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## On a reverse Kohler-Jobin inequality

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**Abstract.** In this paper, we consider the shape optimization problems for the quantities  $\lambda(\Omega)T^q(\Omega)$ , where  $\Omega$  varies among open sets of  $\mathbb{R}^d$  with a prescribed Lebesgue measure. While the characterization of the infimum is completely clear, the same does not happen for the maximization in the case  $q > 1$ . We prove that for  $q$  large enough a maximizing domain exists among quasi-open sets and that the ball is optimal among *nearly spherical domains*.

### 1. Introduction

In the present paper, we consider two well-known quantities that appear in the study of elliptic equations in the Euclidean space  $\mathbb{R}^d$ ,  $d \geq 2$ . The first one is usually called *torsional rigidity*, and is defined, for every nonempty open set  $\Omega \subset \mathbb{R}^d$  with finite Lebesgue measure (in the following, a *domain*), as

$$T(\Omega) = \int w_\Omega dx,$$

where  $w_\Omega$  is the unique solution of the PDE

$$-\Delta u = 1 \quad \text{in } \Omega, \quad u \in H_0^1(\Omega).$$

Equivalently, we may define  $T(\Omega)$  as

$$T(\Omega) = \max \left\{ \frac{(\int u dx)^2}{\int |\nabla u|^2 dx} : u \in H_0^1(\Omega) \setminus \{0\} \right\}.$$

In the integrals above and in the following, we use the convention that integrals without the indicated domain are intended over the entire space  $\mathbb{R}^d$ . The quantity  $T(\Omega)$  satisfies the scaling property

$$T(t\Omega) = t^{d+2} T(\Omega) \quad \text{for every } t > 0;$$

in addition, the maximum of  $T(\Omega)$  among domains with prescribed measure is reached by the ball (*Saint-Venant's inequality*), which can be written in the scaling free formulation as

$$|\Omega|^{-(d+2)/d} T(\Omega) \leq |B|^{-(d+2)/d} T(B),$$

for every domain  $\Omega$  and for every ball  $B \subset \mathbb{R}^d$ .

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The second quantity is the *first eigenvalue*  $\lambda(\Omega)$  of the Dirichlet Laplacian, defined as the smallest  $\lambda$  such that the PDE

$$-\Delta u = \lambda u \quad \text{in } \Omega, \quad u \in H_0^1(\Omega),$$

admits a nonzero solution. Equivalently,  $\lambda(\Omega)$  can be defined through the minimization of the Rayleigh quotient

$$\lambda(\Omega) = \min \left\{ \frac{\int |\nabla u|^2 dx}{\int u^2 dx} : u \in H_0^1(\Omega) \setminus \{0\} \right\}.$$

The quantity  $\lambda(\Omega)$  satisfies the scaling property

$$\lambda(t\Omega) = t^{-2} \lambda(\Omega) \quad \text{for every } t > 0;$$

in addition, the minimum of  $\lambda(\Omega)$  among domains with prescribed measure is reached by the ball (the *Faber–Krahn inequality*), which can be written in the scaling free formulation as

$$|\Omega|^{2/d} \lambda(\Omega) \geq |B|^{2/d} \lambda(B),$$

for every domain  $\Omega$  and for every ball  $B \subset \mathbb{R}^d$ .

The study of relations between  $T(\Omega)$  and  $\lambda(\Omega)$  was performed in several papers (see for instance [1–4, 6, 12, 13, 18, 21–23]), where some important inequalities were established. In particular,

- the Kohler-Jobin inequality

$$\lambda(\Omega) T^q(\Omega) \geq \lambda(B) T^q(B),$$

valid for every  $q \in [0, 2/(d+2)]$  and for every domain  $\Omega$ , where  $B$  is any ball in  $\mathbb{R}^d$  with  $|B| = |\Omega|$ ;

- the Pólya inequality

$$0 < \frac{\lambda(\Omega) T(\Omega)}{|\Omega|} < 1,$$

valid for every domain  $\Omega$  of  $\mathbb{R}^d$ .

In the present paper, we consider the scaling free shape functional

$$F_q(\Omega) = \frac{\lambda(\Omega) T^q(\Omega)}{|\Omega|^{\alpha_q}}, \quad \text{with } \alpha_q = \frac{-2 + q(d+2)}{d},$$

and the two quantities

$$m_q = \inf \{ F_q(\Omega) : \Omega \text{ domain} \} \quad \text{and} \quad M_q = \sup \{ F_q(\Omega) : \Omega \text{ domain} \}.$$

While the situation for  $m_q$  is fully clear, and by the Kohler-Jobin inequality, together with the Saint-Venant inequality, we have

$$m_q = \begin{cases} F_q(B) & \text{if } q \leq 2/(d+2), \\ 0 & \text{if } q > 2/(d+2), \end{cases}$$

the characterization of  $M_q$  is not yet complete. The results available up to now are the following (see [1] and [3]):

- $M_q = \infty$  for every  $q < 1$ ;
- $M_q = 1$  when  $q = 1$ , with the upper bound 1 not reached by any domain  $\Omega$ ;
- $M_q < \infty$  for every  $q > 1$ .

We investigate here this last case. The maximal expectation would be having the following result (the reverse Kohler-Jobin inequality):

- for every  $q > 1$ , the supremum  $M_q$  is attained on an optimal domain  $\Omega_q$ ;
- for every  $q > 1$ , the free boundary  $\partial\Omega_q$  is a smooth  $d - 1$  surface;
- there exists a threshold  $q^* > 1$  such that for every  $q \geq q^*$ , the supremum  $M_q$  is attained by a ball<sup>1</sup>.

We are unable to prove the results in the strong form above. We prove here the weaker results below, for which we need to extend the functional  $F_q$  to the set of capacity measures (see Section 2):

- for every  $q > 1$ , the supremum  $M_q$  is reached on a capacity measure  $\mu_q$  (Theorem 4.3);
- there exists a threshold  $q_0 > 1$  such that, for every  $q \geq q_0$ , the supremum  $M_q$  is reached by a quasi-open set  $\Omega_q$  (Theorem 5.3);
- there exists another threshold  $q_1$  such that, for every  $q \geq q_1$ , the ball is a maximizer for the shape functional  $F_q$  among *nearly spherical domains* (Theorem 6.2).

## 2. Capacity measures

The concept of capacity measure and the related properties shall be a very useful tool for our purposes. When dealing with sequences of PDEs of the form

$$-\Delta u = f \quad \text{in } \Omega_n, \quad u \in H_0^1(\Omega_n),$$

a natural question is to establish if the sequence  $u_{n,f}$  of solutions, or a subsequence of it, converges in  $L^2$  to some function  $u_f$ , and to determine in this case the PDE that the function  $u_f$  solves. Starting from the pioneering papers [14, 15], it is now well understood that the right framework to treat such a kind of questions is that of capacity measures. Below we recall the main results and definitions following [10] and [24]. For further information we refer the reader to the monographs [8] and [20], and references therein.

