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Improved Beckner's inequality for axially symmetric functions on $\mathbb{S}^4$

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Abstract. We show that axially symmetric solutions on $\mathbb{S}^4$ to a constant Q-curvature type equation (it may also be called fourth order mean field equation) must be constant, provided that the parameter α in front of the Paneitz operator belongs to the interval $[\frac{473+\sqrt{209329}}{1800} \approx 0.517, 1)$. This is in contrast to the case $\alpha = 1$, where there exists a family of solutions, known as standard bubbles. The phenomenon resembles the Gaussian curvature equation on $\mathbb{S}^2$. As a consequence, we prove an improved Beckner's inequality on $\mathbb{S}^4$ for axially symmetric functions with their centers of mass at the origin. Furthermore, we show uniqueness of axially symmetric solutions when $\alpha = 1/5$ by exploiting Pohozaev-type identities, and prove the existence of a non-constant axially symmetric solution for $\alpha \in (1/5, 1/2)$ via a bifurcation method.

1. Introduction

We shall consider the constant Q-curvature type equation

$$\alpha P_4 u + 6\Big(1 - \frac{e^{4u}}{\int_{\mathbb{S}^4} e^{4u}\, dw}\Big) = 0, \quad \xi \in \mathbb{S}^4. \tag{1.1}$$

Here, $\mathbb{S}^4$ is the 4-dimensional sphere,

$$P_4 = \Delta^2 - 2\Delta$$

is the Paneitz operator on $\mathbb{S}^4$, and α is a positive constant. The volume form dw is normalized so that $\int_{\mathbb{S}^4} dw = 1$.

The corresponding energy functional is defined in $H^2(\mathbb{S}^4)$ as

$$J_\alpha(u) = \frac{\alpha}{2}\Big(\int_{\mathbb{S}^4} |\Delta u|^2\, dw + 2\int_{\mathbb{S}^4} |\nabla u|^2\, dw\Big) + 6\int_{\mathbb{S}^4} u\, dw - \frac{3}{2}\ln \int_{\mathbb{S}^4} e^{4u}\, dw.$$

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When $\alpha = 1$, (1.1) corresponds to the constant Q-curvature equation on $\mathbb{S}^4$, and it is well known that there holds the following Beckner's inequality, a higher order Moser–Trudinger type inequality:

$$J_1(u) \geq 0, \quad u \in H^2(\mathbb{S}^4). \tag{1.2}$$

Furthermore, J_1 is invariant under the conformal transformation

$$u(\xi) \ \to \ v(\tau\xi) + \frac{1}{4}\ln(|\det(d\tau)(\xi)|),$$

where τ is an element of the conformal group of $\mathbb{S}^4$ and $|\det(\cdot)|$ is the modulus of the corresponding Jacobian determinant. Equality in (1.2) is only attained at functions of the form

$$u(\xi) = -\ln(1 - \zeta\cdot\xi) + C, \quad C \in \mathbb{R},$$

where $\zeta \in B^5 := \{\varsigma \in \mathbb{R}^5, |\varsigma| < 1\}$ (see [1,5]). This in particular implies that (1.1) has a family of axially symmetric solutions $u(\xi) = -\ln(1 - a\xi_1)$, $\quad \xi \in \mathbb{S}^4$ for $a \in (-1, 1)$.

On the other hand, an improved Aubin-type inequality holds, as shown in Lemma 4.6 of [5] or Lemma 4.3 of [2]: for any $\alpha > 1/2$, there exists a constant $C(\alpha) \geq 0$ such that $J_\alpha(u) \geq -C(\alpha)$ if u belongs to the set of functions with center of mass at the origin:

$$\mathfrak{L} = \Big\{ v \in H^2(\mathbb{S}^4) : \int_{\mathbb{S}^4} e^{4v}\xi_j\, dw = 0,\ j = 1, 2, \ldots, 5 \Big\}.$$

This leads to the existence of a minimizer u_0 of J_α in $\mathfrak{L}$, and u_0 satisfies the corresponding Euler–Lagrange equation

$$\alpha P_4 u + 6\Big(1 - \frac{e^{4u}}{\int_{\mathbb{S}^4} e^{4u}\, dw}\Big) = \sum_{i=1}^{5} a_i\,\xi_i\, e^{4u}, \quad \xi \in \mathbb{S}^4, \tag{1.3}$$

for some constants $a_1, \ldots, a_5$.

Equation (1.1) can be regarded as the following 4-dimensional counterpart of the constant Gaussian curvature type equation, or the mean field equation on $\mathbb{S}^2$:

$$-\alpha\,\Delta u + \Big(1 - \frac{e^{2u}}{\int_{\mathbb{S}^2} e^{2u}\, dw}\Big) = 0, \quad \xi \in \mathbb{S}^2. \tag{1.4}$$

For (1.4), there is a vast literature. See, e.g., [4], [14] and references therein.

Similar to the prescribing Gaussian curvature equation on $\mathbb{S}^2$, the Kazdan–Warner obstruction also works for the prescribing Q-curvature equation

$$P_4 u + 6 - Qe^{4u} = 0, \quad \xi \in \mathbb{S}^4.$$

Indeed, it is shown in Remarks (3) (ii) for Corollary 5.4 of [5] and Corollary 2.1 in [4], by using the invariance of J_1 under the conformal transformation, that the following Kazdan–Warner condition holds:

$$\int_{\mathbb{S}^4} \langle \nabla Q, \nabla \xi_i \rangle e^{4u}\, dw = 0, \quad i = 1, 2, \ldots, 5. \tag{1.5}$$

It is an immediate consequence that $a_i = 0$ for $i = 1, 2, \ldots, 5$ in (1.3), just as in the $\mathbb{S}^2$ case in [4]. See also [29], where (1.5) is proved for all dimensions greater than or equal to three. The interested reader is referred to [6, 8, 9, 13, 17, 18, 22–24, 30] for literature on equations that have conformal structure.

In what follows, we shall consider axially symmetric functions that are only dependent on ξ_1, and show that (1.1) under axially symmetric setting admits only constant solutions when $\alpha \in [\frac{473+\sqrt{209329}}{1800}, 1)$. As a consequence, we obtain an improved Aubin-type inequality for axially symmetric functions in $\mathfrak{L}$.

Considering solutions axially symmetric about ξ_1-axis, and denoting ξ_1 by x, we can reduce (1.1) to

$$\alpha\,[(1-x^2)^2 u']''' + 6 - \frac{8e^{4u}}{\int_{-1}^{1}(1-x^2)\,e^{4u}} = 0, \quad x \in (-1,1), \tag{1.6}$$

or equivalently,

$$\alpha\,[(1-x^2)^3 u'']'' - 6\alpha\,[(1-x^2)^2 u']' + 6\,(1-x^2) - \frac{8\,(1-x^2)\,e^{4u}}{\int_{-1}^{1}(1-x^2)\,e^{4u}} = 0.$$

One can refer to Section 2 for the detailed derivation of (1.6). By direct computations, we see that the corresponding functional $I_\alpha(u)$ can be expressed as follows:

$$\begin{aligned} I_\alpha(u) = \frac{\alpha}{2}\int_{-1}^{1}\big((1-x^2)\,|(1-x^2)\,u''|^2 + 6\,|(1-x^2)\,u'|^2\big) \\ + 6\int_{-1}^{1}(1-x^2)\,u - 2\ln\Big(\frac{3}{4}\int_{-1}^{1}(1-x^2)\,e^{4u}\Big). \end{aligned}$$

Here the function space is $H^2(-1,1)$, which is the restriction of $H^2(\mathbb{S}^4)$ in the set of functions axially symmetric about ξ_1-axis and $\xi_1 = x$.

The set $\mathfrak{L}$ is replaced by

$$\mathfrak{L}_r = \Big\{u \in H^2(\mathbb{S}^4) : u = u(x) \text{ and } \int_{-1}^{1} x\,(1-x^2)\,e^{4u} = 0\Big\}. \tag{1.7}$$

Now we state the main results.

