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Representations of Witt Algebras

Dedicated to Kunal Kadam

By

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Abstract

In this paper we construct new families of representations for the Lie-algebra of diffeomorphisms of the torus T^d and describe its sub representations.

§ 0. Introduction

In this paper we construct a continuous family of representations for the Lie-algebra of Diff(T^d). (See [L] for more details and references on Diff(T^d). Here T^d is d-dimensional torus. The case $d=1$ is extensively studied. For example see [KR]). It can also be obtained as the Lie-algebra of derivations on $A=\boldsymbol{C}[t_1^{\pm1}, t_2^{\pm1}, \cdots, t_d^{\pm1}]$. We denote it by Der A and it has a basis $D^i(r)=t_1^{r_1}t_2^{r_2}\cdots t_i^{r_i+1}\cdots t_d^{r_d}$, $1\leqq i\leqq d$. Let h be the linear span of $D^i(0)$, $1\leqq i\leqq d$. Then h is an abelian subalgebra of Der A and Der A decomposes under h.

We construct three types of h weight modules for Der A with d, d^2 and 1 dimensional weight spaces. We investigate the submodule structure of these modules. Let α be a d tuple of complex numbers and b be a complex number. Then we define a Der A module $\pi(\alpha, b)$ whose weight spaces are d-dimensional. In proposition 1.4 we prove that $\pi(\alpha, b)$ is irreducible Der A module whenever $b\neq 0$. When $b=0$ but somecomponent of α is not an integer then we prove that there is a unique (irreducible) submodule. The case $b=0$ and all components of α are integers, we prove that $\pi(\alpha, b)$ is isomorphic to the well known *modules of differentials* Ω_A.

In section 2 we calculate the dual module of $\pi(\alpha, b)$.

In section 3 we construct modules $\hat{\pi}(\alpha, \beta)$ whose weight spaces are d^2 dimensional. In Proposition 3.1 we prove that $\hat{V}(\alpha, b)=W_1\oplus W_0$ where W_1 is a submodule with one dimensional weight spaces and W_0 is a submodule with d^2-1 dimensional weight spaces. We further prove that W_0 is *irreducible*. In

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the next section we give conditions under which W_1 is irreducible.

In section 4 we construct modules $\pi_1(\alpha, b)$ whose weight spaces are *one* dimensional. In Proposition 4.1 we prove $\pi_1(\alpha, b)$ is irreducible Der A module unless α is a d tuple of integers and $b \in \{0, 1\}$.

In section 5 we construct modules $\pi(k, S, \alpha, b)$ whose weight spaces are of d^k dimensional. We leave it as an open problem to describe its submodules.

§1. Modules with d-dimensional Weight Spaces

Let V be a d-dimensional vector space over complex numbers C and basis $e_1, e_2, \cdots, e_d$. Let $(\,,)$ be a non-degenerate symmetric bilinear form on V defined by $(e_i, e_j)=\delta_{i,j}$. Let $\Gamma=\bigoplus_{i=1}^{d} \boldsymbol{Z} e_i$ the $\boldsymbol{Z}$ linear combinations of $e_1, e_2, \cdots, e_d$, be a lattice of V. *We will assume that* $d \geqq 2$.

Let $A=\boldsymbol{C}[t_1^{\pm 1}, t_2^{\pm 1}, \cdots, t_d^{\pm 1}]$ be the algebra of Laurent polynomials functions of the torus $C^X \times C^X \times \cdots \times C^X$. It is well known that Der A, the Lie-algebra of derivations on A is given by the linear span of

$$D^i(r)=t_1^{r_1} t_2^{r_2} \cdots t_i^{r_i+1} \cdots t_d^{r_d} \frac{d}{dt_i}, \qquad r=\sum_i r_i e_i \in \Gamma .$$

Define derivation $D(u, r)$, $u \in V$, $r \in \Gamma$ by

$$D(u, r)=\sum_{i=1}^{d} u_i D^i(r), \qquad u=\sum u_i e_i .$$

(1.1) Then $[D(u, r), D(v, s)]=D(w, r+s)$, where $w=v(u, s)-u(v, r)$.

