(R+\overline{R})+\frac{1}{4}(R+\overline{R})^2-\frac{1}{4}\Gamma.$$

Proof. We compute

$$
\begin{aligned}
D_1\overline{D}_1&=(\bar z_j\,\partial_{m+j}-\bar z_{m+j}\,\partial_j)(z_i\,\bar\partial_{m+i}-z_{m+i}\,\bar\partial_i)\\
&=z_i\,\bar z_j\,\bar\partial_{m+i}\,\partial_{m+j}-\bar z_i\,\bar\partial_i-\bar z_j\,z_{m+i}\,\bar\partial_i\,\partial_{m+j}-\bar z_{m+i}\,\bar\partial_{m+i}\\
&\quad-\bar z_{m+j}\,z_i\,\partial_j\,\bar\partial_{m+i}+\bar z_{m+j}\,z_{m+i}\,\partial_j\,\bar\partial_i
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{D}_1D_1&=(z_j\,\bar\partial_{m+j}-z_{m+j}\,\bar\partial_j)(\bar z_i\,\partial_{m+i}-\bar z_{m+i}\,\partial_i)\\
&=z_j\,\bar z_i\,\bar\partial_{m+j}\,\partial_{m+i}-z_i\,\partial_i-z_j\,\bar z_{m+i}\,\bar\partial_{m+j}\,\partial_i-z_{m+i}\,\partial_{m+i}\\
&\quad-z_{m+j}\,\bar z_i\,\bar\partial_j\,\partial_{m+i}+z_{m+j}\,\bar z_{m+i}\,\bar\partial_j\,\partial_i
\end{aligned}
$$

and so

$$-2D_1\overline{D}_1 - 2\overline{D}_1 D_1 = 2(R+\overline{R}) - 4\sum_{j,i=1}^{m}\left(z_i\,\bar z_j\,\bar\partial_{m+i}\,\partial_{m+j} + z_{m+i}\,\bar z_{m+j}\,\bar\partial_i\,\partial_j\right)$$
$$+ 4\sum_{i,j=1}^{m}\left(z_i\,\bar z_{m+j}\,\bar\partial_{m+i}\,\partial_j + z_{m+i}\,\bar z_j\,\bar\partial_i\,\partial_{m+j}\right).$$

A straightforward computation provides

$$\begin{aligned}\frac{1}{2}[\Delta'_\alpha, R+\overline{R}] &= \Delta'_0 - (R+\overline{R}) + R\overline{R} + \sum_{i,j=1}^{m}\bar z_{m+i}\,z_{m+j}\,\partial_i\,\bar\partial_j + \bar z_i\,z_j\,\partial_{m+i}\,\bar\partial_{m+j}\\ &\quad - \sum_{i,j=1}^{m}\bar z_i\,z_{m+j}\,\partial_{m+i}\,\bar\partial_j + \bar z_{m+i}\,z_j\,\partial_i\,\bar\partial_{m+j}\\ &= \Delta'_0 - (R+\overline{R}) + R\overline{R} + \frac{1}{4}\left(2D_1\overline{D}_1 + 2\overline{D}_1 D_1 + 2(R+\overline{R})\right)\\ &= \Delta'_0 - \frac{1}{2}(R+\overline{R}) + R\overline{R} + \frac{1}{2}\left(D_1\overline{D}_1 + \overline{D}_1 D_1\right)\\ &= \Delta'_0 + R\overline{R} - \frac{1}{2}(R+\overline{R}) + \frac{1}{4}\left((R-\overline{R})^2 - \Gamma\right).\end{aligned}$$

The results follows. ∎

By Lemma 3.4, it is easy to check that

$$\begin{aligned}[\Delta'_\alpha, \Delta'_\beta] &= (\alpha-\beta)\,[R+\overline{R}, \Delta'_0]\\ &= 2(\beta-\alpha)\left(\Delta'_0 - \frac{1}{2}(R+\overline{R}) + \frac{1}{4}(R+\overline{R})^2 - \frac{1}{4}\Gamma\right).\end{aligned}$$

We shall frequently use the fact

$$[\Gamma, \Delta'_\alpha] = \Gamma\,\Delta'_\alpha - \Delta'_\alpha\,\Gamma = 0.$$

Lemma 3.5. *There holds*

$$\begin{aligned}&\Delta'_{(1-k-a)/2}\left\{\left[\Delta'_{(2-a)/2} + \frac{(k-1)^2}{4}\right]^2 - \frac{(k-1)^2}{4}\{\Gamma+1\}\right\} f\\ &\qquad = \left\{\left[\Delta'_{(1-a)/2} + \frac{k^2}{4}\right]^2 - \frac{k^2}{4}\{\Gamma+1\}\right\}\Delta'_{(3-k-a)/2} f.\end{aligned}$$

Proof. We compute, by using (3.1) and Lemma 3.4,

$$\begin{aligned}&\Delta'_{(1-k-a)/2}\left[\Delta'_{(2-a)/2} + \frac{(k-1)^2}{4}\right]\\ &= \left(\Delta'_{(1-a)/2} + \frac{k^2}{4} - \frac{k}{2}(R+\overline{R}) + \frac{k}{2}(2-a-k)\right)\end{aligned}$$

$$
\begin{aligned}
&\times\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}+\frac{1}{2}(R+\overline{R})+\frac{a-2-k}{2}\Big)\\
&=\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)^2+\frac{1}{2}\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)(R+\overline{R})-\frac{k}{2}(R+\overline{R})\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)\\
&\quad+\frac{-k^2+(1-a)k+a-2}{2}\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)+\frac{k(2-a)}{2}(R+\overline{R})\\
&\quad+\frac{k(2-a-k)(a-2-k)}{4}\\
&=\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)^2+\frac{1-k}{2}\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)(R+\overline{R})+\frac{k}{2}\Big[\Delta'_{(1-a)/2}+\frac{k^2}{4},R+\overline{R}\Big]\\
&\quad-\frac{k}{4}(R+\overline{R})^2+\frac{-k^2+(1-a)k+a-2}{2}\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)\\
&\quad+\frac{k(2-a)}{2}(R+\overline{R})+\frac{k(2-a-k)(a-2-k)}{4}\\
&=\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)^2+\frac{1-k}{2}\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)(R+\overline{R})+\frac{k}{2}\big[\Delta'_0,R+\overline{R}\big]\\
&\quad-\frac{k}{4}(R+\overline{R})^2+\frac{-k^2+(1-a)k+a-2}{2}\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)\\
&\quad+\frac{k(2-a)}{2}(R+\overline{R})+\frac{k(2-a-k)(a-2-k)}{4}\\
&=\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)^2+\frac{1-k}{2}\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)(R+\overline{R})\\
&\quad+k\Big(\Delta'_0-\frac{1}{2}(R+\overline{R})-\frac{1}{4}\Gamma\Big)+\frac{-k^2+(1-a)k+a-2}{2}\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)\\
&\quad+\frac{k(2-a)}{2}(R+\overline{R})+\frac{k(2-a-k)(a-2-k)}{4}\\
&=\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)^2+\frac{1-k}{2}\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)(R+\overline{R})+k\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)\\
&\quad-\frac{k}{4}(\Gamma+1)+\frac{-k^2+(1-a)k+a-2}{2}\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)\\
&=\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)^2+\frac{1-k}{2}\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)(R+\overline{R})\\
&\quad-\frac{k}{4}(\Gamma+1)+\frac{-k^2+(3-a)k+a-2}{2}\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)\\
&=\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)^2+\frac{1-k}{2}\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)(R+\overline{R})-\frac{k}{4}(\Gamma+1)\\
&\quad-\frac{(k-1)(k+a-2)}{2}\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)\\
&=\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)^2+\frac{1-k}{2}\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)(R+\overline{R}+k+a-2)-\frac{k}{4}(\Gamma+1).
\end{aligned}
$$

Therefore, we have

$$\begin{aligned}
&\Delta'_{(1-k-a)/2}\Big[\Delta'_{(2-a)/2}+\frac{(k-1)^2}{4}\Big]^2-\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)^2\Delta'_{(3-k-a)/2}\\
&\quad=\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)^2\Big(\Delta'_{(2-a)/2}+\frac{(k-1)^2}{4}-\Delta'_{(3-k-a)/2}\Big)\\
&\qquad+\frac{1-k}{2}\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)(R+\overline{R}+k+a-2)\Big[\Delta'_{(2-a)/2}+\frac{(k-1)^2}{4}\Big]\\
&\qquad-\frac{k}{4}(\Gamma+1)\Big[\Delta'_{(2-a)/2}+\frac{(k-1)^2}{4}\Big]\\
&\quad=\frac{k-1}{2}\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)^2(R+\overline{R}+k+a-4)\\
&\qquad-\frac{k-1}{2}\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)(R+\overline{R}+k+a-2)\Big[\Delta'_{(2-a)/2}+\frac{(k-1)^2}{4}\Big]\\
&\qquad-\frac{k}{4}(\Gamma+1)\Big[\Delta'_{(2-a)/2}+\frac{(k-1)^2}{4}\Big]\\
&\quad=\frac{k-1}{2}\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)\Big\{\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)(R+\overline{R}+k+a-4)\\
&\qquad-(R+\overline{R}+k+a-2)\Big[\Delta'_{(2-a)/2}+\frac{(k-1)^2}{4}\Big]\Big\}\\
&\qquad-\frac{k}{4}(\Gamma+1)\Big[\Delta'_{(2-a)/2}+\frac{(k-1)^2}{4}\Big].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)(R+\overline{R}+k+a-4)-(R+\overline{R}+k+a-2)\Big[\Delta'_{(2-a)/2}+\frac{(k-1)^2}{4}\Big]\\
&\quad=\Big[\Delta'_{(1-a)/2}+\frac{k^2}{4},R+\overline{R}+k+a-4\Big]+(R+\overline{R}+k+a-4)\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)\\
&\qquad-(R+\overline{R}+k+a-2)\Big[\Delta'_{(2-a)/2}+\frac{(k-1)^2}{4}\Big]\\
&\quad=[\Delta'_0,R+\overline{R}]+(R+\overline{R}+k+a-2)\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}-\Delta'_{(2-a)/2}-\frac{(k-1)^2}{4}\Big)\\
&\qquad-2\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)\\
&\quad=[\Delta'_0,R+\overline{R}]-2\Delta'_0-\frac{1}{2}(R+\overline{R})^2+(R+\overline{R})-\frac{1}{2}=-\frac{1}{2}(\Gamma+1).
\end{aligned}$$

Combing both above inequalities yields

$$\begin{aligned}\Delta'_{(1-k-a)/2}\Big[\Delta'_{(2-a)/2}+\frac{(k-1)^2}{4}\Big]^2-\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)^2\Delta'_{(3-k-a)/2}\\ =\frac{\Gamma+1}{4}\Big[-(k-1)\Big(\Delta'_{(1-a)/2}+\frac{k^2}{4}\Big)-k\Big(\Delta'_{(2-a)/2}+\frac{(k-1)^2}{4}\Big)\Big]\\ =\frac{\Gamma+1}{4}\big[(k-1)^2\Delta'_{(1-k-a)/2}-k^2\,\Delta'_{(3-k-a)/2}\big].\end{aligned}$$

■

4. Funk–Hecke formulas

4.1. The Funk–Hecke formula for the quaternionic hyperbolic space

The Funk–Hecke formula on the CR sphere was established by Frank and Lieb in [26]. Beckner, following Geller [28], used an independent calculation for the Funk–Hecke formula for bigraded spherical harmonics in his treatment of radial functions on the Heisenberg group [10].

The main source for the following is [14, 15], where they extend Frank and Lieb's formula. We begin by recalling the Funk–Hecke formulas for the quaternionic case. We recall that $L^2(S^{4m-1})$ may be decomposed into the $U(2m)$-irreducibles decomposition

$$L^2(S^{4m-1})=\bigoplus_{j,k\geq 0}\mathcal{H}_{j,k},$$

where $\mathcal{H}_{j,k}$ consists of the Euclidean harmonic homogeneous polynomials in the complex variables $(z,\bar z)$ and of bidegree (j,k). Recalling that $H^m_{\mathbb{Q}}=\mathrm{Sp}(m,1)/\mathrm{Sp}(m)\times\mathrm{Sp}(1)$, the appropriate irreducible decomposition is into $\mathrm{Sp}(m)\times\mathrm{Sp}(1)$-irreducibles, and is given by

$$L^2(S^{4m-1})=\bigoplus_{j\geq k\geq 0}V_{j,k}, \tag{4.1}$$

where $V_{j,k}\subset\mathcal{H}_{j,k}$ are the so-called (j,k)-bispherical harmonic spaces generated by the $\mathrm{Sp}(m)\times\mathrm{Sp}(1)$ action on a zonal harmonic polynomial (see Theorem 3.1 (4) in [44]).

We recall the following quaternionic Funk–Hecke formula of Christ, Liu and Zhang (Lemma 5.4 in [14]). In the following, $P_k^{\alpha,\beta}(t)$ denotes a Jacobi polynomial of degree k.

Theorem G. *Let K be an L^1 integrable function on the unit ball $\mathbb{B}^1_{\mathbb{Q}}$ in $\mathbb{Q}$. Then, any integral operator on S^{4n+3} with kernel given by $K(\langle\zeta,\bar\eta\rangle_{\mathbb{Q}})$ is diagonal with respect to the decomposition (4.1), and the eigenvalue $\lambda_{j,k}(K)$ on $V_{j,k}$ is given by*

$$\begin{aligned}\lambda_{j,k}(K)&=\frac{2\pi^{2n}\,k!}{(j-k+1)!\,(k+2n-1)!}\\&\quad\times\int_0^{\pi/2}(\sin\theta)^{4n-1}(\cos\theta)^{j-k+3}\,P_k^{(2n-1,j-k+1)}(\cos 2\theta)\,d\theta\\&\quad\times\int_{S^3}K(\cos\theta u)\,\frac{\sin(j-k+1)\phi}{\sin\phi}\,du,\end{aligned} \tag{4.2}$$

where $\Re u=\cos\phi$ (with $\phi\in[0,\pi]$), and du is the round measure on $S^3=\partial B^1_{\mathbb{Q}}$.

Using Theorem G, and inspired by the proof of Lemma 5.5 of [14], we obtain the following integral formula, which will be used later.