**Definition 2.1.** We say that a nonnegative Borel regular measure  $\mu$ , possibly taking the value  $\infty$ , is a capacity measure if

$$\mu(E) = 0 \quad \text{whenever } E \text{ is a Borel set with } \text{cap}(E) = 0,$$

being  $\text{cap}(E)$  the capacity

$$\text{cap}(E) = \inf \left\{ \int_{\mathbb{R}^d} |\nabla u|^2 + u^2 dx : u \in H_0^1(\mathbb{R}^d), u = 1 \text{ in a neighborhood of } E \right\}.$$

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<sup>1</sup>After the submission of this paper, we have been informed that D. Bucur and coauthors were working on similar problems. In particular, by proving a new sharp stability inequality for the spectrum of the Dirichlet Laplacian, they are able to show that such a  $q^*$  does exist. Their results are now collected in the preprint [11].

A property  $P(x)$  is said to hold quasi-everywhere (briefly, q.e.) if the set where  $P(x)$  does not hold has zero capacity. A Borel set  $\Omega \subset \mathbb{R}^d$  is said to be quasi-open if there exists a function  $u \in H^1(\mathbb{R}^d)$  such that  $\Omega = \{u > 0\}$  up to a set of capacity zero. A function  $f: \mathbb{R}^d \rightarrow \mathbb{R}$  is said to be quasi-continuous if there is a sequence of open sets  $\omega_n \subset \mathbb{R}^d$  such that  $\lim_{n \rightarrow \infty} \text{cap}(\omega_n) = 0$  and  $f$  is continuous when restricted to  $\mathbb{R}^d \setminus \omega_n$ . It is well known (see for instance [19]) that every Sobolev function has a quasi-continuous representative, and that two quasi-continuous representatives coincide quasi-everywhere. We then identify the space  $H^1(\mathbb{R}^d)$  with the space of quasi-continuous representatives. We recall that a sequence  $u_n \in H^1(\mathbb{R}^d)$  that converges in norm to some  $u \in H^1(\mathbb{R}^d)$ , converges quasi-everywhere (up to a subsequence) to  $u$ .

Given  $\mu$  a capacity measure, we denote by  $H_\mu^1$  the following space:

$$H_\mu^1 = H^1(\mathbb{R}^d) \cap L_\mu^2(\mathbb{R}^d) = \left\{ u \in H^1(\mathbb{R}^d) : \int u^2 d\mu < \infty \right\}.$$

The space  $H_\mu^1$  is a Hilbert space when endowed with  $\|u\|_{H_\mu^1} = \|u\|_{H^1(\mathbb{R}^d)} + \|u\|_{L_\mu^2(\mathbb{R}^d)}$ , where the quantity  $\|u\|_{L_\mu^2(\mathbb{R}^d)}$  is well defined, being Sobolev functions defined up to a set of zero capacity. We always identify two capacity measures  $\mu$  and  $\nu$  for which

$$(2.1) \quad \int u^2 d\mu = \int u^2 d\nu, \quad \text{for every } u \in H^1(\mathbb{R}^d).$$

If instead (2.1) holds with “ $\leq$ ”, we say that  $\mu \leq \nu$ , and in this case we have  $H_\nu^1 \subseteq H_\mu^1$ . We can associate to any open set (or more generally to any quasi-open set)  $\Omega \subset \mathbb{R}^d$  the capacity measure  $I_\Omega$  defined as follows:

$$I_\Omega(E) := \begin{cases} 0 & \text{if } \text{cap}(E \setminus \Omega) = 0, \\ \infty & \text{if } \text{cap}(E \setminus \Omega) > 0. \end{cases}$$

Notice that, if  $\mu = I_\Omega$  for some open set  $\Omega \subset \mathbb{R}^d$ , then  $H_\mu^1 = H_0^1(\Omega)$ .

To extend the notion of torsional rigidity to a capacity measure  $\mu$ , we need to carefully deal with the fact that the embedding  $H_\mu^1 \hookrightarrow L^1(\mathbb{R}^d)$  can be noncompact, and even noncontinuous. Nevertheless, we can follow an approximation argument: for every  $R > 0$ , let  $w_R$  be the solution to the minimization problem

$$\min \left\{ \int |\nabla u|^2 dx + \int u^2 d\mu - \int u dx : u \in H_\mu^1 \cap H_0^1(B_R) \right\}.$$

The torsion function  $w_\mu$  and the torsional rigidity  $T(\mu)$  of the capacity measure  $\mu$  are defined as

$$w_\mu := \sup_{R>0} w_R \quad \text{and} \quad T(\mu) := \int w_\mu dx.$$

The Dirichlet eigenvalue of  $\mu$  can be defined through the following Rayleigh-type quotient:

$$\lambda_1(\mu) = \inf_{u \in H_\mu^1 \setminus \{0\}} \frac{\int |\nabla u|^2 dx + \int u^2 d\mu}{\int u^2 dx}.$$

Clearly, if  $\mu = I_\Omega$  for some domain  $\Omega \subset \mathbb{R}^d$ , we have  $T(\mu) = T(\Omega)$  and  $\lambda(\mu) = \lambda(\Omega)$  (we adopt this notation also if  $\Omega$  is a quasi-open set). For a general capacity measure  $\mu$ , neither  $\lambda(\mu)$  is necessarily attained by some function  $u \in H_\mu^1$  nor  $T(\mu)$  is necessarily finite. However, as shown in [9], it holds the following:

$$w_\mu \in L^1(\mathbb{R}^d) \iff T(\mu) < \infty \implies \lambda_1(\mu) \text{ is attained by some } u \in H_\mu^1.$$

For every capacity measure  $\mu$  with  $T(\mu) < \infty$ , we define the *set of finiteness*  $A_\mu$  as the quasi-open set

$$A_\mu := \{w_\mu > 0\}.$$

In the case when  $\mu = I_\Omega$ , for some domain  $\Omega \subset \mathbb{R}^d$ , we have  $A_\mu = \Omega$ . The set of capacity measures with finite torsion can be endowed with the following notion of distance.