Let h be the abelian sub-algebra spanned by $D^i(0)$, $1 \leqq i \leqq d$. Clearly Der A is h weight module.

We now construct a continuous family of h-weight modules for Der A whose weight spaces are d-dimensional.

For each $r \in \Gamma$, take an isomorphic copy $V(r)$ of V. Denote the isomorphism by $v \mapsto v(r)$. Let $V(\alpha, b)=\bigoplus_{r \in \Gamma} V(r)$ for $\alpha \in V$ and $b \in \boldsymbol{C}$. We define a representation $\pi := \pi(\alpha, b)$ on $V(\alpha, b)$ for the Lie-algebra Der A.

(1.2) $$D(u, r)v(n)=(u, n+\alpha+br)v(n+r)+(u, v)r(n+r).$$

It is straightforward to verify that π defines a representation.

1.3 *Remark.* The module $V(\alpha, b)$ is isomorphic to $V(\alpha+r, b)$, $r \in \Gamma$ by sending $v(n)$ to $v(n+r)$.

1.4 Proposition.

(1) *If* $b \neq 0$ *then* $V(\alpha, b)$ *is irreducible as* der A *module.*

(2) *If* $b=0$ *then the subspace* W *spanned by* $(n+\alpha)(n)$ *is an irreducible submodule of* $V(\alpha, b)$.

(3) *If* $b=0$, $\alpha\notin\Gamma$ *then* W *is the only proper submodule of* $V(\alpha, b)$.

(4) *If* $b=0$, $\alpha\in\Gamma$ *then* $V(\alpha, b)/W$ *is not irreducible and* $V(-\alpha)$ *is a d-dimensional trivial (only) sub-module of* $V(\alpha, b)/W$.

First we prove some lemmas.

1.5 Lemma. *If* $b\neq 0$ *then the module generated by* $v(n)$ *for some non-zero* v *and for some* n *contains* $k(n)$ *for all* k *in* V.

Proof. Let W be the submodule generated by $v(n)$. Observe that

$$D(u, -r)D(u, r)v(n)=(u, n+\alpha+br)(u, n+r+\alpha-br)v(n)-2b(u, v)(u, r)r(n).$$

Choose u such that $(u, v)\neq 0$ and $(u, r)\neq 0$. (Define $\bar{v}=\sum\bar{v}_i e_i$, where $v=\sum v_i e_i$ and $\bar{v}_i$ denote the complex conjugate of v_i. Now if $(\bar{v}, \bar{r})=0$ choose $u=\bar{v}+\bar{r}$. If $(\bar{v}, \bar{r})\neq 0$ then choose $u=\bar{v}$). Then $r(n)$ belongs to W for all $r\in\Gamma$. Now by choosing $r=e_i$, $1\leqq i\leqq d$ we have $e_i(n)\in W$. By taking linear combinations of $e_i(n)$, we conclude W contains $k(n)$ for all k in V.

1.6 Lemma. *If* $b\neq 0$. *Let* W *be defined as in Lemma* 1.5. *Then* $W=V(\alpha, b)$.

Proof. In view of Lemma 1.5 it is sufficient to prove that given $n\in\Gamma$ there is a w in V such that $w(n)\in W$.

Consider

$$D(u, r)v(n)=(u, n+\alpha+br)v(n+r)+(u, v)r(n+r).$$

Now choose u such that $(u, v)\neq 0$ and $(u, n+\alpha+br)=0$. If v is a multiple of $n+\alpha+br$ then choose a different v. It can be done in view of Lemma 1.5. Then $r(n+r)\in W$. This being true for every r we are done.

Lemma 1.6 proves Proposition 1.4 (1).