Proposition 4.1. *If* $-1/2 < \alpha < \infty$ *and* $0 < r < 1$, *then*

$$\int_{S^{4n+3}} \frac{1}{|1-\langle r\xi,\zeta\rangle_{\mathbb{Q}}|^{2\alpha}}\, d\sigma(\eta) = \frac{2\pi^{2n+2}}{(2n+1)!}\, {}_2F_1(\alpha,\alpha-1;2n+2;r^2). \tag{4.3}$$

Proof. Define the kernel $K_r(q) = |1-rq|^{-2\alpha}$ on $\mathbb{B}^1_{\mathbb{Q}}$, and observe that (4.3) may be understood as an integral operator on S^{4n+3} with kernel $K_r(\langle\zeta,\bar\eta\rangle_{\mathbb{Q}})$ applied to the constant function $1 \in V_{0,0}$. Therefore, we may apply the Funk–Hecke formula (4.2) to K_r with $j = k = 0$ to obtain

$$\lambda_{0,0}(K_r) = \frac{8\pi^{2n+1}}{(2n-1)!}\int_0^{\pi/2} \sin^{4n-1}\theta\cos^3\theta\, P_0^{(2n-1,1)}(\cos 2\theta)\, d\theta \\ \times \int_0^{\pi} (1+r^2\cos^2\theta - 2r\cos\phi\cos\theta)^{-\alpha}\sin^2\phi\, d\phi,$$

where we have used that

$$|1-rq|^2 = 1 + r^2|q| - 2\Re q \implies K(\cos\theta u) = |1+r^2\cos^2\theta - 2r\cos\theta\cos\phi|^{-\alpha},$$

and that

$$du = \sin^2\phi\sin\phi'\, d\phi\, d\phi'\, d\phi'', \quad \phi,\phi' \in [0,\pi], \phi'' \in [0,2\pi], \\ \int_0^{\pi}\int_0^{2\pi} \sin\phi'\, d\phi' d\phi'' = 4\pi.$$

Note also that $P_0^{(2n-1,1)} \equiv 1$. Using the cosine integral (see equation (5.11) in [26])

$$\int_{-\pi}^{\pi} (1-2r\cos\phi + r^2)^{-\alpha} e^{i\ell\phi}\, d\phi = \frac{2\pi}{\Gamma^2(\alpha)}\sum_{\mu\ge 0} r^{\ell+2\mu}\frac{\Gamma(\alpha+\mu)\Gamma(\alpha+\ell+\mu)}{\mu!\,(\ell+\mu)!}$$

for $\ell \in \mathbb{N}$, that the integrand in even and that $\sin^2\phi = \frac{1}{2}(1-\cos 2\phi)$, we have

$$\int_0^{\pi} (1+r^2\cos^2\theta - 2r\cos\theta\cos\phi)^{-\alpha}\sin^2\phi\, d\phi \\ = \frac{\pi}{2\Gamma^2(\alpha)}\sum_{\mu\ge 0} r^{2\mu}\cos^{2\mu}\theta\,\frac{\Gamma^2(\alpha)}{(\mu!)^2} - r^{2+2\mu}\cos^{2+2\mu}\theta\,\frac{\Gamma(\alpha)\Gamma(2+\alpha)}{\mu!(\mu+2)!}.$$

Consequently, there holds

$$\lambda_{0,0}(K_r) = \frac{4\pi^{2n+2}}{(2n-1)!\,\Gamma^2(\alpha)}\sum_{\mu\ge 0}\frac{\Gamma(\mu+\alpha)}{\mu!}\Big(r^{2\mu}\frac{\Gamma(\mu+\alpha)}{\mu!}\int_0^{\pi/2}\sin^{4n-1}\theta\cos^{3+2\mu}\theta\, d\theta \\ - r^{2+2\mu}\frac{\Gamma(\mu+\alpha+2)}{(2+\mu)!}\int_0^{\pi/2}\sin^{4n-1}\theta\cos^{5+2\mu}\theta\, d\theta\Big).$$

Letting $t = \cos 2\theta$ and observing that

$$dt = -4\sin\theta\cos\theta\, d\theta, \quad \cos^2\theta = \frac{1}{2}(1+t) \quad \text{and} \quad \sin^2\theta = \frac{1}{2}(1-t),$$

we find

$$\begin{aligned}\int_0^{\pi/2} \sin^{4n-1}\theta \cos^{\ell+3+2\mu}\theta\, d\theta &= \frac{1}{4}\int_{-1}^{1} (\sin^2\theta)^{2n-1} (\cos^2\theta)^{\frac{\ell}{2}+1+\mu}\, dt \\ &= 2^{-2-2n-\mu-\ell/2}\int_{-1}^{1} (1+t)^{\ell/2+2+\mu-1}(1-t)^{2n-1}\, dt \\ &= \frac{1}{2}B\Big(\frac{\ell}{2}+2+\mu, 2n\Big) = \frac{\Gamma(\ell/2+2+\mu)\,\Gamma(2n)}{2\Gamma(\ell/2+2+\mu+2n)},\end{aligned}$$

where $B(x, y)$ is the beta function.

It follows that

$$\begin{aligned}\lambda_{0,0}(K_r) &= \frac{4\pi^{2n+2}}{(2n-1)!\,\Gamma^2(\alpha)}\sum_{\mu\geq 0}\frac{\Gamma(\mu+\alpha)}{\mu!}\Big(r^{2\mu}\frac{\Gamma(\mu+\alpha)}{\mu!}\,\frac{\Gamma(2+\mu)\Gamma(2n)}{2\Gamma(2+\mu+2n)} \\ &\quad - r^{2\mu+2}\frac{\Gamma(\mu+\alpha+2)}{(2+\mu)!}\,\frac{\Gamma(3+\mu)\Gamma(2n)}{2\Gamma(3+\mu+2n)}\Big) \\ &= \frac{2\pi^{2n+2}}{\Gamma^2(\alpha)}\Bigg(\sum_{\mu\geq 1}\Big[\frac{\Gamma^2(\mu+\alpha)(\mu+1)!}{(\mu!)^2(\mu+1+2n)!} - \frac{\Gamma(\mu-1+\alpha)\Gamma(\mu+\alpha+1)}{(\mu-1)!(\mu+1+2n)!}\Big]r^{2\mu} \\ &\quad + \frac{\Gamma^2(\alpha)}{(2n+1)!}\Bigg) \\ &= (\alpha-1)\frac{2\pi^{2n+2}}{\Gamma^2(\alpha)}\sum_{\mu\geq 0}\frac{\Gamma(\mu+\alpha)\Gamma(\mu-1+\alpha)}{\mu!(\mu+1+2n)!}r^{2\mu} \\ &= \frac{2\pi^{2n+2}}{(2n+1)!}\sum_{\mu\geq 0}\frac{(\alpha)_\mu(\alpha-1)_\mu}{(2n+2)_\mu}\frac{r^{2\mu}}{\mu!} = \frac{2\pi^{2n+2}}{(2n+1)!}\,{}_2F_1(\alpha, \alpha-1; 2n+2; r^2).\end{aligned}$$

This is the desired identity. ■

4.2. The Funk–Hecke formula for the Cayley hyperbolic plane

We now discuss the Funk–Hecke formula for the octonionic case. We recall that $L^2(S^{15})$ may be decomposed into the Spin(9)-irreducible decomposition

$$L^2(S^{15}) = \bigoplus_{j\geq k\geq 0} W_{j,k} \tag{4.4}$$

where $W_{j,k}$ is the so-called (j,k)-bispherical harmonic subspace, which is a finite dimensional space spanned by elements from the cyclic action of Spin(9) on zonal harmonics $Z_{j,k}(\zeta)$ (see [44] or equation 2.12 in [15] for a precise formula).

We point out that the Funk–Hecke formula given in [15] assumes the kernel function K is of the form $K(\zeta\cdot\bar\eta)$, where, if $\zeta=(\zeta_1,\zeta_2),\eta=(\eta_1,\eta_2)\in\mathbb{C}a^2$, then $\zeta\cdot\bar\eta=\zeta_1\bar\eta_1+\zeta_2\bar\eta_2$. Consideration of these kinds of kernel functions arose from their consideration of the natural distance function $|1-\zeta\cdot\bar\eta|$ on the sphere S^{15}. However, taking into consideration the geometry of the Cayley plane $H_{\mathbb{C}a}$ and the non-associativity of $\mathbb{C}a$, it is more appropriate for our purposes to consider kernels of the form $K(\Phi_{\mathbb{C}a}(\zeta,\eta))$ or $K(\Psi_{\mathbb{C}a}(\zeta,\eta))$, since $\Phi_{\mathbb{C}a}(\zeta,\eta)$ and $\Psi_{\mathbb{C}a}(\zeta,\eta)$ are octonionic analogues of $|\langle\cdot,\cdot\rangle_{\mathbb{F}}|^2$ and $|1-\langle\cdot,\cdot\rangle_{\mathbb{F}}|^2$, respectively. As a result, we will establish the following Funk–Hecke formulas, which are more suitable for our purposes.

Theorem 4.1. *Suppose $K(\Phi_{\mathbb{C}a}(\zeta,\eta))$ is such that the integral below exists. Then the integral operator with kernel $K(\Phi_{\mathbb{C}a}(\zeta,\eta))$ is diagonal with respect to the bispherical decomposition harmonic decomposition (4.4), and the eigenvalue on $W_{j,k}$ is given by*

$$\begin{aligned}\lambda_{j,k}(K)=\frac{15\pi^4k!}{(k+3)!}\int_0^{\pi/2}&\cos^{j-k+7}\theta\sin^7\theta\,P_k^{(3,3+j-k)}(\cos 2\theta)\,d\theta\\ &\times\int_S K(\Psi_{\mathbb{C}a}((1,0),(\bar u\cos\theta,0)))\big(a_{j,k}^0\cos(j-k)\phi+a_{j,k}^1\cos(j-k+2)\phi\\ &\quad+a_{j,k}^2\cos(j-k+4)\phi+a_{j,k}^3\cos(j-k+6)\phi\big)\,du,\end{aligned}$$

where $\Re u=\cos\phi$ (with $\phi\in[0,\pi)$), du is the standard surface measure on S (the unit sphere in $\mathbb{C}a$), $P_k^{(3,3+j-k)}(z)$ is the Jacobi polynomial of order k associated to the weight $(1-z)^3(1+z)^{3+j-k}$, and

$$\begin{aligned}a_{j,k}^0&=\frac{1}{8}\frac{1}{j-k+3}-\frac{1}{4}\frac{1}{j-k+2}+\frac{1}{8}\frac{1}{j-k+1},\\ a_{j,k}^1&=\frac{3}{8}\frac{1}{j-k+3}-\frac{1}{4}\frac{1}{j-k+4}-\frac{1}{8}\frac{1}{j-k+1},\\ a_{j,k}^2&=-\frac{3}{8}\frac{1}{j-k+3}+\frac{1}{4}\frac{1}{j-k+2}+\frac{1}{8}\frac{1}{j-k+5},\\ a_{j,k}^3&=-\frac{1}{8}\frac{1}{j-k+3}+\frac{1}{4}\frac{1}{j-k+4}-\frac{1}{8}\frac{1}{j-k+5}.\end{aligned}$$

Proof. Since a portion of the proof is the same as the proof of Lemma 3.3 in [15], we shall only point out the needed adaptation.

We have from Schur's lemma and the irreducibility of the $W_{j,k}$ that the integral operator with kernel $K(\Psi_{\mathbb{C}a}(\zeta,\eta))$ is diagonal. Let $\lambda_{j,k}$ denote the eigenvalue corresponding to the subspace $W_{j,k}$. Letting $Y_{j,k}^{\mu}$, $1\le\mu\le\dim W_{j,k}$, be a normalized orthogonal basis of $W_{j,k}$, we then have

$$\int_{S^{15}}K(\Psi_{\mathbb{C}a}(\zeta,\eta))\,Y_{j,k}^{\mu}(\eta)d\sigma=\lambda_{j,k}\,Y_{j,k}^{\mu}(\zeta).$$

Letting

$$Z_{j,k}(\zeta,\eta)=Z_{j,k}(\zeta\cdot\bar\eta)=\sum_{\mu=1}^{\dim W_{j,k}}Y_{j,k}^{\mu}(\zeta)\,\overline{Y_{j,k}^{\mu}(\eta)}$$

be the reproducing kernel of the projection onto $W_{j,k}$, we have

$$\int_{S^{15}} K(\Psi_{\mathbb{C}a}(\zeta,\eta))\, Z_{j,k}(\eta\cdot\bar{\zeta})\delta\eta = \lambda_{j,k}\, Z_{j,k}(1).$$

Here $Z_{j,k}(1)$ denotes the aforementioned zonal harmonic $Z_{j,k}(\zeta)$ evaluated at $\zeta = 1$. All that is needed now is to observe that $K(\Psi_{\mathbb{C}a}(\zeta,\eta))$ and $Z_{j,k}(\eta,\zeta)$ are invariant under the action of Spin(9). Indeed, if this were the case, then we would obtain

$$\begin{aligned}\lambda_{j,k} &= Z_{j,k}(1)^{-1}\int_{S^{15}} K(\Psi_{\mathbb{C}a}(\zeta,\eta))\, Z_{j,k}(\eta\cdot\bar{\zeta})\, d\sigma \\ &= Z_{j,k}(1)^{-1}\int_{S^{15}} K(\Psi_{\mathbb{C}a}((1,0),\eta))\, Z_{j,k}((1,0),\eta)\, d\eta,\end{aligned}$$

The remainder of the proof would follow as the proof of Lemma 3.3 in [15].

That the kernel $K(\Psi_{\mathbb{C}}(\zeta,\eta))$ is Spin(9)-invariant follows from the Spin(9)-invariance of $\Phi_{\mathbb{C}a}(\zeta,\eta)$. Therefore,

$$\int_{S^{15}} Z_{j,k}(A\zeta, A\eta)\, Y^{\mu}_{j,k}(\eta)\, d\eta = \int_{S^{15}} Z_{j,k}(A\zeta,\eta)\, Y^{\mu}_{j,k}(A^{-1}\eta)\, d\eta = Y^{\mu}_{j,k}(\zeta),$$

which shows that $Z_{j,k}(A\zeta, A\eta) = Z_{j,k}(\zeta,\eta)$ by the uniqueness of the representation of a linear functional. ■

Lastly, we state and prove the octonionic analogue of Proposition 4.1.

Proposition 4.2. *If* $-1/2 < \alpha < \infty$ *and* $0 < r < 1$, *then*

$$\int_{S^{4n+3}} \frac{1}{\Psi_{\mathbb{C}a}(r\xi,\zeta)^{\alpha}}\, d\sigma(\eta) = \frac{2\pi^8}{7!}\, {}_2F_1(\alpha, \alpha-3; 8; r^2).$$

Proof. The proof follows similarly to the proof of Proposition 4.1 by applying Theorem 4.1 to the kernel $\Psi_{\mathbb{C}a}(r\xi,\eta)^{-\alpha}$. It should be pointed out that

$$\Psi_{\mathbb{C}a}((r,0),(\bar{u}\cos\theta,0)) = 1 - 2r\cos\phi\cos\theta + r^2\cos^2\theta,$$

since $\Re u = \cos\phi$. ■

5. Kernel estimates

We recall that the heat kernel $e^{t\Delta}$ on $H^m_{\mathbb{Q}}$ is given by the following formula:

$$\begin{aligned} e^{t\Delta} = c_m\, t^{-1/2}\, e^{-(2m+1)^2 t} \int_{\rho}^{\infty} &\frac{\sinh 2r}{\sqrt{\cosh 2r - \cosh 2\rho}}\Big(-\frac{1}{\sinh 2r}\frac{\partial}{\partial r}\Big)^2 \\ &\times \Big(-\frac{1}{\sinh r}\frac{\partial}{\partial r}\Big)^{2m-2} e^{-\frac{r^2}{4t}}\, dr,\end{aligned}$$

where $c_m = 2^{-2m+3/2}\,\pi^{-2m-1/2}$.

The heat kernel $e^{t\Delta}$ on $H_{\mathbb{C}a}$ is given by

$$e^{t\Delta} = c_o\, t^{-1/2}\, e^{-11^2 t} \int_\rho^\infty \frac{\sinh 2r}{\sqrt{\cosh 2r - \cosh 2\rho}} \Big(- \frac{1}{\sinh 2r} \frac{\partial}{\partial r} \Big)^4 \Big(- \frac{1}{\sinh r} \frac{\partial}{\partial r} \Big)^4 e^{-\frac{r^2}{4t}}\, dr,$$

where $c_o = 2^{-9/2} \pi^{-17/2}$.

Letting $h_t(\rho, 2\tilde{m}+1)$ denote the heat kernel on the odd dimensional real hyperbolic space $H_{\mathbb{R}}^{2\tilde{m}+1}$, we recall also that

$$h_t(\rho, 2\tilde{m}+1) = b_{\tilde{m}}\, t^{-1/2}\, e^{-\tilde{m}^2 t} \Big(- \frac{1}{\sinh \rho} \frac{\partial}{\partial \rho} \Big)^{\tilde{m}} e^{-\frac{\rho^2}{4t}}, \tag{5.1}$$

where $b_{\tilde{m}} = 2^{-\tilde{m}-1} \pi^{-\tilde{m}-1/2}$. See for example [21], [5] and [58] for these formulas.