Theorem 1.1. *If* $\frac{473+\sqrt{209329}}{1800} \le \alpha < 1$, *then* (1.6) *admits only constant solutions. As an immediate consequence, we have*

$$\inf_{u \in \mathfrak{L}_r} I_\alpha(u) = 0.$$

We conjecture that Theorem 1.1 holds, in fact, for $1/2 \le \alpha < 1$. Indeed, the lower bound $\frac{473+\sqrt{209329}}{1800}$ can be improved slightly to 0.5145 (see discussions in Section 6). We believe that $J_{1/2}(u) \ge 0$ for $u \in \mathfrak{L}$, given the similar inequality for $\mathbb{S}^2$ as shown in [14]. It is worth pointing out that Wei and Xu proved in [29] that for $u \in \mathfrak{L}$ and $\varepsilon > 0$ small enough, there holds

$$J_\alpha(u) \ge 0, \quad \alpha \in (1-\varepsilon, 1)$$

in all dimensions $n \ge 3$.

Now we define the following first momentum functionals on $H^2(\mathbb{S}^4)$:

$$\mathcal{J}_\alpha(u) = \frac{\alpha}{2}\int_{\mathbb{S}^4} u(P_4 u)\,dw + 6\int_{\mathbb{S}^4} u\,dw - \frac{3}{4}\ln\Big[\Big(\int_{\mathbb{S}^4} e^{4u}\,dw\Big)^2 - \sum_{i=1}^{5}\Big(\int_{\mathbb{S}^4} e^{4u}\xi_i\,dw\Big)^2\Big].$$

Note that $\mathcal{J}_\alpha(u) = J_\alpha(u)$ when $u \in \mathfrak{L}$. As a consequence of Theorem 1.1, for axially symmetric functions on $\mathbb{S}^4$, we find the following sharp inequality arising from the Szegö's limit theorem.

Theorem 1.2. *There holds*

$$\mathcal{J}_{4/5}(u) \geq 0, \quad \forall u \in \{v \in H^2(\mathbb{S}^4) : v(\xi) = v(\xi_1)\}.$$

Concerning the classification of axially symmetric solutions there is another critical parameter $\alpha = 1/5$, which corresponds to the second bifurcation of nontrivial axially symmetric solutions from the constant solutions. We have the following theorem.

Theorem 1.3. *If $\alpha = 1/5$ and u is an axially symmetric solution to* (1.1)*, then u must be constant.*

Using a bifurcation approach and Theorems 1.1–1.3, we can also show the existence of non-constant axially symmetric solution for $\alpha \in (1/5, 1/2)$.

Theorem 1.4. *There exists a non-constant solution u_α to* (1.6) *for $\alpha \in (1/5, 1/2)$. Moreover, there exist a sequence $\alpha_m \in (\frac{1}{5}, \frac{473+\sqrt{209329}}{1800})$ and a sequence of non-constant solutions $(u_{\alpha_m})_{m\geq 1}$ to* (1.6) *such that*

$$\alpha_m \to \frac{1}{2}, \quad \int_{-1}^{1}(1-x^2)\,e^{4u_{\alpha_m}} = \frac{4}{3}, \quad \textit{and} \quad \|u_{\alpha_m}\|_{L^\infty([-1,1])} \to \infty \quad \textit{as } m \to \infty.$$

We also establish the following proposition concerning the centers of mass and first order momentums of solutions to (1.1).

Proposition 1.5. *If u solves* (1.1)*, then*

$$\int_{\mathbb{S}^4} e^{4u}\,\xi_i\,dw = 0 \quad \textit{and} \quad \int_{\mathbb{S}^4} u\,\xi_i\,dw = 0, \quad i = 1, 2, \ldots, 5,$$

whenever $\alpha \neq 1$.

In the course of final revision of this paper, the authors learned that Theorem 1.1 was recently proved under the sharp condition $1/2 \leq \alpha < 1$ in [21].

The paper is organized as follows. First, we list some preliminaries and integral identities in Section 2 which will be substantially used in the later context. Section 3 is devoted to the proof of Theorems 1.1 and 1.2. In Section 4, we derive various Pohozaev-type identities and employ them to validate Theorem 1.3 together with Proposition 1.5. In Section 5, we carry out a bifurcation analysis of (1.6) and its equivalent form, and prove Theorem 1.4 based on Theorems 1.1 and 1.3. The last section is devoted to some discussions of the improvement of the best constant for α.

2. Preliminaries and integral identities

In this section, we state several important preliminaries and integral identities which will be needed in the proof of Theorem 1.1. We begin by introducing some basic facts on spherical geometry of $\mathbb{S}^4$.

Let θ_i, $i = 1, 2, 3, 4$, denote the usual angular coordinates on the sphere, with

$$\theta_i \in [0, \pi] \quad \text{for } i = 1, 2, 3, \quad \text{and} \quad \theta_4 \in [0, 2\pi],$$

and define $x = \xi_1 = \cos(\theta_1)$. Then the metric tensor can be given as follows:

$$g = \begin{pmatrix} (1-x^2)^{-1} & 0 & 0 & 0 \\ 0 & 1-x^2 & 0 & 0 \\ 0 & 0 & (1-x^2)\sin^2\theta_2 & 0 \\ 0 & 0 & 0 & (1-x^2)\sin^2\theta_2 \sin^2\theta_3 \end{pmatrix}.$$

For axially symmetric functions, we have

$$\begin{aligned} \int_{\mathbb{S}^4} e^{4u}\, dw &= \frac{3}{8\pi^2} \int_{-1}^{1}\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi} e^{4u}(1-x^2)\sin^2\theta_2 \sin\theta_3\, d\theta_4\, d\theta_3\, d\theta_2\, dx \\ &= \frac{3}{4}\int_{-1}^{1}(1-x^2)\, e^{4u} \end{aligned}$$

and

$$\begin{aligned} \Delta u &= |g|^{-1/2}\frac{\partial}{\partial x}\Big(|g|^{1/2} g^{11}\frac{\partial u}{\partial x}\Big) = (1-x^2)^{-1}\frac{\partial}{\partial x}\Big[(1-x^2)^2\frac{\partial u}{\partial x}\Big] \\ &= (1-x^2)u'' - 4xu'. \end{aligned}$$

One further has that

$$((1-x^2)\Delta u)'' = -2\Delta u - 4x(\Delta u)' + (1-x^2)(\Delta u)'' = \Delta^2 u - 2\Delta u.$$

Thus, the Paneitz operator on $\mathbb{S}^4$ can be expressed as

$$P_4 u = [(1-x^2)^2 u'' - 4x(1-x^2)u']'' = [(1-x^2)^2 u']''',$$

for $u = u(x)$. Then, we transform the original equation (1.1) on $\mathbb{S}^4$ into the ODE (1.6).

Note that the eigenfunctions associated with the Paneitz operator coincide with those associated with the Laplacian. It is natural to introduce Gegenbauer polynomials (see [25], Chapter 2.4), which can be considered as a family of generalized Legendre polynomials.

Let

$$F_k(x) = \frac{(-1)^k\,\Gamma(2)}{2^k\,\Gamma(k+2)}\,\frac{1}{(1-x^2)}\,\frac{d^k}{dx^k}(1-x^2)^{k+1}$$

be the k-th Gegenbauer polynomials. Then F_k satisfies that

$$(1-x^2)F_k'' - 4xF_k' + \lambda_k F_k = 0, \quad \lambda_k = k(k+3),\ k = 0, 1, \ldots, \tag{2.1}$$

and

$$\int_{-1}^{1}(1-x^2)\, F_k\, F_l = 0 \quad \text{for } k \neq l. \tag{2.2}$$

Here F_k is a sphere harmonic of degree k. Then it is readily checked that for $x \in (-1,1)$,

$$(1-x^2)[(1-x^2)^2 F_k']''' = (\lambda_k^2 + 2\lambda_k)(1-x^2) F_k.$$

Moreover (see [12, 25]),

$$|F_k'(x)| \le \frac{\lambda_k}{4} \quad \text{and} \quad \int_{-1}^{1} (1-x^2) F_k^2 = \frac{8}{(2k+3)(k+1)(k+2)}. \tag{2.3}$$

We will focus on the gradient of u on the sphere throughout the rest of the paper. Define

$$G(x) = (1-x^2) u', \tag{2.4}$$

where $u = u(x)$ is a solution to (1.1). Then we have the following decomposition using the orthogonal polynomials F_k:

$$G = a_0 F_0 + \beta x + a_2 \frac{1}{4}(5x^2 - 1) + \sum_{k=3}^{\infty} a_k F_k(x). \tag{2.5}$$

Define

$$G_2 = \sum_{k=3}^{\infty} a_k F_k(x) \quad \text{and} \quad b_k^2 = a_k^2 \int_{-1}^{1} (1-x^2) F_k^2, \quad k \ge 2.$$

We first derive a lemma concerning the constant term a_0 in (2.5).