Proof of Proposition 1.4 (2). First note that

$$D(u, r)(n+\alpha)(n)=(u, n+\alpha)(n+\alpha+r)(n+r). \tag{1.7}$$

Let W be linear span of $(n+\alpha)(n)$, $n\in\Gamma$. Then from 1.7 it follows that W is an irreducible submodule of $V(\alpha, 0)$. Also it is clear that each weight space is one-dimensional.

Proof of Proposition 1.4 (3). We have $b=0$ and $\alpha\notin\Gamma$. Let W_0 be any submodule different from W. Then W_0 necessarily contains a vector $t(n)$ such that t is not a scalar multiple of $n+\alpha$. Now choose u such that $(u, n+\alpha)=0$ and $(u, t)\neq 0$. Consider

$$D(u, r)t(n)=(u, n+\alpha)t(n)+(u, t)r(n+r).$$

Then it will follow that $r(n+r)\in W_0$.

Now consider

$$D(u', -r)\cdot r(n+r)=(u', n+\alpha+r)r(n)+(u', r)(-r)(n)=(u', n+\alpha)r(n)\in W_0.$$

Since $\alpha\notin\Gamma$, $n+\alpha\neq 0$ for any $n\in\Gamma$. Hence we can choose u' such that $(u', n+\alpha)\neq 0$. Hence $r(n)\in W_0$ for all $r\in\Gamma$. By choosing $r=e_i$, $1\leqq i\leqq d$ we have $e_i(n)\in W_0$.

Now consider

$$D(u', r)e_i(n)=(u', n+\alpha)e_i(n+r)+(u', e_i)r(n+r).$$

Since $r(n+r)\in W_0$ we have $e_i(n+r)\in W_0$. This proves $v(n+r)\in W_0$ for all $v\in V$ and for all $r\in\Gamma$. Hence $W_0=V(\alpha, b)$.

Proof of Proposition 1.4 (4). We have $b=0$, $\alpha\in\Gamma$. First note that

$$D(u, r)e_i(-\alpha)=(u, -\alpha)r(-\alpha+r)\in W. \tag{1.8}$$

Let W_1 be the space spanned by W and $e_i(-\alpha)$, $1\leqq i\leqq d$. Then by 1.8 it will follow that W_1 is a sub module of $V(\alpha, b)$. It will also follow that the space spanned by $e_i(-\alpha)$, $1\leqq i\leqq d$ is a trivial sub representation of $V(\alpha, b)/W$.

Let W_0 be a submodule of $V(\alpha, b)$. Assume that W_0 is *not* a submodule of W_1. Then we *claim* that $W_0=V(\alpha, b)$. Since W_0 is not contained in W_1, W_0 contains a vector $t(n)$ such that $n+\alpha\neq 0$ and t is not a scalar multiple of $n+\alpha$. By the argument in Proof of Proposition 1.4 (3), we can deduce that $W_0=V(\alpha, b)$.

In fact in this case $V(\alpha, b)$ is isomorphic to the well known *module of differentials* Ω_A.

§ 2. Duality

Let V be a vector space of dimension d as in section 1 with non-degenerate bilinear form (,). Let $V^*(\alpha, b)=\bigoplus_{r\in\Gamma} V(r)$, $\alpha\in\Gamma$, $b\in\boldsymbol{C}$ where $V(r)$ is an isomorphic copy of V. We define a representation $\pi^*=\pi^*(\alpha, b)$ on $V^*(\alpha, b)$ for the Lie-algebra Der A.