It will be useful to write $e^{t\Delta}$ in terms of h_t, and this can be done as follows. We consider $H_{\mathbb{Q}}^m$ first. Observe that, if $\tilde{m} = 2m-2$, then

$$e^{-(2m+1)^2 t} = e^{(-12m+3)t}\, e^{-\tilde{m}^2 t},$$

and so,

$$\begin{aligned} e^{t\Delta} = \frac{c_m}{b_{2m-2}} \int_\rho^\infty & \frac{\sinh 2r}{\sqrt{\cosh 2r - \cosh 2\rho}} \Big(- \frac{1}{\sinh 2r} \frac{\partial}{\partial r} \Big)^2 \\ & \times e^{(-12m+3)t}\, h_t(r, 4m-3)\, dr. \end{aligned} \tag{5.2}$$

Similarly, on $H_{\mathbb{C}a}$, there holds (by setting $\tilde{m} = 4$)

$$e^{t\Delta} = \frac{c_o}{b_4} \int_\rho^\infty \frac{\sinh 2r}{\sqrt{\cosh 2r - \cosh 2\rho}} \Big(- \frac{1}{\sinh 2r} \frac{\partial}{\partial r} \Big)^4 e^{-105 t}\, h_t(r, 9)\, dr. \tag{5.3}$$

We now recall the Bessel–Green–Riesz functions. For the sake of notational convenience, we write

$$\begin{aligned} k_{\zeta,\gamma} &= \Big(-\Delta - \frac{Q^2}{4} + \zeta^2 \Big)^{-\gamma/2} && \text{for } 0 < \gamma < \dim_{\mathbb{R}} H_{\mathbb{F}}^m \text{ and } \zeta > 0, \\ k_\gamma &= \Big(-\Delta - \frac{Q^2}{4} \Big)^{-\gamma/2} && \text{for } 0 < \gamma < 3. \end{aligned}$$

In (iii) in page 1083 of [4], Anker and Ji established the following asymptotics for $k_{\zeta,\gamma}$ and k_γ:

$$\begin{aligned} k_{\zeta,\gamma} &\sim \rho^{(\gamma-2)/2}\, e^{-\zeta\rho - Q\rho/2} && \text{for } \rho \ge 1, \\ k_\gamma &\sim \rho^{\gamma-2}\, e^{-Q\rho/2} && \text{for } \rho \ge 1. \end{aligned} \tag{5.4}$$

We will need several technical lemmas to obtain small distance estimates of k_ζ. We state them now. The first estimate is a small distance estimate for the Bessel–Green–Riesz kernel on the real hyperbolic space $H_{\mathbb{R}}^k$ (see Lemma 3.2 in [54]).

Lemma C. *Let* $k \geq 3$ *and* $0 < \gamma < 3$. *If* $0 < \rho < 1$, *then*

$$\Big(-\Delta_{H^k_{\mathbb{R}}} - \Big(\frac{k-1}{2}\Big)^2\Big)^{-\gamma/2} = \frac{1}{\gamma_k(\gamma)}\,\frac{1}{\rho^{k-\gamma}} + O\Big(\frac{1}{\rho^{k-\gamma-1}}\Big),$$

where

$$\gamma_k(\gamma) = \frac{\pi^{k/2}\,2^{\gamma}\,\Gamma(\gamma/2)}{\Gamma((k-\gamma)/2)}.$$

The next lemma is an exact evaluation of a hyperbolic trigonometric integral (see Lemma 4.1 in [63]). We include a sketch of the proof here.

Lemma D. *Let* $\beta > 0$ *and* $\rho > 0$. *Then*

$$\int_\rho^\infty \frac{\cosh r}{(\sinh r)^\beta}\,\frac{1}{\sqrt{\cosh 2r - \cosh 2\rho}}\,dr = \frac{\Gamma(1/2)\,\Gamma(\beta/2)}{2\sqrt{2}\,\Gamma((1+\beta)/2)}\,\frac{1}{(\sinh\rho)^\beta}.$$

Proof. Substituting $t = \cosh 2r - \cosh 2\rho$, we have

$$\begin{aligned}\int_\rho^{+\infty} \frac{\cosh r}{(\sinh r)^\beta\sqrt{\cosh 2r - \cosh 2\rho}}\,dr =& \frac{1}{4}\int_0^{+\infty}\frac{1}{\sqrt{t}}\cdot\frac{1}{(t/2+\sinh^2\rho)^{(1+\beta)/2}}\,dt\\ =& \frac{1}{2\sqrt{2}(\sinh\rho)^\beta}\int_0^{+\infty} s^{-1/2}(1+s)^{-(1+\beta)/2}\,ds.\end{aligned}$$

Using the substitution $t = 2s\sinh^2\rho$, we obtain the desired formula. ∎

We remark here that the above result (and its proof) defines an integral transform that preserves inverse powers of the hyperbolic sine function.

The last lemma pertains to controlling higher order derivatives of $r^{\beta-2}/\sinh r$ for large r (see also Lemma 3.1 in [54] and Corollary 5.14 in [3]).

Lemma 5.1. *Let* $p, q \in \mathbb{N}_{\geq 0}$ *and let* $0 < \gamma < 3$. *If* $0 < r$, *then*

$$\Big(-\frac{1}{\sinh 2r}\frac{\partial}{\partial r}\Big)^q\Big(-\frac{1}{\sinh r}\frac{\partial}{\partial r}\Big)^q\frac{r^{\beta-2}}{\sinh r} \lesssim r^{\beta-2}\,e^{-(p+2q+1)r}.$$

Proof. Using

$$\frac{1}{\sinh r} = \frac{2e^{-r}}{1-e^{-2r}} = 2\sum_{j=0}^\infty e^{-(2j+1)r},$$

it is easy to see that

$$\Big(-\frac{1}{\sinh r}\frac{\partial}{\partial r}\Big)^p\frac{r^{\beta-2}}{\sinh r} \sim r^{\beta-2}\,[e^{-(p+1)r} + e^{-(p+3)r} + \cdots],$$

and, similarly, that

$$\Big(-\frac{1}{\sinh 2r}\frac{\partial}{\partial r}\Big)^q\Big(-\frac{1}{\sinh r}\frac{\partial}{\partial r}\Big)^p\frac{r^{\beta-2}}{\sinh r} \lesssim r^{\beta-2}e^{-(2q+p+1)r},$$

as desired. ∎

In the following subsections, we will prove various kernel estimates for k_γ, $k_{\zeta,\gamma'}$, $k_\gamma * k_{\zeta,\gamma'}$ and $k_\gamma * k_{\zeta,\gamma'} * f$ for smooth compactly supported function on $\mathbb{B}^m_{\mathbb{Q}}$ and $\mathbb{B}_{\mathbb{C}a}$. Along with the Fourier analysis on symmetric spaces (i.e., the Plancherel theorem and the Kunze–Stein phenomenon) and factorization, these estimates form the ingredients of the proofs of the Poincaré–Sobolev and Hardy–Sobolev–Maz'ya inequalities on $H^m_{\mathbb{Q}}$ and $H_{\mathbb{C}a}$.

5.1. Convolution estimates

In order to prove the kernel estimates, we will need asymptotics of certain convolutions. This is contained in Lemmas 5.2, 5.3, 5.4 and 5.5 below. Due to the appearance of $\Psi_{\mathbb{C}a}$ in the automorphisms on $\mathbb{B}_{\mathbb{C}a}$, separate considerations are needed for $\mathbb{B}_{\mathbb{C}a}$, and so we state the convolution estimates for $\mathbb{B}^m_{\mathbb{Q}}$ and $\mathbb{B}_{\mathbb{C}a}$ separately. We mention that, when compared to the complex hyperbolic setting, the hypothesis $\lambda_1 + \lambda_2 > \gamma + \gamma' - 4m + 2$ differs from the reasonably expected $\lambda_1 + \lambda_2 > \gamma + \gamma' - 4m$, and this has to do with the higher dimensional center of $\mathbb{H}^{m-1}_{\mathbb{Q}}$. This is similar for the corresponding hypothesis in Lemma 5.3 for $H_{\mathbb{C}a}$.

We will need the following convolution integral on Euclidean space (see [68]).

Lemma E. *For $0 < \gamma, \gamma' < k$ and $0 < \gamma + \gamma' < k$, there holds*

$$\int_{\mathbb{R}^k} |x|^{\gamma-k} |y-x|^{\gamma'-k}\, dx = \frac{\gamma_k(\gamma)\,\gamma_k(\gamma')}{\gamma_k(\gamma+\gamma')} |y|^{\gamma+\gamma'-k},$$

where

$$\gamma_k(\gamma) = \frac{\pi^{k/2}\, 2^\gamma\, \Gamma(\gamma/2)}{\Gamma((k-\gamma)/2)}.$$

We may now state the main convolution estimate lemma for small distances.

Lemma 5.2. *Let $0 < \gamma < 4m$, $0 < \gamma' < 4m$, and $\lambda_1 + \lambda_2 > \gamma + \gamma' - 4m + 2$. If $0 < \gamma + \gamma' < 4m - 1$ and $0 < \rho < 1$, then on $\mathbb{B}^m_{\mathbb{Q}}$ there holds*

$$\begin{aligned}\frac{1}{(\sinh\rho)^{4m-\gamma}(\cosh\rho)^{\lambda_1}} &* \frac{1}{(\sinh\rho)^{4m-\gamma'}(\cosh\rho)^{\lambda_2}}\\ &\le \frac{\gamma_{4m}(\gamma)\,\gamma_{4m}(\gamma')}{\gamma_{4m}(\gamma+\gamma')}\frac{1}{\rho^{4m-\gamma-\gamma'}} + O\Big(\frac{1}{\rho^{4m-\gamma-\gamma'-1}}\Big).\end{aligned}$$

If $4m - 1 \le \gamma + \gamma' < 4m$, $0 < \varepsilon < 4m - \gamma - \gamma'$ and $0 < \rho < 1$, then on $\mathbb{B}^m_{\mathbb{Q}}$ there holds

$$\begin{aligned}\frac{1}{(\sinh\rho)^{4m-\gamma}(\cosh\rho)^{\lambda_1}} &* \frac{1}{(\sinh\rho)^{4m-\gamma'}(\cosh\rho)^{\lambda_2}}\\ &\le \frac{\gamma_{4m}(\gamma)\gamma_{4m}(\gamma')}{\gamma_{4m}(\gamma+\gamma')}\frac{1}{\rho^{4m-\gamma-\gamma'}} + O\Big(\frac{1}{\rho^{4m-\gamma-\gamma'-\varepsilon}}\Big).\end{aligned}$$

Proof. By Lemma item (iv) of A, and by using that

$$dV = \frac{dz}{(1-|z|^2)^{2m+2}},$$

we compute as follows:

$$
\begin{aligned}
&\frac{1}{(\sinh\rho)^{4m-\gamma}(\cosh\rho)^{\lambda_1}} * \frac{1}{(\sinh\rho)^{4m-\gamma'}(\cosh\rho)^{\lambda_2}}\\
&= \int_{\mathbb{B}^m_{\mathbb{Q}}} \Big(\frac{\sqrt{1-|z|^2}}{|z|}\Big)^{4m-\gamma} (1-|z|^2)^{\lambda_1/2}\Big(\frac{(1-|w|^2)(1-|z|^2)}{|z-w|^2+|\langle z,w\rangle_{\mathbb{Q}}|^2-|z|^2|w|^2}\Big)^{(4m-\gamma')/2}\\
&\quad\times\Big(\frac{(1-|w|^2)(1-|z|^2)}{|1-\langle z,w\rangle_{\mathbb{Q}}|^2}\Big)^{\lambda_2/2}\frac{dz}{(1-|z|^2)^{2m+2}}\\
&= (1-|w|^2)^{(4m-\gamma'+\lambda_2)/2}\int_{\mathbb{B}^m_{\mathbb{Q}}}\frac{1}{|z|^{4m-\gamma}}\Big(\frac{1}{|z-w|^2+|\langle z,w\rangle_{\mathbb{Q}}|^2-|z|^2|w|^2}\Big)^{(4m-\gamma')/2}\\
&\quad\times\frac{1}{|1-\langle z,w\rangle_{\mathbb{Q}}|^{\lambda_2}}\frac{1}{(1-|z|^2)^{(4+\lambda+\lambda'-4m-\lambda_1-\lambda_2)/2}}\,dz\\
&= (\cosh\rho(w))^{-(4m-\gamma'+\lambda_2)}\,(A_5+A_6),
\end{aligned}
$$

where

$$A_5=\int_{\{|z|<1/2\}}\cdots \quad\text{and}\quad A_6=\int_{\{1/2\le|z|<1\}}\cdots.$$

Note that, when $\rho(w)<1$ and $|z|\le 1/2$, there holds

$$|1-\langle z,w\rangle_{\mathbb{Q}}|^{\lambda_2}\,(1-|z|^2)^{(4+\gamma+\gamma'-4m-\lambda_1-\lambda_2)/2}=1+O(|z|).$$

On the other hand, there holds

$$
\begin{aligned}
|\langle z,w\rangle_{\mathbb{Q}}|^2-|z|^2|w|^2 &= \big|\,|z|^2+\langle z,w-z\rangle_{\mathbb{Q}}\,\big|^2-|z|^2|w-z+z|^2\\
&= |\,\langle z,w-z\rangle_{\mathbb{Q}}\,|^2-|z|^2|w-z|^2=|z|^2|w-z|^2\Big[\Big|\Big\langle\frac{z}{|z|},\frac{w-z}{|w-z|}\Big\rangle_{\mathbb{Q}}\Big|-1\Big],
\end{aligned}
$$

and so

$$
\begin{aligned}
|z-w|^2+|\langle z,w\rangle_{\mathbb{Q}}|^2-|z|^2|w|^2 &= |z-w|^2\Big[1+|z|^2\Big[\Big|\Big\langle\frac{z}{|z|},\frac{w-z}{|w-z|}\Big\rangle_{\mathbb{Q}}\Big|^2-1\Big]\Big]\\
&= |z-w|^2\,(1+O(|z|^2)).
\end{aligned}
$$

Since $0<\gamma+\gamma'<4m-1$, we may use Lemma E to compute

$$
\begin{aligned}
A_5 &= \int_{\{|z|\le 1/2\}}\frac{1}{|z|^{4m-\gamma}}\frac{1}{|z-w|^{4m-\gamma'}}\,(1+O(|z|))\,dz\\
&\le \int_{\mathbb{R}^{4m}}\frac{1}{|z|^{4m-\gamma}}\frac{1}{|z-w|^{4m-\gamma'}}\,dz+O\Big(\int_{\mathbb{R}^M}\frac{1}{|z|^{4m-\gamma-1}}\frac{1}{|z-w|^{4m-\gamma'}}dz\Big)\\
&= \frac{\gamma_{4m}(\gamma)\gamma_{4m}(\gamma')}{\gamma_{4m}(\gamma+\gamma')}\frac{1}{|w|^{4m-\gamma-\gamma'}}+O\Big(\frac{1}{|w|^{4m-\gamma-\gamma'-1}}\Big).
\end{aligned}
$$

Similarly, if $0 < \varepsilon < 4m - \gamma - \gamma'$, we obtain

$$A_5 = \frac{\gamma_{4m}(\gamma)\,\gamma_{4m}(\gamma')}{\gamma_{4m}(\gamma + \gamma')}\,\frac{1}{|w|^{4m-\gamma-\gamma'}} + O\Big(\frac{1}{|w|^{4m-\gamma-\gamma'-\varepsilon}}\Big).$$