Lemma 2.1. *If u is a critical point of I_α whenever $1/2 < \alpha < 1$, then the function $G(x)$ belongs to $H^2(-1,1)$ and satisfies that $\int_{-1}^{1}(1-x^2)G = 0$. In other words, $a_0 = 0$.*

Proof. When $1/2 < \alpha < 1$, if u is a critical point of I_α, then u is a smooth function on $\mathbb{S}^4$ by the estimate Lemma 4.6. of [5]. Then $G(x) = \nabla u \cdot \vec{e}_1$ belongs to $H^2(-1,1)$. Indeed, we can obtain an explicit estimate of $H^2(-1,1)$-norm of $G(x)$ in Remark 2.1 below.

In view of equation (1.6), we have

$$\alpha((1-x^2)G)''' + 6 - \frac{8}{\gamma} e^{4u} = 0, \tag{2.6}$$

where

$$\gamma = \int_{-1}^{1} (1-x^2) e^{4u}. \tag{2.7}$$

By differentiating (2.6), we further have

$$\alpha((1-x^2)G)'''' - \frac{32}{\gamma} e^{4u} u' = 0. \tag{2.8}$$

Multiplying (2.8) by $(1-x^2)^2$ and employing (2.6), we have

$$(1-x^2)^2((1-x^2)G)'''' - \frac{24}{\alpha}(1-x^2)G - 4[(1-x^2)G]'''(1-x^2)G = 0. \tag{2.9}$$

Integrating (2.9) over $[-1, 1]$, we have

$$\int_{-1}^{1} (1-x^2)^2 ((1-x^2)G)'''' - \frac{24}{\alpha}\int_{-1}^{1} (1-x^2)G - 4\int_{-1}^{1} ((1-x^2)G)'''(1-x^2)G = 0. \tag{2.10}$$

We use integration by parts for the first term of (2.10):

$$\begin{aligned}
&\int_{-1}^{1} (1-x^2)^2 ((1-x^2)G)'''' \\
&\quad = [(1-x^2)^2((1-x^2)G)''']\big|_{-1}^{1} - \int_{-1}^{1} ((1-x^2)^2)'((1-x^2)G)''' \\
&\quad = ((1-x^2)^2)''((1-x^2)G)'\big|_{-1}^{1} - ((1-x^2)^2)'''((1-x^2)G)\big|_{-1}^{1} \\
&\qquad + \int_{-1}^{1} ((1-x^2)^2)''''(1-x^2)G \\
&\quad = 24\int_{-1}^{1} (1-x^2)G.
\end{aligned} \tag{2.11}$$

Similarly, for the last term in (2.10), one has

$$\begin{aligned}
\int_{-1}^{1} ((1-x^2)G)'''(1-x^2)G &= ((1-x^2)G)''(1-x^2)G\big|_{-1}^{1} - \int_{-1}^{1} (1-x^2)G)'((1-x^2)G)'' \\
&= -\frac{1}{2}((1-x^2)G' - 2xG)^2\Big|_{-1}^{1} = 0.
\end{aligned} \tag{2.12}$$

We conclude from (2.10)–(2.12) that

$$\left(24 - \frac{24}{\alpha}\right)\int_{-1}^{1} (1-x^2)G = 0,$$

which implies that $\int_{-1}^{1}(1-x^2)G = 0$, since $\alpha \neq 1$. ■

Remark 2.1. To obtain an estimate of the norm of $G(x) \in H^2(-1,1)$ (with the natural weight $(1-x^2)$), we first use the Bochner–Lichnerowicz–Weitzenböck formula to get

$$\begin{aligned}
\int_{\mathbb{S}^4} |\nabla^2 G|^2\, dw &= \int_{\mathbb{S}^4} |\Delta G|^2\, dw - \int_{\mathbb{S}^4} \mathrm{Ric}_g(\nabla G, \nabla G)\, dw \\
&\leq \int_{\mathbb{S}^4} (P_4 G)G\, dw + \int_{\mathbb{S}^4} G^2\, dw,
\end{aligned}$$

which together with (3.9) leads to an estimate of the norm of G in $H^2(-1,1)$ (which also shows directly that $G(x) \in H^2(-1,1)$). We note that in the subsequent context, $H^2(-1,1)$ represents the Sobolev space with the natural weight function $(1-x^2)$ inherited from $H^2(\mathbb{S}^4)$. Please see Lemma 3.2 for a detailed proof of (3.9).

Next, we state some important integral identities which will be used frequently in the proof of Theorem 1.1.

Lemma 2.2. *The following equalities hold for $G(x) = (1-x^2)u'$, where u is a solution of* (1.6) *and $\alpha > 0$:*

$$\int_{-1}^{1} (1-x^2)\, F_1\, G = \frac{4}{15}\,\beta, \tag{2.13}$$

$$\int_{-1}^{1} (1-x^2)^2\, \frac{e^{4u}}{\gamma} = \frac{4}{5}\,(1-\alpha\beta), \tag{2.14}$$

$$\int_{-1}^{1} (1-x^2)\, F_k\, G = -\frac{8}{\alpha\,(\lambda_k^2 + 2\lambda_k)} \int_{-1}^{1} \frac{e^{4u}}{\gamma}\,(1-x^2)^2\, F_k', \quad k \geq 2, \tag{2.15}$$

$$\int_{-1}^{1} |[(1-x^2)\,G]'|^2 = \frac{16}{15}\left(5 - \frac{1}{\alpha}\right)\beta, \tag{2.16}$$

recalling that β is defined in (2.5) *and γ is given in* (2.7).

Proof. Indeed, we have

$$F_0(x) = 1, \quad F_1(x) = x \quad \text{and} \quad F_2(x) = \frac{1}{4}\,(5x^2 - 1).$$

Then it follows from (2.9) that

$$\int_{-1}^{1} (1-x^2)\, F_1\, G = \beta \int_{-1}^{1} (1-x^2)\, x^2 = \frac{4}{15}\,\beta.$$

This finishes the proof of (2.13).

For (2.14), multiplying (2.6) by $\int_{-1}^{x} (1-s^2) F_k(s) ds$ with $k \geq 1$ and integrating over the interval $[-1, 1]$, one has

$$\int_{-1}^{1} \int_{-1}^{x} (1-s^2)\, F_k(s) \left[\alpha((1-x^2)\,G)''' + 6 - \frac{8}{\gamma}\, e^{4u}\right] ds\, dx = 0. \tag{2.17}$$

It is easy to see that

$$\begin{aligned}
&\int_{-1}^{1} \int_{-1}^{x} (1-s^2)\, F_k(s)\, ((1-x^2)\,G)'''\, ds\, dx \\
&\quad = [((1-x^2)\, F_k(x))'\,(1-x^2)\,G]\,\big|_{-1}^{1} - \int_{-1}^{1} ((1-x^2)\, F_k)''\, ((1-x^2)\,G) \\
&\quad = -\int_{-1}^{1} G(1-x^2)\left[(1-x^2)\, F_k'' - 4x F_k' - 2F_k\right] \\
&\quad = (\lambda_k + 2) \int_{-1}^{1} (1-x^2)\, G\, F_k.
\end{aligned} \tag{2.18}$$

Moreover,

$$\int_{-1}^{1}\int_{-1}^{x}(1-s^2)F_k(s)\,ds\,dx = \left(x\int_{-1}^{x}(1-s^2)\,F_k(s)\right)\Big|_{-1}^{1} - \int_{-1}^{1}(1-x^2)\,x\,F_k$$

$$= \int_{-1}^{1}(1-s^2)\,F_k(s) - \delta_{1k}\int_{-1}^{1}(1-x^2)\,x^2 = -\frac{4}{15}\,\delta_{1k}. \tag{2.19}$$

By (2.1), we see that for $k \geq 1$, there holds

$$[(1-x^2)^2\,F_k']' = (1-x^2)^2\,F_k'' - 4x\,(1-x^2)\,F_k' = -\lambda_k\,(1-x^2)\,F_k. \tag{2.20}$$

Entailing from (2.20), we deduce that

$$\int_{-1}^{x}(1-s^2)\,F_k(s) = -\frac{1}{\lambda_k}\,(1-x^2)^2\,F_k'(x).$$

Letting $k = 1$, we have

$$\int_{-1}^{1}\int_{-1}^{x}(1-s^2)\,F_1(s)\,ds\,e^{4u}\,dx = -\frac{1}{4}\int_{-1}^{1}(1-x^2)^2\,e^{4u}. \tag{2.21}$$

Keep in mind that

$$(\lambda_k+2)\int_{-1}^{1}(1-x^2)\,GF_k = 6\beta\int_{-1}^{1}(1-x^2)\,x^2 = \frac{8}{5}\,\beta. \tag{2.22}$$

Then, (2.14) follows from (2.17)–(2.22).