$$D(u, r)v(n)=(u, n+\alpha+br)v(n+r)-(r, v)u(n+r). \tag{2.1}$$

2.2 Definition. Let W be a der A module with finite dimensional weight spaces. That is $W=\bigoplus_{n\in\Gamma} W_n$ where each W_n is finite dimensional. Then $W^*=\bigoplus_{n\in\Gamma} W_n^*$ (where W_n^* is a vector space dual). The dual model is defined as

$$D(u, r)w^*\cdot v=w^*(D(u, -r)v).$$

2.3 Proposition. *The dual module of* $\pi(\alpha, b)$ *is isomorphic to* $\pi^*(\alpha, 1-b)$.

Proof. Define $e_i^*(n)$ of W^* by

$$e_i^*(n)e_j(m)=\delta_{i,j}\delta_{m,n}\,.$$

For $v\in V$, define $v^*(n)=\sum v_i e_i^*(n)$ where $v=\sum_i v_i e_i(n)$. Consider

$$D^j(r)e_i^*(n)v(n+r)=e_i^*(n)(D^j(-r)v(n+r))$$

$$=e_i^*(n)((n_j+r_j+\alpha_j-br_j)v(n)-v_j r(n))$$

$$=(n_j+\alpha_j-(b-1)r_j)e_i^*(n+r)v(n+r)-r_i e_j^*(n+r)v(n+r)$$

$$D^j(r)\tilde{v}^*(n)\cdot v(n+r)=n_j+\alpha_j-(b-1)r_j\tilde{v}^*(n+r)\cdot v(n+r)-(r,\ \tilde{v})e_j^*(n+r)v(n+r).$$

Let $D(u,\ r)=\sum u_j D^j(n)$. Then

$$D(u,\ r)\tilde{v}^*(n)=(u,\ n+\alpha-(b-1)r)\tilde{v}^*(n+r)-(r,\ v)u^*(n+r).$$

This proves the proposition.

§ 3. Modules with d^2-dimensional Weight Spaces

Let V be a vector space of dimension d as in section 1 with non-degenerate bilinear form (,). Let $\hat{V}(\alpha,\ b)=\oplus_{r\in\Gamma} V\otimes V(r)(\alpha\in V,\ b\in \boldsymbol{C})$ where $V\otimes V(r)$ is an isomorphic copy of $V\otimes V$. We define a representation $\hat{\pi}(\alpha,\ b)=\hat{\pi}$ on $\hat{V}(\alpha,\ b)$ for the Lie-algebra Der A.

$$D(u,\ r)k\otimes t(n)=(u,\ n+\alpha+br)k\otimes t(n+r)$$

$$-(r,\ t)k\otimes u(n+r)$$

$$+(k,\ u)r\otimes t(n+r)$$

clearly $\hat{V}(\alpha,\ b)$ is a weight module where each weight space is d^2 dimensional.

3.1 Proposition. (1) *Let* $k(n)=\sum_{i=1}^d e_i\otimes e_i(n)$ *and let* W_1 *be linear span of* $k(n)$, $n\in\Gamma$. *Then* W_1 *is a submodule of* $\hat{V}(\alpha,\ b)$ *whose weight spaces are one dimensional.*

(2) *Let* $W_0=\{\sum_{i=1}^d k_i\otimes t_i(n),\ n\in\Gamma,\ \sum(k_i,\ t_i)=0\}$.

Then W_0 *is an irreducible submodule whose weight spaces are of* d^2-1 *dimensional.*

(3) $\hat{V}(\alpha,\ b)=W_1\oplus W_0$.

3.2 *Remark*. Let $W_0(n)$ be a weight space of W_0 of weight n. Then observe that $e_i\otimes e_j(n)$, $i\neq j$ and $e_i\otimes e_i(n)-e_{i+1}\otimes e_{i+1}(n)$ $1\leq i\leq d-1$ is a vectorspace basis of $W_0(n)$. In particular $W_0(n)$ is of d^2-1 dimension and any vector $v\in W_0(n)$ can be written as

$$\sum_{i\neq j} a_{ij}e_i\otimes e_j(n)+\sum_{i=1}^{d} a_i e_i\otimes e_i(n), \qquad \sum_{i=1}^{d} a_i=0\,.$$

We first prove some lemmas.