We are left with estimating A_6: since

$$\frac{4 + \gamma + \gamma' - 4m - \lambda_1 - \lambda_2}{2} < 1 \quad \text{is equivalent to} \quad \gamma + \gamma' - 4m + 2 < \lambda_1 + \lambda_2,$$

we find

$$\begin{aligned} A_6 &= \int_{\{\frac{1}{/}2\le|z|\le 1\}} \frac{1}{|z|^{4m-\gamma}}\Big(\frac{1}{|z-w|^2 + |\langle z,w\rangle_{\mathbb{Q}}|^2 - |z|^2\,|w|^2}\Big)^{(4m-\gamma')/2} \\ &\quad \times \frac{1}{|1-\langle z,w\rangle_{\mathbb{Q}}|^{\lambda_2}}\,\frac{1}{(1-|z|^2)^{(4+\gamma+\gamma'-4m-\lambda_1-\lambda_2)/2}}\,dz \\ &\sim \int_{\{1/2\le|z|\le 1\}} \frac{1}{(1-|z|^2)^{(4+\gamma+\gamma'-4m-\lambda_1-\lambda_2)/2}}\,dz \\ &\sim \int_{1/2}^{1} \frac{r}{(1-r^2)^{(4+\gamma+\gamma'-4m-\lambda_1-\lambda_2)/2}}\,dr < \infty. \end{aligned}$$

In conclusion, since $\cosh r \sim 1$ as $r \to 0$, we find

$$\begin{aligned} &\frac{1}{(\sinh\rho)^{4m-\gamma}(\cosh\rho)^{\lambda_1}} * \frac{1}{(\sinh\rho)^{4m-\gamma'}(\cosh\rho)^{\lambda_2}} \\ &\qquad\le \frac{\gamma_{4m}(\gamma)\,\gamma_{4m}(\gamma')}{\gamma_{4m}(\gamma+\gamma')}\,\frac{1}{|w|^{4m-\gamma-\gamma'}} + O\Big(\frac{1}{|w|^{4m-\gamma-\gamma'-1}}\Big), \end{aligned}$$

and the result follows since

$$\rho(w) = \frac{1}{2}\log\frac{1+|w|}{1-|w|} = |w| + O(|w|^3) \quad \text{as } |w| \to 0. \qquad \blacksquare$$

Lemma 5.3. *Let* $0 < \gamma < 16$, $0 < \gamma' < 16$, *and* $\lambda_1 + \lambda_2 > \gamma + \gamma' - 10$. *If* $0 < \gamma + \gamma' < 15$ *and* $0 < \rho < 1$, *then on* $\mathbb{B}_{\mathbb{C}a}$ *there holds*

$$\begin{aligned} &\frac{1}{(\sinh\rho)^{16-\gamma}(\cosh\rho)^{\lambda_1}} * \frac{1}{(\sinh\rho)^{16-\gamma'}(\cosh\rho)^{\lambda_2}} \\ &\qquad\le \frac{\gamma_{16}(\gamma)\,\gamma_{16}(\gamma')}{\gamma_{16}(\gamma+\gamma')}\,\frac{1}{\rho^{16-\gamma-\gamma'}} + O\Big(\frac{1}{\rho^{15-\gamma-\gamma'}}\Big). \end{aligned}$$

If $15 \le \gamma + \gamma' < 16$, $0 < \varepsilon < 16 - \gamma - \gamma'$ *and* $0 < \rho < 1$, *then on* $\mathbb{B}_{\mathbb{C}a}$ *there holds*

$$\begin{aligned} &\frac{1}{(\sinh\rho)^{16-\gamma}(\cosh\rho)^{\lambda_1}} * \frac{1}{(\sinh\rho)^{16-\gamma'}(\cosh\rho)^{\lambda_2}} \\ &\qquad\le \frac{\gamma_{16}(\gamma)\gamma_{16}(\gamma')}{\gamma_{16}(\gamma+\gamma')}\,\frac{1}{\rho^{16-\gamma-\gamma'}} + O\Big(\frac{1}{\rho^{16-\gamma-\gamma'-\varepsilon}}\Big). \end{aligned}$$

Proof. By Lemma B (iv), and by using that $dV = dz/(1-|z|^2)^{12}$, we compute

$$\begin{aligned}
&\frac{1}{(\sinh\rho)^{16-\gamma}(\cosh\rho)^{\lambda_1}} * \frac{1}{(\sinh\rho)^{16-\gamma'}(\cosh\rho)^{\lambda_2}} \\
&= \int_{\mathbb{B}^m_{\mathbb{Q}}} \Big(\frac{\sqrt{1-|z|^2}}{|z|}\Big)^{16-\gamma} (1-|z|^2)^{\lambda_1/2} \Big(\frac{(1-|w|^2)(1-|z|^2)}{\Psi_{\mathbb{C}a}(z,w)-(1-|z|^2)(1-|w|^2)}\Big)^{(16-\gamma')/2} \\
&\quad\times \Big(\frac{(1-|w|^2)(1-|z|^2)}{\Psi_{\mathbb{C}a}(z,w)}\Big)^{\lambda_2/2} \frac{dz}{(1-|z|^2)^{12}} \\
&= (1-|w|^2)^{(16-\gamma'+\lambda_2)/2} \int_{\mathbb{B}^m_{\mathbb{Q}}} \frac{1}{|z|^{16-\gamma}} \Big(\frac{1}{\Psi_{\mathbb{C}a}(z,w)-(1-|z|^2)(1-|w|^2)}\Big)^{(16-\gamma')/2} \\
&\quad\times \frac{1}{\Psi_{\mathbb{C}a}(z,w)^{\lambda_2/2}} \frac{1}{(1-|z|^2)^{(\gamma+\gamma'-\lambda_1-\lambda_2-8)/2}}\, dz \\
&= (\cosh\rho(w))^{-(16-\gamma'+\lambda_2)}\,(A_5'+A_6'),
\end{aligned}$$

where

$$A_5' = \int_{\{|z|<1/2\}} \cdots \quad \text{and} \quad A_6' = \int_{\{1/2\le|z|<1\}} \cdots .$$

Note that, when $\rho(w) < 1$ and $|z| \le 1/2$, there holds

$$\Psi_{\mathbb{C}a}(z,w)^{\lambda_2/2}\,(1-|z|^2)^{(\gamma+\gamma'-\lambda_1-\lambda_2-8)/2} = 1 + O(|z|).$$

Next, we have

$$\begin{aligned}
\Psi_{\mathbb{C}a}(z,a) - (1-|z|^2)(1-|w|^2) &= \Phi_{\mathbb{C}a}(z,a) - 2\langle z,a\rangle_{\mathbb{R}} + |z|^2 + |a|^2 - |z|^2|a|^2 \\
&= \Phi_{\mathbb{C}a}(z,a) + |z-a|^2 - |z|^2|a|^2 = |z-a|^2\Big(1 + \frac{\Phi_{\mathbb{C}a}(z,a)-|z|^2|a|^2}{|z-a|^2}\Big).
\end{aligned}$$

Moreover, it is not hard to see that

$$\frac{\Phi_{\mathbb{C}a}(z,w)-|z|^2|w|^2}{|z-w|^2} = O(|z|^2).$$

Indeed, using invariance of distance ρ, we can assume $w = (w_1, w_2)$ with $\Re w_1 = c \in \mathbb{R}$ and all other components are zero. Then

$$\Phi_{\mathbb{C}a}(z,a) - |z|^2|a|^2 = -c^2|z_2|^2,$$

and clearly $|z_2|^2/|z-a|^2$ is bounded as $z \to a$. Therefore, using also that

$$\Phi_{\mathbb{C}a}(z,w) \le |z|^2|w|^2,$$

we obtain

$$\Psi_{\mathbb{C}a}(z,a) - (1-|z|^2)(1-|w|^2) = |z-a|^2\,(1+O(|z|^2)).$$

The remainder of the proof is analogous to the proof of Lemma 5.2, and is thus omitted. ∎

Next, we will state and prove the main convolution lemma for large distances. In preparation, we recall some properties and definitions of certain special functions. First, recall the generalized hypergeometric function

$$ {}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z)=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(n_q)_k}\frac{z^k}{k!}. $$

Second, we recall the following hypergeometric integral (see equation 7.512.5 in [30]): supposing the complex parameters α, β, γ, ρ and σ satisfy

$$ \Re\rho>0,\quad \Re\sigma>0,\quad \Re(\gamma+\sigma-\alpha-\beta)>0, $$

there holds

$$ \int_0^1 x^{\rho-1}(1-x)^{\sigma-1}\,{}_2F_1(\alpha,\beta;\gamma;x)\,dx=\frac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\rho+\sigma)}\,{}_3F_2(\alpha,\beta,\rho;\gamma,\rho+\sigma;1). \tag{5.5} $$

Lemma 5.4. *Let $0<\gamma<4m$, $0<\gamma'<4m$, and $\lambda_1+\lambda_2>\gamma+\gamma'-4m+2$. If $\lambda_2-\gamma'<\lambda_1-\gamma$ and $1\le\rho$, then on $\mathbb{B}^m_{\mathbb{Q}}$ there holds*

$$ \frac{1}{(\sinh\rho)^{4m-\gamma}(\cosh\rho)^{\lambda_1}}*\frac{1}{(\sinh\rho)^{4m-\lambda'}(\cosh\rho)^{\lambda_2}}\sim e^{-(4m-\gamma'+\lambda_2)\,\rho}. $$

Proof. By the proof of Lemma 5.2, we have

$$ \begin{aligned} &\frac{1}{(\sinh\rho)^{4m-\gamma}(\cosh\rho)^{\lambda_1}}*\frac{1}{(\sinh\rho)^{4m-\gamma'}(\cosh\rho)^{\lambda_2}}\\ &=(\cosh\rho(w))^{-(4m-\gamma'+\lambda_2)}\int_{\mathbb{B}^m_{\mathbb{Q}}}\frac{1}{|z|^{4m-\gamma}}\Big(\frac{1}{|z-w|^2+|\langle z,w\rangle_{\mathbb{Q}}|^2-|z|^2|w|^2}\Big)^{(4m-\gamma')/2}\\ &\quad\times\frac{1}{|1-\langle z,w\rangle_{\mathbb{Q}}|^{\lambda_2}}\frac{1}{(1-|z|^2)^{(4+\lambda+\lambda'-4m-\lambda_1-\lambda_2)/2}}\,dz. \end{aligned} $$

Setting

$$ F(w)=\int_{S^{4m-1}}\Big(\frac{1}{|z-w|^2+|\langle z,w\rangle_{\mathbb{Q}}|^2-|z|^2|w|^2}\Big)^{(4m-\gamma')/2}\frac{1}{|1-\langle z,w\rangle_{\mathbb{Q}}|^{\lambda_2}}\,d\sigma, $$

we see that $F(w)=F(|w|)$. Moreover, by Proposition 4.1, we find

$$ \begin{aligned} \lim_{|w|\to1^-}F(w)&=\lim_{|w|\to1^-}\int_{S^{4m-1}}|1-\langle z,w\rangle_{\mathbb{Q}}|^{-(4m-\gamma'+\lambda_2)}\,d\sigma\\ &=\frac{2\pi^{2m}}{\Gamma(2m)}\,{}_2F_1\Big(\frac{4m-\gamma'+\lambda_2}{2},\frac{4m-\gamma'+\lambda_2-2}{2};2m;|z|^2\Big). \end{aligned} $$

Consequently, there holds

$$\begin{aligned}\lim_{|w|\to 1^-}\int_{\mathbb{B}^m_{\mathbb{Q}}} &\frac{1}{|z|^{4m-\gamma}}\Big(\frac{1}{|z-w|^2+|\langle z,w\rangle_{\mathbb{Q}}|^2-|z|^2|w|^2}\Big)^{(4m-\gamma')/2}\\ &\times\frac{1}{|1-\langle z,w\rangle_{\mathbb{Q}}|^{\lambda_2}}\frac{1}{(1-|z|^2)^{(4+\lambda+\lambda'-4m-\lambda_1-\lambda_2)/2}}\,dz\\ &=\frac{2\pi^{2m}}{\Gamma(2m)}\int_0^1 r^{\gamma-1}(1-r^2)^{-(4+\gamma+\gamma'-4m-\lambda_1-\lambda_2)/2}\\ &\quad\times{}_2F_1\Big(\frac{4m-\gamma'+\lambda_2}{2},\frac{4m-\gamma'+\lambda_2-2}{2};2m;r^2\Big)\,dr\\ &=\frac{2\pi^{2m}}{\Gamma(2m)}\int_0^1 t^{\gamma/2-1}\,(1-t)^{-(4+\gamma+\gamma'-4m-\lambda_1-\lambda_2)/2}\\ &\quad\times{}_2F_1\Big(\frac{4m-\gamma'+\lambda_2}{2},\frac{4m-\gamma'+\lambda_2-2}{2};2m;t\Big)\,dt,\end{aligned}$$

where the change of variable $r^2=t$ was used in the last equality. Now, using (5.5), we have

$$\begin{aligned}\lim_{|w|\to 1^-}\int_{\mathbb{B}^m_{\mathbb{Q}}} &\frac{1}{|z|^{4m-\gamma}}\Big(\frac{1}{|z-w|^2+|\langle z,w\rangle_{\mathbb{Q}}|^2-|z|^2|w|^2}\Big)^{(4m-\gamma')/2}\\ &\times\frac{1}{|1-\langle z,w\rangle_{\mathbb{Q}}|^{\lambda_2}}\frac{1}{(1-|z|^2)^{(4+\lambda+\lambda'-4m-\lambda_1-\lambda_2)/2}}\,dz\\ &=\frac{\pi^{2m}}{\Gamma(2m)}\frac{\Gamma(\gamma/2)\,\Gamma\big(\frac{4m+\lambda_1+\lambda_2-\gamma-\gamma'-2}{2}\big)}{\Gamma\big(\frac{4m+\lambda_1+\lambda_2-\gamma'-2}{2}\big)}\\ &\quad\times{}_3F_2\Big(\frac{4m-\gamma'+\lambda_2}{2},\frac{4m-\gamma'+\lambda_2-2}{2},\frac{\gamma}{2};2m,\frac{4m+\lambda_1+\lambda_2-\gamma'-2}{2};1\Big).\end{aligned}$$

At last, using $\cosh r\sim e^r$ for $1\le r$, we have proved

$$\begin{aligned}\frac{1}{(\sinh\rho)^{4m-\gamma}(\cosh\rho)^{\lambda_1}}*\frac{1}{(\sinh\rho)^{4m-\gamma'}(\cosh\rho)^{\lambda_2}}&\sim(\cosh\rho)^{-(4m-\gamma'+\lambda_2)}\\ &\sim e^{-(4m-\gamma'+\lambda_2)\rho}.\end{aligned}$$ ■

Lemma 5.5. *Let $0<\gamma<16$, $0<\gamma'<16$, and $\lambda_1+\lambda_2>\gamma+\gamma'-10$. If $\lambda_2-\gamma'<\lambda_1-\gamma$ and $1\le\rho$, then*

$$\frac{1}{(\sinh\rho)^{16-\gamma}(\cosh\rho)^{\lambda_1}}*\frac{1}{(\sinh\rho)^{16-\lambda'}(\cosh\rho)^{\lambda_2}}\sim e^{-(16-\gamma'+\lambda_2)\rho}.$$

Proof. By the proof of Lemma 5.3, we have

$$\begin{aligned}&\frac{1}{(\sinh\rho)^{16-\gamma}(\cosh\rho)^{\lambda_1}}*\frac{1}{(\sinh\rho)^{16-\gamma'}(\cosh\rho)^{\lambda_2}}\\&=(\cosh\rho(w))^{-(16-\gamma'+\lambda_2)}\int_{\mathbb{B}_{\mathbb{C}a}}\frac{1}{|z|^{16-\gamma}}\Big(\frac{1}{\Psi_{\mathbb{C}a}(z,w)-(1-|z|^2)(1-|w|^2)}\Big)^{(16-\gamma')/2}\\&\quad\times\frac{1}{\Psi_{\mathbb{C}a}(z,w)^{\lambda_2/2}}\frac{1}{(1-|z|^2)^{(\gamma+\gamma'-\lambda_1-\lambda_2-8)/2}}\,dz.\end{aligned}$$

