



Almost complex structures, transverse complex structures, and transverse Dolbeault cohomology

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Abstract. We define a transverse Dolbeault cohomology associated to any almost complex structure j on a smooth manifold M . This we do by extending the notion of transverse complex structure and by introducing a natural j -stable involutive limit distribution with such a transverse complex structure. We relate this transverse Dolbeault cohomology to the generalized Dolbeault cohomology of (M, j) introduced by Cirici and Wilson in 2001, showing that the $(p, 0)$ cohomology spaces coincide. This study of transversality leads us to suggest a notion of minimally non-integrable almost complex structure.

Introduction

Dolbeault cohomology is defined for a manifold endowed with an integrable almost complex structure. There have been various ways to extend this cohomology to a manifold endowed with a non integrable almost complex structure. An almost complex structure j is a smooth field of endomorphisms of the tangent bundle whose square is minus the identity ($j^2 = -\text{Id}$). It induces a splitting of the complexified tangent bundle $TM^{\mathbb{C}} = T_j^{1,0} \oplus T_j^{0,1}$ into $\pm i$ eigenspaces for j , a corresponding dual splitting of the complexified cotangent bundle, and a splitting of complex valued k -forms on M :

$$\Omega^k(M, \mathbb{C}) = \bigoplus_{p+q=k} \Omega_j^{p,q}.$$

The exterior differential d has the property that

$$d\Omega_j^{p,q} \subset \Omega_j^{p-1,q+2} \oplus \Omega_j^{p,q+1} \oplus \Omega_j^{p+1,q} \oplus \Omega_j^{p+2,q-1},$$

and splits accordingly as

$$d = \bar{\mu}_j \oplus \bar{\partial}_j \oplus \partial_j \oplus \mu_j.$$

An almost complex structure j is integrable if and only if $d = \bar{\partial}_j \oplus \partial_j$, i.e., $\bar{\mu}_j = \mu_j = 0$, which is equivalent to the vanishing of the Nijenhuis tensor (also called torsion) N^j of j . The operator $\bar{\partial}_j$ (often the subscript j is omitted) defines then the Dolbeault cohomology

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of the complex manifold (M, j) . An important feature about this cohomology comes from Hodge theory [9], which relates it to harmonic forms. This requires the choice of a Riemannian metric g compatible with j . Many works are devoted to the study of properties of $\bar{\partial}$ -harmonic (p, q) -forms or variations of those when j is not integrable; see for instance [17, 18], the recent review [19] by Zhang of Hodge theory for almost complex manifolds, and papers quoted there. In Hirzebruch's 1954 problem list [8], a question attributed to Kodaira and Spencer concerning these spaces asks whether the dimension of the space of $\bar{\partial}$ -harmonic (p, q) -forms depends on the choice of the Riemannian metric, or only on the almost complex structure j . In 2020, Holt and Zhang gave in [10] examples showing that the dimension may depend on the choice of the metric. This raises the interest of a cohomology depending only on the almost complex structure j .

Li and Zhang introduced j -invariant and j -anti-invariant cohomologies in [12]; this was studied on 4-dimensional manifolds in [6]. J. Cirici and S. Wilson gave in 2021 in [4] the definition of a generalized Dolbeault cohomology associated to an almost complex structure j . It is defined as the cohomology of the operator induced by $\bar{\partial}_j$ on the cohomology spaces for the operator $\bar{\mu}_j$. Some of these cohomology spaces are infinite dimensional in the non integrable case [5]. Sillari and Tomassini compared the two constructions in [14], and have recently defined Bott–Chern and Aeppli cohomologies associated to an almost complex structure [15].

In the present paper, we suggest another Dolbeault cohomology defined only in terms of the almost complex structure j , which we call transverse Dolbeault cohomology. It is defined as the (usual) Dolbeault cohomology of a natural transverse complex structure induced by the given almost complex structure.

We consider the involutive j stable (generalized) distribution defined by the real part of the limit of the derived flag of distributions associated to the $+i$ eigenspace of j . It is a natural involutive limit of a sequence of nested real distributions $\mathcal{D}_j^\infty = \bigcup_k \mathcal{D}_j^{(k)}$ associated to j , which contains the image of the Nijenhuis tensor N^j . We extend the notion of transverse complex structure to that setting. The cohomology we define is the cohomology of the $\bar{\partial}_j$ operator restricted to a subspace of forms which are clearly in the kernel of $\bar{\mu}_j$: the forms whose contraction and Lie derivative with respect to a vector field in \mathcal{D}_j^∞ vanish.

Equivalently, it is the cohomology of the operator $\bar{\partial}_j$ restricted to the largest subspace $\Omega_{\mathcal{D}_j^\infty}(M)$ of smooth complex forms on the manifold, which has the property that it consists of forms vanishing whenever contracted with a vector field in the image of the Nijenhuis tensor of j , which is stable under the differential d and which splits into (p, q) components relatively to j (i.e., for ω a k -form in the subspace, $\omega \circ j_r$ is in the subspace for any $r \leq k$, where $\circ j_r$ indicates the precomposition with j acting on the r -th argument of ω).

If one of the derived distribution is involutive and has constant rank, and if the space of leaves of its real part has a manifold structure making the canonical projection a submersion, then j induces a complex structure on this quotient manifold and the cohomology we define coincides with the Dolbeault cohomology of this space of leaves.

An almost complex structure is maximally non-integrable [2] when the image of the Nijenhuis tensor at each point p spans the whole tangent space at p . For a maximally non-integrable almost complex structure, all our transverse Dolbeault cohomology spaces

vanish. The first geometrical non-integrable almost complex structure was given by Eells and Salamon in [7]; it arises on a twistor space when flipping the sign of the vertical part of the standard integrable almost complex structure on this space and it is maximally non-integrable; this property remains true for a similar construction on many twistor spaces [3]. Maximally non integrable almost complex structures are generic in high dimension: R. Coelho, G. Placini, and J. Stelzig prove in [5] that in dimension $2n \geq 10$ any almost complex structure on a $2n$ -dimensional manifold is homotopic to a maximally non-integrable one.

A minimality condition for a non-integrable almost complex structure is given by asking the involutivity of the first derived distribution of the $+i$ eigenspace of j . This corresponds to the existence of a maximal transverse complex structure. Examples are given by complex line bundles over complex manifolds with no holomorphic structure with the j naturally defined when choosing a connection. Another example, coming from an invariant situation, is given by the Kodaira–Thurston 4-dimensional compact symplectic manifold with no Kähler structure, endowed with an almost complex structure which is compatible and positive with respect to the symplectic structure, and which is induced by an invariant one on the covering group. We compute the transverse Dolbeault cohomology of this example.

In Section 1, we describe the derived flag of distributions associated to the $+i$ eigenspace of an almost complex structure j and the involutive real limit distribution \mathcal{D}_j^∞ .

In Section 2, we define \mathcal{D} -transverse objects when $\mathcal{D} = \Gamma^\infty(D)$ is the space of smooth sections of a real smooth involutive distribution D (not necessarily of constant rank), or an involutive limit of an increasing sequence of spaces of sections such as \mathcal{D}_j^∞ . The definitions are chosen so that, in the case where $\mathcal{D} = \Gamma^\infty(D)$ with D of constant dimension and such that the space of leaves for D , denoted by M/D , has a manifold structure making the canonical projection $p: M \rightarrow M/D$ a submersion, \mathcal{D} -transverse objects correspond to objects on the space of leaves M/D . We define in the general context \mathcal{D} -transverse vector fields, \mathcal{D} -transverse almost complex and complex structures, \mathcal{D} -transverse forms and \mathcal{D} -transverse Dolbeault cohomology.

In Section 3, we define the transverse Dolbeault cohomology associated to an almost complex structure j : it is the \mathcal{D} -transverse Dolbeault cohomology in the sense of Section 2 when \mathcal{D} is the real limit distribution \mathcal{D}_j^∞ defined in Section 1. We prove that the $(p, 0)$ -cohomology spaces for this transverse cohomology coincide with the $(p, 0)$ -cohomology spaces for the Dolbeault cohomology introduced by Cirici and Wilson.

In Section 4, we suggest a minimality condition for a non-integrable almost complex structure, we write this condition in a homogeneous framework, and we study the example of the Kodaira–Thurston 4-dimensional manifold.

1. Derived distributions associated to an almost complex structure

The *Nijenhuis tensor*, also called *torsion*, associated to a smooth field k of endomorphisms of the tangent bundle is the tensor of type $(1, 2)$ defined by

$$(1.1) \quad N^k(X, Y) := [kX, kY] - k[kX, Y] - k[X, kY] + k^2[X, Y], \quad \forall X, Y \in \mathfrak{X}(M),$$

where $\mathfrak{X}(M)$ is the Lie algebra of C^∞ vector fields on M .

The Newlander–Nirenberg theorem asserts that an almost complex structure j on a manifold M is integrable if and only if its Nijenhuis tensor N^j vanishes identically.

Given an almost complex structure j on a manifold M , the *image distribution* on M , denoted $\text{Im } N^j$, has for value at a point $x \in M$ the subspace $(\text{Im } N^j)_x$ of the tangent space $T_x M$ spanned by all values $N_x^j(X, Y)$. Since $N_x^j(jX, Y) = -jN_x^j(X, Y)$, this distribution is stable by j . It is smooth (in the sense that in the neighborhood of any point, there exists a finite number of smooth vector fields whose values at any point linearly generate the distribution at that point) but the dimension of the distribution may not be constant. C. Bozetti and C. Medori give in [1] a classification of 4-manifolds endowed with a non integrable almost complex structure such that $\text{Im } N^j$ is everywhere of dimension 2, with a non degeneracy condition, and assuming a condition of local homogeneity. The rank of the Nijenhuis tensor is studied by L. Sillari and A. Tomassini in [16].