**Definition 2.2.** Given two capacity measures  $\mu$  and  $\nu$  such that  $w_\mu, w_\nu \in L^1(\mathbb{R}^d)$ , we define the  $\gamma$ -distance between them as

$$d_\gamma(\mu, \nu) = \|w_\mu - w_\nu\|_{L^1(\mathbb{R}^d)}.$$

We say that a sequence  $\mu_n$   $\gamma$ -converges to  $\mu$  if  $d_\gamma(\mu_n, \mu) \rightarrow 0$  as  $n \rightarrow \infty$ . When  $I_{\Omega_n} \xrightarrow{\gamma} \mu$ , we simply write  $\Omega_n \xrightarrow{\gamma} \mu$ .

We summarize the main properties of the  $\gamma$ -distance below:

- The space  $(\{\mu : \mu \text{ capacity measure with } w_\mu \in L^1(\mathbb{R}^d)\}, d_\gamma)$  is a complete metric space, and the set  $\{I_\Omega : \Omega \subset \mathbb{R}^d \text{ open set with } w_\Omega \in L^1(\mathbb{R}^d)\}$  is a dense subset of it.
- The functionals  $\mu \mapsto \lambda(\mu)$  and  $\mu \mapsto T(\mu)$  are  $\gamma$ -continuous.
- The map  $\mu \mapsto |A_\mu|$ , or more generally integral functionals as  $\int_{A_\mu} f(x) dx$  with  $f \geq 0$  and measurable, are lower semicontinuous with respect to the  $\gamma$ -convergence.
- The  $\gamma$ -convergence of  $\mu_n$  to  $\mu$  implies the  $\Gamma$ -convergence in  $L^2(\mathbb{R}^d)$  of the functionals  $\|\cdot\|_{H_{\mu_n}^1} : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  defined by

$$\|u\|_{H_{\mu_n}^1} = \begin{cases} \|u\|_{H^1(\mathbb{R}^d)} + \int u^2 d\mu_n & \text{if } u \in H_{\mu_n}^1, \\ \infty & \text{if } u \notin H_{\mu_n}^1, \end{cases}$$

to the functional  $\|\cdot\|_{H_\mu^1} : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ ,

$$\|u\|_{H_\mu^1} = \begin{cases} \|u\|_{H^1(\mathbb{R}^d)} + \int u^2 d\mu & \text{if } u \in H_\mu^1, \\ \infty & \text{if } u \notin H_\mu^1. \end{cases}$$

- For a given capacity measures  $\mu$  with finite torsion, we call resolvent of  $\mu$  the linear compact and self-adjoint operator

$$R_\mu : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d), \quad R_\mu(f) = w_{\mu,f},$$

where  $w_{\mu,f}$  is the solution of the problem

$$w_{\mu,f} \in H_\mu^1, \quad -\Delta w_{\mu,f} + w_{\mu,f} \mu = f,$$

in the sense that

$$w_{\mu,f} \in H_{\mu}^1, \quad \int \nabla w_{\mu,f} \cdot \nabla \phi \, dx + \int w_{\mu,f} \phi \, d\mu = \int f \phi \, dx \quad \text{for every } \phi \in H_{\mu}^1.$$

The  $\gamma$ -convergence of  $\mu_n$  to  $\mu$  implies the norm convergence of  $R_{\mu_n}$  to  $R_{\mu}$ , i.e.,

$$\lim_{n \rightarrow \infty} \|R_{\mu_n} - R_{\mu}\|_{\mathcal{L}(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d))} = 0.$$

- If  $\mu_n$  is a sequence of capacity measures whose set of finiteness have uniformly bounded measures  $|A_{\mu_n}|$ , then

$$\mu_n \xrightarrow{\gamma} \mu \iff \|R_{\mu_n} - R_{\mu}\|_{\mathcal{L}(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d))} \rightarrow 0 \iff \|u\|_{H_{\mu_n}^1} \xrightarrow{\Gamma} \|u\|_{H_{\mu}^1} \text{ on } L^2(\mathbb{R}^d).$$

The classical concentration-compactness principle of P. L. Lions was extended to sequences of open sets in [7]. Notably, the following result holds.

**Theorem 2.3.** *Let  $\Omega_n$  be a sequence of open sets (more generally, quasi-open sets) with uniformly bounded measures. Then there exists a subsequence (still denoted with the same indices  $n$ ) such that one of the following situations occurs.*

- (Compactness). *There exists a sequence  $x_n \subset \mathbb{R}^d$  such that the sequence of capacity measures  $x_n + \Omega_n$   $\gamma$ -converges.*
- (Vanishing). *The sequence  $R_{I_{\Omega_n}}$  converges in norm to 0. Moreover, we have that  $\|w_{\Omega_n}\|_{L^{\infty}} \rightarrow 0$  and  $\lambda(\Omega_n) \rightarrow \infty$ , as  $n \rightarrow \infty$ .*
- (Dichotomy). *There exist two sequences of quasi-open sets  $\Omega_n^1, \Omega_n^2 \subset \Omega_n$  such that*
  - $\text{dist}(\Omega_n^1, \Omega_n^2) \rightarrow \infty$ , as  $n \rightarrow \infty$ ;
  - $d_{\gamma}(I_{\Omega_n}, I_{\Omega_n^1 \cup \Omega_n^2}) \rightarrow 0$ , as  $n \rightarrow \infty$ ;
  - $\liminf_{n \rightarrow \infty} T(\Omega_n^1) > 0$  and  $\liminf_{n \rightarrow \infty} T(\Omega_n^2) > 0$ .

The proof of the theorem above can be deduced by combining Theorem 2.2 of [7] and Theorem 3.5 of [10].

### 3. Relaxation of $F_q$

In this section, we characterize the relaxation of the functional  $F_q$  to the set of capacity measures. We define the set  $\mathcal{M}_{\text{ad}}$  of admissible capacity measures as

$$\mathcal{M}_{\text{ad}} = \{\mu : \mu \text{ capacity measure with } 0 < |A_{\mu}| < \infty\}.$$

For  $\mu \in \mathcal{M}_{\text{ad}}$ , we define the relaxed form of our functional  $F_q$  as

$$F_q(\mu) = \sup \left\{ \limsup_n F_q(\Omega_n) : \Omega_n \subset \mathbb{R}^d \text{ domain such that } \Omega_n \xrightarrow{\gamma} \mu \right\},$$

so that

$$M_q = \sup\{F_q(\mu) : \mu \in \mathcal{M}_{\text{ad}}\}.$$

**Lemma 3.1.** *Let  $\mu \in \mathcal{M}_{\text{ad}}$  and let  $\Omega_n$  be a sequence of domains such that  $\Omega_n \xrightarrow{\gamma} \mu$ . If  $|A_{\mu}| < \infty$ , then  $\Omega_n \cap A_{\mu} \xrightarrow{\gamma} \mu$ .*

*Proof.* Being the sequence  $\Omega_n \cap A_\mu$  of uniformly bounded measure, by the properties of  $\gamma$ -convergence seen above, we have to show that

$$\|u\|_{H_{\mu_n}^1} \xrightarrow{\Gamma} \|u\|_{H_\mu^1} \quad \text{on } L^2(\mathbb{R}^d),$$

where we set  $\mu_n = I_{\Omega_n \cap A_\mu}$ .