Setting

$$F(w)=\int_{S^{4m-1}}\Big(\frac{1}{\Psi_{\mathbb{C}a}(z,w)-(1-|z|^2)(1-|w|^2)}\Big)^{(16-\gamma')/2}\frac{1}{\Psi_{\mathbb{C}a}(z,w)^{\lambda_2/2}}\,d\sigma,$$

we see that $F(w)=F(|w|)$. Moreover, by Proposition 4.2, we find

$$\begin{aligned}\lim_{|w|\to1^-}F(w)&=\lim_{|w|\to1^-}\int_{S^{4m-1}}\Psi_{\mathbb{C}a}(z,w)^{-(16-\gamma'+\lambda_2)/2}\,d\sigma\\&=\frac{2\pi^8}{7!}\,{}_2F_1\Big(\frac{16-\gamma'+\lambda_2}{2},\frac{10-\gamma'+\lambda_2}{2};8;r^2\Big).\end{aligned}$$

Consequently, there holds

$$\begin{aligned}&\lim_{|w|\to1^-}\int_{\mathbb{B}_{\mathbb{C}a}}\frac{1}{|z|^{16-\gamma}}\Big(\frac{1}{\Psi_{\mathbb{C}a}(z,w)-(1-|z|^2)(1-|w|^2)}\Big)^{(16-\gamma')/2}\\&\qquad\times\frac{1}{\Psi_{\mathbb{C}a}(z,w)^{\lambda_2/2}}\frac{1}{(1-|z|^2)^{(\gamma+\gamma'-\lambda_1-\lambda_2-8)/2}}\,dz\\&=\frac{2\pi^8}{7!}\int_0^1 r^{\gamma-1}(1-r^2)^{-(\gamma+\gamma'-\lambda_1-\lambda_2-8)/2}\,{}_2F_1\Big(\frac{16-\gamma'+\lambda_2}{2},\frac{16-\gamma'+\lambda_2-6}{2};8;r^2\Big)dr\\&=\frac{2\pi^8}{7!}\int_0^1 t^{\gamma/2-1}(1-t)^{-(\gamma+\gamma'-\lambda_1-\lambda_2-8)/2}\,{}_2F_1\Big(\frac{16-\gamma'+\lambda_2}{2},\frac{10-\gamma'+\lambda_2}{2};8;r^2\Big)\,dt,\end{aligned}$$

where the change of variable $r^2=t$ was used in the last equality. Now, using (5.5), we have

$$\begin{aligned}&\lim_{|w|\to1^-}\int_{\mathbb{B}_{\mathbb{C}a}}\frac{1}{|z|^{16-\gamma}}\Big(\frac{1}{\Psi_{\mathbb{C}a}(z,w)-(1-|z|^2)(1-|w|^2)}\Big)^{(16-\gamma')/2}\\&\qquad\times\frac{1}{\Psi_{\mathbb{C}a}(z,w)^{\lambda_2/2}}\frac{1}{(1-|z|^2)^{(\gamma+\gamma'-\lambda_1-\lambda_2-8)/2}}\,dz\\&\qquad=\frac{\pi^8}{7!}\frac{\Gamma(\gamma/2)\,\Gamma((10+\lambda_1+\lambda_2-\gamma-\gamma')/2)}{\Gamma((10+\lambda_1+\lambda_2-\gamma')/2)}\\&\qquad\quad\times{}_3F_2\Big(\frac{16-\gamma'+\lambda_2}{2},\frac{10-\gamma'+\lambda_2}{2};\frac{\gamma}{2};7,\frac{10+\lambda_1+\lambda_2-\gamma'}{2};1\Big).\end{aligned}$$

At last, using $\cosh r\sim e^r$ for $1\le r$, we have proved

$$\begin{aligned}\frac{1}{(\sinh\rho)^{16-\gamma}(\cosh\rho)^{\lambda_1}}*\frac{1}{(\sinh\rho)^{16-\gamma'}(\cosh\rho)^{\lambda_2}}&\sim(\cosh\rho)^{-(16-\gamma'+\lambda_2)}\\&\sim e^{-(16-\gamma'+\lambda_2)\rho}.\end{aligned}$$

∎

5.2. Estimates for k_γ

In this subsection, we obtain the asymptotics for k_γ. Note that the large distance asymptotics ($1 \le \rho$) are already contained in (5.4).

Lemma 5.6. *Let* $0 < \gamma < 3$ *and let* $N = \dim_{\mathbb{R}} H^m_{\mathbb{F}}$. *If* $0 < \rho < 1$, *then*

$$k_\gamma \le \frac{1}{\gamma_N(\gamma)} \frac{1}{\rho^{N-\gamma}} + O\Big(\frac{1}{\rho^{N-\gamma-1}}\Big).$$

If $1 \le \rho$, *then*

$$k_\gamma \sim \rho^{\gamma-2} e^{-Q\rho/2}.$$

Proof. Since the large distance asymptotics ($1 \le \rho$) are already contained in (5.4), we only need to prove the estimate for $0 < \rho < 1$.

By using (5.2), we will write k_γ in terms of a Bessel–Green–Riesz kernel on $H^n_{\mathbb{R}}$, where $n = 2\tilde{m} + 1$ and $\tilde{m} = 2m - 2$ if $\mathbb{F} = \mathbb{Q}$, and $\tilde{m} = 4$ if $\mathbb{F} = \mathbb{C}a$. Recall that $h_t(\rho, n)$ denotes the heat kernel on $H^n_{\mathbb{R}}$ (see (5.1)). Finally, let c denote c_m (respectively, c_o) from (5.2) (respectively, (5.3)), and let $\mu = 2$ (respectively, $\mu = 4$) when $\mathbb{F} = \mathbb{Q}$ (respectively, $\mathbb{F} = \mathbb{C}a$).

Then, by the Mellin transform and (5.2) and (5.3), we have

$$\begin{aligned}
k_\gamma(\rho) &= \frac{1}{\Gamma(\gamma/2)} \int_0^\infty t^{\gamma/2-1} e^{t(\Delta + Q^2/4)}\, dt \\
&= \frac{c}{b_{\tilde{m}}} \int_\rho^\infty \frac{\sinh 2r}{\sqrt{\cosh 2r - \cosh 2\rho}} \Big(-\frac{1}{\sinh 2r}\frac{\partial}{\partial r}\Big)^\mu \\
&\quad \times \frac{1}{\Gamma(\gamma/2)} \int_0^\infty t^{\gamma/2-1} e^{Q^2 t/4} e^{-\tilde{m}^2 t} h_t(r, n)\, dt\, dr \\
&= \frac{c}{b_{\tilde{m}}} \int_\rho^\infty \frac{\sinh 2r}{\sqrt{\cosh 2r - \cosh 2\rho}} \Big(-\frac{1}{\sinh 2r}\frac{\partial}{\partial r}\Big)^\mu \Big(-\Delta_{H^n_{\mathbb{R}}} - \Big(\frac{n-1}{2}\Big)^2\Big)^{-\gamma/2} dr \\
&= A_1 + A_2,
\end{aligned}$$

where

$$A_1 = \int_\rho^1 \cdots \quad \text{and} \quad A_2 = \int_1^\infty \cdots.$$

We begin by estimating A_1. Using Lemma C, it is easy to see that, for $0 < r < 1$, there holds

$$\begin{aligned}
&\Big(-\frac{1}{\sinh 2r}\frac{\partial}{\partial r}\Big)^2 \Big(-\Delta_{H^n_{\mathbb{R}}} - \Big(\frac{n-1}{2}\Big)^2\Big)^{-\gamma/2} \\
&\qquad = \Big(-\frac{1}{\sinh 2r}\frac{\partial}{\partial r}\Big)^2 \Big(\frac{1}{\gamma_n(\gamma)} \frac{1}{r^{n-\gamma}} + O\Big(\frac{1}{r^{n-\gamma-1}}\Big)\Big) \\
&\qquad = \frac{1}{\gamma_n(\gamma)} \frac{(n-\gamma)(n+2-\gamma)}{4} \frac{1}{r^{n+4-\gamma}} + O\Big(\frac{1}{r^{n+3-\gamma}}\Big),
\end{aligned}$$

and similarly,

$$\Big(-\frac{1}{\sinh 2r}\frac{\partial}{\partial r}\Big)^4\Big(-\Delta_{H^n_{\mathbb{R}}}-\Big(\frac{n-1}{2}\Big)^4\Big)^{-\gamma/2}$$
$$=\frac{1}{\gamma_n(\gamma)}\frac{(n-\gamma)(n+2-\gamma)(n+4-\gamma)(n+6-\gamma)}{16}\frac{1}{r^{n+8-\gamma}}+O\Big(\frac{1}{r^{n+7-\gamma}}\Big).$$

Consequently, in the quaternionic case, there holds

$$\sinh 2r\Big(-\frac{1}{\sinh 2r}\frac{\partial}{\partial r}\Big)^2\Big(-\Delta_{H^n_{\mathbb{R}}}-\Big(\frac{n-1}{2}\Big)^2\Big)$$
$$=\frac{1}{\gamma_n(\gamma)}\frac{(n-\gamma)(n+2-\gamma)}{2}\frac{1}{r^{n+3-\gamma}}+O\Big(\frac{1}{r^{n+2-\gamma}}\Big)$$

and, in the octonionic case, there holds

$$\sinh 2r\Big(-\frac{1}{\sinh 2r}\frac{\partial}{\partial r}\Big)^4\Big(-\Delta_{H^n_{\mathbb{R}}}-\Big(\frac{n-1}{2}\Big)^2\Big)$$
$$=\frac{1}{\gamma_n(\gamma)}\frac{(n-\gamma)(n+2-\gamma)(n+4-\gamma)(n+6-\gamma)}{8}\frac{1}{r^{n+7-\gamma}}+O\Big(\frac{1}{r^{n+6-\gamma}}\Big).$$

Now, using Lemma D, we compute in the quaternionic case that

$$\begin{aligned}
A_1&=\frac{c_m}{b_{2m-2}}\frac{(n-\gamma)(n+2-\gamma)}{2\gamma_n(\gamma)}\int_\rho^1\frac{1}{\sqrt{\cosh 2r-\cosh 2\rho}}\Big[\frac{1}{r^{n+3-\gamma}}+O\Big(\frac{1}{r^{n+2-\gamma}}\Big)\Big]dr\\
&\le\frac{c_m}{b_{2m-2}}\frac{(n-\gamma)(n+2-\gamma)}{2\gamma_n(\gamma)}\int_\rho^1\frac{\cosh r}{\sqrt{\cosh 2r-\cosh 2\rho}}\\
&\quad\times\Big[\frac{1}{(\sinh r)^{n+3-\gamma}}+O\Big(\frac{1}{(\sinh r)^{n+2-\gamma}}\Big)\Big]dr\\
&=\frac{c_m(n-\gamma)(n+2-\gamma)}{2\gamma_n(\gamma)b_{2m-2}}\frac{\Gamma(1/2)\Gamma\big(\frac{n+3-\gamma}{2}\big)}{2\sqrt{2}\Gamma\big(\frac{n+4-\gamma}{2}\big)}\frac{1}{(\sinh\rho)^{n+3-\gamma}}+O\Big(\frac{1}{(\sinh\rho)^{n+2-\gamma}}\Big)\\
&=\frac{1}{\gamma_{4m}(\gamma)}\frac{1}{(\sinh\rho)^{4m-\gamma}}+O\Big(\frac{1}{(\sinh\rho)^{4m-\gamma-1}}\Big),
\end{aligned}$$

where we have computed

$$\frac{c_m(n-\gamma)(n+2-\gamma)}{2\gamma_n(\gamma)\,b_{2m-2}}\frac{\Gamma(1/2)\,\Gamma((n+3-\gamma)/2)}{2\sqrt{2}\,\Gamma((n+4-\gamma)/2)}=\frac{1}{\gamma_{4m}(\gamma)}.$$

Similarly, we have in the octonionic case that

$$A_1=\frac{1}{\gamma_{16}(\gamma)}\frac{1}{(\sinh\rho)^{16-\gamma}}+O\Big(\frac{1}{(\sinh\rho)^{15-\gamma}}\Big).$$

Concerning estimating A_2, it is clear from Lemma 5.1 that $A_2\lesssim 1$ for both the quaternionic and octonionic cases, and so

$$k_\gamma(\rho)=A_1+A_2\le\frac{1}{\gamma_N(\gamma)}\frac{1}{\rho^{N-\gamma}}+O\Big(\frac{1}{\rho^{N-\gamma-1}}\Big),$$

as desired. ∎

5.3. Estimate for $k_{\zeta,\gamma}$

In this subsection, we obtain the asymptotics for $k_{\zeta,\gamma}$ for $0<\gamma<4m$ and $0<\zeta$. Note that the large distance asymptotics ($1\le\rho$) are already contained in (5.4).

Lemma 5.7. *Let $N=\dim_{\mathbb{R}} H^m_{\mathbb{F}}$ and let $0<\gamma<N$, $0<\zeta$ and $0<\varepsilon<\min\{1,N-\gamma\}$. If $0<\rho<1$, then*

$$k_{\zeta,\gamma}\le\frac{1}{\gamma_N(\gamma)}\,\frac{1}{\rho^{N-\gamma}}+O\Big(\frac{1}{\rho^{N-\gamma-\varepsilon}}\Big).$$

If $1\le\rho$, then

$$k_{\zeta,\gamma}\sim\rho^{(\gamma-2)/2}\,e^{-\zeta\rho-Q\rho/2}.$$

Proof. As mentioned above, we only need to prove the estimate for $0<\rho<1$.

As before, let $n=2\tilde m+1$ with $\tilde m$ as above, and choose $\tilde\gamma$ and ℓ such that $0<\tilde\gamma<3$, $0\le\ell<n-1$ and $\gamma=\tilde\gamma+\ell$. Then

$$k_{\zeta,\gamma}=k_{\zeta,\tilde\gamma}*k_{\zeta,\ell}.$$

Using Lemmas 5.2 and 5.3, it will be sufficient to estimate $k_{\zeta,\tilde\gamma}$ and $k_{\zeta,\ell}$ separately.

To estimate $k_{\zeta,\tilde\gamma}$, note that, by Lemma 5.6, there holds

$$\begin{aligned}
k_{\zeta,\tilde\gamma}&=\Big(-\Delta-\frac{Q^2}{4}+\zeta^2\Big)^{-\tilde\gamma/2}=\frac{1}{\Gamma(\tilde\gamma/2)}\int_0^\infty t^{\tilde\gamma/2-1}\,e^{t\,(\Delta+Q^2/4-\zeta^2)}\,dt\\
&\le\frac{1}{\Gamma(\tilde\gamma/2)}\int_0^\infty t^{\tilde\gamma/2-1}\,e^{t\,(\Delta+Q^2/4)}\,dt=\Big(-\Delta-\frac{Q^2}{4}\Big)^{-\tilde\gamma/2}=k_{\tilde\gamma}\\
&\le\frac{1}{\gamma_N(\tilde\gamma)}\,\frac{1}{\rho^{N-\tilde\gamma}}+O\Big(\frac{1}{\rho^{N-\tilde\gamma-1}}\Big).
\end{aligned}$$

We see that, if $\ell=0$, then we are done, and so we assume without loss of generality that $0<\ell$.