For any almost complex structure j , one has the splitting of the complexified tangent bundle induced by j :

$$TM^{\mathbb{C}} = T_j^{1,0} \oplus T_j^{0,1}$$

into $\pm i$ eigenspaces for j . We shall denote by $\mathcal{T}_j^{1,0}$ and $\mathcal{T}_j^{0,1}$ the sections of the corresponding distributions. We have the $C^\infty(M, \mathbb{R})$ -linear bijections

$$A^+ : \mathfrak{X}(M) \rightarrow \mathcal{T}_j^{1,0} : X \mapsto \frac{1}{2}(X - ijX) \quad \text{and} \quad A^- : \mathfrak{X}(M) \rightarrow \mathcal{T}_j^{0,1} : X \mapsto \frac{1}{2}(X + ijX).$$

Observe that, for any $X, W \in \mathfrak{X}(M)$ one has

$$\begin{aligned} X + iW &= \frac{1}{2}((X + jW) - ij(X + jW)) + \frac{1}{2}((X - jW) + ij(X - jW)) \\ &= A^+(X + jW) + A^-(X - jW), \end{aligned}$$

and

$$\begin{aligned} [X - i\varepsilon jX, Y - i\varepsilon' jY] &= [X, Y] - \varepsilon\varepsilon'[jX, jY] - i(\varepsilon[jX, Y] + \varepsilon'[X, jY]) \\ &= A^+([X, Y] - \varepsilon\varepsilon'[jX, jY] - \varepsilon j[jX, Y] - \varepsilon' j[X, jY]) \\ &\quad + A^-([X, Y] - \varepsilon\varepsilon'[jX, jY] + \varepsilon j[jX, Y] + \varepsilon' j[X, jY]). \end{aligned}$$

With $\varepsilon = \varepsilon' = 1$, the above shows that the projection on $\mathcal{T}_j^{0,1}$ of $[X - ijX, Y - ijY]$ is given by $A^-(-N^j(X, Y))$. Hence the bracket of two sections in $\mathcal{T}_j^{1,0}$ is always an element of $\mathcal{T}_j^{1,0}$ if and only if the Nijenhuis tensor is zero. A rephrasing of Newlander–Nirenberg's theorem is that j is integrable if and only if the distribution $\mathcal{T}_j^{1,0}$ (and hence $\mathcal{T}_j^{0,1}$) is involutive.

When a distribution is not involutive, one extends it to make it involutive in the following way.

Definition 1.1. Given a smooth real (respectively, complex) distribution \mathcal{D} whose sections are denoted \mathcal{D} , one defines the *derived flag of the distribution* as the nested sequence of distributions defined inductively by

$$\mathcal{D}^{(0)} = \mathcal{D}, \quad \mathcal{D}^{(1)} = \mathcal{D} + [\mathcal{D}, \mathcal{D}], \quad \mathcal{D}^{(i+1)} = \mathcal{D}^{(i)} + [\mathcal{D}^{(i)}, \mathcal{D}^{(i)}].$$

The limit $\mathcal{D}^\infty := \bigcup_k \mathcal{D}^{(k)}$ is called *the involutive limit distribution*; it is a $C^\infty(M, \mathbb{R})$ submodule (respectively, $C^\infty(M, \mathbb{C})$ submodule) of real (respectively, complex) vector fields, and is “involutive” in the sense that it is a Lie subalgebra of vector fields (i.e., closed under bracket of vector fields).

The computation made above shows the following.

Proposition 1.2. *The first derived distribution of $\mathcal{T}_j^{1,0}$ is given by*

$$(\mathcal{T}_j^{1,0})^{(1)} = \mathcal{T}_j^{1,0} + [\mathcal{T}_j^{1,0}, \mathcal{T}_j^{1,0}] = \mathcal{T}_j^{1,0} \oplus A^-(\mathcal{I}m N^j) = \mathcal{T}_j^{1,0} + (\mathcal{I}m N^j)^\mathbb{C},$$

where $\mathcal{I}m N^j$ denotes the sections of $\text{Im } N^j$.

An almost complex structure j is said to be *maximally non-integrable* if $\text{Im } N^j = TM$. This happens if and only if the first derived distribution of $\mathcal{T}_j^{1,0}$ consist of all complex valued vector fields on M .

Proposition 1.3. *The k -th derived distribution of $\mathcal{T}_j^{1,0}$ can be written as*

$$(\mathcal{T}_j^{1,0})^{(k)} =: \mathcal{T}_j^{1,0} \oplus A^-(\mathcal{D}_j^{(k)}) = \mathcal{T}_j^{1,0} + (\mathcal{D}_j^{(k)})^\mathbb{C},$$

where $\mathcal{D}_j^{(1)} = \mathcal{I}m N^j$, and the real smooth distributions $\mathcal{D}_j^{(k)}$ are stable under j and defined inductively by

$$\mathcal{D}_j^{(k+1)} = \mathcal{D}_j^{(k)} + \sum_{U \in \mathcal{D}_j^{(k)}} (\mathcal{I}m \mathcal{L}_U j) + [\mathcal{D}_j^{(k)}, \mathcal{D}_j^{(k)}],$$

where $\mathcal{I}m \mathcal{L}_U j = \{(\mathcal{L}_U j)X \mid X \in \mathfrak{X}(M)\}$ denotes the image of the smooth field of endomorphisms of the tangent bundle given by the Lie derivative of the almost complex structure j in the direction of the vector field U .

Proof. Using again that $X + iW = A^+(X + jW) + A^-(X - jW)$, we have

$$\begin{aligned} [X - ijX, U] &= [X, U] - i[jX, U] = A^+([X, U] - j[jX, U]) + A^-([X, U] + j[jX, U]) \\ &= A^+([X, U] - j[jX, U]) + A^-((\mathcal{L}_U j)(jX)). \end{aligned}$$

One proves inductively that the distributions $\mathcal{D}_j^{(k)}$ are stable by j , observing that j anti-commutes with $\mathcal{L}_U j$ since $j^2 = -\text{Id}$, and $j[U, U'] = [U, jU'] - (\mathcal{L}_U j)(U')$. ■

Proposition 1.4. *The k -th derived distribution $(\mathcal{T}_j^{1,0})^{(k)} =: \mathcal{T}_j^{1,0} \oplus A^-(\mathcal{D}_j^{(k)})$ is involutive if and only if $\mathcal{D}_j^{(k)}$ is involutive and has the property that $\mathcal{I}m \mathcal{L}_U j \subset \mathcal{D}_j^{(k)}$ for each $U \in \mathcal{D}_j^{(k)}$.*

The complex involutive limit distribution

$$(\mathcal{T}_j^{1,0})^\infty := \bigcup_k (\mathcal{T}_j^{1,0})^{(k)} = \mathcal{T}_j^{1,0} \oplus A^-\left(\bigcup_k \mathcal{D}_j^{(k)}\right)$$

is involutive (in the sense that the bracket of two elements in $(\mathcal{T}_j^{1,0})^\infty$ is again in $(\mathcal{T}_j^{1,0})^\infty$).

The real limit distribution

$$(1.2) \quad \mathcal{D}_j^\infty := \bigcup_k \mathcal{D}_j^{(k)}$$

is a $C^\infty(M, \mathbb{R})$ submodule of real vector fields; it is involutive, and has the property that

$$\mathcal{I}m \mathcal{L}_U j \subset \mathcal{D}_j^\infty, \quad \forall U \in \mathcal{D}_j^\infty.$$

The first derived distribution of $\mathcal{T}_j^{1,0}$, $(\mathcal{T}_j^{1,0})^{(1)} = \mathcal{T}_j^{1,0} + (\mathcal{I}m N^j)^\mathbb{C}$, is involutive if and only if

$$(1.3) \quad [N, jX] - j[N, X] = (\mathcal{L}_N j)X \in \mathcal{I}m N^j, \quad \forall X \in \mathfrak{X}(M), \forall N \in \mathcal{I}m N^j.$$

Proof. All statements except the last one are direct consequences of the former proposition. Equation (1.3) is necessary for $(\mathcal{T}_j^{1,0})^{(1)}$ to be involutive; it will be sufficient if and only if it implies that $\mathcal{I}m N^j$ is involutive. This is true, because, for any $N \in \mathcal{I}m N^j$ and any $X, Y \in \mathfrak{X}(M)$,

$$\begin{aligned} & [N^j(X, Y), N] \\ &= [[jX, jY], N] - [j[jX, Y], N] - [j[X, jY], N] - [[X, Y], N] \\ &= [[jX, N], jY] + [jX, [jY, N]] - j[[jX, Y], N] + (\mathcal{L}_N j)([jX, Y]) \\ &\quad - j[[X, jY], N] + (\mathcal{L}_N j)([X, jY]) - [[X, N], Y] - [X, [Y, N]] \\ &= (\mathcal{L}_N j)([jX, Y]) + (\mathcal{L}_N j)([X, jY]) \\ &\quad + [[jX, N], jY] + [jX, [jY, N]] - j[[jX, N], Y] - j[jX, [Y, N]] \\ &\quad - j[[X, N], jY] - j[X, [jY, N]] - [[X, N], Y] - [X, [Y, N]] \\ &= (\mathcal{L}_N j)([jX, Y]) + (\mathcal{L}_N j)([X, jY]) + [j[X, N], jY] - [(\mathcal{L}_N j)(X), jY] \\ &\quad + [jX, j[Y, N]] - [jX, (\mathcal{L}_N j)(Y)] - j[j[X, N], Y] + j[(\mathcal{L}_N j)(X), Y] \\ &\quad - j[jX, [Y, N]] - j[[X, N], jY] - j[X, j[Y, N]] + j[X, (\mathcal{L}_N j)(Y)] \\ &\quad - [[X, N], Y] - [X, [Y, N]] \\ &= (\mathcal{L}_N j)([jX, Y]) + (\mathcal{L}_N j)([X, jY]) - [(\mathcal{L}_N j)(X), jY] + j[(\mathcal{L}_N j)(X), Y] \\ &\quad - [jX, (\mathcal{L}_N j)(Y)] + j[X, (\mathcal{L}_N j)(Y)] + N^j([X, N], Y) + N^j(X, [Y, N]) \\ &= (\mathcal{L}_N j)([jX, Y]) + (\mathcal{L}_N j)([X, jY]) - (\mathcal{L}_{(\mathcal{L}_N j)(X)} j)(Y) + (\mathcal{L}_{(\mathcal{L}_N j)(Y)} j)(X) \\ &\quad + N^j([X, N], Y) + N^j(X, [Y, N]), \end{aligned}$$

which obviously belongs to $\mathcal{I}m N^j$, when $(\mathcal{L}_N j)(X) \in \mathcal{I}m N^j$ for any $N \in \mathcal{I}m N^j$. ■

Remark 1.5. If the k -th derived distribution $(\mathcal{T}_j^{1,0})^{(k)} =: \mathcal{T}_j^{1,0} \oplus A^-(\mathcal{D}_j^{(k)})$ is involutive and regular, then $\mathcal{D}_j^\infty = \mathcal{D}_j^{(k)}$ defines a foliation.

In a homogeneous context, with a G -invariant almost complex structure j on a G -homogeneous space M , each derived distribution is G -invariant and regular, so there is always an integer k such that $(\mathcal{T}_j^{1,0})^{(k)}$ is involutive. In that context, $\mathcal{D}_j^\infty = \mathcal{D}_j^{(k)}$ is a smooth real regular involutive distribution which defines a foliation on M .