3.3 Lemma. *Let $\widetilde{W}$ be some submodule of W_0. Then $\widetilde{W}$ contains a vector*

$$w=\sum_{i\neq j} a_{ij}e_i\otimes e_j(n)+\sum a_i e_i\otimes e_i(n), \qquad \sum a_i=0$$

and $a_{lk}\neq 0$ for some $l\neq k$.

Proof. $\widetilde{W}$ is a weight module being a submodule of a weight module W_0. Hence $\widetilde{W}$ contains weight vectors. Let $v\in\widetilde{W}$ be a weight vector and write

$$v=\sum_{i\neq j} a_{ij}e_i\otimes e_j(n)+\sum_{i} a_i e_i\otimes e_i(n), \qquad \sum a_i=0\,.$$

If $a_{ij}\neq 0$ for some $i\neq j$ then we are done. So we can assume

$$v=\sum a_i e_i\otimes e_i(n), \qquad \sum a_i=0\,.$$

Consider

$$\begin{aligned} D(u,\ r)v=&(u,\ n+\alpha+br)\sum_{i} a_i e_i\otimes e_i(n+r)\\ &-\sum_{i}(r,\ e_i)a_i e_i\otimes u(n+r)\\ &+\sum(u,\ e_i)a_i r\otimes e_i(n+r)\,. \end{aligned}$$

Choose $u=e_k$, $r=e_l$ for some $k\neq l$. Then

$$\begin{aligned} D(u,\ r)v=&(u,\ n+\alpha+br)\sum a_i e_i\otimes e_i(n+r)\\ &-a_l e_l\otimes e_k(n+r)+a_k e_l\otimes e_k(n+r)\,. \end{aligned}$$

Suppose $a_k=a_l$ for all $k\neq l$. Then it is a contradiction to $\sum a_i=0$. Hence $a_k\neq a_l$ for some $k\neq l$. This proves the Lemma.

3.4 Lemma.

$$\begin{aligned} D(u,\ -r)D(u,\ r)k\otimes t(n)=&(u,\ n+\alpha+r-br)(u,\ n+\alpha+br)k\otimes t(n)\\ &+(2b-2)(r,\ t)(r,\ u)k\otimes u(n)\\ &-2b(k,\ u)(u,\ r)r\otimes t(n)\\ &+2(k,\ u)(r,\ t)r\otimes u(n)\,. \end{aligned}$$

Proof. Direct checking.

3.5 Lemma. *Let $\widetilde{W}$ be some submodule of W_0 and let w be a vector of $\widetilde{W}$ such that*

$$w=\sum_{i\neq j} a_{ij}e_i\otimes e_j(n)+\sum a_i e_i\otimes e_i(n), \qquad \sum a_i=0$$

and $a_{ij}\neq 0$ *for some* $i\neq j$. *Then* $e_i\otimes e_j(n)\in\widetilde{W}$.

Proof. Let $u=e_i$, $r=e_j$ so that $(u, r)=0$. Now by Lemma 3.4 we have

$$D(u, -r)\cdot D(u, r)w=(u, n+r+\alpha-br)(u, n+\alpha+br)w+a_{ij}e_i\otimes e_j(n)\in\widetilde{W}.$$

This implies $e_i\otimes e_j(n)\in\widetilde{W}$.

3.6 Lemma. *Let* $\widetilde{W}$ *be some submodule of* W_0. *Assume that* $e_i\otimes e_j(n)\in\widetilde{W}$ *for some* n *and for some* $i\neq j$. *Then*

(i) $e_l\otimes e_k(n)\in\widetilde{W}$ *for all* $l\neq k$.

(ii) $e_l\otimes e_l(n)-e_k\otimes e_k(n)\in\widetilde{W}$ *for all* l *and* k.