We now estimate $k_{\zeta,\ell}$. As in the previous proof, let $\mu=2$ for the quaternionic case and $\mu=4$ for the octonionic case, and let c denote c_m or c_o in the respective cases. We compute

$$\begin{aligned}
k_{\zeta,\ell}&=\frac{1}{\Gamma(\ell/2)}\int_0^\infty t^{\ell/2-1}\,e^{t\,(\Delta+Q^2/4-\zeta^2)}\,dt\\
&=\frac{c}{b_{\tilde m}}\int_\rho^\infty\frac{\sinh 2r}{\sqrt{\cosh 2r-\cosh 2\rho}}\Big(-\frac{1}{\sinh 2r}\frac{\partial}{\partial r}\Big)^2\Big(-\Delta_{H^n_{\mathbb{R}}}-\Big(\frac{n-1}{2}\Big)^2+\zeta^2\Big)^{-\ell/2}dr\\
&=A_7+A_8,
\end{aligned}$$

where

$$A_7=\int_\rho^1\cdots\quad\text{and}\quad A_8=\int_1^\infty\cdots.$$

From Proposition 2.5 in [53], we have that

$$\Big(-\Delta-\Big(\frac{n-1}{2}\Big)^2+\zeta^2\Big)^{-\ell/2}=\frac{1}{\gamma_n(\gamma)}\,\frac{1}{\rho^{n-\ell}}+O\Big(\frac{1}{\rho^{n-\ell-1}}\Big),$$

and by similar computations to those given in the proof of Lemma 5.6, we have

$$\sinh 2r\Big(-\frac{1}{\sinh 2r}\frac{\partial}{\partial r}\Big)^2\Big(-\Delta_{H^n_{\mathbb{R}}}-\Big(\frac{n-1}{2}\Big)^2\Big) = \frac{(n-\ell)(n+2-\ell)}{2\gamma_n(\gamma)}\,\frac{1}{r^{n+3-\ell}}+O\Big(\frac{1}{r^{n+2-\ell}}\Big)$$

and

$$\sinh 2r\Big(-\frac{1}{\sinh 2r}\frac{\partial}{\partial r}\Big)^4\Big(-\Delta_{H^n_{\mathbb{R}}}-\Big(\frac{n-1}{2}\Big)^2\Big) = \frac{(n-\ell)(n+2-\ell)(n+4-\ell)(n+6-\ell)}{8\gamma_n(\gamma)}\,\frac{1}{r^{n+7-\ell}}+O\Big(\frac{1}{r^{n+6-\ell}}\Big).$$

Consequently, using $1\le\cosh r$ and Lemma D, we find for the quaternionic case that

$$\begin{aligned} A_7 &\le \frac{c_m}{b_{2m-2}}\,\frac{(n-\ell)(n+2-\ell)}{2\gamma_n(\gamma)}\int_\rho^\infty \cosh r\,\frac{\sinh 2r}{\sqrt{\cosh 2r-\cosh 2\rho}} \\ &\quad\times\Big[\frac{1}{(\sinh r)^{n+3-\ell}}+O\Big(\frac{1}{(\sinh r)^{n+2-\ell}}\Big)\Big]\,dr \\ &= \frac{1}{\gamma_{4m}(\ell)}\,\frac{1}{(\sinh\rho)^{4m-\ell}}+O\Big(\frac{1}{(\sinh\rho)^{4m-\ell-1}}\Big) \end{aligned}$$

where we have computed

$$\frac{c_m(n-\ell)(n+2-\ell)}{2\gamma_n(\ell)\,b_{2m-2}}\,\frac{\Gamma(1/2)\,\Gamma((n+3-\ell)/2)}{2\sqrt{2}\,\Gamma((n+4-\ell)/2)}=\frac{1}{\gamma_{4m}(\ell)}.$$

Similarly, we have for that octonionic case that

$$A_7\le\frac{1}{\gamma(\ell)(\gamma)}\,\frac{1}{(\sinh\rho)^{16-\ell}}+O\Big(\frac{1}{(\sinh\rho)^{15-\ell}}\Big).$$

Again, using Lemma 5.1, we have that $A_8\lesssim 1$, and so we have proved to two estimates

$$\begin{aligned} k_{\zeta,\ell} &\le \frac{1}{\gamma_N(\ell)}\,\frac{1}{(\sinh\rho)^{N-\ell}}+O\Big(\frac{1}{(\sinh\rho)^{N-\ell-1}}\Big), \\ k_{\zeta,\tilde\gamma} &\le \frac{1}{\gamma_N(\tilde\gamma)}\,\frac{1}{(\sinh\rho)^{N-\tilde\gamma}}+O\Big(\frac{1}{(\sinh\rho)^{N-\tilde\gamma-1}}\Big). \end{aligned}$$

Now, using (5.4), we have, for any $0<\zeta'<\zeta$, $0<\alpha$ and $1\le\rho$, there holds

$$k_{\zeta,\alpha}\sim\rho^{(\alpha-2)/2}\,e^{-\zeta\rho-Q/2}\lesssim_\alpha e^{-\zeta'\rho-Q\rho/2}.$$

Therefore, using this estimate and that $\cosh r\sim e^r$ and $\sinh r\sim e^r$ for $r>1$, and $\sinh r\sim r$ and $\cosh r\sim 1$ for $0<r<1$, we obtain the following global estimates (i.e., for $0<\rho$):

$$\begin{aligned} k_{\zeta,\ell} &\le \frac{1}{\gamma_N(\ell)}\,\frac{(\cosh\rho)^{N-Q/2-\ell-\zeta'}}{(\sinh\rho)^{N-\ell}}+O\Big(\frac{(\cosh\rho)^{N-Q/2-\ell-\zeta'-1}}{(\sinh\rho)^{N-\ell-1}}\Big), \\ k_{\zeta,\tilde\gamma} &\le \frac{1}{\gamma_N(\tilde\gamma)}\,\frac{(\cosh\rho)^{N-Q/2-\tilde\gamma-\zeta'}}{(\sinh\rho)^{N-\tilde\gamma}}+O\Big(\frac{(\cosh\rho)^{N-Q/2-\tilde\gamma-\zeta'-1}}{(\sinh\rho)^{N-\tilde\gamma-1}}\Big). \end{aligned}$$

Finally, using Lemmas 5.2 and 5.3 and letting $0 < \varepsilon < \min\{1, N-\gamma\}$, we obtain

$$k_{\zeta,\gamma} = k_{\zeta,\ell} * k_{\zeta,\tilde{\gamma}} \le \frac{1}{\gamma_N(\ell+\tilde{\gamma})} \frac{1}{\rho^{N-\tilde{\gamma}-\ell}} + O\Big(\frac{1}{\rho^{N-\tilde{\gamma}-\ell-\varepsilon}}\Big),$$

which gives the desired estimate since $\gamma = \ell + \tilde{\gamma}$. ∎

5.4. Estimates for $k_\gamma * k_{\zeta,\gamma'}$

In this subsection, we obtain the asymptotics for $k_\gamma * k_{\zeta,\gamma}$ for $0 < \gamma < 3$, $0 < \gamma' < N-\gamma$ and $0 < \zeta$.

Lemma 5.8. *Let* $0 < \gamma < 3$, $0 < \gamma' < N-\gamma$, $0 < \zeta$ *and* $0 < \varepsilon < \min\{1, N-\gamma-\gamma', \zeta/2\}$. *If* $0 < \rho < 1$, *then*

$$k_\gamma * k_{\zeta,\gamma'} \le \frac{1}{\gamma_N(\gamma+\gamma')} \frac{1}{\rho^{N-\gamma-\gamma'}} + O\Big(\frac{1}{\rho^{N-\gamma-\gamma'-\varepsilon}}\Big).$$

If $1 \le \rho$, *then*

$$k_\gamma * k_{\zeta,g'} \lesssim e^{(\varepsilon-Q/2)\rho}.$$

Proof. By Lemma 5.6, we have for $0 < \rho < 1$ the estimate

$$k_\gamma \le \frac{1}{\gamma_N(\gamma)} \frac{1}{(\sinh\rho)^{N-\gamma}} + O\Big(\frac{1}{(\sinh\rho)^{N-\gamma-1}}\Big),$$

and, by (5.4), we have for any $0 < \varepsilon$ and $1 \le \rho$ the estimate

$$k_\gamma \sim \rho^{\gamma-2} e^{-Q\rho/2} \lesssim_\gamma e^{(\varepsilon-Q\rho/2)}.$$

Consequently, we obtain the following global estimate (i.e., for $0 < \rho$):

$$k_\gamma \le \frac{1}{\gamma_N(\gamma)} \frac{(\cosh\rho)^{N-Q/2-\gamma+\varepsilon}}{(\sinh\rho)^{N-\gamma}} + O\Big(\frac{(\cosh\rho)^{N-Q/2-\gamma+\varepsilon-1}}{(\sinh\rho)^{N-\gamma-1}}\Big).$$

Similarly, we have for $0 < \rho$ the global estimate

$$k_{\zeta,\gamma'} \le \frac{1}{\gamma_N(\gamma')} \frac{(\cosh\rho)^{N-Q/2-\gamma'-\zeta+\varepsilon}}{(\sinh\rho)^{N-\gamma'}} + O\Big(\frac{(\cosh\rho)^{N-Q/2+\varepsilon-\gamma'-\zeta-1}}{(\sinh\rho)^{N-\gamma'-1}}\Big).$$

Therefore, by Lemmas 5.2 and 5.3, there holds

$$k_\gamma * k_{\zeta,\gamma'} \le \frac{1}{\gamma_N(\gamma+\gamma')} \frac{1}{\rho^{N-\gamma-\gamma'}} + O\Big(\frac{1}{\rho^{N-\gamma-\gamma'-\varepsilon}}\Big)$$

for $0 < \rho < 1$.

Similarly, using Lemmas 5.4 and 5.5 we have

$$k_\gamma * k_{\zeta,\gamma'} \lesssim e^{(\varepsilon-Q\rho/2)}.$$

∎

Lemma 5.9. *Let* $0 < \gamma < 3$, $0 < \gamma' < N - \gamma$, $0 < \zeta$ *and* $0 < \zeta' < \zeta$. *If* $1 \le \rho$, *then*

$$k_\gamma * k_{\zeta,\gamma'} \lesssim e^{-(\zeta' + Q\rho/2)} + \rho^{\gamma-2}\, e^{-Q\rho/2} * k_{\zeta,\gamma'}.$$

Proof. Using (5.4), we have

$$\begin{aligned} k_\gamma * k_{\zeta,\gamma'} &= \int_{\{z\in\mathbb{B}^m_{\mathbb{F}}\,:\,\rho(z)<1/2\}} k_\gamma(\rho(z))\, k_{\zeta,\gamma'}(\rho(z,w))\, dV(z) \\ &\quad + \int_{\{z\in\mathbb{B}^m_{\mathbb{F}}\,:\,1/2\le\rho(z)<1\}} k_\gamma(\rho(z))\, k_{\zeta,\gamma'}(\rho(z,w))\, dV(z) \\ &\lesssim \int_{\{z\in\mathbb{B}^m_{\mathbb{F}}\,:\,\rho(z)<1/2\}} k_\gamma(\rho(z))\, k_{\zeta,\gamma'}(\rho(z,w))\, dV(z) \\ &\quad + \int_{\{z\in\mathbb{B}^m_{\mathbb{F}}\,:\,1/2\le\rho(z)<1\}} \rho(z)^{\gamma-2}\, e^{-Q/2\rho(z)}\, k_{\zeta,\gamma'}(\rho(z,w))\, dV(z) \\ &\le \int_{\{z\in\mathbb{B}^m_{\mathbb{F}}\,:\,\rho(z)<1/2\}} k_\gamma(\rho(z))\, k_{\zeta,\gamma'}(\rho(z,w))\, dV(z) + \rho^{\gamma-2}\, e^{-Q\gamma/2} * k_{\zeta,\gamma'}. \end{aligned}$$

Thus we need only show that, for $1 \le \rho$, there holds

$$\int_{\{z\in\mathbb{B}^m_{\mathbb{F}}\,:\,\rho(z)<1/2\}} k_\gamma(\rho(z))\, k_{\zeta,\gamma'}(\rho(z,w))\, dV(z) \lesssim e^{-(\zeta'+Q/2)\rho}.$$

By Lemma 5.6, we have that, for $\rho(z) < 1/2$, there holds

$$k_\gamma(\rho(z)) \lesssim \frac{1}{\rho(z)^{N-\gamma}} \sim \frac{1}{|z|^{N-\gamma}}.$$

Next, observing that $1 \le \rho(w)$ and $\rho(z) < 1/2$ imply $1/2 \le \rho(w) - \rho(z) \le \rho(z,w)$, we have by (5.4) that, for $0 < \zeta' < \zeta$, there holds

$$k_{\zeta,\gamma'}(\rho(z,w)) \lesssim e^{-\zeta'\rho(z,w) - \frac{Q}{2}\rho(z,w)} \sim (\cosh\rho(z,w))^{-(\zeta'+Q/2)}.$$

Combining these estimates with Lemma A, we compute

$$\begin{aligned} &\int_{\{z\in\mathbb{B}^m_{\mathbb{Q}}\,:\,\rho(z)<1/2\}} k_\gamma(\rho(z))\, k_{\zeta,\gamma'}(\rho(z,w))\, dV(z) \\ &\lesssim \int_{\{z\in\mathbb{B}^m_{\mathbb{Q}}\,:\,\rho(z)<1/2\}} \frac{1}{|z|^{4m-\gamma}} \left(\frac{\sqrt{(1-|w|^2)(1-|z|^2)}}{|1-\langle z,w\rangle_{\mathbb{Q}}|}\right)^{2m+1+\zeta'} \left(\frac{1}{1-|z|^2}\right)^{2m+2} dz \\ &\sim (1-|w|^2)^{(2m+1+\zeta')/2} \int_{\{z\in\mathbb{B}^m_{\mathbb{Q}}\,:\,\rho(z)<1/2\}} \frac{1}{|z|^{4m-\gamma}}\, dz \\ &\sim (\cosh\rho)^{-(2m+1+\zeta')} \sim e^{-(\zeta'+2m+1)\rho}. \end{aligned}$$

Similarly,

$$\int_{\{z\in\mathbb{B}_{\mathbb{C}a}:\rho(z)<1/2\}} k_\gamma(\rho(z))\,k_{\zeta,\gamma'}(\rho(z,w))\,dV(z)$$
$$\lesssim \int_{\{z\in\mathbb{B}_{\mathbb{C}a}:\rho(z)<1/2\}} \frac{1}{|z|^{16-\gamma}}\Big(\frac{(1-|w|^2)(1-|z|^2)}{\Psi_{\mathbb{C}a}(z,w)}\Big)^{(11+\zeta')/2}\Big(\frac{1}{1-|z|^2}\Big)^{12}dz$$
$$\sim (1-|w|^2)^{(11+\zeta')/2}\int_{\{z\in\mathbb{B}_{\mathbb{C}a}:\rho(z)<1/2\}}\frac{dz}{|z|^{16-\gamma}} \sim (\cosh\rho)^{-(11+\zeta')} \sim e^{-(\zeta'+11)\rho}. \quad \blacksquare$$

6. Rearrangement estimates

We first collect known results about nonincreasing rearrangements and Lorentz spaces on the hyperbolic spaces $\mathbb{X}$. These results will be used to prove estimates on $k_\gamma * k_{\zeta,\gamma'} * f$ for $f \in C_0^\infty(\mathbb{X})$.

To begin, let $f:\mathbb{X}\to\mathbb{R}$, and define

$$f^*(t) = \inf\{s>0 : \lambda_f(s)\le t\}$$
$$\lambda_f(s) = |\{z\in\mathbb{X} : |f(z)|>s\}| = \int_{z\in\mathbb{X}:|f(z)|>s} dV.$$

Next, for a domain $\Omega\subset\mathbb{X}$, we recall that the Lorentz spaces $L^{p,q}(\Omega)$ consist of functions for which the following norm is finite:

$$\|f\|_{L^{p,q}(\Omega)} = \begin{cases} \big\|t^{1/p-1/q}f^*(t)\big\|_{L^q(0,|\Omega|)} & \text{if } 1\le q<\infty,\\ \sup_{t>0} t^{1/p}f^*(t) & \text{if } q=\infty.\end{cases}$$

Define next $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds$ and

$$\|f\|^*_{L^{p,q}(\Omega)} = \begin{cases} \big\|t^{1/p-1/q}f^{**}(t)\big\|_{L^q(0,|\Omega|)|} & \text{if } 1\le q<\infty,\\ \sup_{t>0} t^{1/p}f^{**}(t) & \text{if } q=\infty.\end{cases}$$

Let $1<r, p_1, p_2<\infty$ and $1\le s, q_1, q_2\le\infty$ satisfy

$$\frac{1}{p_1}+\frac{1}{p_2}-1=\frac{1}{r} \quad\text{and}\quad \frac{1}{q_1}+\frac{1}{q_2}\ge\frac{1}{s},$$

and assume $f\in L^{p_1,q_1}(\mathbb{X})$ and $g\in L^{p_2,q_2}(\mathbb{X})$. The generalized Young inequality (see Theorem 2.6 in [66]),

$$\|f*g\|_{L^{r,s}} \le C\,\|f\|_{L^{p_1,q_1}}\,\|g\|_{L^{p_2,q_2}},$$

and the norm equivalence (see [66] for $1\le r<\infty$ and Theorem 3.4 in [75] for $0<r<1$)

$$\|f*g\|_{L^{q,r}} \le \|f*g\|^*_{L^{q,r}} \le \frac{q}{q-1}\|f*g\|_{L^{q,r}}$$

give the following lemma.