2. \mathcal{D} -transverse structures

Let $\mathcal{D} = \Gamma^\infty(D)$ be the space of smooth sections of a real smooth involutive distribution D (not necessarily of constant rank) on a manifold M , or an involutive limit of an increasing sequence of such spaces of sections $\mathcal{D} = \bigcup_k \mathcal{D}^{(k)}$, with $\mathcal{D}^{(k)} = \Gamma^\infty(D^{(k)})$ and $\mathcal{D}^{(k)} + [\mathcal{D}^{(k)}, \mathcal{D}^{(k)}] \subset \mathcal{D}^{(k+1)}$, as, for example, the space \mathcal{D}_j^∞ defined in (1.2).

We are going to define \mathcal{D} -transverse objects. We start with the particular ideal situation where \mathcal{D} is the space of sections of a distribution D which is regular (i.e., of constant dimension) and such that the space of leaves for D , denoted by M/D , has a manifold structure making the canonical projection $p: M \rightarrow M/D$ a submersion. In that case, \mathcal{D} -transverse objects translate, at the level of M , corresponding objects on the space of leaves M/D . We then extend the definitions to our more general setting for \mathcal{D} .

2.1. \mathcal{D} -transverse vector fields

In the particular ideal situation described above, one can consider the pullback of the tangent bundle $T(M/D)$, and clearly

$$p^*T(M/D) \simeq TM/D =: Q.$$

The bundle Q is called the normal bundle, and one denotes by $\Pi: TM \rightarrow Q$ the canonical projection. There is an action of \mathcal{D} on sections of Q : for $F \in \mathcal{D}$ and $u \in \Gamma^\infty(M, Q)$,

$$L_F^Q u := \Pi([F, U])$$

if $U \in \mathfrak{X}(M)$ is a lift of u in the sense that $\Pi \circ U = u$. A vector field on M/D can be viewed as a section u of Q which is ‘‘constant’’ along the leaves in the sense that $L_F^Q u = 0$ for $F \in \mathcal{D}$.

In our general setting, we define a transverse vector field as follows.

Definition 2.1. A \mathcal{D} -transverse vector field is an equivalence class $[U]$ of a vector field $U \in \mathfrak{X}(M)$ which is \mathcal{D} -foliated in the sense that

$$[F, U] \in \mathcal{D} \quad \text{for any } F \in \mathcal{D},$$

the equivalence of foliated vector fields being defined by $U \sim U'$ if and only if $U - U' \in \mathcal{D}$.

2.2. \mathcal{D} -Transverse almost complex structures

In the ideal situation, an almost complex structure \hat{j} on M/D gives a section \tilde{j} of $\text{End}(Q)$ which squares to $-\text{Id}$ and which is ‘‘constant’’ along the leaves in the sense that

$$L_F^{\text{End}(Q)} \tilde{j} = 0 \quad \text{for any } F \in \mathcal{D}, \text{ where } L_F^{\text{End}(Q)} \tilde{j} = L_F^Q \circ \tilde{j} - \tilde{j} \circ L_F^Q.$$

A lift of \tilde{j} is a section $k \in \text{End}(TM)$ such that $\tilde{j}(\Pi(U)) = \Pi(kU)$; notice that

- $\Pi(kF) = 0$ for all $F \in \mathcal{D}$ if and only if $k(\mathcal{D}) \subset \mathcal{D}$,
- $\tilde{j}^2(\Pi(U)) = -\Pi(U)$ if and only if $\Pi(k^2U) = -\Pi(U)$, if and only if $k^2U + U \in \mathcal{D}$, for all $U \in \mathfrak{X}(M)$;
- $(L_F^Q \circ \tilde{j} - \tilde{j} \circ L_F^Q)\Pi(U) = 0 = \Pi([F, kU] - k[F, U])$, for all $U \in \mathfrak{X}(M)$.

In our general setting, we define a transverse almost complex structure as follows.

Definition 2.2. A \mathcal{D} -transverse almost complex structure is the equivalence class $[k]$ of a section k of $\text{End}(TM)$ such that $k(\mathcal{D}) \subset \mathcal{D}$, $k^2U + U \in \mathcal{D}$ for all $U \in \mathfrak{X}(M)$, and

$$(\mathcal{L}_F k)(U) = [F, kU] - k[F, U] \in \mathcal{D} \quad \text{for all } F \in \mathcal{D}, U \in \mathfrak{X}(M),$$

the equivalence being defined by $k \sim k'$ if and only if $\text{Im}(k - k') \subset \mathcal{D}$.

2.3. \mathcal{D} -transverse complex structures

In the ideal situation, an almost complex structure \hat{j} on M/D is integrable if and only if its Nijenhuis tensor vanishes. This will be true if and only if the torsion $N^{\tilde{j}}$ corresponding to the section \tilde{j} of $\text{End}(Q)$,

$$N^{\tilde{j}}(u, v) := [\tilde{j}u, \tilde{j}v] - \tilde{j}[\tilde{j}u, v] - \tilde{j}[u, \tilde{j}v] - [u, v],$$

vanishes for any u, v sections of Q corresponding to vector fields on M/D , i.e., such that $L_F^Q u = 0$ and $L_F^Q v = 0$ for $F \in \mathcal{D}$. Another way to formulate this condition is that a lift $k \in \text{End}(TM)$ of \tilde{j} must satisfy

$$\Pi([kU, kV] - k[kU, V] - k[U, kV] + k^2[U, V]) = \Pi(N^k(U, V)) = 0$$

for any U, V foliated vector fields. Since the torsion N^k of k is a tensor, and since the value at a point of foliated vector fields generate the whole tangent space to M at that point in this ideal situation, it is equivalent to ask that N^k takes its values in \mathcal{D} .

In our general setting, we define a transverse complex structure as follows.

Definition 2.3. A \mathcal{D} -transverse complex structure is a \mathcal{D} -transverse almost complex structure $[k]$ (i.e., k is a section of $\text{End}(TM)$, such that $k(\mathcal{D}) \subset \mathcal{D}$, $k^2U + U \in \mathcal{D}$, and $[F, kU] - k[F, U] \in \mathcal{D}$ for all $F \in \mathcal{D}$ and all $U \in \mathfrak{X}(M)$) which has the property that

$$N^k(U, V) = [kU, kV] - k[kU, V] - k[U, kV] + k^2[U, V] \in \mathcal{D} \quad \text{for all } U, V \in \mathfrak{X}(M).$$

2.4. \mathcal{D} -transverse forms

Definition 2.4. A \mathcal{D} -transverse – or basic – real or complex p -form is a p -form ω on M such that

$$\iota(F)\omega = 0 \quad \text{and} \quad \mathcal{L}_F\omega = 0, \quad \forall F \in \mathcal{D}.$$

If the section k of $\text{End}(TM)$ defines a \mathcal{D} -transverse almost complex structure, a complex \mathcal{D} -transverse 1-form ω is of type $(1, 0)$ (respectively, $(0, 1)$) if and only if

$$\omega(U + ikU) = 0 \quad (\text{respectively, } \omega(U - ikU) = 0), \quad \forall U \in \mathfrak{X}(M).$$

Proposition 2.5. Given a \mathcal{D} -transverse almost complex structure $[k]$, there is a splitting of complex \mathcal{D} -transverse p -forms as a direct sum

$$\Omega_{\mathcal{D}}^{\ell}(M) = \bigoplus_{p+q=\ell} \Omega_{\mathcal{D}, [k]}^{p,q}.$$

Proof. Observe that any complex \mathcal{D} -transverse 1-form ω splits as

$$\omega = \frac{1}{2} (\omega - i \omega \circ k) + \frac{1}{2} (\omega + i \omega \circ k),$$

and this decomposition does not depend on the element $k \in [k]$. The form $\omega \circ k$ is \mathcal{D} -transverse since $\omega \circ k(F) = 0$ for $F \in \mathcal{D}$, because $kF \in \mathcal{D}$ and

$$\mathcal{L}_F(\omega \circ k) = \mathcal{L}_F(\omega) \circ k + \omega \circ \mathcal{L}_F k = 0$$

since $\mathcal{L}_F k(U) \in \mathcal{D}$ for any $U \in \mathfrak{X}(M)$. Furthermore,

$$(\omega \mp i \omega \circ k)(U \pm ikU) = \omega(U + k^2U) + i\omega(\mp kU \pm kU) = 0, \quad \text{for all } U \in \mathfrak{X}(M).$$

The same applies for complex \mathcal{D} -transverse ℓ -forms. ■

2.5. \mathcal{D} -transverse Dolbeault cohomology

Assume again that the section k of $\text{End}(TM)$ defines a \mathcal{D} -transverse almost complex structure. The differential of a \mathcal{D} -transverse form is again \mathcal{D} -transverse, and one has

$$d\Omega_{\mathcal{D},[k]}^{p,q} \subset \Omega_{\mathcal{D},[k]}^{p+2,q-1} \oplus \Omega_{\mathcal{D},[k]}^{p+1,q} \oplus \Omega_{\mathcal{D},[k]}^{p,q+1} \oplus \Omega_{\mathcal{D},[k]}^{p-1,q+2}.$$

Indeed, for $\omega \in \Omega_{\mathcal{D},[k]}^{p,q}$ and for complex vector fields Z_i ,

$$\begin{aligned} d\omega(Z_0, \dots, Z_p) &= \sum_r (-1)^r Z_r(\omega(Z_0, \dots, \widehat{Z}_r, \dots, Z_p)) \\ &\quad + \sum_{r < s} (-1)^{r+s} \omega([Z_r, Z_s], Z_0, \dots, \widehat{Z}_r, \dots, \widehat{Z}_s, \dots, Z_p), \end{aligned}$$

so that, for elements $Z_r = A_k^\pm Y_r := Y_r \mp ikY_r$, it vanishes if there are not at least $p-1$ elements of the form $Z_r = A_k^+ Y_r$ and $q-1$ elements of the form $Z_r = A_k^- Y_r$.