Proof. Claim 1. $e_j\otimes e_i(n)$, $e_i\otimes e_i(n)-e_j\otimes e_j(n)\in\widetilde{W}$.
Let $u=e_i+e_j$, $r=e_j+e_i$. Then by Lemma 3.4 we have

$$\begin{aligned} D(u, -r)D(u, r)e_i\otimes e_j(n)=&(u, n+r+\alpha-br)(u, n+\alpha+br)e_i\otimes e_j(n)\\ &+(4b-2)(e_i\otimes e_i(n)-e_j\otimes e_j(n))\\ &+2(e_j\otimes e_i(n)-e_i\otimes e_j(r))\in\widetilde{W}. \end{aligned}$$

Now by Lemma 3.5 we have

$$e_j\otimes e_i(r)\in\widetilde{W}.$$

Also we have

$$(4b-2)(e_i\otimes e_i(n)-e_j\otimes e_j(n))\in\widetilde{W} \qquad \text{———} \qquad A1$$

Now take $r=e_j$ and $u=e_i+e_j$. Then by Lemma 3.4 we have

$$\begin{aligned} D(u, -r)D(u, r)e_i\otimes e_j(n)=&(u, n+r+\alpha-br)(u, n+\alpha+br)e_i\otimes e_j(n)\\ &+(2b-2)(e_i\otimes e_i(n)-e_j\otimes e_j(n))\\ &+2e_j\otimes e_i(n)+(2b-2)e_i\otimes e_j(n)\in\widetilde{W}. \end{aligned}$$

Since $e_i\otimes e_j(n)$, $e_j\otimes e_i(n)\in\widetilde{W}$, it will follow that

$$(2b-2)(e_i\otimes e_i(n)-e_j\otimes e_j(n))\in\widetilde{W} \qquad \text{———} \qquad A2$$

From $A1$ and $A2$ we have

$$e_i\otimes e_i(n)-e_j\otimes e_j(n)\in\widetilde{W}.$$

This completes the proof of Claim 1. Now to see the proof of Lemma 3.5, the case $d=2$ follows from Claim 1. Hence we can assume $d\geqq 3$. Let $k\neq i$ and $k\neq j$ and take $u=e_k+e_i$, $r=e_k+e_j$.

Consider

$$D(u, -r)D(u, r)e_i \otimes e_j(n) = (u, n+r+\alpha-br)(u, n+\alpha+br)e_i \otimes e_j(n)$$
$$+2b(e_i \otimes e_i(n) - e_j \otimes e_j(n))$$
$$+2(e_k \otimes e_k(n) - e_i \otimes e_i(n))$$
$$+2(b-1)e_i \otimes e_k(n)$$
$$-2be_k \otimes e_j(n) + 2e_k \otimes e_i(n) + 2e_j \otimes e_i(n)$$
$$+2e_j \otimes e_k(n) \in \widetilde{W} .$$

Now by Lemma 3.5 we have $e_k \otimes e_i(n)$, $e_j \otimes e_i(n)$ and $e_j \otimes e_k(n) \in \widetilde{W}$. By replacing the above argument for vectors $e_k \otimes e_i(n)$, $e_j \otimes e_i(n)$, $e_j \otimes e_k(n)$ in place of $e_i \otimes e_j(n)$ and l in place of k, we conclude that $e_k \otimes e_l \in \widetilde{W}$ for any $k \neq l$. This completes the first part of the Lemma. We also have

$$2b(e_i \otimes e_i(n) - e_j \otimes e_j(n)) + 2(e_k \otimes e_k(n) - e_i \otimes e_i(n)) \in \widetilde{W} .$$

But by Claim 1 we know that $e_i \otimes e_i(n) - e_j \otimes e_j(n) \in \widetilde{W}$. Hence $e_k \otimes e_k - e_i \otimes e_i(n) \in \widetilde{W}$. This completes the second part of the Lemma.

Proof of the Proposition 3.1.

(1) It is easy to verify that W_1 is an invariant subspace of $\hat{V}(\alpha, b)$. It is clear that each weight space is *one* dimensional.

(2) Let $\widetilde{W}$ be a non-zero submodule of W_0.