Lemma F. *Let* $1 < r, p_1, p_2 < \infty$ *and* $1 \le s, q_1, q_2 \le \infty$. *If*

$$\frac{1}{p_1} + \frac{1}{p_2} - 1 = \frac{1}{r} \quad \textit{and} \quad \frac{1}{q_1} + \frac{1}{q_2} \ge \frac{1}{s},$$

then for $f \in L^{p_1,q_1}(\mathbb{X})$ *and* $g \in L^{p_2,q_2}(\mathbb{X})$, *we have*

$$\|f * g\|_{L^{r,s}} \le C\, \|f\|_{L^{p_1,q_1}}\, \|g\|_{L^{p_2,q_2}}\,.$$

In this section, we collect the kernel estimates obtained above and state the corresponding estimates for their nonincreasing rearrangements. We also prove that the square integrability of the rearrangement $[k_\zeta * k_{\zeta,\gamma'}]^*$ on any interval of the form (c,∞), $0 < c$.

In preparation of obtaining the rearrangement estimates, we first estimate the volume of the geodesic ball B_ρ centered at the origin and with radius ρ. For $H^m_{\mathbb{Q}}$, we may use

$$|B_\rho| = \omega_{4m-1} \int_0^\rho (\sinh r)^{4m-1} (\cosh \rho)^3\, dr$$

to obtain

$$|B_\rho| = \frac{\omega_{4m-1}}{4m}\,\rho^{4m} + O(\rho^{4m+2}) \quad \text{if } 0 < \rho < 1$$

and

$$|B_\rho| \sim e^{(4m+2)\rho} \quad \text{if } 1 \le \rho.$$

Similarly, for $H_{\mathbb{C}a}$, we may use

$$|B_\rho| = \omega_{15} \int_0^\rho (\sinh r)^{15}\, (\cosh \rho)^7\, dr$$

to obtain

$$|B_\rho| = \frac{\omega_{15}}{16}\rho^{16} + O(\rho^{18}) \quad \text{if } 0 < \rho < 1$$

and

$$|B_\rho| \sim e^{22\rho} \quad \text{if } 1 \le \rho.$$

Next, we collect the kernel estimates established above. On $H^m_{\mathbb{F}}$ with $N = \dim_{\mathbb{R}} H^m_{\mathbb{F}}$, there holds the following.

- Let $0 < \zeta$. If $0 < \gamma < N$, $0 < \varepsilon < \min\{1, N-\gamma\}$ and $0 < \rho < 1$, then

$$k_{\zeta,\gamma} \le \frac{1}{\gamma_{N(\gamma)}}\, \frac{1}{\rho^{N-\gamma}} + O\Big(\frac{1}{\rho^{N-\gamma-\varepsilon}}\Big).$$

  If $0 < \gamma$ and $1 \le \rho$, then

$$k_{\zeta,\gamma} \sim \rho^{(\gamma-2)/2}\, e^{-(\zeta+Q/2)\rho}.$$

- Let $\zeta = 0$. If $0 < \gamma < 3$ and $0 < \rho < 1$, then

$$k_\gamma \le \frac{1}{\gamma_N(\gamma)}\, \frac{1}{\rho^{N-\gamma}} + O\Big(\frac{1}{\rho^{N-\gamma-1}}\Big).$$

  If $0 < \gamma < 3$ and $1 \le \rho$, then

$$k_\gamma \sim \rho^{\gamma-2}\, e^{-Q\rho/2}.$$

- Let $0 < \zeta$. If $0 < \gamma < 3$, $0 < \gamma' < N - \gamma$, $0 < \varepsilon < \min\{1, N - \gamma - \gamma', \zeta/2\}$ and $0 < \rho < 1$, then

$$k_\gamma * k_{\zeta,\gamma'} \leq \frac{1}{\gamma_N(\gamma+\gamma')} \frac{1}{\rho^{N-\gamma-\gamma'}} + O\Big(\frac{1}{\rho^{N-\gamma-\gamma'-\varepsilon}}\Big).$$

  If $1 \leq \rho$, then

$$k_\gamma * k_{\zeta,g'} \lesssim e^{(\varepsilon - Q/2)\rho}.$$

  If $0 < \zeta' < \zeta$ and $1 \leq \rho$, then

$$k_\gamma * k_{\zeta,\gamma'} \lesssim e^{-(\zeta'+Q/2)\rho} + \rho^{\gamma-2} e^{-Q\rho/2} * k_{\zeta,\gamma'}.$$

  The corresponding estimates for their rearrangements are listed now.

- Let $0 < \zeta$. If $0 < \gamma < N$, $0 < \varepsilon < \min\{1, N - \gamma\}$ and $0 < t < 2$, then

$$[k_{\zeta,\gamma}]^* \leq \frac{1}{\gamma_{N(\gamma)}} \Big(\frac{N}{\omega_{N-1}}\, t\Big)^{(\gamma-N)/N} + O\big(t^{(\gamma+\varepsilon-N)/N}\big).$$

  If $0 < \gamma$ and $2 \leq t$, then

$$[k_{\zeta,\gamma}]^* \sim t^{-1/2-\zeta/N} (\ln t)^{(\gamma-2)/2}$$

- Let $\zeta = 0$. If $0 < \gamma < 3$ and $0 < t < 2$, then

$$[k_\gamma]^* \leq \frac{1}{\gamma_N(\gamma)} \Big(\frac{N}{\omega_{N-1}}\, t\Big)^{(\gamma-N)/N} + O\big(t^{(\gamma+1-N)/N}\big).$$

  If $0 < \gamma < 3$ and $2 \leq t$, then

$$[k_\gamma]^* \sim t^{-1/2} (\ln t)^{\gamma-2}.$$

- Let $0 < \zeta$. If $0 < \gamma < 3$, $0 < \gamma' < N - \gamma$, $0 < \varepsilon < \min\{1, N - \gamma - \gamma', \zeta/2\}$ and $0 < t < 2$, then

$$[k_\gamma * k_{\zeta,\gamma'}]^* \leq \frac{1}{\gamma_N(\gamma+\gamma')} \Big(\frac{N}{\omega_{N-1}}\, t\Big)^{(\gamma+\gamma'-N)/N} + O\big(t^{(\gamma+\gamma'+\varepsilon-N)/N}\big). \tag{6.1}$$

  If $2 \leq t$, then

$$[k_\gamma * k_{\zeta,g'}]^* \lesssim t^{(\varepsilon-Q/2)/N}. \tag{6.2}$$

  Moreover, using Lemma 5.9, we have, for $c > 0$,

$$\int_c^\infty |[k_\alpha * k_{\zeta,\beta}]^*(t)|^2\, dt < \infty. \tag{6.3}$$

  The proof of (6.3) is similar to that given in Lemma 4.1 of [54], and we omit it.

6.1. Estimates for $k_\gamma * k_{\zeta,\gamma'} * f$

In this section, we prove an L^p-$L^{p'}$ inequality for $k_\gamma * k_{\zeta,\gamma'} * f$, which is dual to the Poincaré–Sobolev inequality. We will need to make use of the Kunze–Stein phenomenon. The Kunze–Stein phenomenon is important in harmonic analysis (see [16–18, 39, 47, 48, 67, 69]), and is closely related to geometric and functional inequalities, as has been explored by Beckner, along with symmetry in Fourier analysis, see e.g., [9, 10]. In particular, in [9] and [7], Beckner identified for the first time the sharp constants for the Kunze–Stein inequalities on $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$ and the Lorentz groups, among other things.

We begin by recalling relevant results. The proofs of Lemmas G and H may be found in [63].

Cowling, Giulini and Meda (see [16–18]) established the following sharp version on the Lorentz space (see [38,66]) of the Kunze–Stein phenomenon for connected real simple groups G of real rank one with finite center:

$$L^{p,q_1}(G) * L^{p,q_2} \subset L^{p,q_3}(G)$$

provided $1 < p < 2$, $1 \le q_1, q_2, q_3 \le \infty$ and $1 + 1/q_3 \le 1/q_1 + 1/q_2$. In particular, this applies to $\mathrm{Sp}(m,1)$ and F_4, and by following [63], we can obtain similar phenomenon on $H^m_{\mathbb{Q}}$ and $H^2_{\mathbb{Q}}$. To be more precise, let $L^p(G)$ and $L^{p,q}(G)$ denote the usual Lebesgue and Lorentz spaces, respectively, and let $L^{p,q}(G/K)$, $L^{p,q}(K\backslash G)$ and $L^{p,q}(K\backslash G/K)$ denote the closed subspaces of $L^{p,q}(G)$ of the right-K-invariant, left-K-invariant and K-bi-invariant functions, respectively. Following [63], we can show:

Lemma G. *For $p \in (1,2)$, there holds*

$$L^p\,(K\backslash G) * L^p\,(G/K) \subset L^{p,\infty}(K\backslash G/K).$$

Lemma H. *For $p \in (1,2)$ and $p' = p/(p-1)$, there holds*

$$L^{p',1}\,(K\backslash G/K) * L^p(G/K) \subset L^{p'}(G/K)$$

and, if $f \in L^{p,1}(K\backslash G/K)$ and $h \in L^p(G/K)$, then there is a constant $C > 0$ such that

$$\|f * h\|_{L^{p'}(G/K)} \le C\,\|f\|_{L^{p',1}(K\backslash G/K)}\,\|h\|_{L^p(G/K)}.$$

Using Lemma H, we prove the following estimate on $k_\gamma * k_{\zeta,\gamma'} * f$.

Lemma 6.1. *Let $0 < \gamma < 3$, $0 < \gamma' < N - \gamma$, $0 < \zeta$ and $\frac{2N}{N+\gamma+\gamma'} \le p < 2$. Then, for $f \in C_0^\infty(\mathbb{B}^m_{\mathbb{F}})$, there holds*

$$\|k_\gamma * k_{\zeta,\gamma'} * f\|_{p'} \le C\,\|f\|_p.$$

Proof. Define the cut off functions

$$\eta_1(\rho) = \begin{cases} k_\gamma * k_{\zeta,\gamma'} & \text{for } 0 < \rho < 1, \\ 0 & \text{for } 1 \le \rho, \end{cases} \quad \text{and} \quad \eta_2(\rho) = k_\gamma * k_{\zeta,\gamma'} - \eta_1(\rho).$$

By (6.1), there exists $t_0 > 0$ such that, for $0 < t \le t_0$, there holds

$$\eta_1^*(t) \lesssim t^{(\gamma+\gamma'-N)/N},$$

and, for $t_0 < t$, there holds

$$\eta_1^*(t) = 0.$$

Next, by Lemma F, there holds

$$\|\eta_1 * f\|_{L^{p'}} = \|\eta_1 * f\|_{L^{p',p'}} \le C\,\|\eta_1\|_{L^{p'/2,\infty}}\,\|f\|_{L^p}.$$

But

$$\|\eta_1\|_{L^{p'/2,\infty}} = \sup_{0<t<\infty} t^{2/p'}\,\eta_1^*(t) \lesssim \sup_{0<t<t_0} t^{2/p'+(\gamma+\gamma'-N)/N} < \infty,$$

provided

$$\frac{2}{p'} + \frac{\gamma+\gamma'-N}{N} > 0,$$

which is equivalent to

$$p > \frac{2N}{\gamma+\gamma'+N},$$

as it is assumed. Consequently, there holds

$$\|\eta_1 * f\|_{L^{p'}} \lesssim \|f\|_{L^p}.$$

Next, by (6.2), there exists $0 < t_0$ such that, for $0 < t \le t_0$, there holds

$$\eta_2(t) \lesssim 1,$$

and, for $t_0 < t$ and $0 < \varepsilon < \min\{1, N-\gamma-\gamma', \zeta/2\}$, there holds

$$\eta_2^*(t) \lesssim t^{(\varepsilon-Q/2)/N}.$$

Consequently, we find that, for $0 < \varepsilon < Q/2 + N/p$,

$$\|\eta_2\|_{L^{p',1}} = \int_0^\infty t^{1/p'-1}\,\eta_2^*(t)\,dt < \infty.$$

Finally, by Lemma H, we obtain

$$\|\eta_2 * f\|_{L^{p'}} \le C\,\|f\|_{L^p},$$

and therefore

$$\|k_\gamma * k_{\zeta,\gamma'} * f\|_{L^p} \le \|\eta_1 * f\|_{L^{p'}} + \|\eta_2 * f\|_{L^p} \le C\,\|f\|_{L^p},$$

as desired. ∎

7. Proofs of Theorems 1.3 and 1.4

With all the kernel estimates proved in Section 5, we are ready to prove the Poincaré–Sobolev inequality of Theorem 1.3, and the Hardy–Sobolev–Maz'ya inequality of Theorem 1.4. For the reader's convenience, we restate these theorems before their respective proofs.