Assume now that the section k of $\text{End}(TM)$ defines a \mathcal{D} -transverse complex structure. Then the projections of $d\omega$ on $\Omega_{\mathcal{D},[k]}^{p+2,q-1}$ and $\Omega_{\mathcal{D},[k]}^{p-1,q+2}$ vanish; indeed,

$$\begin{aligned} &d\omega(A_k^+ X_1, \dots, A_k^+ X_{p-1}, A_k^- Y_1, \dots, A_k^- Y_{q+2}) \\ &= \sum \pm \omega([A_k^- Y_r, A_k^- Y_s], A_k^+ X_1, \dots, A_k^+ X_{p-1}, A_k^- Y_1, \dots, \widehat{A_k^- Y_r}, \dots, \widehat{A_k^- Y_s}, \dots, A_k^- Y_{q+2}) \\ &= 0 \end{aligned}$$

because

$$\begin{aligned} [Y_r + ikY_r, Y_s + ikY_s] &= [Y_r, Y_s] - [kY_r, kY_s] + i[Y_r, kY_s] + i[kY_r, Y_s] \\ &= A_k^+([Y_r, Y_s] - [kY_r, kY_s] + k[Y_r, kY_s] + k[kY_r, Y_s]) \\ &\quad + A_k^-([Y_r, Y_s] - [kY_r, kY_s] - k[Y_r, kY_s] - k[kY_r, Y_s]) \\ &\quad + i(k^2 + \text{Id})([Y_r, kY_s] + [kY_r, Y_s]) \\ &= A_k^+(-N^k(Y_r, Y_s)) + A_k^-(\dots) + F, \quad \text{with } F \in \mathcal{D}^{\mathbb{C}}, \end{aligned}$$

and $N^k(Y_r, Y_s)$ is in \mathcal{D} when k defines a complex transverse structure.

Hence

$$d\Omega_{\mathcal{D},[k]}^{p,q} \subset \Omega_{\mathcal{D},[k]}^{p+1,q} \oplus \Omega_{\mathcal{D},[k]}^{p,q+1},$$

and we denote by $\partial_{\mathcal{D},[k]}$ and $\bar{\partial}_{\mathcal{D},[k]}$ the corresponding projections.

Definition 2.6. The \mathcal{D} -transverse Dolbeault cohomology induced by $[k]$ is

$$H_{\mathcal{D},[k],\bar{\partial}}^{p,q}(M) = \text{Ker } \bar{\partial}_{\mathcal{D},[k]}|_{\Omega_{\mathcal{D},[k]}^{p,q}} / \text{Im } \bar{\partial}_{\mathcal{D},[k]}|_{\Omega_{\mathcal{D},[k]}^{p,q-1}}.$$

Remark 2.7. In the ideal situation where D is regular and there is a manifold structure on the space of leaves M/D such that $p: M \rightarrow M/D$ is a submersion, a \mathcal{D} -transverse complex structure $[k]$ corresponds to a complex structure \hat{j} on M/D , and the transverse Dolbeault cohomology for $[k]$ on M is the usual Dolbeault cohomology for \hat{j} on the space of leaves:

$$H_{\mathcal{D},[k],\bar{\partial}}^{p,q}(M) = H^{p,q}(M/D, \bar{\partial}_{\hat{j}}).$$

3. Transverse Dolbeault cohomology

3.1. Transverse complex structure induced by an almost complex structure

Let j be an almost complex structure on the manifold M and let \mathcal{D} be a real generalized involutive distribution, stable under j ; by this we mean that \mathcal{D} can be the space of smooth sections of a smooth involutive distribution D (not necessarily of constant rank), stable under j , or –as before– can be the involutive limit of a sequence of nested spaces of sections.

Following Definitions 2.2 and 2.3, this j yields a \mathcal{D} -transverse almost complex structure if and only if

$$[F, jU] - j[F, U] \in \mathcal{D}, \quad \forall F \in \mathcal{D}, \forall U \in \mathfrak{X}(M),$$

and a complex \mathcal{D} -transverse structure if and only if, furthermore, \mathcal{D} contains the image of N^j .

Proposition 3.1. *The structure j defines a complex \mathcal{D} -transverse structure if and only if*

$$\mathcal{T}_j^{1,0} + \mathcal{D} = \mathcal{T}_j^{1,0} \oplus A^-(\mathcal{D}) = \mathcal{T}_j^{1,0} + \mathcal{D}^{\mathbb{C}} \quad \text{is involutive.}$$

Then $\mathcal{D} \supset \mathcal{D}_j^\infty = \bigcup_k \mathcal{D}_j^{(k)} \supset \text{Im } N^j$.

Proof. This results directly from the computations made in the proof of Proposition 1.3. ■

In particular, j defines a complex structure transverse to \mathcal{D}_j^∞ and a corresponding \mathcal{D}_j^∞ -transverse Dolbeault cohomology. The splitting of \mathcal{D}_j^∞ -transverse forms corresponds to the usual splitting of forms on M relatively to j , $\Omega^k(M, \mathbb{C}) = \bigoplus_{p+q=k} \Omega_j^{p,q}$, restricted to \mathcal{D}_j^∞ -transverse forms, and the operator $\bar{\partial}_{\mathcal{D}_j^\infty, [j]}|_{\Omega_{\mathcal{D}_j^\infty, [j]}^{p,q}}$ is the restriction of $\bar{\partial}_j$ to the forms in $\Omega_j^{p,q}$ which are \mathcal{D}_j^∞ -transverse.

Remark 3.2. In 1986, O. Muskarov made the link [13] between the existence of holomorphic functions for an almost complex structure and the existence of an involutive subbundle of the (real) tangent bundle which contains the image of the Nijenhuis tensor.

3.2. Transverse Dolbeault cohomology and $(p, 0)$ -spaces.

Definition 3.3. The j -transverse Dolbeault cohomology is the \mathcal{D}_j^∞ -transverse Dolbeault cohomology induced by j :

$$H_{j\text{-trans}}^{p,q}(M) := H_{\mathcal{D}_j^\infty, [j], \bar{\partial}}^{p,q}(M) = \text{Ker } \bar{\partial}_j|_{\Omega_{\mathcal{D}_j^\infty, [j]}^{p,q}} / \text{Im } \bar{\partial}_j|_{\Omega_{\mathcal{D}_j^\infty, [j]}^{p,q-1}};$$

the space of forms considered are

$$\Omega_{\mathcal{D}_j^\infty, [j]}^{p,q} = \{\omega \in \Omega_j^{p,q} \mid \iota(X)\omega = 0, \mathcal{L}_X\omega = 0, \forall X \in D_j^\infty\}$$

and, as before, for ω in $\Omega_{\mathcal{D}_j^\infty, [j]}^{p,q}$, $\bar{\partial}_j$ is the projection of $d\omega$ on $\Omega_j^{p+1,q}$.

Remark 3.4. If \mathcal{D}_j^∞ is regular, it defines a foliation with transverse complex structure induced by j . If its space of leaves has a manifold structure making the canonical projection a submersion, then the j transverse Dolbeault cohomology is the Dolbeault cohomology of this space of leaves. Recall that for a G -invariant almost complex structure j on a G -homogeneous space M , each derived distribution is G -invariant and regular, so \mathcal{D}_j^∞ always defines a foliation.

Proposition 3.5. Let \mathcal{D} be any regular involutive j -stable distribution such that the space of leaves M/D has a manifold structure with $p: M \rightarrow M/D$ a submersion, and such that j induces a complex structure on M/D . The \mathcal{D} -transverse Dolbeault cohomology induced by j coincides with the Dolbeault cohomology of the space of leaves, and maps into the j -transverse Dolbeault cohomology.

Proof. The first part follows from the definition of the transverse Dolbeault cohomology. The fact that the \mathcal{D} -transverse Dolbeault cohomology induced by j maps into the j -transverse Dolbeault cohomology follows from the fact that $\mathcal{D} \supset \mathcal{D}_j^\infty$ by Proposition 3.1, the space of \mathcal{D} -transverse (p, q) -forms is thus contained in the space of \mathcal{D}_j^∞ -transverse (p, q) -forms, and the cohomologies are both induced by the restriction of $\bar{\partial}_j$ to those spaces. ■

Proposition 3.6. The space $\Omega_{\mathcal{D}_j^\infty}(M) := \bigoplus_{p,q} \Omega_{\mathcal{D}_j^\infty, [j]}^{p,q}$ is the largest subspace of the space $\Omega(M, \mathbb{C})$ of smooth complex forms on the manifold such that

- each form ω in it vanishes whenever contracted with a vector field in the image of the Nijenhuis tensor of j ,

$$\iota(X)\omega = 0, \quad \text{for all } X \in \mathfrak{Im } N^j;$$

- it is stable under the differential d ;
- it splits into (p, q) components relatively to j , in the sense that for ω a k -form in the subspace, $\omega \circ j_r$ is in the subspace for any $r \leq k$, where $\circ j_r$ indicates the precomposition with j acting on the r -th argument of ω .

Proof. All conditions are clearly satisfied by

$$\Omega_{\mathcal{D}_j^\infty}(M) = \bigoplus_{p,q} \{\omega \in \Omega_j^{p,q} \mid \iota(X)\omega = 0, \mathcal{L}_X\omega = 0, \forall X \in D_j^\infty\}.$$

Reciprocally, one proceeds by induction, showing that a space of forms satisfying all three conditions is included, for all k , in

$$\{\omega \in \Omega_j^{p,q} \mid \iota(X)\omega = 0, \mathcal{L}_X\omega = 0, \forall X \in \mathcal{D}_j^{(k)}\}.$$

For a given k , let us consider a subspace of forms such that $\iota(X)\omega = 0$ for all $X \in \mathcal{D}_j^{(k)}$, recalling that $\mathcal{D}_j^{(1)} = \mathcal{I}m N^j$.

The second condition implies that $\iota(X)d\omega = \mathcal{L}_X\omega = 0$ for all $X \in \mathcal{D}_j^{(k)}$. Thus, one also has that $\iota(Y)\omega = 0$ for any $Y \in \mathcal{D}_j^{(k)} + [\mathcal{D}_j^{(k)}, \mathcal{D}_j^{(k)}]$ (using the fact that $\iota([X, X'])\omega = (\iota(X) \circ \mathcal{L}_{X'} - \mathcal{L}_{X'} \circ \iota(X))\omega$).

The third condition implies then that $\omega \circ \mathcal{L}_X j_r = 0$, which in turns implies that $\iota(Y)\omega = 0$ for all $Y \in \mathcal{I}m \mathcal{L}_X j$ when $X \in \mathcal{D}_j^{(k)}$.