The “ $\Gamma$ -liminf” inequality readily follows from the fact that  $H_{\mu_n}^1 = H_0^1(\Omega_n \cap A_\mu) \subseteq H_0^1(\Omega_n)$ , and from the  $\Gamma$ -convergence of  $\|\cdot\|_{H_0^1(\Omega_n)}$  to  $\|\cdot\|_{H_\mu^1}$  in  $L^2(\mathbb{R}^d)$ .

To prove the “ $\Gamma$ -limsup” inequality, we can suppose without loss of generality that  $u \in H_\mu^1$ . Since  $\Omega_n \xrightarrow{\gamma} \mu$ , there exists a sequence  $u_n \in H_0^1(\Omega_n)$  such that

$$u_n \rightarrow u \text{ strongly } L^2(\mathbb{R}^d) \quad \text{and} \quad \lim_{n \rightarrow \infty} \left( \int |\nabla u_n|^2 dx \right) = \int |\nabla u|^2 dx + \int |u|^2 d\mu.$$

We denote respectively by  $u_n^+$  and  $u_n^-$  the positive and negative part of  $u_n$ . Since we have

$$\int |\nabla(u_n^+ - u_n^-)|^2 dx = \int |\nabla u_n^+|^2 dx + \int |\nabla u_n^-|^2 dx,$$

and  $u_n = u_n^+ - u_n^-$ , by possibly passing to a subsequence (still indexed by  $n$ ), we can suppose that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \left( \int |\nabla u_n^+|^2 dx \right) + \limsup_{n \rightarrow \infty} \left( \int |\nabla u_n^-|^2 dx \right) \\ (3.1) \quad & = \lim_{n \rightarrow \infty} \left( \int |\nabla(u_n^+ - u_n^-)|^2 dx \right) = \int |\nabla u|^2 dx + \int u^2 d\mu. \end{aligned}$$

We define

$$v_n^+ = u_n^+ \wedge u^+ \in H^1(\mathbb{R}^d) \quad \text{and} \quad v_n^- = u_n^- \wedge u^- \in H^1(\mathbb{R}^d).$$

Since  $u \in H_\mu^1$  and  $u_n \in H_0^1(\Omega_n)$ , we have  $u = 0$  q.e. on  $A_\mu^c$  and  $u_n = 0$  q.e. on  $\Omega_n^c$ . This implies that both  $v_n^+$  and  $v_n^-$  vanish q.e. on  $(\Omega_n \cap A_\mu)^c$ , and consequently that  $v_n^+, v_n^- \in H_0^1(\Omega_n \cap A_\mu)$ . Moreover, it is easy to show that

$$v_n^+ - v_n^- \longrightarrow u \quad \text{strongly } L^2(\mathbb{R}^d).$$

Therefore the thesis is achieved if we show that

$$(3.2) \quad \limsup_{n \rightarrow \infty} \left( \int |\nabla(v_n^+ - v_n^-)|^2 dx \right) \leq \lim_{n \rightarrow \infty} \left( \int |\nabla(u_n^+ - u_n^-)|^2 dx \right).$$

We have

$$\begin{aligned} (3.3) \quad & \int |\nabla v_n^+|^2 dx = \int_{\{u_n^+ \leq u^+\}} |\nabla u_n^+|^2 dx + \int_{\{u_n^+ > u^+\}} |\nabla u^+|^2 dx \\ & = \int |\nabla u_n^+|^2 dx - \int (|\nabla u_n^+|^2 - |\nabla u^+|^2) 1_{\{u_n^+ > u^+\}} dx. \end{aligned}$$

By lower semicontinuity, we have

$$(3.4) \quad \liminf_n \int (|\nabla u_n^+|^2 - |\nabla u^+|^2) 1_{\{u_n^+ > u^+\}} dx \geq 0.$$

Indeed, to show the inequality above, it is enough to write

$$\begin{aligned} \int (|\nabla u_n^+|^2 - |\nabla u^+|^2) 1_{\{u_n^+ > u^+\}} dx &= \int (|\nabla u_n^+|^2 - |\nabla u^+|^2) 1_{\{u_n^+ \geq u^+\}} dx \\ &= \int |\nabla(u_n^+ \vee u^+)|^2 - |\nabla u^+|^2 dx, \end{aligned}$$

and to notice that  $u_n^+ \rightharpoonup u^+$  weakly in  $H^1(\mathbb{R}^d)$  implies  $u_n^+ \vee u^+ \rightharpoonup u^+$  weakly in  $H^1(\mathbb{R}^d)$  and so, by lower semicontinuity,

$$\liminf_n \int |\nabla(u_n^+ \vee u^+)|^2 - |\nabla u^+|^2 dx \geq 0.$$

Combining (3.3) and (3.4), we deduce that

$$(3.5) \quad \limsup_{n \rightarrow \infty} \left( \int |\nabla v_n^+|^2 dx \right) \leq \limsup_{n \rightarrow \infty} \left( \int |\nabla u_n^+|^2 dx \right).$$

Similarly we have

$$(3.6) \quad \limsup_{n \rightarrow \infty} \left( \int |\nabla v_n^-|^2 dx \right) \leq \limsup_{n \rightarrow \infty} \left( \int |\nabla u_n^-|^2 dx \right).$$

Combining (3.1), (3.5) and (3.6), we finally deduce (3.2), and this concludes the proof of the lemma.  $\blacksquare$

**Remark 3.2.** By Lemma 3.1, for every measure  $\mu \in \mathcal{M}_{\text{ad}}$  there exists a sequence of quasi-open sets  $\Omega_n$  (that can be taken open by a standard approximation procedure) such that  $\Omega_n \xrightarrow{\gamma} \mu$  and for which

$$|\Omega_n| \rightarrow |A_\mu| \quad \text{as } n \rightarrow \infty.$$

To show this fact, we take  $\Omega_n = A_\mu \cap O_n$ , where  $O_n$  is any sequence of domains  $\gamma$ -converging to  $\mu$ . We then have  $|\Omega_n| \leq |A_\mu|$  and, by the lower semicontinuity of the Lebesgue measure with respect to  $\gamma$ -convergence (see the properties recalled in Section 2), we have  $|A_\mu| \leq \liminf_n |\Omega_n|$ . This in turns implies that the set

$$\{I_\Omega : \Omega \subset \mathbb{R}^d \text{ domain}\}$$

is  $\gamma$ -dense in  $\mathcal{M}_{\text{ad}}$ . Furthermore, we can extend the Saint-Venant, Faber–Krahn and Pólya inequalities to any capacity measure. That is,

$$(3.7) \quad |A_\mu|^{-(d+2)/d} T(\mu) \leq |B|^{-(d+2)/d} T(B), \quad |A_\mu|^{2/d} \lambda(\mu) \geq |B|^{2/d} \lambda(B),$$

and

$$(3.8) \quad 0 < |A_\mu|^{-1} \lambda(\mu) T(\mu) < 1$$

for every measure  $\mu \in \mathcal{M}_{\text{ad}}$  and every ball  $B \subset \mathbb{R}^d$ .