Theorem 1.3. *Let $0 < \gamma < 3$, $0 < \gamma'$, $2 < p$ and $0 < \zeta$. Denote by $N = \dim \mathbb{X}$. If $0 < \gamma' < N - \gamma$, suppose further that $2 < p \leq \frac{2N}{N-(\gamma+\gamma')}$. Then there exists a constant $C > 0$ such that, for all $u \in C_0^\infty(\mathbb{X})$, there holds*

$$\|u\|_p \leq C \left\| (-\Delta_{\mathbb{X}} - \rho_{\mathbb{X}}^2 + \zeta^2)^{\gamma'/4} (-\Delta - \rho_{\mathbb{X}}^2)^{\gamma/4} u \right\|_2.$$

Proof. By Lemma 6.1, we have

$$\left\| (-\Delta_{\mathbb{X}} - \rho_{\mathbb{X}}^2 + \zeta^2)^{-\gamma'/4} (-\Delta - \rho_{\mathbb{X}}^2)^{-\gamma/4} u \right\|_{L^{p'}} \leq C \|u\|_{L^p}. \tag{7.1}$$

Consulting the Lemma in the Appendix of [8], we have that (7.1) is equivalent to

$$\|u\|_{L^p} \leq C \left\| (-\Delta_{\mathbb{X}} - \rho_{\mathbb{X}}^2 + \zeta^2)^{\gamma'/4} (-\Delta - \rho_{\mathbb{X}}^2)^{\gamma/4} u \right\|_{L^2},$$

thereby proving the theorem. ∎

Proof of Theorem 1.4. We need only prove the inequality in case

$$\lambda = \prod_{j=1}^{k} \frac{(a-k+2j-2)^2}{4}.$$

We will use the factorization theorem (Theorem 1.1), and so we set

$$u = \varrho^{(k-(2m+1)-a)/2} f,$$

and obtain

$$\begin{aligned}
4^k \int_{\mathbb{H}_{\mathbb{Q}}^{m-1}} \int_0^\infty u \prod_{j=1}^{k} \left[-\varrho \partial_{\varrho\varrho} - a\partial_\varrho - \varrho \Delta_Z - \mathcal{L}_0 + i(k+1-2j)\sqrt{-\Delta_Z}\, \right] u \,\frac{dx\,dz\,d\varrho}{\varrho^{1-a}} \\
= \int_{\mathbb{H}_{\mathbb{Q}}^{m-1}} \int_0^\infty f \prod_{j=1}^{k} \left[-\Delta - (2m+1)^2 + (a-k+2j-2)^2 \right] f \,\frac{dx\,dz\,d\varrho}{\varrho^{2m+2}} \\
= 4 \int_{\mathcal{U}^m} f \prod_{j=1}^{k} \left[-\Delta - (2m+1)^2 + (a-k+2j-2)^2 \right] f \,dV.
\end{aligned}$$

Next, using that $\operatorname{spec}(-\Delta) = [(2m+1)^2, \infty)$, we have the following sharp inequality:

$$\int_{\mathcal{U}^m} f \prod_{j=1}^{k} \left[-\Delta - (2m+1)^2 + (a-k+2j-2)^2 \right] f \,dV \geq \prod_{j=1}^{k} (a-k+2j-2)^2 \int_{\mathcal{U}^m} f^2 \,dV.$$

Applying Plancherel's theorem, there holds

$$\int_{\mathcal{U}^m} f\prod_{j=1}^{k}\left[-\Delta-(2m+1)^2+(a-k+2j-2)^2\right] f\,dV-\prod_{j=1}^{k}(a-k+2j-2)^2\int_{\mathcal{U}^m} f^2\,dV$$
$$= C_m\int_{-\infty}^{\infty}\int_{S^{4m-1}}\left[\prod_{j=1}^{k}(\lambda^2+(a-k+2j-2)^2)-\prod_{j=1}^{k}(a-k+2j-2)^2\right] \times|\widehat{f}(\lambda,\varsigma)|^2\,|\mathrm{c}(\lambda)|^{-2}\,d\lambda\,d\sigma(\varsigma).$$

Choosing $0<\delta$ so that

$$\prod_{j=1}^{k}(\lambda^2+(a-k+2j-2)^2)-\prod_{j=1}^{k}(a-k+2j-2)^2\geq\lambda^2(\lambda^2+\delta)^{k-1},$$

applying Theorem 1.3, and applying the Plancherel theorem, we obtain

$$\int_{\mathcal{U}^m} f\prod_{j=1}^{k}\left[-\Delta-(2m+1)^2+(a-k+2j-2)^2\right] f\,dV-\prod_{j=1}^{k}(a-k+2j-2)^2\int_{\mathcal{U}^m} f^2\,dV$$
$$\geq C_m\int_{-\infty}^{\infty}\int_{S^{4m-1}}\lambda^2(\lambda^2+\delta)^{k-1}\,|\widehat{f}(\lambda,\varsigma)|^2\,|\mathrm{c}(\lambda)|^{-2}\,d\lambda\,d\sigma(\varsigma)$$
$$=\int_{\mathcal{U}^m} f(-\Delta-(2m+1)^2)\,(-\Delta-(2m+1)^2+\delta)^{k-1} f\,dV\geq C\|f\|_{L^p}^2.$$

This proves the first inequality. The proof of the second inequality is similar, and we omit it. ■

8. Proofs of Theorems 1.5 and 1.6

Proof of Theorem 1.5. Set $\Omega(u)=\{x\in\mathbb{B}_{\mathbb{C}}^n:|u(x)|\geq 1\}$. Then by Theorem 1.3, we have, for $p>2$,

$$|\Omega(u)|=\int_{\Omega(u)}dV\leq\int_{\mathbb{X}}|u|^p\,dV\lesssim 1.$$

Therefore, $|\Omega(u)|\leq\Omega_0$ for some constant Ω_0 independent of u. We write

$$\int_{\mathbb{B}_{\mathbb{C}}^n}(e^{\beta_0(N/2,N)u^2}-1-\beta_0(N/2,N)u^2)\,dV$$
$$=\int_{\Omega(u)}(e^{\beta_0(N/2,N)u^2}-1-\beta_0(N/2,N)u^2)\,dV$$
$$+\int_{\mathbb{X}\setminus\Omega(u)}(e^{\beta_0(N/2,N)u^2}-1-\beta_0(N/2,N)u^2)\,dV$$
$$\leq\int_{\Omega(u)}e^{\beta_0(N/2,N)u^2}\,dV+\int_{\mathbb{X}\setminus\Omega(u)}(e^{\beta_0(n,2n)u^2}-1-\beta_0(N/2,N)u^2)\,dV. \tag{8.1}$$

The second part of right-hand of (8.1) is bounded. In fact, we have

$$\int_{\mathbb{X}\setminus\Omega(u)} (e^{\beta_0(N/2,N)u^2} - 1 - \beta_0(N/2,N)u^2)\, dV = \int_{\mathbb{X}\setminus\Omega(u)} \sum_{n=2}^{\infty} \frac{(\beta_0(N/2,N)u^2)^n}{n!}\, dV$$

$$\leq \int_{\mathbb{X}\setminus\Omega(u)} \sum_{n=2}^{\infty} \frac{(\beta_0(N/2,N))^n u^4}{n!}\, dV \leq \sum_{n=2}^{\infty} \frac{(\beta_0(N/2,N))^n}{n!} \int_{\mathbb{X}} |u(x)|^4\, dV \leq C.$$

Here we use the fact $|u(z)| < 1$, $z \in \mathbb{X} \setminus \Omega(u)$, and Theorem 1.3.

Next we shall show that

$$\int_{\Omega(u)} e^{\beta_0(N/2,N)u^2}\, dV$$

is also bounded by some universal constant. Set

$$v = (-\Delta_{\mathbb{X}} - \rho_{\mathbb{X}}^2 + \zeta^2)^{(2n-\alpha)/4} (-\Delta_{\mathbb{X}} - \rho_{\mathbb{X}}^2)^{\alpha/4} u.$$

Then

$$\int_{\mathbb{X}} |v|^2\, dV \leq 1,$$

and we can write u as a potential,

$$u = (-\Delta_{\mathbb{X}} - \rho_{\mathbb{X}}^2 + \zeta^2)^{-(2n-\alpha)/4} (-\Delta_{\mathbb{X}} - \rho_{\mathbb{X}}^2)^{-\alpha/4} v = v * \phi,$$

where

$$\phi = (-\Delta_{\mathbb{X}} - \rho_{\mathbb{X}}^2 + \zeta^2)^{-(2n-\alpha)/4} (-\Delta_{\mathbb{X}} - \rho_{\mathbb{X}}^2)^{-\alpha/4} = k_{\zeta,(N-\alpha)/2} * k_{\alpha/2}.$$

By (6.1) and (6.3),

$$\phi^*(t) \leq \frac{1}{\gamma_N(N/2)} \cdot \left(\frac{Nt}{\omega_{N-1}}\right)^{-1/2} + O(t^{(\varepsilon-n)/(2n)}), \quad \text{for } 0 < t < 2,$$

and

$$\int_c^{\infty} |\phi^*(t)|^2\, dt < \infty, \quad \forall c > 0.$$

Closely following the proof of Theorem 1.7 in [53], we have that there exists a constant C, which is independent of u and $\Omega(u)$, such that

$$\int_{\Omega(u)} e^{\beta_0(N/2,N)u^2}\, dV = \int_0^{|\Omega(u)|} \exp(\beta_0(N/2,N)\,|u^*(t)|^2)\, dt$$

$$\leq \int_0^{\Omega_0} \exp(\beta_0(N/2,N)\,|u^*(t)|^2)\, dt \leq C.$$

The proof of Theorem 1.5 is thereby completed. ■

Proof of Theorem 1.6. It is enough to show that in terms of the ball model, for some $\zeta > 0$, there holds

$$\begin{aligned}&\big\|(-\Delta_{\mathbb{X}} - \rho_{\mathbb{X}}^2 + \zeta^2)^{(2m-1)/2}(-\Delta_{\mathbb{X}} - \rho_{\mathbb{X}}^2)^{1/2}[(1-|z|^2)^{(a+1)/2}u]\big\|_2 \\ &\le 4^{2m}\int_{\mathbb{B}_{\mathbb{Q}}^n} u\prod_{j=1}^{2m}\Big[\Delta'_{(1-a-(2m+1))/2} + \frac{(2m+1-2j)^2}{4} - i\,\frac{2m+1-2j}{2}\sqrt{\Gamma+1}\,\Big] \\ &\qquad\qquad \times u\,\frac{dz}{(1-|z|^2)^{1-a}} \\ &\quad -\prod_{j=1}^{2m}(a-2m+2j-2)^2\int_{\mathbb{B}_{\mathbb{Q}}^n}\frac{u^2}{(1-|z|^2)^{2m+1-a}}\,dz,\end{aligned}$$

and in terms of the Siegel domain,

$$\begin{aligned}&\big\|(-\Delta_{\mathbb{X}} - \rho_{\mathbb{X}}^2 + \zeta^2)^{(2m-1)/2}(-\Delta_{\mathbb{X}} - \rho_{\mathbb{X}}^2)^{1/2}[\varrho^{(a+1)/2}u]\big\|_2 \\ &\le 4^{2m}\int_{\mathbb{H}_{\mathbb{Q}}^{m-1}}\int_0^\infty u\prod_{j=1}^{n}\big[-\varrho\partial_{\varrho\varrho} - a\partial_\varrho - \varrho\Delta_Z + \mathcal{L}_0 + i(k+1-2j)\sqrt{-\Delta_Z}\,\big] \\ &\qquad\qquad \times u\,\frac{dx\,dz\,d\varrho}{\varrho^{1-a}} \\ &\quad -\prod_{j=1}^{2m}(a-n+2j-2)^2\int_{\mathbb{H}_{\mathbb{Q}}^{-1}}\int_0^\infty\frac{u^2}{\varrho^{2m+1-a}}\,dx\,dz\,d\varrho.\end{aligned}$$

The proof is similar to that given in the proof of Theorem 1.4 via Plancherel's formula, and we omit it. ■

A. Proofs of Theorems 1.7 and 1.8

In this appendix, we will outline the proofs of the Adams inequalities, namely Theorems 1.7 and 1.8 for the convenience of the reader. We refer the interested reader to [53, 54, 60, 63] for all the details.

Proof of Theorem 1.7. Let $f = (-\Delta_{\mathbb{X}} - \rho_{\mathbb{X}}^2 + \zeta^2)^{\alpha/2}u$. Then $\|f\|_p \le 1$ and

$$u = (-\Delta_{\mathbb{X}} - \rho_{\mathbb{X}}^2 + \zeta^2)^{-\alpha/2} f = f * k_{\zeta,\alpha}$$

Using O'Neil's lemma ([66]), we have for $t > 0$,

$$u^*(t) \le \frac{1}{t}\int_0^t f^*(s)\,ds\int_0^t k_{\zeta,\alpha}^*(s)\,ds + \int_t^\infty f^*(s)\,k_{\zeta,\alpha}^*(s)\,ds.$$

Using the rearrangement estimates of $[k_{\zeta,\alpha}]^*$, it is easy to check that

$$\begin{aligned}&[k_{\zeta,\alpha}]^*(t) \le \frac{1}{\gamma_N(\alpha)}\Big(\frac{Nt}{\omega_{2n-1}}\Big)^{(\alpha-N)/N} + O\big(t^{(\alpha+\varepsilon-N)/N}\big), \quad \text{for } 0 < t < 2; \\ &\int_c^\infty |[k_{\zeta,\alpha}]^*(t)|^{p'}\,dt < \infty, \quad \forall c > 0.\end{aligned}$$

Closely following the proof of Theorem 1.13 in [54], we have that there exists a constant C, which is independent of u, such that

$$\frac{1}{|E|}\int_E \exp(\beta_0(\alpha,N)\,|u|^{p'})\,dV \le \frac{1}{|E|}\int_0^{|E|} \exp(\beta_0(\alpha,N)|u^*(t)|^{p'})\,dt$$
$$\le \frac{1}{|E|}\int_0^{|E|} \exp\Big(\beta_0(\alpha,N)\Big|\frac{1}{t}\int_0^t f^*(s)\,ds\int_0^t k^*_{\zeta,\alpha}(s)\,ds + \int_t^\infty f^*(s)\,k^*_{\zeta,\alpha}(s)\,ds\Big|^{p'}\Big)\,dt$$
$$\le C.$$

The sharpness of the constant $\beta_0(\alpha,N)$ can be verified by a process similar to that in [1,46], and thus the proof of Theorem 1.7 is completed. ■

Using the symmetrization-free argument from the local inequalities to global ones developed by Lam and the second author in [49,50], and subsequently used, e.g., in [12, 51,52,55], etc., we can conclude the:

Proof of Theorem 1.8. Let $u \in W^{\alpha,p}(\mathbb{X})$ with

$$\int_{\mathbb{X}} |(-\Delta_{\mathbb{X}} - \rho_{\mathbb{X}}^2 + \zeta^2)^{\alpha/2} u|^p\,dV \le 1.$$

By a Hörmander–Mikhlin type multiplier theorem (see [2]), we have

$$\int_{\mathbb{X}} |u|^p\,dV \lesssim \int_{\mathbb{X}} |(-\Delta_{\mathbb{X}} - \rho_{\mathbb{X}}^2 + \zeta^2)^{\alpha/2} u|^p\,dV \le 1$$

provided $\zeta > 2\rho_{\mathbb{X}}|1/2 - 1/p|$. Set $\Omega(u) = \{z \in \mathbb{X} : |u(z)| \ge 1\}$. Then we have

$$|\Omega(u)| = \int_{\Omega(u)} dV \le \int_{\mathbb{X}} |u|^p\,dV \le \Omega_0,$$

where Ω_0 is a constant independent of u. We write

$$\int_{\mathbb{X}} \Phi_p(\beta_0(\alpha,N)\,|u|^{p'})\,dV$$
$$= \int_{\Omega(u)} \Phi_p(\beta_0(\alpha,N)\,|u|^{p'})\,dV + \int_{\mathbb{X}\setminus\Omega(u)} \Phi_p(\beta_0(\alpha,N)\,|u|^{p'})\,dV.$$

Notice that on the domain $\mathbb{X}\setminus\Omega(u)$, we have $|u(z)| < 1$. Thus,

$$\int_{\mathbb{X}\setminus\Omega(u)} \Phi_p(\beta_0(\alpha,N)\,|u|^{p'})\,dV \le \sum_{k=j_p-1}^{\infty} \frac{\beta_0(\alpha,N)^k}{k!}\int_{\mathbb{X}\setminus\Omega(u)} \sum_{n=2}^{\infty} |u|^{p'k}\,dV$$
$$\le \sum_{k=j_p-1}^{\infty} \frac{\beta_0(\alpha,N)^k}{k!}\int_{\mathbb{X}\setminus\Omega(u)} \sum_{n=2}^{\infty} |u|^{p}\,dV \tag{A.1}$$
$$\le \sum_{k=j_p-1}^{\infty} \frac{\beta_0(\alpha,N)^k}{k!}\,\|u\|_p^p \le C.$$

Moreover, by Theorem 1.7, if ζ satisfies $\zeta > 0$ if $1 < p < 2$ and $\zeta > 2n\,|1/p - 1/2|$ if $p \geq 2$, then

$$\int_{\Omega(u)} \Phi_p(\beta_0(\alpha, N)\,|u|^{p'})\,dV \leq \int_{\Omega(u)} \exp(\beta_0(\alpha, N)\,|u|^{p'})\,dV \leq C. \tag{A.2}$$

Combining (A.1) and (A.2) yields

$$\begin{aligned}\int_{\mathbb{X}} &\Phi_p(\beta_0(\alpha, N)\,|u|^{p'})\,dV \\ &= \int_{\Omega(u)} \Phi_p(\beta_0(\alpha, N)\,|u|^{p'})\,dV + \int_{\mathbb{X}\setminus\Omega(u)} \Phi_p(\beta_0(\alpha, N)\,|u|^{p'})\,dV \leq C\end{aligned}$$

provided that ζ satisfies $\zeta > 2\rho_{\mathbb{X}}\,|1/p - 1/2|$.

The sharpness of the constant $\beta_0(\alpha, N)$ can be verified by a process similar to that in [54]. ■

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