This shows that $\iota(Y)\omega = 0$ for all $Y \in \mathcal{D}_j^{(k+1)}$, and one proceeds inductively. \blacksquare

Proposition 3.7. *The j -transverse Dolbeault cohomology is given in degree $(p, 0)$ by*

$$H_{j\text{-trans}}^{p,0}(M) := \{\omega \in \Omega^p(M, \mathbb{C}) \mid \iota(Z)\omega = 0, \mathcal{L}_Z\omega = 0, \\ \forall Z \in (\mathcal{T}_j^{0,1})^\infty = \mathcal{T}_j^{0,1} \oplus A^-(\mathcal{D}_j^\infty)\}.$$

Proof.

$$\begin{aligned} H_{j\text{-trans}}^{p,0}(M) &= H_{\mathcal{D}_j^\infty, [j], \bar{\partial}}^{p,0}(M) = \{\omega \in \Omega_{\mathcal{D}_j^\infty, [j]}^{p,0} \mid \bar{\partial}_{\mathcal{D}_j^\infty, [j]}\omega = 0\} \\ &= \{\omega \in \Omega^p(M, \mathbb{C}) \mid \iota(F)\omega = 0, \mathcal{L}_F\omega = 0 \forall F \in \mathcal{D}_j^\infty, \\ &\quad \iota(U + ijU)\omega = 0 \forall U \in \mathfrak{X}(M), \bar{\partial}_{\mathcal{D}_j^\infty, [j]}\omega = 0\} \\ &= \{\omega \in \Omega^p(M, \mathbb{C}) \mid \iota(Z)\omega = 0, \forall Z \in \mathcal{T}_j^{0,1} + \mathcal{D}_j^\infty, \mathcal{L}_F\omega = 0, \forall F \in \mathcal{D}_j^\infty, \\ &\quad d\omega(U + ijU, V_1 - ijV_1, \dots, V_{p-1} - ijV_{p-1}) = 0\} \\ &= \{\omega \in \Omega^p(M, \mathbb{C}) \mid \iota(Z)\omega = 0, \forall Z \in \mathcal{T}_j^{0,1} + \mathcal{D}_j^\infty, \mathcal{L}_F\omega = 0, \forall F \in \mathcal{D}_j^\infty, \\ &\quad (A^-U)(\omega(A^+V_1, \dots, A^+V_{p-1})) \\ &\quad - \sum_k \omega(A^+V_1, \dots, [A^-U, A^+V_k], \dots, A^+V_{p-1}) = 0\} \\ &= \{\omega \in \Omega^p(M, \mathbb{C}) \mid \iota(Z)\omega = 0, \mathcal{L}_Z\omega = 0, \forall Z \in (\mathcal{T}_j^{0,1})^\infty = \mathcal{T}_j^{0,1} + \mathcal{D}_j^\infty\}. \quad \blacksquare \end{aligned}$$

3.3. Comparison with the generalized Dolbeault cohomology of an almost complex structure

J. Cirici and S Wilson introduced in [4] a generalized Dolbeault cohomology associated to an almost complex structure j on a manifold M in the following way.

One decomposes as before the complexified tangent bundle $TM^{\mathbb{C}} = T_j^{1,0} \oplus T_j^{0,1}$ into $\pm i$ eigenspaces for j , and the dual decomposition of the complexified cotangent

bundle $T^*M^{\mathbb{C}} = (T_j^*)^{1,0} \oplus (T_j^*)^{0,1}$ leads to the usual decomposition of the space of complex valued k -forms on M into

$$\Omega^k(M, \mathbb{C}) = \bigoplus_{p+q=k} \Omega_j^{p,q}.$$

Then

$$d\Omega_j^{p,q} \subset \Omega_j^{p-1,q+2} \oplus \Omega_j^{p,q+1} \oplus \Omega_j^{p+1,q} \oplus \Omega_j^{p+2,q-1},$$

and the differential splits accordingly as

$$d = \bar{\mu}_j \oplus \bar{\partial}_j \oplus \partial_j \oplus \mu_j.$$

The fact that $d^2 = 0$ is equivalent to

$$(3.1) \quad \bar{\mu}_j^2 = 0,$$

$$(3.2) \quad \bar{\mu}_j \circ \bar{\partial}_j + \bar{\partial}_j \circ \bar{\mu}_j = 0,$$

$$(3.3) \quad \bar{\mu}_j \circ \partial_j + \partial_j \circ \bar{\mu}_j + \bar{\partial}_j^2 = 0,$$

$$\mu_j \circ \bar{\mu}_j + \bar{\mu}_j \circ \mu_j + \partial_j \circ \bar{\partial}_j + \bar{\partial}_j \circ \partial_j = 0,$$

$$\mu_j \circ \bar{\partial}_j + \bar{\partial}_j \circ \mu_j + \partial_j^2 = 0,$$

$$\mu_j \circ \partial_j + \partial_j \circ \mu_j = 0,$$

$$\mu_j^2 = 0.$$

Equation (3.1) shows that one can define the $\bar{\mu}_j$ cohomology spaces:

$$(3.4) \quad H_{\bar{\mu}_j}^{(p,q)} = \text{Ker } \bar{\mu}_j|_{\Omega_j^{p,q}} / \text{Im } \bar{\mu}_j|_{\Omega_j^{p+1,q-2}}.$$

Equation (3.2) shows that $\bar{\partial}_j$ induces a map $\widetilde{\bar{\partial}}_j$ on those $\bar{\mu}_j$ cohomology spaces

$$(3.5) \quad \widetilde{\bar{\partial}}_j : H_{\bar{\mu}_j}^{(p,q)} \rightarrow H_{\bar{\mu}_j}^{(p,q+1)} : \omega + \text{Im } \bar{\mu}_j \mapsto \bar{\partial}_j \omega + \text{Im } \bar{\mu}_j,$$

and equation (3.3) shows that $(\widetilde{\bar{\partial}}_j)^2 = 0$, so one can look at the corresponding cohomology spaces

$$(3.6) \quad H_{\text{Dol}}^{(p,q)}(M) = \text{Ker } \widetilde{\bar{\partial}}_j|_{H_{\bar{\mu}_j}^{(p,q)}} / \text{Im } \widetilde{\bar{\partial}}_j|_{H_{\bar{\mu}_j}^{(p,q-1)}}$$

Those are the spaces of the generalized Dolbeault cohomology.

Proposition 3.8. *The generalized Dolbeault and the transverse Dolbeault cohomology spaces coincide in degrees $(p, 0)$:*

$$H_{\text{Dol}}^{(p,0)}(M) = H_{j\text{-trans}}^{p,0}(M).$$

Proof. We know that $H_{j\text{-trans}}^{p,0}(M) = H_{\mathcal{D}_j^\infty, [\bar{\mu}_j, \bar{\partial}]}^{p,0}(M)$ and, by Proposition 3.7, we have

$$H_{j\text{-trans}}^{p,0}(M) := \{\omega \in \Omega^p(M, \mathbb{C}) \mid \iota(Z)\omega = 0, \mathcal{L}_Z\omega = 0, \forall Z \in (\mathcal{T}_j^{0,1})^\infty = \mathcal{T}_j^{0,1} \oplus A^-(D_j^\infty)\}.$$

On the other hand,

$$H_{\text{Dol}}^{(p,0)}(M) = \{\omega \in \Omega_j^{p,0} \mid \bar{\mu}_j\omega = 0, \bar{\partial}_j\omega = 0\}.$$

Now, for $\omega \in \Omega_j^{p,0}$, one has

$$\begin{aligned} \bar{\mu}_j\omega = 0 &\iff d\omega(A^-(Y), A^-(Z), A^+(X_1), \dots, A^+(X_{p-1})) = 0 \\ &\iff \omega([Y + ijY, Z + ijZ], X_1 - ijX_1, \dots, X_{p-1} - ijX_{p-1}) = 0 \\ &\iff \iota(N^j(Y, Z))\omega = 0. \end{aligned}$$

So

$$H_{\bar{\mu}_j}^{(p,0)} = \{\omega \in \Omega^p(M, \mathbb{C}) \mid \iota(W)\omega = 0, \forall W \in (\mathcal{T}_j^{0,1})^{(1)}\}.$$

We have, for $\omega \in H_{\bar{\mu}_j}^{(p,0)} \subset \Omega_j^{p,0}$,

$$\begin{aligned} \bar{\partial}_j\omega = 0 &\iff d\omega(Y + ijY, A^+(X_1), \dots, A^+(X_p)) = 0 \\ &\iff (Y + ijY)\omega(A^+(X_1), \dots, A^+(X_p)) \\ &\quad - \sum_i \omega(A^+(X_1), \dots, [Y + ijY, A^+(X_i)], \dots, A^+(X_p)) = 0 \\ &\iff \mathcal{L}_Z\omega = 0, \quad \forall Z \in \mathcal{T}_j^{0,1}. \end{aligned}$$

Using the fact that $[\mathcal{L}_Z, \mathcal{L}_{Z'}] = \mathcal{L}_{[Z, Z']}$ and $[\iota(W), \mathcal{L}_Z] = \iota([W, Z])$, we get

$$\begin{aligned} H_{\text{Dol}}^{(p,0)}(M) &= \{\omega \in \Omega^p(M, \mathbb{C}) \mid \iota(W)\omega = 0 \text{ and } \mathcal{L}_Z\omega = 0, \forall Z \in \mathcal{T}_j^{0,1}, \forall W \in (\mathcal{T}_j^{0,1})^{(1)}\} \\ &= \{\omega \in \Omega^p(M, \mathbb{C}) \mid \iota(Z)\omega = 0 \text{ and } \mathcal{L}_Z\omega = 0, \forall Z \in (\mathcal{T}_j^{0,1})^\infty = \bigcup_k (\mathcal{T}_j^{0,1})^{(k)}\} \\ &= H_{j\text{-trans}}^{p,0}(M). \end{aligned} \quad \blacksquare$$

Remark 3.9. Since any element in

$$\Omega_{\mathcal{D}_j^\infty}(M) = \bigoplus_{p,q} \Omega_{\mathcal{D}_j^\infty, [\bar{\mu}_j]}^{p,q}(M)$$

is in the kernel of $\bar{\mu}_j$, there is always a map from $\Omega_{\mathcal{D}_j^\infty}(M)$ to $H_{\bar{\mu}_j}^{(p,q)}$ mapping an element ω to $[\omega]$, and this induces a map in cohomology

$$H_{j\text{-trans}}^{p,q}(M) \rightarrow H_{\text{Dol}}^{(p,q)}(M)$$

mapping the class in $H_{j\text{-trans}}^{p,q}(M)$ of a $\bar{\partial}_j$ -closed (p, q) -form ω in $\Omega_{\mathcal{D}_j^\infty}(M)$ to the class in $H_{\text{Dol}}^{(p,q)}(M)$ of the $\widetilde{\bar{\partial}_j}$ -closed element $[\omega]$ in $H_{\bar{\mu}_j}^{(p,q)}$.