**Proposition 3.3.** *Let  $\mu \in \mathcal{M}_{\text{ad}}$ . Then we have*

$$(3.9) \quad |A_\mu| = \inf \left\{ \liminf_n |\Omega_n| : \Omega_n \text{ domain, } \Omega_n \xrightarrow{\gamma} \mu \right\}.$$

The quantity  $|A_\mu|$  is then the relaxation, in the  $\gamma$ -convergence, of the Lebesgue measure  $|\Omega|$ . As a consequence, if  $\alpha_q > 0$ , we have

$$(3.10) \quad F_q(\mu) = \frac{\lambda(\mu) T^q(\mu)}{|A_\mu|^{\alpha_q}}.$$

*Proof.* The inequality  $\leq$  in (3.9) follows from the  $\gamma$ -lower semicontinuity of the map  $\mu \mapsto |A_\mu|$  seen above. The opposite inequality follows at once by Remark 3.2. Since  $T(\mu)$  and  $\lambda(\mu)$  are  $\gamma$ -continuous, the proof of (3.10) is achieved by a similar argument. ■

The scaling properties of the shape functionals  $|\Omega|$ ,  $\lambda(\Omega)$ ,  $T(\Omega)$  and  $F_q(\Omega)$  extend to their relaxations  $|A_\mu|$ ,  $\lambda(\mu)$ ,  $T(\mu)$  and  $F_q(\mu)$  in  $\mathcal{M}_{\text{ad}}$ . More precisely, setting for  $t > 0$ ,

$$\mu_t(E) = t^{d-2} \mu(E/t),$$

we have

$$|A_{\mu_t}| = t^d |A_\mu|, \quad \lambda(\mu_t) = t^{-2} \lambda(\mu), \quad T(\mu_t) = t^{d+2} T(\mu) \quad \text{and} \quad F_q(\mu_t) = F_q(\mu).$$

#### 4. Existence of an optimal measure for $q > 1$

In [3], it is proved that the supremum  $M_1 = 1$  is not attained in the class of domains. In the next proposition, we point out that the same occurs even in the class  $\mathcal{M}_{\text{ad}}$ .

**Proposition 4.1** (Nonexistence for  $q = 1$  of an optimal measure). *The problem*

$$\sup\{F_1(\mu) : \mu \in \mathcal{M}_{\text{ad}}\}$$

*does not have a maximizer.*

*Proof.* The proof follows at once by exploiting Theorem 1.1 in [3], which asserts that there exists a dimensional constant  $c_d > 0$  for which

$$(4.1) \quad F_1(\Omega) \leq 1 - \frac{c_d T(\Omega)}{|\Omega|^{1+2/d}},$$

for every domain  $\Omega$ . Then, for every  $\mu \in \mathcal{M}_{\text{ad}}$ , by Remark 3.2 we can select a sequence  $\Omega_n \xrightarrow{\gamma} A_\mu$  for which

$$F_1(\Omega_n) \rightarrow F_1(\mu), \quad T(\Omega_n) \rightarrow T(\mu), \quad |\Omega_n| \leq |A_\mu| \quad \text{as } n \rightarrow \infty.$$

Thus, using (4.1) with  $\Omega = \Omega_n$  and then passing to the limit as  $n \rightarrow \infty$ , we get that  $F_1(\mu) < 1 = M_1$ . ■

To prove the main result of this section, we need the following elementary lemma.

**Lemma 4.2.** *Let  $0 < c_1 < c_2 < \infty$  and  $1 < \alpha_1 < \alpha_2 < \infty$ . Then there exists  $\beta < 1$  such that, for every  $a, b, c, d \in (c_1, c_2)$ , it holds*

$$\frac{(a+b)^{\alpha_1}}{(c+d)^{\alpha_2}} \leq \beta \max \left\{ \frac{a^{\alpha_1}}{c^{\alpha_2}}, \frac{b^{\alpha_1}}{d^{\alpha_2}} \right\}.$$

*Proof.* Letting  $x = b/a$  and  $y = d/c$ , is enough to prove that

$$\frac{(1+x)^{\alpha_1}}{(1+y)^{\alpha_2}} \leq \beta \max \left\{ 1, \frac{x^{\alpha_1}}{y^{\alpha_2}} \right\}.$$

Suppose that  $x \leq y$ . Since  $x \geq c_1/c_2$ , it holds

$$(4.2) \quad (1+x)^{\alpha_1} = (1+x)^{\alpha_2} (1+x)^{\alpha_1-\alpha_2} \leq (1+y)^{\alpha_2} \left(1 + \frac{c_1}{c_2}\right)^{\alpha_1-\alpha_2}.$$

Similarly, if  $x > y$ , since  $x \leq c_2/c_1$ , it holds

$$(4.3) \quad \left(1 + \frac{1}{x}\right)^{\alpha_1} \leq \left(1 + \frac{1}{y}\right)^{\alpha_2} \left(1 + \frac{1}{x}\right)^{\alpha_1-\alpha_2} \leq \left(1 + \frac{1}{y}\right)^{\alpha_2} \left(1 + \frac{c_1}{c_2}\right)^{\alpha_1-\alpha_2}.$$

Eventually, we achieve the thesis by letting

$$\beta = \left(1 + \frac{c_1}{c_2}\right)^{\alpha_1-\alpha_2}$$

and combining (4.2) and (4.3). ■

**Theorem 4.3** (Existence for  $q > 1$  of an optimal measure). *For every  $q > 1$ , there exists a measure  $\mu^* \in \mathcal{M}_{\text{ad}}$  such that*

$$F_q(\mu^*) = \sup\{F_q(\mu) : \mu \in \mathcal{M}_{\text{ad}}\}.$$

*Proof.* We select a sequence  $\mu_n \in \mathcal{M}_{\text{ad}}$  such that  $F_q(\mu_n) \rightarrow M_q$ , as  $n \rightarrow \infty$ . By density, we can suppose that  $\mu_n = I_{\Omega_n}$ , for some sequence of open sets  $\Omega_n$ . Further, being  $F_q$  scaling free, we can also assume  $|\Omega_n| = 1$ . Hence, we can apply Theorem 2.3.