Similarly, letting $k \geq 2$, we conclude that

$$(\lambda_k+2)\,\alpha\int_{-1}^{1}(1-x^2)\,GF_k = -\frac{8}{\gamma\lambda_k}\int_{-1}^{1}(1-x^2)^2\,F_k'\,e^{4u}.$$

So (2.15) holds.

For (2.16), multiplying (2.9) by x and integrating from -1 to 1, we get

$$\int_{-1}^{1}\Big[x\,(1-x^2)^2\,((1-x^2)\,G)'''' - \frac{24}{\alpha}\,x\,(1-x^2)\,G - 4[(1-x^2)\,G]'''\,(1-x^2)\,x\,G\Big] = 0. \tag{2.23}$$

For the first term in (2.23), we have

$$\int_{-1}^{1}x\,(1-x^2)^2\,((1-x^2)\,G)'''' = \int_{-1}^{1}(x\,(1-x^2)^2)''''\,(1-x^2)\,G$$

$$= 120\int_{-1}^{1}(1-x^2)\,x\,G = 32\beta. \tag{2.24}$$

For the second term, one has

$$\frac{24}{\alpha}\int_{-1}^{1}x\,(1-x^2)\,G = \frac{24\beta}{\alpha}\int_{-1}^{1}(1-x^2)\,x^2 = \frac{32}{5\alpha}\,\beta. \tag{2.25}$$

For the last term, we find

$$4\int_{-1}^{1}[(1-x^2)G]'''(1-x^2)xG = 6\int_{-1}^{1}|[(1-x^2)G]'|^2. \tag{2.26}$$

Therefore, (2.16) follows from (2.23)–(2.26). ∎

3. Proof of Theorem 1.1

Inspired by [10] and [15], our basic strategy is to assume $\beta \neq 0$, and show that it leads to a contradiction with the range of α. It is fairly easy to see from (2.16) that if $\beta = 0$, then $\nabla u = 0$, which implies that u is a constant. One important new ingredient is the surprising a priori estimate in Lemma 3.1 regarding the derivative of the gradient of u.

We now give the key estimate on the derivative of G, which was defined in (2.4). Note that the lemma is true for general $\alpha > 0$.

Lemma 3.1. *Let* $M = \max_{x\in[-1,1]} G'(x)$. *Then we have*

$$M \le \frac{1}{\alpha}, \tag{3.1}$$

i.e.,

$$G'(x) \le \frac{1}{\alpha}$$

for all $x \in [-1,1]$.

Proof. Take $M = G'(x_0)$ for some $x_0 \in [-1,1]$. We first prove the lemma if $x_0 \in (-1,1)$. After some calculations, (2.6) becomes

$$\alpha[-6G' - 6xG'' + (1-x^2)G'''] + 6 - \frac{8}{\gamma}e^{4u} = 0, \quad x \in (-1,1). \tag{3.2}$$

At $x = x_0$, we have

$$G'(x_0) = M, \quad G''(x_0) = 0 \quad \text{and} \quad G'''(x_0) \le 0.$$

Consequently, $M \le 1/\alpha$.

If $x_0 \notin (-1,1)$, then we may assume without loss of generality that

$$\sup_{x\in(-1,1)} G'(x) = \lim_{x_n \to 1} G'(x_n)$$

for some sequence $\{x_n\} \subset (-1,1)$. Let $r = |x'| = \sqrt{1-x^2}$. Then we write

$$G(x) = \bar{G}(r) \quad \text{and} \quad u(x) = \bar{u}(r), \quad \text{for } r \in [0,1) \text{ and } x \in (0,1].$$

It is well known that $\bar{u}(r)$ can be extended evenly and $\bar{u}(r) \in \mathcal{C}^\infty(-1/2,1/2)$. For $r \in (0,1/2)$,

$$u'(x) = \bar{u}_r(r)\frac{dr}{dx} = -\frac{\bar{u}_r(r)}{r}\sqrt{1-r^2},$$

then, one has

$$G(x) = (1-x^2)u'(x) = -r\,\bar{u}_r(r)\sqrt{1-r^2} = \bar{G}(r), \quad r \in (0, 1/2). \tag{3.3}$$

A direct calculation shows that

$$G'(x) = \bar{G}_r(r)\,\frac{dr}{dx} = \left(\bar{u}_{rr}(r) + \frac{\bar{u}_r(r)}{r}\right)(1-r^2) - r\,\bar{u}_r(r).$$

It is easy to see that $\lim_{x\to 1} G'(x) = 2\bar{u}_{rr}(0)$. Note that

$$\lim_{r\to 0} \bar{u}_r(r) = \bar{u}_r(0) = 0 \quad \text{and} \quad \lim_{r\to 0} \frac{\bar{u}_r}{r} = \bar{u}_{rr}(0).$$

We can write

$$\bar{u}_r(r) = t_1 r + t_2 r^3 + O(r^5) \quad \text{near } r = 0. \tag{3.4}$$

Then

$$G'(1) = \lim_{x\to 1} G'(x) = 2t_1 = \sup_{x\in(-1,1)} G'(x) > 0. \tag{3.5}$$

The last inequality is ensured by the fact that (3.3) implies

$$G(1) = G(-1) = \lim_{x\to \pm 1} G(x) = 0.$$

Furthermore, by (3.4),

$$G'(x) = 2t_1 + (4t_2 - 3t_1)r^2 + O(r^4) \quad \text{near } r = 0.$$

It follows from (3.5) that

$$4t_2 - 3t_1 \le 0.$$

By similar arguments, we obtain that near $r = 0$,

$$G''(x) = \frac{dG'(x)}{dr}\,\frac{dr}{dx} = -2(4t_2 - 3t_1 + O(r^2))\sqrt{1-r^2}.$$

Therefore,

$$\lim_{x\to 1} -x\,G''(x) \le 0. \tag{3.6}$$

By similar calculations again, near $r = 0$,

$$(1-x^2)G'''(x) = r^2\,\frac{dG''(x)}{dr}\,\frac{dr}{dx} = O(r^2).$$

This ensures that

$$\lim_{x\to 1} (1-x^2)G'''(x) = 0. \tag{3.7}$$

Using (3.2) together with (3.5)–(3.7), we have

$$G'(1) = \sup_{x\in(-1,1)} G'(x) \le \frac{1}{6\alpha}\left(6 - \frac{8}{\gamma}\,e^{4u}\right) \le \frac{1}{\alpha}\,. \tag{3.8}$$

■

Remark 3.1. When $\alpha = 1$, there is a family of solutions $u = -\ln(1 - ax)$ to (1.6) for any $a \in (0, 1)$. Straightforward computations show that the estimate in Lemma 3.1 is indeed optimal in general. However, given some extra information, the estimate may be improved slightly (see the discussion in Section 6 below for details).