4. A notion of minimal non-integrability

Definition 4.1. We say that a non-integrable almost complex structure j on a manifold M is minimally non-integrable if the first derived distribution of $\mathcal{T}_j^{1,0}$, $(\mathcal{T}_j^{1,0})^{(1)} = \mathcal{T}_j^{1,0} + (\mathcal{I}m N^j)^{\mathbb{C}}$, is involutive. By Proposition 1.4, this will be true if and only if

$$[N, jX] - j[N, X] = (\mathcal{L}_N j)X \in \mathcal{I}m N^j, \quad \forall X \in \mathfrak{X}(M), \forall N \in \mathcal{I}m N^j.$$

In that case, $\mathcal{D}_j^\infty = \mathcal{I}m N^j$ and the transverse Dolbeault cohomology is defined using the operator $\bar{\partial}_j$ restricted to

$$\Omega_{\mathcal{D}_j^\infty, [j]}^{p,q} = \{\omega \in \Omega_j^{p,q} \mid \iota(X)\omega = 0, \mathcal{L}_X\omega = 0, \forall X \in \mathcal{I}m N^j\}.$$

We could ask furthermore that $\dim \mathcal{I}m N^j = 2$ everywhere. Then we have a foliation, with two-dimensional leaves carrying a complex structure, and with a transverse complex structure.

As we have seen in Proposition 3.1, minimal non-integrability for j means that it has the largest possible transverse complex structure.

4.1. Minimally non-integrable invariant almost complex structure on a Lie group.

Let j be a left invariant almost complex structure on a Lie group G . Denoting by \mathfrak{g} the Lie algebra of G and by J the endomorphism of \mathfrak{g} given by the value of j at the neutral element $e \in G$, the Nijenhuis tensor N^J is left invariant and its value at e is given by

$$N^J(X, Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y], \quad \text{for all } X, Y \in \mathfrak{g}.$$

$\text{Im } N^J$ is a smooth left invariant regular distribution whose value at e is the J -invariant subspace $\text{Im } N^J$ of \mathfrak{g} .

Proposition 4.2. Consider a left invariant almost complex structure j on a Lie group G such that $\text{Im } N^J$ satisfies

$$(4.1) \quad [N, JX] - J[N, X] \in \text{Im } N^J, \quad \forall X \in \mathfrak{g}, \forall N \in \text{Im } N^J.$$

Then $\text{Im } N^J$ defines a foliation, the leaves carry an induced almost complex structure, and j induces a transverse complex structure.

If the subgroup H corresponding to the subalgebra $\text{Im } N^J$ is closed, one has a principal fiber bundle $p: G \rightarrow G/H$, whose base manifold (which is the leaf space) is complex, the fibers (leaves) are almost complex, and the projection is pseudo-holomorphic. The fibers are complex if, furthermore, $N^J(N, N') = 0$ for all $N, N' \in \text{Im } N^J$; this is always true if $\dim \text{Im } N^J = 2$.

Remark that condition (4.1) implies that $\text{Im } N^J$ is a subalgebra of \mathfrak{g} and is automatically satisfied if $\text{Im } N^J$ is an ideal in \mathfrak{g} .

4.2. Minimally non-integrable invariant almost complex structure on a homogeneous space

Let $G \times M \rightarrow M: (g, x) \mapsto g \cdot x =: \rho(g)x$ denote the action of a Lie group G on a manifold M . Assume this action is transitive. Choosing a base point x_0 , its stabilizer will be denoted by H , so $H = \{g \in G \mid g \cdot x_0 = x_0\}$ is a closed subgroup of G and the manifold M is diffeomorphic to G/H . We denote by $\pi: G \rightarrow M: g \mapsto g \cdot x_0$ the canonical projection. Thus $T_{x_0}M$ identifies with $\mathfrak{g}/\mathfrak{h}$. For any element $h \in H$, the action $\rho(h)_{*x_0}$ on $T_{x_0}M$ identifies with the action induced by $\text{Ad}(h)$ and denoted $\widetilde{\text{Ad}}(h)$ on the quotient $\mathfrak{g}/\mathfrak{h}$. We denote by A^* the fundamental vector field on M defined by an element $A \in \mathfrak{g}$ (i.e., $A_x^* = \frac{d}{dt} \exp -tA \cdot x|_{t=0}$); clearly $A_{x_0}^* = -\pi_* A$.

Assume that there exists an almost complex structure j on M which is G -invariant, i.e., $\rho(g)_{*x} \circ j_x = j_{g \cdot x} \circ \rho(g)_{*x}$ for all $g \in G$. Invariance implies

$$L_{A^*} j = 0, \quad \text{i.e.,} \quad [A^*, jX] = j[A^*, X], \quad \forall X \in \mathfrak{X}(M), \forall A \in \mathfrak{g}.$$

The Nijenhuis tensor N^j is invariant under the action of G . Its value at the base point is obtained as follows

Proposition 4.3 (Theorem 6.4 on p. 217 of [11]). *Let M be a G -homogeneous manifold endowed with a G -invariant almost complex structure j . We choose a base point $x_0 \in M$ and a linear map $J: \mathfrak{g} \rightarrow \mathfrak{g}$ such that $j_{x_0} A^* = (JA)_{x_0}^*$ for all $A \in \mathfrak{g}$. Then*

$$N_{x_0}^j(A^*, B^*) = (-N^J(A, B))_{x_0}^*,$$

where N^J is defined in terms of the Lie bracket in \mathfrak{g} by

$$N^J(A, B) := [JA, JB] - J[JA, B] - J[A, JB] - [A, B], \quad \forall A, B \in \mathfrak{g}.$$

Proof. Using the invariance of j , we have, for all $A, B \in \mathfrak{g}$,

$$N^j(A^*, B^*) = [jA^*, jB^*] + j[B^*, jA^*] - j[A^*, jB^*] - [A^*, B^*] = [jA^*, jB^*] + [A^*, B^*].$$

We introduce an auxiliary torsion-free linear connexion ∇ on the manifold M . At the base point x_0 , one has

$$\begin{aligned} [jA^*, jB^*]_{x_0} &= (\nabla_{jA^*} jB^* - \nabla_{jB^*} jA^*)_{x_0} = (\nabla_{(JA)^*} jB^* - \nabla_{(JB)^*} jA^*)_{x_0} \\ &= ([(JA)^*, jB^*] + \nabla_{jB^*} (JA)^* - [(JB)^*, jA^*] - \nabla_{jA^*} (JB)^*)_{x_0} \\ &= (j[(JA)^*, B^*] + \nabla_{(JB)^*} (JA)^* - j[(JB)^*, A^*] - \nabla_{(JA)^*} (JB)^*)_{x_0} \\ &= (j[JA, B]^* + [(JB)^*, (JA)^*] - j[JB, A]^*)_{x_0} \\ &= (J[JA, B] + [JB, JA] - J[JB, A])_{x_0}^*. \end{aligned}$$

Hence,

$$N^j(A^*, B^*)_{x_0} = (-[JA, JB] + J[JA, B] + J[A, JB] + [A, B]^*)_{x_0}^* = -N^J(A, B)_{x_0}^*. \quad \blacksquare$$

Proposition 4.4. *Let M be a G -homogeneous manifold endowed with a G -invariant almost complex structure j . We choose a base point $x_0 \in M$ and a linear map $J: \mathfrak{g} \rightarrow \mathfrak{g}$ such that $j_{x_0} A^* = (JA)_{x_0}^*$ for all $A \in \mathfrak{g}$. The distribution $\text{Im } N^J$ is involutive and gives a foliation with transverse complex structure if and only if*

$$(4.2) \quad [JA, N^J(B, C)] - J[A, N^J(B, C)] \in \text{Im } N^J + \mathfrak{h}, \quad \forall A, B, C \in \mathfrak{g},$$

with N^J defined as above: $N^J(A, B) = [JA, JB] - J[JA, B] - J[A, JB] - [A, B]$.

Proof. Since j in G -invariant, the tensor N^J is G -invariant, hence, for all $A \in \mathfrak{g}$,

$$L_{A^*} N^J = 0,$$

i.e.,

$$[A^*, N^J(X, Y)] = N^J([A^*, X], Y) + N^J(X, [A^*, Y]), \quad \forall X, Y \in \mathfrak{X}(M).$$

The condition $[jX, N] - j[X, N] \in \mathfrak{Im } N^J$ for all $X \in \mathfrak{X}(M)$ and for all $N \in \mathfrak{Im } N^J$ will be satisfied at every point if it is satisfied at the base point. Since $[jX, N] - j[X, N]$ is tensorial in X , it is enough to verify it for X in the space of fundamental vector fields; since it remains true if one multiplies $N \in \mathfrak{Im } N^J$ by a function, the condition will be satisfied if and only if

$$(4.3) \quad ([jA^*, N^J(B^*, C^*)] - j[A^*, N^J(B^*, C^*)])_{x_0} \in \text{Im } N_{x_0}^J, \quad \forall A, B, C \in \mathfrak{g}.$$

Introducing again an auxiliary torsion-free linear connexion ∇ on the manifold M , we have

$$\begin{aligned} ([jA^*, N^J(B^*, C^*)])_{x_0} &= ((JA)^*, N^J(B^*, C^*)) - \nabla_{N^J(B^*, C^*)}(jA^* - (JA)^*)_{x_0} \\ &= ((JA)^*, N^J(B^*, C^*)) + \nabla_{(N^J(B, C))^*}(jA^* - (JA)^*)_{x_0} \\ &= [(JA)^*, N^J(B^*, C^*)]_{x_0} + [N^J(B, C)^*, jA^* - (JA)^*]_{x_0} \\ &= ((JA)^*, N^J(B^*, C^*)) + j[N^J(B, C)^*, A^*] - [N^J(B, C), JA]^*_{x_0} \\ &= (N^J([JA, B]^*, C^*) + N^J(B^*, [JA, C]^*))_{x_0} + (J[N^J(B, C), A] - [N^J(B, C), JA])^*_{x_0} \\ &= (-N^J([JA, B], C) - N^J(B, [JA, C]) + J[N^J(B, C), A] - [N^J(B, C), JA])^*_{x_0}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} (j[A^*, N^J(B^*, C^*)])_{x_0} &= (jN^J([A, B]^*, C^*) + jN^J(B^*, [A, C]^*))_{x_0} \\ &= (-jN^J([A, B], C) - jN^J(B, [A, C]))^*_{x_0}. \end{aligned}$$

Hence

$$\begin{aligned} &([jA^*, N^J(B^*, C^*)] - j[A^*, N^J(B^*, C^*)])_{x_0} \\ &= (-N^J([JA, B], C) - N^J(B, [JA, C]) + J[N^J(B, C), A] - [N^J(B, C), JA] \\ &\quad + jN^J([A, B], C) + jN^J(B, [A, C]))^*_{x_0}, \end{aligned}$$

and this is in $\text{Im } N_{x_0}^J = (\text{Im } N^J)^*_{x_0}$ if and only if

$$[JA, N^J(B, C)] - J[A, N^J(B, C)] \in \text{Im } N^J + \mathfrak{h}. \quad \blacksquare$$

4.3. The example of Kodaira–Thurston 4-dimensional manifold

Thurston gave in 1976 a first example of a 4-dimensional compact symplectic manifold with no Kähler structure. It is a compact nilmanifold, i.e., the quotient $M = \Gamma \backslash G$ of a nilpotent Lie group G by a lattice Γ (i.e., a discrete subgroup acting on the left cocompactly); the group G is the 4-dimensional nilpotent group which is the direct product of \mathbb{R} and the 3-dimensional Heisenberg group $G = \mathbb{R}^4 = \mathbb{R} \times \mathbb{H}_3$, with multiplication defined by

$$(a^1, a^2, a^3, a^4) \cdot (x^1, x^2, y^1, y^2) = (a^1 + x^1, a^2 + x^2 + a^3 y^2, a^3 + y^1, a^4 + y^2),$$

and the lattice is $\Gamma = \mathbb{Z}^4$.