If dichotomy occurs, then there exist two sequences of quasi-open sets  $\Omega_n^1, \Omega_n^2 \subset \Omega_n$  such that

$$\Omega_n^1 \cap \Omega_n^2 = \emptyset, \quad d_\gamma(I_{\Omega_n}, I_{\Omega_n^1 \cup \Omega_n^2}) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Taking into account the Saint-Venant inequality and the fact that  $|\Omega_n| = 1$ , there exist constants  $c_1, c_2 > 0$ , which depend only on the dimension, such that

$$c_1 < \inf_n |T(\Omega_n^i)| \leq \sup_n |T(\Omega_n^i)| < c_2, \quad c_1 < \inf_n |\Omega_n^i| \leq \sup_n |\Omega_n^i| < c_2, \quad \text{for } i = 1, 2.$$

Since  $\lambda$  is decreasing with respect to set inclusion, we have

$$(4.4) \quad \lambda(\Omega_n) \leq \min\{\lambda(\Omega_n^1), \lambda(\Omega_n^2)\}.$$

Lemma 4.2 together with (4.4) gives

$$\frac{\lambda(\Omega_n)(T(\Omega_n^1 \cup \Omega_n^2))^q}{|\Omega_n|^{\alpha_q}} \leq \frac{\lambda(\Omega_n)(T(\Omega_n^1) + T(\Omega_n^2))^q}{(|\Omega_n^1| + |\Omega_n^2|)^{\alpha_q}} \leq \beta \max_{i=1,2} \frac{\lambda(\Omega_n^i)T^q(\Omega_n^i)}{|\Omega_n^i|^{\alpha_q}}.$$

By taking the limit as  $n \rightarrow \infty$  in the latter inequality, we obtain the contradiction

$$\sup_{\mu \in \mathcal{M}_{\text{ad}}} F_q(\mu) < \sup_{\mu \in \mathcal{M}_{\text{ad}}} F_q(\mu),$$

and hence dichotomy cannot occur. Now, the maximality condition on the sequence  $\Omega_n$ , together with Pólya's inequality, give that for  $n$  large enough,

$$(4.5) \quad \lambda(B) \frac{T^q(B)}{|B|^{\alpha_q}} \leq \lambda(\Omega_n) T^q(\Omega_n) = \lambda(\Omega_n) T(\Omega_n) \cdot T^{q-1}(\Omega_n) \leq T^{q-1}(\Omega_n),$$

where  $B$  is any ball of  $\mathbb{R}^d$ . In particular, it cannot be  $\lim_{n \rightarrow \infty} T(\Omega_n) = 0$ , and this rules out the vanishing case.

Therefore, compactness holds and there exist a capacity measure  $\mu^*$  and a sequence  $x_n \in \mathbb{R}^d$  such that  $I_{x_n + \Omega_n}$   $\gamma$ -converges to  $\mu^*$ .

By (4.5), we deduce that  $T(\mu^*) > 0$ , which by (3.7) implies  $|A_{\mu^*}| > 0$ , and hence that  $\mu^*$  belongs to  $\mathcal{M}_{\text{ad}}$ . Clearly, the measure  $\mu^*$  maximizes the functional  $F_q$  on  $\mathcal{M}_{\text{ad}}$ , and this concludes the proof.  $\blacksquare$

## 5. Optimal measures are quasi-open sets for large $q$

We are now interested to prove that, when  $q$  is large enough, optimal measures  $\mu$  coming from Theorem 4.3 can be represented as quasi-open sets. We begin by recalling the following result, see [17] and Proposition 3.83 in [24].

**Theorem 5.1.** *Let  $\mu$  be a capacity measure with finite torsion. Then the eigenfunctions  $u \in L^2(\mathbb{R}^d)$  of the operator  $-\Delta + \mu$  with unitary  $L^2$  norm are in  $L^\infty(\mathbb{R}^d)$ , and satisfy*

$$\|u\|_\infty \leq e^{1/(8\pi)} \lambda(\mu)^{d/4}.$$

We also use the following lemma.

**Lemma 5.2.** *For every  $q > 1$ , let  $\mu_q \in \mathcal{M}_{\text{ad}}$  be a maximal measure for the functional  $F_q$ , such that  $|A_{\mu_q}| = 1$ . Then*

$$\liminf_{q \rightarrow \infty} T(\mu_q) > 0.$$

*Proof.* Let  $q_n$  be a diverging sequence and let  $B \subset \mathbb{R}^d$  be a ball of unitary measure. By the density of domains among capacity measures, we can select a sequence  $\Omega_n \subset \mathbb{R}^d$  of open sets such that  $|\Omega_n| = 1$  for every  $n$ , and

$$(5.1) \quad |F_{q_n}(\Omega_n) - F_{q_n}(\mu_{q_n})| = o(T^{q_n}(B)) \quad \text{as } n \rightarrow \infty.$$

Notice that  $T^{q_n}(B) \rightarrow 0$  as  $n \rightarrow \infty$ . Then we can apply Theorem 2.3 to the sequence  $\Omega_n$ . Dichotomy can be ruled out by the same argument as that of the proof of Theorem 4.3 once noticed that (3.8) implies

$$F_{q_n}^{1/q_n}(\mu_{q_n}) \leq T^{(q_n-1)/q_n}(B) \rightarrow T(B) \quad \text{as } n \rightarrow \infty.$$

The vanishing case can be excluded too by following again the proof of Theorem 4.3. Indeed, for  $n$  large enough, Pólya's inequality and (5.1) imply

$$T^{q_n-1}(\Omega_n) \geq F_{q_n}(\Omega_n) \geq F_{q_n}(\mu_{q_n}) - |F_{q_n}(\Omega_n) - F_{q_n}(\mu_{q_n})| \geq F_{q_n}(B) + o(T^{q_n}(B)).$$

Hence we deduce

$$\liminf_{n \rightarrow \infty} T^{(1-1/q_n)}(\Omega_n) > 0,$$

which implies that it cannot be  $T(\Omega_n) \rightarrow 0$ , as  $n \rightarrow \infty$ . Therefore, compactness holds true and the sequence  $\Omega_n$  has a subsequence (still denoted by the same indices) that  $\gamma$ -converges to some  $\mu \in \mathcal{M}_{\text{ad}}$  up to translations.