Lemma 3.2. *Concerning a semi-norm of G, we have the following estimate*:

$$\lfloor G \rfloor^2 \le \left(\frac{4}{\alpha} - 6\right) \int_{-1}^{1} |[(1-x^2)G]'|^2 + \frac{16}{\alpha} \int_{-1}^{1} (1-x^2) G^2, \tag{3.9}$$

where

$$\lfloor G \rfloor^2 = \frac{4}{3} \int_{\mathbb{S}^4} (P_4 G) G \, dw = \int_{-1}^{1} (1-x^2) \, [(1-x^2)^2 G']''' G. \tag{3.10}$$

Proof. Multiplying (2.9) by G and integrating over $[-1, 1]$, we have

$$\begin{aligned} &\int_{-1}^{1} ((1-x^2)G)'''' (1-x^2)^2 G - \frac{24}{\alpha} \int_{-1}^{1} (1-x^2) G^2 \\ &\qquad - 4 \int_{-1}^{1} ((1-x^2)G)''' (1-x^2) G^2 = 0. \end{aligned} \tag{3.11}$$

For the first term in (3.11), after integration by parts, one has

$$\begin{aligned} &\int_{-1}^{1} ((1-x^2)G)'''' (1-x^2)^2 G \\ &\quad = -\int_{-1}^{1} ((1-x^2)G)''' \, [(1-x^2)^2 G' - 4x(1-x^2)G] \\ &\quad = \int_{-1}^{1} (1-x^2) \, [(1-x^2)^2 G']''' G + 4 \int_{-1}^{1} [((1-x^2)G)''' x (1-x^2) G] \end{aligned}$$

and

$$\begin{aligned} \int_{-1}^{1} ((1-x^2)G)''' x (1-x^2) G &= -\int_{-1}^{1} ((1-x^2)G)'' \, [(1-x^2)G + x((1-x^2)G)'] \\ &= \frac{3}{2} \int_{-1}^{1} |((1-x^2)G)'|^2. \end{aligned}$$

It follows that

$$\begin{aligned} &\int_{-1}^{1} ((1-x^2)G)'''' (1-x^2)^2 G \\ &\qquad = \int_{-1}^{1} (1-x^2)((1-x^2)^2 G')''' G + 6 \int_{-1}^{1} |((1-x^2)G)'|^2. \end{aligned} \tag{3.12}$$

For the last term in (3.11), we obtain

$$\begin{aligned}\int_{-1}^{1} &((1-x^2)G)'''(1-x^2)G^2 \\ &= \int_{-1}^{1}[-6G'+(1-x^2)G'''](1-x^2)G^2 - 3\int_{-1}^{1}(1-x^2)^2 G G' G'' \\ &\quad - \frac{3}{2}\int_{-1}^{1}(1-x^2)^2 G^2 G''' \\ &= -2\int_{-1}^{1}(1-x^2)(G^3)' - \frac{1}{2}\int_{-1}^{1}(1-x^2)^2[G^2G''' + 6GG'G''] \\ &= -4\int_{-1}^{1} xG^3 + \frac{1}{6}\int_{-1}^{1}[(1-x^2)^2]'''G^3 + \int_{-1}^{1}(1-x^2)^2(G')^3 \\ &= \int_{-1}^{1}(1-x^2)^2(G')^3.\end{aligned}$$

This and (3.12) show that (3.11) is equivalent to

$$\begin{aligned}\int_{-1}^{1}&(1-x^2)((1-x^2)^2G')'''G + 6\int_{-1}^{1}|((1-x^2)G)'|^2 \\ &- \frac{24}{\alpha}\int_{-1}^{1}(1-x^2)G^2 - 4\int_{-1}^{1}(1-x^2)^2(G')^3 = 0.\end{aligned}\tag{3.13}$$

Note that

$$\begin{aligned}6\int_{-1}^{1}&|((1-x^2)G)'|^2 - 4\int_{-1}^{1}(1-x^2)^2(G')^3 \\ &= 6\int_{-1}^{1}(1-x^2)^2(G')^2\left[1-\frac{2}{3}G'\right] + 12\int_{-1}^{1}(1-x^2)G^2.\end{aligned}\tag{3.14}$$

We deduce from (3.1), (3.13) and (3.14) that

$$\left(\frac{24}{\alpha}-12\right)\int_{-1}^{1}(1-x^2)G^2 \geq \lfloor G\rfloor^2 + \left(6-\frac{4}{\alpha}\right)\int_{-1}^{1}(1-x^2)^2(G')^2.\tag{3.15}$$

It is easy to see that

$$\int_{-1}^{1}|[(1-x^2)G]'|^2 - \int_{-1}^{1}(1-x^2)^2(G')^2 = 2\int_{-1}^{1}(1-x^2)G^2.\tag{3.16}$$

Then (3.15) and (3.16) lead to that

$$\lfloor G\rfloor^2 \leq \left(\frac{4}{\alpha}-6\right)\int_{-1}^{1}|[(1-x^2)G]'|^2 + \frac{16}{\alpha}\int_{-1}^{1}(1-x^2)G^2.\tag{3.17}$$

■

Corollary 3.3. *If* $2/3 < \alpha < 1$, *any axially symmetric solution to* (1.1) *must be constant.*

Proof. Using the facts that the first eigenvalue of Laplacian on $\mathbb{S}^4$ is $\lambda_1 = 4$ as in (2.1) and the first eigenvalue of P_4 is $\lambda_1(\lambda_1+2) = 24$, we obtain from Lemma 3.2 immediately that when $\alpha > 2/3$, G must be constant 0 and hence u must be constant. ■

Proof of Theorem 1.1. We shall use higher order eigenfunctions in (2.1) to gain better estimate for α and prove the main theorem. We first define the following quantity:

$$D := \sum_{k=3}^{\infty}\Big[\lambda_k(\lambda_k+2)-\Big(10-\frac{4}{3\alpha}\Big)(\lambda_k+2)-\frac{16}{\alpha}\Big]b_k^2. \tag{3.18}$$

From (2.16), Lemma 3.2 and the definition of G_2, we can check that

$$\begin{aligned}
D &= \int_{-1}^{1}(1-x^2)((1-x^2)^2G_2')'''G_2-\Big(10-\frac{4}{3\alpha}\Big)\int_{-1}^{1}|((1-x^2)G_2)'|^2\\
&\quad-\frac{16}{\alpha}\int_{-1}^{1}(1-x^2)G_2^2\\
&\le \lfloor G\rfloor^2-\Big(10-\frac{4}{3\alpha}\Big)\int_{-1}^{1}|[(1-x^2)G]'|^2-\frac{16}{\alpha}\int_{-1}^{1}(1-x^2)G^2\\
&\quad+\Big(36+\frac{8}{\alpha}\Big)\beta^2\int_{-1}^{1}(1-x^2)F_1^2\\
&\le \Big(\frac{16}{3\alpha}-16\Big)\int_{-1}^{1}|[(1-x^2)G]'|^2+\Big(36+\frac{8}{\alpha}\Big)\frac{4\beta^2}{15}\\
&\le \frac{16\beta}{15}\Big[\Big(9+\frac{2}{\alpha}\Big)\beta+\Big(\frac{16}{3\alpha}-16\Big)\Big(5-\frac{1}{\alpha}\Big)\Big].
\end{aligned}$$

In what follows, we assume that $\beta\neq 0$. From (2.13) and (2.14), one has

$$0<\beta<\frac{1}{\alpha}. \tag{3.19}$$

By (2.3) and (2.15),

$$\begin{aligned}
b_k^2 &= a_k^2\int_{-1}^{1}(1-x^2)F_k^2=\frac{1}{\int_{-1}^{1}(1-x^2)F_k^2}\Big[\frac{8}{\alpha(\lambda_k^2+2\lambda_k)}\int_{-1}^{1}\frac{e^{4u}}{\gamma}(1-x^2)^2F_k'\Big]^2\\
&\le \frac{(2k+3)(k+1)(k+2)}{8}\Big[\frac{8}{\alpha(\lambda_k^2+2\lambda_k)}\frac{\lambda_k}{4}\frac{4}{5}(1-\alpha\beta)\Big]^2.
\end{aligned}$$

Hence one has

$$b_k^2\le\frac{8(2k+3)}{25(\lambda_k+2)}\Big(\frac{1}{\alpha}-\beta\Big)^2,\quad k\ge 2. \tag{3.20}$$