Claim: (1) $e_i \otimes e_j(n) \in \widetilde{W}$ *for all* $i \neq j$ *and for all* $n \in \Gamma$

(2) $e_i \otimes e_i(n) - e_j \otimes e_j(n) \in \widetilde{W}$ *for all* $i \neq j$ *and for all* $n \in \Gamma$.

To prove the claim, in view of Lemma 3.6, it is sufficient to prove that there exists i and j, $i \neq j$ such that $e_i \otimes e_j(n) \in \widetilde{W}$ for all $m \in \Gamma$. But by Lemmas 3.3 and 3.5, $\widetilde{W}$ contains $e_i \otimes e_j(n)$ for some n and for some $i \neq j$.

Subclaim. $e_i \otimes e_j(m) \in \widetilde{W}$, *for all* $m \in \Gamma$.

Consider

$$D(u, r)e_i \otimes e_j(n) + D(u, -r)e_i \otimes e_j(n) = (u, n+\alpha)e_i \otimes e_j(n+r) \in \widetilde{W} .$$

Suppose $n+\alpha \neq 0$. Then choose u such that $(u, n+\alpha) \neq 0$. Then subclaim follows. Suppose $n+\alpha = 0$. Then consider

$$D(u, r)e_i \otimes e_j(n) = b(u, r)e_i \otimes e_j(n+r)$$
$$-(r, e_j)e_i \otimes u(n+r)$$
$$+(e_i, u)r \otimes e_j(n+r).$$

Choose $r = e_j + e_i$, $u = e_j - e_i$. Then $(u, r) = 0$, $(r, e_j) = 1$, $(e_i, u) = -1$ and

$$D(u,r)e_i\otimes e_j(n)=-(e_i\otimes(-e_i+e_j)(n+r))-(e_j+e_i)\otimes e_j(n+r)$$
$$=e_i\otimes e_i(n+r)-e_j\otimes e_j(n+r)-2e_i\otimes e_j(n+r).$$

Now by Lemma 3.4, $e_i\otimes e_j(n+r)\in\widetilde{W}$. This proves the sub claim.

In view of Remark 3.2 and the claim we prove that $\widetilde{W}=W_0$. Also by Remark 3.2 each weight space of W_0 is d^2-1 dimensional.

(3) It is clear that $W_0\cap W_1=\{0\}$. Now $\hat{V}=W_0\oplus W_1$ as the dimensions of weight spaces match.

§ 4. Modules with 1-dimensional Weight Spaces

In this section we construct modules for Der A whose weight spaces are one-dimensional. Let V_1 be one dimensional vector space with basis v. For each $r\in\Gamma$, take an isomorphic copy $V_1(r)$ of V_1. Denote the isomorphism by $v\mapsto v(r)$. Let $W(\alpha,b)=\oplus_{r\in\Gamma}V_1(r)$ for $\alpha\in V$ and $c\in C$. Define Der A module $\pi_1(\alpha,b)$ in the following way.

$$D(u,r)v(n)=(u,n+\alpha+br)v(n+r).$$

It is straightforward to verify that $W(\alpha,b)$ is a Der A module.

4.1 Proposition.

(1) *$W(\alpha,b)$ is irreducible* Der A*-module unless $\alpha\in\Gamma$ and $b\in\{0,1\}$.*

(2) *If $\alpha\in\Gamma$ and $b=0$ then $\boldsymbol{C}v(-\alpha)$ is the only non-zero* Der A *proper submodule of $W(\alpha,b)$.*

(3) *If $\alpha\in\Gamma$ and $b=1$ then $W(\alpha,b)-\boldsymbol{C}v(-\alpha)$ is the only* Der A *proper (irreducible) submodule of $W(\alpha,b)$.*

Proof. It can easily be deduced from the following well known Proposition A.