By the maximality of  $\mu_{q_n}$ , it holds

$$F_{q_n}^{1/q_n}(B) \leq F_{q_n}^{1/q_n}(\mu_{q_n}) = T(\Omega_n)(\lambda(\Omega_n) + o(1))^{1/q_n},$$

and we deduce, passing to the limit as  $n \rightarrow \infty$ ,

$$T(B) \leq T(\mu) = \lim_{n \rightarrow \infty} T(\Omega_n).$$

Since the sequence  $q_n$  was arbitrary, we obtain the conclusion. ■

**Theorem 5.3.** *Let  $\mu \in \mathcal{M}_{\text{ad}}$  be an optimal measure for  $F_q$  with  $q > 1$ . There exists  $q_0 > 1$  such that for  $q > q_0$  we have  $\mu = I_{A_\mu}$ . In particular, the optimal measure can be represented by a quasi-open set.*

*Proof.* Since  $F_q$  is scaling free, we can suppose that  $|A_\mu| = 1$ . Let  $\varepsilon > 0$  be a small parameter, and let  $\mu_\varepsilon$  be the capacitary measure defined by

$$\mu_\varepsilon(E) = (1 - \varepsilon)\mu(E).$$

Being  $A_\mu = A_{\mu_\varepsilon}$ , we have  $\mu_\varepsilon \in \mathcal{M}_{\text{ad}}$ . We assume by contradiction that  $\mu \neq I_{A_\mu}$  (notice that this implies  $\mu_\varepsilon \neq \mu$ ). For the sake of brevity, we denote respectively by  $w$  and  $w_\varepsilon$  the torsion functions of  $\mu$  and  $\mu_\varepsilon$ . It is easy to verify that, as  $\varepsilon \rightarrow 0$ ,

$$\|\cdot\|_{H_{\mu_\varepsilon}^1} \xrightarrow{\Gamma} \|\cdot\|_{H_\mu^1}, \quad \text{on } L^2(\mathbb{R}^d),$$

and therefore we have  $\mu_\varepsilon \xrightarrow{\gamma} \mu$  and  $w_\varepsilon \rightarrow w$  in  $L^1(\mathbb{R}^d)$ , as  $\varepsilon \rightarrow 0$ . Let us denote by  $t(\varepsilon)$ ,  $l(\varepsilon)$  and  $f_q(\varepsilon)$  the real functions

$$\varepsilon \mapsto t(\varepsilon) = T(\mu_\varepsilon), \quad \varepsilon \mapsto l(\varepsilon) = \lambda(\mu_\varepsilon) \quad \text{and} \quad \varepsilon \mapsto f_q(\varepsilon) = F_q(\mu_\varepsilon),$$

and by  $t'_+(0)$ ,  $l'_+(0)$  and  $(f_q)'_+(0)$  the limits for  $\varepsilon \rightarrow 0$  of the respective different quotients.

By writing  $w_\varepsilon = w + \varepsilon \xi_\varepsilon$  for some  $\xi_\varepsilon \in L^1(\mathbb{R}^d)$  and using the fact that  $w$  and  $w_\varepsilon$ , respectively, weakly solve the PDEs

$$-\Delta w + w\mu = 1$$

and

$$(5.2) \quad -\Delta w_\varepsilon + w_\varepsilon \mu_\varepsilon = 1,$$

we deduce that  $\xi_\varepsilon$  weakly solves the PDE

$$(5.3) \quad -\Delta \xi_\varepsilon + \xi_\varepsilon \mu_\varepsilon = w\mu.$$

This allows us to compute the derivative

$$t'_+(0) = \lim_{\varepsilon \rightarrow 0} \left( \int \xi_\varepsilon dx \right) = \lim_{\varepsilon \rightarrow 0} \left( \int \nabla w_\varepsilon \nabla \xi_\varepsilon dx + \int w_\varepsilon \xi_\varepsilon d\mu_\varepsilon \right) = \lim_{\varepsilon \rightarrow 0} \left( \int w w_\varepsilon d\mu \right),$$

where we test (5.2) with  $\xi_\varepsilon$  and we use (5.3) tested with  $w_\varepsilon$ . Since as  $\varepsilon \rightarrow 0$ ,  $w_\varepsilon \rightarrow w$  in  $L^1(\mathbb{R}^d)$ , we obtain

$$(5.4) \quad t'_+(0) = \int w^2 d\mu.$$

We can treat with a similar argument the eigenvalue. Let  $u$  and  $u_\varepsilon$  be the first eigenfunctions (with unitary  $L^2$  norm), respectively, of the operators  $-\Delta + \mu_\varepsilon$  and  $-\Delta + \mu$ , and let  $v_\varepsilon \in L^2(\mathbb{R}^d)$  be such that  $u_\varepsilon = u + \varepsilon v_\varepsilon$ . Since

$$-\Delta u + u\mu = \lambda(\mu)u \quad \text{and} \quad -\Delta u_\varepsilon + u_\varepsilon \mu_\varepsilon = \lambda(\mu_\varepsilon)u_\varepsilon,$$

we have

$$-\Delta v_\varepsilon + v_\varepsilon \mu - u\mu - \varepsilon v_\varepsilon \mu = \left( \frac{\lambda(\mu_\varepsilon) - \lambda(\mu)}{\varepsilon} \right) u + \lambda(\mu_\varepsilon) v_\varepsilon.$$

By testing the PDE above with  $u \in H_\mu^1$ , and since  $\int u^2 dx = 1$ , we obtain

$$\begin{aligned} \left( \frac{\lambda(\mu_\varepsilon) - \lambda(\mu)}{\varepsilon} \right) &= \int \nabla v_\varepsilon \nabla u dx + \int v_\varepsilon u d\mu - \int u^2 d\mu \\ &\quad - \varepsilon \int v_\varepsilon u d\mu - \lambda(\mu_\varepsilon) \int v_\varepsilon u dx. \end{aligned}$$

By taking the limit as  $\varepsilon \rightarrow 0$  and exploiting the fact that  $u_\varepsilon \rightarrow u$  weakly in  $H_\mu^1$  and  $\lambda(\mu_\varepsilon) \rightarrow \lambda(\mu)$ , we get

$$(5.5) \quad l'_+(0) = - \int u^2 d\mu.$$

By combining (5.4) and (5.5), we get

$$(f_q)'_+(0) = l'_+(0) T^q(\mu) + q\lambda(\mu) T^{q-1}(\mu) t'_+(0) = F_q(\mu) \int \left( -\frac{u^2}{\lambda(\mu)} + q \frac{w^2}{T(\mu)} \right) d\mu.$$