In particular, we obtain

$$\frac{5}{7}|a_2|\le\Big(\frac{1}{\alpha}-\beta\Big). \tag{3.21}$$

It follows from $\alpha>1/2$ and (3.18)–(3.19) that

$$\beta\ge\frac{16}{13}\Big(1-\frac{1}{3\alpha}\Big)\Big(5-\frac{1}{\alpha}\Big)\ge\frac{16}{13} \tag{3.22}$$

and

$$\Big(9+\frac{2}{\alpha}\Big)\frac{1}{\alpha}+\Big(\frac{16}{3\alpha}-16\Big)\Big(5-\frac{1}{\alpha}\Big)\geq 0,$$

which indicates that

$$\alpha < 0.5732. \tag{3.23}$$

From (2.16), (3.18) and (3.21), we derive that

$$\begin{aligned}
\frac{16\beta}{15}&\Big[\Big(9+\frac{2}{\alpha}\Big)\beta+\Big(\frac{16}{3\alpha}-16\Big)\Big(5-\frac{1}{\alpha}\Big)\Big]\\
&\geq D\geq\Big[\lambda_3-\Big(10-\frac{4}{3\alpha}\Big)-\frac{16}{\alpha(\lambda_3+2)}\Big]\sum_{k=3}^{\infty}(\lambda_k+2)\,b_k^2\\
&=8\Big(1+\frac{1}{15\alpha}\Big)\int_{-1}^{1}|[(1-x^2)\,G_2]'|^2\\
&=8\Big(1+\frac{1}{15\alpha}\Big)\Big[\int_{-1}^{1}|[(1-x^2)\,G]'|^2-\frac{8}{5}\beta^2-\frac{8}{7}a_2^2\Big]\\
&\geq 8\Big(1+\frac{1}{15\alpha}\Big)\Big[\frac{16\beta}{15}\Big(5-\frac{1}{\alpha}\Big)-\frac{8}{5}\beta^2-\frac{56}{25}\Big(\frac{1}{\alpha}-\beta\Big)^2\Big].
\end{aligned}$$

Then one has

$$\begin{aligned}
2\beta&\Big(\frac{16}{3\alpha}-16\Big)\Big(5-\frac{1}{\alpha}\Big)-\frac{16\beta}{15}\Big(5-\frac{1}{\alpha}\Big)\Big(15+\frac{1}{\alpha}\Big)\\
&\geq -2\Big[\Big(9+\frac{2}{\alpha}\Big)+\frac{4}{5}\Big(15+\frac{1}{\alpha}\Big)\Big]\beta^2-\frac{56}{25}\Big(15+\frac{1}{\alpha}\Big)\Big(\frac{1}{\alpha}-\beta\Big)^2.
\end{aligned}$$

After some straightforward computations, we obtain

$$\begin{aligned}
2\beta&\Big[\Big(5-\frac{1}{\alpha}\Big)\Big(\frac{24}{5\alpha}-24\Big)+\frac{1}{\alpha}\Big(\frac{14}{5\alpha}+21\Big)\Big]\\
&\geq\Big(\frac{1}{\alpha}-\beta\Big)\Big[2\beta\Big(\frac{14}{5\alpha}+21\Big)-\frac{56}{25}\Big(15+\frac{1}{\alpha}\Big)\Big(\frac{1}{\alpha}-\beta\Big)\Big]\geq 0.
\end{aligned}\tag{3.24}$$

From (3.19) and (3.22), we conclude that

$$0\leq\Big(5-\frac{1}{\alpha}\Big)\Big(\frac{24}{5\alpha}-24\Big)+\frac{1}{\alpha}\Big(\frac{14}{5\alpha}+21\Big)\leq 10. \tag{3.25}$$

The first inequality suggests that

$$\alpha < 0.5444.$$

Furthermore, it follows from (3.22)–(3.25) that

$$\frac{1}{\alpha}-\beta\leq 20\Big(\frac{2002}{25}\beta-\frac{10136}{325\alpha}\Big)^{-1}\beta\leq\frac{325}{588}\beta. \tag{3.26}$$

Next, we fix an integer $n \geq 3$. After some computations, we get

$$
\begin{aligned}
&\sum_{k=3}^{n}\Big(\lambda_k-\lambda_{n+1}-\frac{4}{15\alpha}\Big)(\lambda_k+2)\,b_k^2\\
&\qquad+\Big(\lambda_{n+1}-10+\frac{4}{5\alpha}\Big)\Big[\frac{16\beta}{15}\Big(5-\frac{1}{\alpha}\Big)-\frac{8}{5}\beta^2-\frac{8}{7}a_2^2\Big]\\
&\qquad\leq\sum_{k=3}^{\infty}\Big[\lambda_k(\lambda_k+2)-\Big(10-\frac{4}{3\alpha}\Big)(\lambda_k+2)-\frac{16}{\alpha}\Big]b_k^2\\
&\qquad\leq\frac{16\beta}{15}\Big[\Big(9+\frac{2}{\alpha}\Big)\beta+\Big(\frac{16}{3\alpha}-16\Big)\Big(5-\frac{1}{\alpha}\Big)\Big].
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
0\leq{}&\frac{16\beta}{15}\Big(5-\frac{1}{\alpha}\Big)\Big(\frac{68}{15\alpha}-6-\lambda_{n+1}\Big)+\frac{8}{15}\beta^2\Big(3\lambda_{n+1}-12+\frac{32}{5\alpha}\Big)\\
&+\frac{8}{7}\Big(\lambda_{n+1}-10+\frac{4}{5\alpha}\Big)a_2^2+\sum_{k=3}^{n}\Big(\lambda_{n+1}-\lambda_k+\frac{4}{15\alpha}\Big)(\lambda_k+2)\,b_k^2.
\end{aligned}
$$

By (3.20), one further has

$$
\begin{aligned}
0\leq{}&\frac{16\beta}{15}\Big(5-\frac{1}{\alpha}\Big)\Big(\frac{68}{15\alpha}-6-\lambda_{n+1}\Big)+\frac{8}{15}\beta^2\Big(3\lambda_{n+1}-12+\frac{32}{5\alpha}\Big)\\
&+\frac{8}{25}\Big(\frac{1}{\alpha}-\beta\Big)^2\Big[7\Big(\lambda_{n+1}-10+\frac{4}{5\alpha}\Big)+c_n\Big],
\end{aligned}
\tag{3.27}
$$

where

$$
c_n:=\sum_{k=3}^{n}\Big(\lambda_{n+1}-\lambda_k+\frac{4}{15\alpha}\Big)(2k+3).
$$

A straightforward calculation shows

$$
c_n=\frac{1}{2}\lambda_{n+1}^2+\Big(\frac{4}{15\alpha}-14\Big)\lambda_{n+1}+90-(n+16)\,\frac{4}{15\alpha}\cdot
$$

Therefore, (3.27) is equivalent to

$$
0\leq 10\beta\Big(5-\frac{1}{\alpha}\Big)\Big(\frac{68}{15\alpha}-6-\lambda_{n+1}\Big)+5\beta^2\Big(3\lambda_{n+1}-12+\frac{32}{5\alpha}\Big)+3\,\bar{c}_n\Big(\frac{1}{\alpha}-\beta\Big)^2.
$$

Here

$$
\bar{c}_n=\frac{1}{2}\lambda_{n+1}^2-7\lambda_{n+1}+20+\frac{4}{15\alpha}(\lambda_{n+1}+5-n).
$$

We use the same technique as in (3.24) to obtain

$$
\begin{aligned}
&\frac{5\beta}{3}\Big[-\frac{8}{\alpha^2}+(15\lambda_{n+1}+136)\,\frac{1}{\alpha}-180-30\lambda_{n+1}\Big]\\
&\qquad\geq\Big(\frac{1}{\alpha}-\beta\Big)\Big[5\beta\Big(3\lambda_{n+1}-12+\frac{32}{5\alpha}\Big)-3\,\bar{c}_n\Big(\frac{1}{\alpha}-\beta\Big)\Big].
\end{aligned}
\tag{3.28}
$$

When $n = 3$, we derive from (3.26) that

$$0 < \Big(\frac{1}{\alpha} - \beta\Big)\Big(\beta + 18\,\frac{\beta}{\alpha}\Big) \le \text{RHS of (3.28)}.$$

A direct calculation suggests that

$$\alpha < \frac{139 + \sqrt{17281}}{510} := \alpha_3. \tag{3.29}$$

Next, we consider $\alpha \in [\alpha_{n+1}, \alpha_n)$ with $f_n(\alpha_n) = 0$ and $\alpha_n \in (1/2, 1)$, where

$$f_n(\alpha) = -\frac{8}{\alpha^2} + 136\,\frac{1}{\alpha} - 180 - 15\lambda_{n+1}\Big(2 - \frac{1}{\alpha}\Big). \tag{3.30}$$

It is readily checked that

$$f_n(\alpha_{n+1}) > 0 \quad \text{and} \quad f_{n+1}(\alpha_n) < 0.$$

We now claim that there exists some $d_n > 0$ for $n = 3$ or 4, such that for $\alpha \in [\alpha_{n+1}, \alpha_n)$,

$$\begin{cases} \dfrac{1}{\alpha} - \beta \le \dfrac{d_n}{\lambda_{n+2}}, \\ 15(\lambda_{n+2} - 4)\beta + \dfrac{32\beta}{\alpha} - \dfrac{3\bar{c}_{n+1} d_n}{\lambda_{n+2}} > 0. \end{cases} \tag{3.31}$$

Note that $f_{n+1}(\alpha) \le f_{n+1}(\alpha_{n+1}) = 0$ when $\alpha \in [\alpha_{n+1}, \alpha_n)$. We see from (3.28) and (3.31) that

$$0 \ge \frac{5\beta}{3}\, f_{n+1}(\alpha) \ge \Big(\frac{1}{\alpha} - \beta\Big)\Big[15\beta\,(\lambda_{n+2} - 4) + \frac{32\beta}{\alpha} - 3\bar{c}_{n+1}\Big(\frac{1}{\alpha} - \beta\Big)\Big] > 0. \tag{3.32}$$

There is a contradiction.