A basis of left invariant vector fields on G is given by $\{\partial_{x^1}, \partial_{x^2}, \partial_{y^1}, \partial_{y^2} + y^1 \partial_{x^2}\}$ and the dual basis of left-invariant 1-forms on G is $\{dx^1, dx^2 - y^1 dy^2, dy^1, dy^2\}$.

The manifold M can be seen as a fibration in tori over the torus $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$ with the projection $p: M \rightarrow \mathbb{T}^2$ mapping the left coset $\mathbb{Z}^4(x^1, x^2, y^1, y^2)$ on $(x^1, y^1) + \mathbb{Z}^2$. The fibration appears as

$$M = \mathbb{Z}^4 \backslash (\mathbb{R} \times \mathbb{H}_3) \simeq \mathbb{R}^2 \times_{\mathbb{Z}^2, \alpha} \mathbb{T}^2, \quad \mathbb{Z}^4(x^1, x^2, y^1, y^2) \simeq [(x^1, y^1), (x^2, y^2) + \mathbb{Z}^2]_{\alpha},$$

where $\mathbb{R}^2 \times_{\mathbb{Z}^2, \alpha} \mathbb{T}^2$ is the set of equivalence classes in $\mathbb{R}^2 \times \mathbb{T}^2$ for the equivalence relation $((x^1, y^1), z) \sim_{\alpha} ((n^1 + x^1, n^3 + y^1), \alpha(n^1, n^3)z)$ for each $(n^1, n^3) \in \mathbb{Z}^2$ (with $z \in \mathbb{T}^2$) defined by the homomorphism $\alpha: \mathbb{Z}^2 \rightarrow \text{Diff}(\mathbb{T}^2)$ into the group of diffeomorphisms of the torus given by

$$\alpha(n^1, n^3)((x^2, y^2) + \mathbb{Z}^2) := ((x^2 + n^3 y^2, y^2) + \mathbb{Z}^2).$$

The group G is endowed with the left invariant symplectic structure

$$\tilde{\omega} = dx^1 \wedge dy^1 + dx^2 \wedge dy^2,$$

and the symplectic structure on $M = \Gamma \backslash G$ is the one lifting to $\tilde{\omega}$ on G :

$$\pi^* \omega = \tilde{\omega}$$

with $\pi: G \rightarrow M$ the natural projection. Any G -invariant almost complex structure \tilde{j} on G induces an almost complex structure j on M via

$$j \circ \pi_* = \pi_* \circ \tilde{j},$$

and the Nijenhuis tensor of j , N^j , is induced by the Nijenhuis tensor of \tilde{j} , $N^{\tilde{j}}$, via

$$N^j(\pi_* X, \pi_* Y) = \pi_*(N^{\tilde{j}}(X, Y)).$$

We take a G -invariant almost complex structure on G which is compatible with $\tilde{\omega}$ and positive. It yields an almost complex structure j on M which is compatible with the symplectic structure ω and positive, hence non integrable by Thurston's result. The Nijenhuis tensor of \tilde{j} is invariant by G and does not vanish; its value at the neutral element $e \in G$ is given by

$$N^{\tilde{j}}(X, Y) := [\tilde{J}X, \tilde{J}Y] - \tilde{J}[\tilde{J}X, Y] - \tilde{J}[X, \tilde{J}Y] - [X, Y], \quad \forall X, Y \in \mathfrak{g},$$

where \mathfrak{g} is the Lie algebra of G and $\tilde{J}: \mathfrak{g} \rightarrow \mathfrak{g}$ is the value of \tilde{j} at the neutral element.

The Lie algebra $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{S}_3 = \mathbb{R}^4$ has brackets given by

$$[(A^1, A^2, B^1, B^2), (X^1, X^2, Y^1, Y^2)] = (0, B^1 Y^2 - B^2 Y^1, 0, 0).$$

Since the derived ideal is 1-dimensional, the image of $N^{\tilde{J}}$ is spanned by $e_2 := (0, 1, 0, 0)$ and $\tilde{J}(e_2)$. The image of $N^{\tilde{J}}$ is an abelian ideal in \mathfrak{g} . Thus the invariant almost complex structure \tilde{j} on the group G is minimally non-integrable, $\text{Im } N^{\tilde{J}}$ defines a foliation of G , the leaf through $g \in G$ is the coset gH where $H = \mathbb{R}e_2 \oplus \mathbb{R}\tilde{J}(e_2) \simeq \mathbb{R}^2$ is an abelian normal subgroup of G ; each leaf carries an induced almost complex structure, and \tilde{j} induces a transverse complex structure on $G/H \simeq \mathbb{R}^2$. Projecting on M gives the following:

The almost complex structure j on the manifold M , induced by a G -invariant almost complex structure \tilde{j} on G which is compatible with $\tilde{\omega}$ and positive, is minimally non-integrable; the distribution D_j^∞ is the distribution defined by $\text{Im } N^j$; the leaf of $\text{Im } N^j$ through $\pi(g)$ is the projection of the leaf of $\text{Im } N^{\tilde{J}}$ through g ; it is given by $\pi(gH) = \pi(Hg)$ and its structure depends on the intersection of H with Γ ; the set of leaves is in bijection with the double coset space $\Gamma \backslash G/H$ which may not have a manifold structure.

The so-called standard almost complex structure j_0 on M corresponds to the construction with

$$\tilde{J}_0(A^1, A^2, B^1, B^2) = (-B^1, -B^2, A^1, A^2),$$

and hence $\tilde{J}_0(e_2) = e_4 := (0, 0, 0, 1)$. For this standard structure j_0 , and more generally, for any positive compatible j defined via a \tilde{J} such that $\tilde{J}(e_2)$ is in the plane spanned by e_2 and e_4 , the leaf of $\text{Im } N^j$ through $\pi(x^1, x^2, y^1, y^2)$ is given by the torus $\pi((x^1, x^2, y^1, y^2)(0, \mathbb{R}, 0, \mathbb{R}))$. Each leaf contains exactly one element in $\pi(x^1, 0, y^1, 0)$, the space of leaves is diffeomorphic to the torus \mathbb{T}^2 and the projection on the space of leaves is given by the projection of the fibration defined above

$$p : M \rightarrow \mathbb{T}^2 : \mathbb{Z}^4(x^1, x^2, y^1, y^2) \mapsto (x^1, y^1) + \mathbb{Z}^2.$$

The structure j induces a complex structure \bar{j} on the space of leaves such that p is pseudo-holomorphic; thus the transverse Dolbeault cohomology for (M, j) is the usual Dolbeault cohomology for (\mathbb{T}^2, \bar{j}) .

The transverse Dolbeault cohomology for the standard almost complex structure on Kodaira–Thurston manifold $M = \mathbb{Z}^4 \backslash (\mathbb{R} \times \mathbb{H}_3)$ is the usual Dolbeault cohomology of the torus $\mathbb{T}^2 = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$.

We shall now study the transverse Dolbeault cohomology for any almost complex structure j induced by a G -invariant almost complex structure \tilde{j} on G which is compatible with $\tilde{\omega}$ and positive.

Write $\tilde{J}(e_2) = (a, b, c, 1)\ell$, with $\ell > 0$ by positivity. The leaf through $\pi(x^1, x^2, y^1, y^2)$ is the image under the projection π of $\{(x^1 + at, x^2 + s, y^1 + ct, y^2 + t) \mid s, t \in \mathbb{R}\}$. The transverse Dolbeault cohomology is computed using the operator ∂_j restricted to forms in

$$\Omega_{\mathcal{D}_j^\infty, [j]}^{p,q} = \{\omega \in \Omega_j^{p,q} \mid \iota(X)\omega = 0, \mathcal{L}_X\omega = 0, \forall X \in \mathcal{I}m N^j\}.$$

A form $\alpha \in \Omega_{\mathcal{D}_j^\infty, [j]}^{p,q}$ lifts to a-form $\tilde{\alpha} = \pi^*\alpha \in \Omega_{\tilde{j}}^{p,q}(\mathbb{R}^4)$ which is invariant by Γ and such that $i(\tilde{X})\tilde{\alpha} = 0$ and $L_{\tilde{X}}\tilde{\alpha} = 0$ for all vector fields \tilde{X} with values in the $\mathcal{I}m N^{\tilde{J}}$; since

$$\tilde{j}(\partial_{x^2}) = \ell(a\partial_{x^1} + c\partial_{y^1} + \partial_{y^2}) + \ell(b + y^1)\partial_{x^2},$$

this is equivalent to

$$i(\partial_{x^2})\tilde{\alpha} = 0, \quad L(\partial_{x^2})\tilde{\alpha} = 0, \quad i(a\partial_{x^1} + c\partial_{y^1} + \partial_{y^2})\tilde{\alpha} = 0, \quad L(a\partial_{x^1} + c\partial_{y^1} + \partial_{y^2})\tilde{\alpha} = 0.$$

Smooth functions lift to Γ -invariant smooth functions on \mathbb{R}^4 which are constant along the leaves; since those functions do not depend on the second variable x^2 , they can be seen as \mathbb{Z}^4 -invariant functions on \mathbb{R}^4 for the usual action of \mathbb{Z}^4 on \mathbb{R}^4 given by translations, hence as pullbacks of functions on the torus $\mathbb{T}^4 = \mathbb{R}^4/\mathbb{Z}^4$ and constant on the leaves.