Now, the optimality condition on  $\mu$  implies  $(f_q)'_+(0) \leq 0$ , and hence that

$$(5.6) \quad \int \left( \frac{u^2}{\lambda(\mu)} - q \frac{w^2}{T(\mu)} \right) d\mu \geq 0.$$

We claim that

$$(5.7) \quad \frac{u^2}{\lambda(\mu)} - q \frac{w^2}{T(\mu)} < 0 \quad \text{q.e. on } \mathbb{R}^d$$

for  $q$  large enough. Indeed, by an application of Theorem 5.1 together with a comparison principle, we have

$$u \leq e^{1/(8\pi)} \lambda^{d/4+1}(\mu) w \quad \text{q.e. on } \mathbb{R}^d,$$

and so by the Pólya inequality,

$$u^2 \leq e^{1/(4\pi)} \lambda^{d/2}(\mu) \frac{\lambda(\mu)}{T(\mu)} w^2 \quad \text{q.e. on } \mathbb{R}^d.$$

This implies that

$$\frac{u^2}{\lambda(\mu)} - q \frac{w^2}{T(\mu)} \leq \frac{w^2}{T(\mu)} (e^{1/(4\pi)} \lambda^{d/2}(\mu) - q) \quad \text{q.e. on } \mathbb{R}^d.$$

Therefore, for every  $q$  such that

$$\sup_{v \text{ optimal}} e^{1/(4\pi)} \lambda^{d/2}(v) < q,$$

the inequality (5.7) is satisfied. Notice that the supremum in the inequality above is finite as a consequence of Lemma 5.2 combined again with Pólya's inequality.

To conclude, it is now enough to notice that (5.7) contradicts (5.6). ■

## 6. Optimality for nearly spherical domains

In the following, we consider the classes  $\mathcal{S}_{\delta, \gamma}$  of *nearly spherical domains*. Let  $B_1$  be the unitary ball of  $\mathbb{R}^d$ . A domain  $\Omega$  such that

$$|\Omega| = |B_1| \quad \text{and} \quad \int_{\Omega} x \, dx = 0$$

belongs to the class  $\mathcal{S}_{\delta, \gamma}$  if there exists  $\phi \in C^{2, \gamma}(\partial B_1)$  with  $\|\phi\|_{L^\infty(\partial B_1)} \leq 1/2$  and such that

$$\partial\Omega = \{x \in \mathbb{R}^d : x = (1 + \phi(y))y, \, y \in \partial B_1\} \quad \text{and} \quad \|\phi\|_{C^{2, \gamma}(\partial B_1)} \leq \delta.$$

We recall the following result.

**Theorem 6.1.** *Let  $\gamma \in (0, 1)$ . There exists  $\delta = \delta(d, \gamma) > 0$  such that, if  $\Omega \in \mathcal{S}_{\delta, \gamma}$ , then*

$$T(B_1) - T(\Omega) \geq C_1 \|\phi\|_{H^{1/2}(\partial B_1)}^2 \quad \text{and} \quad \lambda(\Omega) - \lambda(B_1) \leq C_2 \|\phi\|_{H^{1/2}(\partial B_1)}^2$$

for suitable constants  $C_1$  and  $C_2$  depending only on the dimension  $d$ .

*Proof.* The inequality for the torsional rigidity follows from Theorem 3.3 in [5]. The inequality for the eigenvalue follows by combining Theorem 1.4 and Lemma 2.8 of [16]. ■

**Theorem 6.2.** *Let  $\gamma \in (0, 1)$ . There exist  $\delta > 0$  and  $q_1 > 1$  such that, for every  $q \geq q_1$  and every  $\Omega \in \mathcal{S}_{\gamma, \delta}$ , it holds*

$$\lambda(B_1) T^q(B_1) \geq \lambda(\Omega) T^q(\Omega).$$

*Proof.* For every domain  $\Omega$  we have

$$\lambda(B_1) T^q(B_1) - \lambda(\Omega) T^q(\Omega) = \lambda(B_1) (T^q(B_1) - T^q(\Omega)) + T^q(\Omega) (\lambda(B_1) - \lambda(\Omega)),$$

which, by the elementary inequality

$$x^q - y^q \geq q y^{q-1} (x - y), \quad \text{for every } x, y \geq 0, q > 1,$$

implies

$$(6.1) \quad \begin{aligned} \lambda(B_1) T^q(B_1) - \lambda(\Omega) T^q(\Omega) \\ \geq T^{q-1}(\Omega) [q \lambda(B_1) (T(B_1) - T(\Omega)) - T(\Omega) (\lambda(\Omega) - \lambda(B_1))]. \end{aligned}$$

Let  $\delta$  be the constant determined by Theorem 6.1 and assume  $\Omega \in \mathcal{S}_{\gamma, \delta}$ . Since  $2^{-1} B_1 \subset \Omega \subset 2 B_1$ , we get

$$2^{-(2+d)} T(B_1) \leq T(\Omega) \leq 2^{2+d} T(B_1).$$

Combining Theorem 6.1 and inequality (6.1) we get

$$\begin{aligned} \lambda(B_1) T^q(B_1) - \lambda(\Omega) T^q(\Omega) \\ \geq (2^{-(2+d)} T(B_1))^{q-1} (q C_1 - 2^{2+d} C_2 T(B_1)) \|\phi\|_{H^{1/2}(\partial B_1)}^2. \end{aligned}$$

Hence, if  $q$  is such that

$$q \geq 2^{d+2} \frac{C_2}{C_1} T(B_1),$$

we obtain

$$\lambda(B_1) T^q(B_1) \geq \lambda(\Omega) T^q(\Omega),$$

and this concludes the proof. ■

**Remark 6.3.** Although for large  $q$  we expect the ball to be optimal for the functional  $F_q$ , it is easy to see that this does not occur when  $q$  approaches 1. Indeed, if the ball maximizes  $F_q$  for every  $q > 1$ , passing to the limit as  $q \rightarrow 1$ , this would happen also for  $q = 1$ , which is not true, even in the class of convex domains. To see this, it is enough to notice that

$$F_1(B_1) = \frac{\lambda(B_1)}{d(d+2)} \leq \frac{d+4}{2(d+2)},$$

where the last inequality follows simply by taking  $u(x) = 1 - |x|^2$  as a test function for  $\lambda(B_1)$ . On the other hand, taking as  $\Omega_\varepsilon$  the thin slab  $(0, 1)^{d-1} \times (0, \varepsilon)$  gives

$$\lim_{\varepsilon \rightarrow 0} F_1(\Omega_\varepsilon) = \frac{\pi^2}{12},$$

and

$$\frac{\pi^2}{12} > \frac{d+4}{2(d+2)} \quad \text{for every } d \geq 2.$$

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