We are ready to prove assertion (3.31). First, we study more accurately the bound in (3.26) when $\alpha \in [\alpha_{n+1}, \alpha_n)$. Let

$$h(\alpha) = \frac{16}{13}\Big(1 - \frac{1}{3\alpha}\Big)\Big(5 - \frac{1}{\alpha}\Big)$$

and

$$\bar{h}(\alpha) = \Big(5 - \frac{1}{\alpha}\Big)\Big(\frac{24}{5\alpha} - 24\Big) + \frac{1}{\alpha}\Big(\frac{14}{5\alpha} + 21\Big).$$

From (3.22)–(3.25), it follows that

$$\begin{aligned} \frac{1}{\alpha} - \beta &\le \Big[2\beta\Big(\frac{98}{25\alpha} + \frac{189}{5}\Big) - \frac{56}{25}\Big(15 + \frac{1}{\alpha}\Big)\frac{1}{\alpha}\Big]^{-1} 2\bar{h}(\alpha_{n+1})\beta \\ &\le \Big[2h(\alpha_{n+1})\Big(\frac{98}{25\alpha_n} + \frac{189}{5}\Big) - \frac{56}{25}\Big(15 + \frac{1}{\alpha_{n+1}}\Big)\frac{1}{\alpha_{n+1}}\Big]^{-1} 2\bar{h}(\alpha_{n+1})\beta \\ &:= \gamma_n\beta. \end{aligned} \tag{3.33}$$

Then we derive from (3.28), (3.30) and (3.33) that

$$\begin{aligned}\frac{5\beta}{3}\, f_n(\alpha) &\geq \Big(\frac{1}{\alpha}-\beta\Big)\Big[15\beta\,(\lambda_{n+1}-4)+\frac{32\,\beta}{\alpha}-3\,\bar{c}_n\Big(\frac{1}{\alpha}-\beta\Big)\Big]\\ &\geq \beta\Big(\frac{1}{\alpha}-\beta\Big)\Big[15\,(\lambda_{n+1}-4)+\frac{32}{\alpha}-3\,\bar{c}_n\,\gamma_n\Big]\geq \omega_n\,\beta\Big(\frac{1}{\alpha}-\beta\Big),\end{aligned} \tag{3.34}$$

where

$$\begin{aligned}\omega_n &= 15\,(\lambda_{n+1}-4)+\frac{32}{\alpha_n}-\gamma_n\Big(\frac{3}{2}\,\lambda_{n+1}^2-21\,\lambda_{n+1}+60+\frac{4}{5\alpha_{n+1}}\,(\lambda_{n+1}+5-n)\Big)\\ &:= A_n - B_n\,\gamma_n.\end{aligned}$$

Thus,

$$\frac{1}{\alpha}-\beta \leq \frac{5\,f_n(\alpha_{n+1})}{3\omega_n}. \tag{3.35}$$

One can use (3.35) to prove the first claim in (3.31). Then the other one is ensured by some calculations.

More precisely, if $n = 3$, then $\alpha \in [\alpha_4, \alpha_3)$. After some computations, we find that $\gamma_3 < 0.186$, and so

$$\frac{1}{\alpha}-\beta \leq \frac{25.553\times 0.137}{\lambda_5} \leq \frac{3.51}{\lambda_5}. \tag{3.36}$$

Furthermore,

$$A_4 - \frac{3.51}{\lambda_5\,h(\alpha_4)}\,B_4 \geq 600.3 - 1682.9\times 0.0064 \geq 589.5 > 0. \tag{3.37}$$

Combining (3.32) and (3.37), we find

$$0 = \frac{5\beta}{3}\, f_4(\alpha_4) \geq \frac{5\beta}{3}\, f_4(\alpha) \geq \Big(\frac{1}{\alpha}-\beta\Big)\Big[A_4 - \frac{3.51}{\lambda_5\,h(\alpha_4)}\,B_4\Big] > 0, \tag{3.38}$$

which yields that $\alpha < \alpha_4$.

Similarly, if $n = 4$, then $\alpha \in [\alpha_5, \alpha_4)$. Here

$$\alpha_5 = \frac{473+\sqrt{209329}}{1800}.$$

One has $\gamma_4 < 0.249$, so

$$\frac{1}{\alpha}-\beta \leq \frac{23.0\times 0.298}{\lambda_6} \leq \frac{6.855}{\lambda_6}$$

and

$$A_5 - \frac{6.855}{\lambda_6\,h(\alpha_5)}\,B_5 \geq 811.27 - 3383.58\times 0.095 \geq 489.5 > 0.$$

The previous arguments show that $\alpha < \alpha_5$.

This proves Theorem 1.1 with $\alpha \in (\frac{473+\sqrt{209329}}{1800} \approx 0.51695, 1)$. The range of α for Theorem 1.1 to hold can be slightly improved to $0.5145 \leq \alpha < 1$, see Section 6 for discussions. ∎

Remark 3.2. The approach used in the case $n = 3$ or 4 for (3.31) does not work for $n \geq 5$. The main obstacle is that $\bar{c}_n$ contains a term involving $-\lambda_{n+1}^2$, so we cannot guarantee that the value of ω_n is positive for $n \geq 5$. Let us take $n = 5$ as an example. Some computations indicate that $\gamma_5 \leq 0.2994$ and then

$$\omega_5 = A_5 - \gamma_5 B_5 \approx -632,$$

which shows that there does not exist such a d_5 that the assertion (3.31) holds for $n = 5$. Therefore, it seems impossible to get a contradiction similar to (3.38).

Next we shall show Theorem 1.2 as an immediate consequence of Theorem 1.1 and invariance of $\mathcal{J}_{4/5}$ under a family of conformal transformations $\phi_{P,t}$, $P \in \mathbb{S}^4$, $t > 0$, of $\mathbb{S}^4$.

Following [5], we define $\phi_{P,t}$, for $P \in \mathbb{S}^4$ and $t > 0$, to be

$$\phi_{P,t}(\xi) = \tilde{\xi} := \pi_P^{-1}(ty),$$

where $y = \pi_P(\xi)$ is the stereographic project of $\mathbb{S}^4$ from P as the north pole to the equatorial plane. In particular, we denote $\phi_t = \phi_{P_0,t}$, where $P_0 = (1, 0, \ldots, 0)$.

Given $u \in H^2(\mathbb{S}^4)$ and $t > 0$, let

$$v(\xi) = u(\phi_t(\xi)) + \frac{5}{4^2} \ln|\det(d\phi_t)|, \quad \xi \in \mathbb{S}^4.$$

We have the following invariance property of $\mathcal{J}_{4/5}$ under the transformation $u \to v$.