Computation of $H_{j\text{-trans}}^{0,0}(M)$

A function f satisfying $\bar{\partial}_j f = 0$ lifts to a function \tilde{f} such that $\bar{\partial}_{\tilde{j}} \tilde{f} = 0$. It also satisfies $\bar{\partial}_{j'} \tilde{f} = 0$, where j' is the \mathbb{R}^4 -invariant complex structure on \mathbb{R}^4 (\mathbb{R}^4 acting by translations) whose value at 0 is J . Indeed, on the one hand, since $d\tilde{f}(\partial_{x^2}) = 0$, we have that

$$\begin{aligned} d\tilde{f}(\partial_{x^2} - i\tilde{j}\partial_{x^2}) &= 0 \\ \iff d\tilde{f}(\partial_{x^2} - i\ell(a\partial_{x^1} + b\partial_{x^2} + c\partial_{y^1} + \partial_{y^2})) &= d\tilde{f}(\partial_{x^2} - ij'\partial_{x^2}) = 0. \end{aligned}$$

On the other hand, the Ω -orthogonal to $\text{Im } N^j$ is J -stable, spanned by $\{e_1 - ce_2, e_3 + ae_2\}$ and the corresponding G -invariant vector fields coincide with the vector fields invariant by translations, hence

$$(\partial_{x^1} - c\partial_{x^2}) - i\tilde{j}(\partial_{x^1} - c\partial_{x^2}) = (\partial_{x^1} - c\partial_{x^2}) - ij'(\partial_{x^1} - c\partial_{x^2}).$$

The lift of a function f satisfying $\bar{\partial}_j f = 0$ is thus holomorphic for j' and \mathbb{Z}^4 -periodic, which implies that it is constant. Hence $H_{j\text{-trans}}^{0,0}(M) = \mathbb{C}$.

Computation of $H_{j\text{-trans}}^{1,0}(M)$

The two 1-forms on \mathbb{R}^4 defined by $dx^1 - ady^2$ and $dy^1 - cdy^2$ form a basis of G -invariant forms which vanish on $\mathfrak{Im } N^{\tilde{j}}$; there is thus a complex number ρ such that

$$dx^1 - ady^2 + \rho(dy^1 - cdy^2)$$

is a G -invariant $(1, 0)$ -form for \tilde{j} . Hence a $(1, 0)$ -form on M lifts as

$$\tilde{f}(x^1, x^2, y^1, y^2)(dx^1 - ady^2 + \rho(dy^1 - cdy^2)),$$

with $\tilde{f}(x^1, x^2, y^1, y^2) = \tilde{f}(x^1 + at, x^2, y^1 + ct, y^2 + t)$ for all $t \in \mathbb{R}$, $\partial_{x^2} \tilde{f} = 0$ and \tilde{f} \mathbb{Z}^4 -periodic. Such a form is $\bar{\partial}_{\tilde{j}}$ -closed if and only if $\bar{\partial}_{\tilde{j}} \tilde{f} = 0$, if and only if, as above, f is constant. Hence $H_{j\text{-trans}}^{1,0}(M) = \mathbb{C}$.

Computation of $H_{j\text{-trans}}^{0,1}(M)$

A $(0, 1)$ -form lifts as

$$\tilde{f}(x^1, x^2, y^1, y^2)(dx^1 - ady^2 + \bar{\rho}(dy^1 - cdy^2)),$$

with \tilde{f} constant along the leaves and \mathbb{Z}^4 -periodic; it is always $\bar{\partial}_j$ closed. It is $\bar{\partial}_j$ exact if and only if its lift is $\bar{\partial}_{\tilde{j}}$ -exact and the boundary of a Γ -invariant function which is constant along the leaves. A function \tilde{f} is constant along the leaves if and only if, seen as a function of the 4-variables ($x'^1 := x^1 - ay^2, x'^2 := x^2, y'^1 := y^1 - cy^2, y'^2 := y^2$), it does not depend of x'^2 and y'^2 . We distinguish cases to study the Γ -invariance.

Case 1. If a, c and c/a are in $\mathbb{R} \setminus \mathbb{Q}$, a function which is constant along the leaves and \mathbb{Z}^4 -periodic is necessarily constant, because the projection of the line $\{(x^1 + at, y^1 + ct, y^2 + t) \mid t \in \mathbb{R}\}$ on the torus $\mathbb{R}^3/\mathbb{Z}^3$ has a dense image. Hence $H_{j\text{-trans}}^{0,1}(M) = \mathbb{C}$.

Case 2. If two of the three numbers a, c and c/a are in $\mathbb{R} \setminus \mathbb{Q}$ and one of them, say $c/a = p/q$, is in \mathbb{Q} , then a function \tilde{f} which is constant along the leaves and \mathbb{Z}^4 -periodic is just a function of one variable $\tilde{f}(x^1, x^2, y^1, y^2) = g(x^1 - q/p y^1)$ which is periodic of period $1/p$. Observe that

$$(dx^1 - ady^2 + \bar{\rho}(dy^1 - cdy^2)) = \rho' d(x^1 - q/p y^1) - i d(x^1 - q/p y^1) \circ \tilde{j}$$

for some $\rho' \in \mathbb{C}$, so that

$$\begin{aligned} \tilde{f}(x^1, x^2, y^1, y^2)(dx^1 - ady^2 + \bar{\rho}(dy^1 - cdy^2)) \\ = \rho' g(x^1 - q/p y^1)(d(x^1 - q/p y^1) - i d(x^1 - q/p y^1) \circ \tilde{j}) \end{aligned}$$

is the lift of a $\bar{\partial}_{\tilde{j}}$ exact $(0, 1)$ -form if and only if $g(x') dx'$ is the differential of a periodic function of period $1/p$; this is true if and only if the integral over a period of g vanishes, hence $H_{j\text{-trans}}^{0,1}(M) = \mathbb{C}$.

Case 3. If a and c are in \mathbb{Q} , write $a = p/m$ and $c = q/m$ with p, q, m in $\mathbb{Z}, m > 0$ minimum. The leaf through $\pi(x^1, x^2, y^1, y^2)$ is the projection of $\{(x^1 + pt, x^2 + s, y^1 + qt, y^2 + mt) \mid s, t \in \mathbb{R}\}$ and is isomorphic to $S^1 \times S^1$; the projection

$$\mathbb{R} \times \mathbb{H}_3 \rightarrow \mathbb{R}^2/\mathbb{Z}^2 : (x^1, x^2, y^1, y^2) \mapsto (x^1 - y^2 p/m, y^1 - y^2 q/m) + \mathbb{Z}^2$$

induces a diffeomorphism between the leaf space and $\mathbb{Z}_m \backslash \mathbb{T}^2$, with the action of $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ on \mathbb{T}^2 defined by

$$(k + m\mathbb{Z}) \cdot ((x^1, y^1) + \mathbb{Z}^2) := (x^1 + kp/m, y^1 + kq/m) + \mathbb{Z}^2.$$

The almost complex structure j induces a complex structure \hat{j} on this quotient $\mathbb{Z}_m \backslash \mathbb{T}^2$; this lifts to the invariant complex structure \hat{j}' on the torus $\mathbb{R}^2/\mathbb{Z}^2$ induced by \tilde{j} . The transverse cohomology for the almost complex structure j on M is the usual Dolbeault cohomology of $(\mathbb{Z}_m \backslash \mathbb{T}^2, \hat{j})$. The group \mathbb{Z}_m being a finite group acting holomorphically on \mathbb{T}^2 , and each cohomology class in the Dolbeault cohomology of (\mathbb{T}^2, \hat{j}') containing a form which is invariant by \mathbb{Z}_m , the Dolbeault cohomology of $(\mathbb{Z}_m \backslash \mathbb{T}^2, \hat{j})$ coincides with the Dolbeault cohomology of (\mathbb{T}^2, \hat{j}') .

Computation of $H_{j\text{-trans}}^{1,1}(M)$

A $(1, 1)$ -form lifts as

$$\tilde{f}(x^1, x^2, y^1, y^2)(dx^1 - ady^2) \wedge (dy^1 - cdy^2),$$

with \tilde{f} constant along the leaves and \mathbb{Z}^4 -periodic. We continue as above.

If a, c and c/a are in $\mathbb{R} \setminus \mathbb{Q}$, a function which is constant along the leaves and \mathbb{Z}^4 -periodic is necessarily constant, so $H_{j\text{-trans}}^{1,1}(M) = \mathbb{C}$.

If two of the three numbers a, c and c/a are in $\mathbb{R} \setminus \mathbb{Q}$ and one of them, say $c/a = p/q$, is in \mathbb{Q} , a function \tilde{f} which is constant along the leaves and \mathbb{Z}^4 -periodic is just a function of one variable $\tilde{f}(x^1, x^2, y^1, y^2) = g(x^1 - q/p y^1)$ which is periodic of period $1/p$; we can write

$$\begin{aligned} \tilde{f}(x^1, x^2, y^1, y^2)(dx^1 - a dy^2) \wedge (dy^1 - c dy^2) \\ = \rho'' g(x^1 - q/p y^1)(dx^1 - q/p y^1) \wedge d(x^1 - q/p y^1) \circ \tilde{j} \end{aligned}$$

for some $\rho'' \in \mathbb{C}$. The $(1, 1)$ -form is $\bar{\partial}_j$ exact if and only if $g(x') dx'$ is the differential of a periodic function of period $1/p$; this is true if and only if the integral over a period of g vanishes, hence $H_{j\text{-trans}}^{1,1}(M) = \mathbb{C}$.

Theorem 4.5. *On the Kodaira–Thurston manifold $M = \mathbb{Z}^4 \backslash (\mathbb{R} \times H_3)$, the only non vanishing transverse cohomology spaces for any almost complex structure j induced by a left invariant almost complex structure \tilde{j} on $G = \mathbb{R} \times H_3$ which is compatible with the invariant symplectic structure $\tilde{\omega}$ and positive, are*

$$H_{j\text{-trans}}^{0,0}(M) = \mathbb{C}, \quad H_{j\text{-trans}}^{1,0}(M) = \mathbb{C}, \quad H_{j\text{-trans}}^{0,1}(M) = \mathbb{C}, \quad H_{j\text{-trans}}^{1,1}(M) = \mathbb{C}.$$

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