Let $d_n=t^{n+1}d/dt$ be a derivation on $\boldsymbol{C}[t,t^{-1}]$ the Laurent polynomials in one variable. Let L be the Lie-algebra spanned by d_n, $n\in\boldsymbol{Z}$ with Lie structure

$$[d_n,d_m]=(m-n)d_{n+m}.$$

For any complex numbers a, b define

$$V_{a,b}=\oplus_{k\in\boldsymbol{Z}}\boldsymbol{C}v_k.$$

Now we define $L-$ module structure $V_{a,b}$ depending on a and b.

$$d_nv_k=(k+a+bn)v_{n+k}.$$

Proposition. $\boldsymbol{A}$ [KR].

(1) *$V_{a,b}$ is reducible as $L-$ module if and only if $a\in\boldsymbol{Z}$ and $b\in\{1,0\}$.*

(2) *If $b=0$ and $a\in\boldsymbol{Z}$ then Cv_{-a} is the only proper submodule.*

(3) *If* $b=1$ *and* $a\in\boldsymbol{Z}$ *then* $V_{a,b}\setminus\{Cv_{-a}\}$ *is the only proper (irreducible) submodule.*

§ 5. Modules with d^k-dimensional Weight Spaces

In this section, for every positive integer k, we construct a continuous family of modules whose weight spaces are d^k dimensional. Let V be a vector space of dimension d with non-degenerate bilinear form $(\,,)$ defined in section 1. Let $W=V\otimes\cdots\otimes V$($k$ times). Let $V(k,\alpha,b)=\oplus_{n\in\Gamma}W(n)$ where $W(n)$ is an isomorphic copy of W. Let S be *any* subset of $\{1,2,\cdots,k\}$. Define Der A module $\pi(k,S,\alpha,b)$ in the following way.

$$\begin{aligned}D(u,r)v_1\otimes\cdots\otimes v_k(n)&=(u,n+\alpha+br)v_1\otimes\cdots\otimes v_k(n+r)\\&+\sum_{i\in S}(u,v_i)v_1\otimes\cdots v_{i-1}\otimes r\otimes v_{i+1}\cdots\otimes v_k(n+r)\\&-\sum_{i\notin S}(r,v_i)v_1\otimes\cdots v_{i-1}\otimes u\otimes v_{i+1}\cdots\otimes v_k(n+r).\end{aligned}$$

It is straightforward to verify that $\pi(k,S,\alpha,b)$ defines Der A module. We leave it as an open problem to describe its submodules.

The above modules are motivated by vertex operator representations constructed in [EM].

5.1 *Remark.* (1) It can easily be verfiied that the dual of $\pi(k,S,\alpha,b)$ is isomorphic to $\pi(k,S^{\perp},\alpha,1-b)$ where $S^{\perp}=\{1,2,\cdots,k\}\setminus S$.

(2) Let σ be a permutation of $\{1,2,\cdots,k\}$ such that $\sigma(S)\subseteq S$. σ acts on W in an obvious way and commutes with π action. So that each σ-eigenspace of W is a subrepresentation. In particular W is reducible. Such σ exists in all cases except in two cases (and its duals) discussed in section 1 and 3.

5.2 *Remark.* The modules considered in Proposition A are the only known modules for L with property.

(1) irreducibility

(2) dimension of weight spaces are bounded by uniform constant. The other known modules for L are highest weight and lowest weight modules. It has been conjectured by Kac in [K] that these are the only modules for L with finite dimensional weight spaces.

Now coming to the Der A the only known modules are the one constructed in this paper. The concept of highest weight modules does not go through for Der A as there is no canonical positive and negative subalgebras.

In [EM] some modules for an abelian extension (infinite) for Der A are constructed. It may be said that Der A has no non-trivial central extensions. [RSS].

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Added in proof: Recently, we have become aware of a paper by T. A. Larsson, Conformal fields: A class of Representations of Vect(N), Internat. J. Modern Phys. A, **7**, No. 26 (1992), 6493-6508 where he defines modules similar to our §5.