Proposition 3.4.
$$\mathcal{J}_{4/5}(u) = \mathcal{J}_{4/5}(v), \quad \forall u \in H^2(\mathbb{S}^4),\ t > 0.$$

Proof. The invariance of

$$\frac{2}{5}\int_{\mathbb{S}^4} u(P_4 u)\, dw + 6\int_{\mathbb{S}^4} u\, dw$$

can be proven similarly as part (a) of Theorem 4.1 in [5]. We only need to check that

$$\left(\int_{\mathbb{S}^4} e^{4v}\, dw\right)^2 - \sum_{i=1}^{5}\left(\int_{\mathbb{S}^4} e^{4v}\xi_i\, dw\right)^2 = \left(\int_{\mathbb{S}^4} e^{4u}\, dw\right)^2 - \sum_{i=1}^{5}\left(\int_{\mathbb{S}^4} e^{4u}\tilde{\xi}_i\, dw\right)^2. \tag{3.39}$$

Indeed, after a proper rotation, we may assume that $P = P_0$. Letting

$$a = \frac{1 - t^2}{1 + t^2},$$

we have

$$\xi_1 = \frac{a + \tilde{\xi}_1}{1 + a\tilde{\xi}_1}, \quad \xi_i = \frac{\sqrt{1 - a^2}\,\tilde{\xi}_i}{1 + a\tilde{\xi}_1}, \quad i = 2, 3, 4, 5,$$

and

$$|\det(d\phi_t)|^{1/4}(\xi) = \frac{\sqrt{1 - a^2}}{1 - a\xi_1} = |\det(d\phi_t^{-1})|^{-1/4}(\tilde{\xi}) = \frac{1 + a\tilde{\xi}_1}{\sqrt{1 - a^2}}.$$

Hence,

$$\begin{aligned}
&\Big(\int_{\mathbb{S}^4} e^{4v}\,dw\Big)^2-\Big(\int_{\mathbb{S}^4} e^{4v}\xi_1\,dw\Big)^2\\
&=\Big(\int_{\mathbb{S}^4} e^{4u(\phi_t(\xi))}|\det(d\phi_t)(\xi)|^{1+1/4}\,dw\Big)^2-\Big(\int_{\mathbb{S}^4} e^{4u(\phi_t(\xi)}|\det(d\phi_t)(\xi)|^{1+1/4}\xi_1\,dw\Big)^2\\
&=\Big(\int_{\mathbb{S}^4} e^{4u}|\det(d\phi_t^{-1})|^{-1/4}(\tilde\xi)\,dw\Big)^2-\Big(\int_{\mathbb{S}^4} e^{4u}|det(d\phi_t^{-1})|^{-1/4}(\tilde\xi)\xi_1\,dw\Big)^2\\
&=\Big(\int_{\mathbb{S}^4} e^{4u}|\det(d\phi_t^{-1})|^{-1/4}(\tilde\xi)\,(1-\xi_1)\,dw\Big)\Big(\int_{\mathbb{S}^4} e^{4u}|\det(d\phi_t^{-1})|^{-1/4}(\tilde\xi)\,(1+\xi_1)\,dw\Big)\\
&=\Big(\int_{\mathbb{S}^4} e^{4u}(1-\tilde\xi_1)\,dw\Big)\Big(\int_{\mathbb{S}^4} e^{4u}(1+\tilde\xi_1)\,dw\Big)\\
&=\Big(\int_{\mathbb{S}^4} e^{4u}\,dw\Big)^2-\Big(\int_{\mathbb{S}^4} e^{4u}\,\tilde\xi_1\,dw\Big)^2,
\end{aligned}$$

and for $i=2,3,4,5,$

$$\int_{\mathbb{S}^4} e^{4v}\,\xi_i\,dw=\int_{\mathbb{S}^4} e^{4u(\phi_t(\xi))}\,|\det(d\phi_t)(\xi)|^{1+1/4}\,\xi_i\,dw=\int_{\mathbb{S}^4} e^{4u}\,\tilde\xi_i\,dw.$$

Therefore, (3.39) holds. This completes the proof. ∎

When $P=P_0$ is chosen to coincide with the direction of the center of mass of e^{4u}, we also observe from the above proof that

$$\int_{\mathbb{S}^4} e^{4v}\,\xi_i\,dw=\int_{\mathbb{S}^4} e^{4u}\,\tilde\xi_i\,dw=0,\quad i=2,3,4,5,$$

and

$$\int_{\mathbb{S}^4} e^{4v}\,\xi_1\,dw=\frac{1}{\sqrt{1-a^2}}\int_{\mathbb{S}^4} e^{4u}\,(a+\tilde\xi_1)\,dw=0$$

if we also choose

$$a=-\frac{\int_{\mathbb{S}^4} e^{4u}\,\tilde\xi_1\,dw}{\int_{\mathbb{S}^4} e^{4u}\,dw}.$$

For any $u\in H^2(\mathbb{S}^4)$, there is a $\phi_{P,t}$ such that

$$v(\xi)=u(\phi_{P,t}(\xi))+\frac{5}{4^2}\ln|\det(d\phi_{P,t})|,\quad \xi\in\mathbb{S}^4,$$

belongs to $\mathfrak{L}$. Moreover, we have that $\mathcal{J}_\alpha(u)=J_\alpha(v)$ for $v\in\mathfrak{L}$. Then Theorem 1.2 follows immediately from Theorem 1.1 and Proposition 3.4.

We note that a similar but more general Szegö limit theorem for $u\in H^1(\mathbb{S}^2)$ is proven in [3] using a variational method with a mass center constraint, in combination with the improved Moser–Trudinger inequality in [14]. In general, a similar Szegö limit theorem should be true for $\mathbb{S}^n$, $n\geq 5$ with $\alpha=4/5$ replaced by $\alpha=n/(n+1)$, provided that an improved Beckner's inequality could be proven for $\alpha\leq n/(n+1)$. Note that a counter part of Proposition 3.4 always holds for general $\mathbb{S}^n$.

4. Pohozaev-type identities and classification result

Pohozaev-type identities are very powerful tools in studying the symmetry of solutions to semilinear elliptic equations. They play a vital role in proving classification results (see, e.g., [11, 27]). Recently, Shi et. al. [28] obtain several Pohozaev-type identities and apply them to prove the uniqueness of axially symmetric solution of mean field equation on $\mathbb{S}^2$ for α. In this section, we first list several useful Pohozaev-type identities corresponding to solutions of (1.1), and then we prove Theorem 1.3 based on these identities.

We now prove Proposition 1.5. Motivated by [4, 19], since (1.1) is invariant under adding a constant, we can normalize $\int_{\mathbb{S}^4} e^{4u}\, dw = 1$. Then, (1.1) becomes

$$\alpha P_4 u + 6(1 - e^{4u}) = 0, \quad \xi \in \mathbb{S}^4. \tag{4.1}$$

Multiply (4.1) by ξ_i, $i = 1, 2, \ldots, 5$, and integrate to get

$$\alpha \int_{\mathbb{S}^4} (P_4 u)\, \xi_i \, dw = 6 \int_{\mathbb{S}^4} e^{4u}\, \xi_i \, dw.$$

Note that $-\Delta \xi_i = \lambda_1 x_1 = 4x_1$ and $P_4 \xi_i = \lambda_1(\lambda_1 + 2)\xi_i$, for $i = 1, 2, \ldots, 5$. We further have

$$4\alpha \int_{\mathbb{S}^4} u\, \xi_i \, dw = \int_{\mathbb{S}^4} e^{4u}\, \xi_i \, dw.$$

On the other hand, let

$$Q = \frac{6}{\alpha} - \Big(\frac{1}{\alpha} - 1\Big) 6 e^{-4u}.$$

Then (4.1) can be written as

$$P_4 u + 6 = Q e^{4u}. \tag{4.2}$$

By the Kazdan–Warner condition (1.5), one obtains

$$0 = 24\Big(\frac{1}{\alpha} - 1\Big) \int_{\mathbb{S}^4} \langle \nabla u, \nabla \xi_i \rangle \, dw = -24\Big(\frac{1}{\alpha} - 1\Big) \int_{\mathbb{S}^4} u \Delta \xi_i \, dw = 96\Big(\frac{1}{\alpha} - 1\Big) \int_{\mathbb{S}^4} u\, \xi_i \, dw.$$

Therefore,

$$\int_{\mathbb{S}^4} u\, \xi_i \, dw = 0 \quad \text{and} \quad \int_{\mathbb{S}^4} e^{4u}\, \xi_i \, dw = 0, \quad i = 1, 2, \ldots, 5, \tag{4.3}$$

whenever $\alpha \neq 1$. Hence, we can conclude Proposition 1.5.

Remark 4.1. The identities in (4.3) hold true for

$$\alpha P_n u + (n-1)!\Big(1 - \frac{e^{nu}}{\int_{\mathbb{S}^n} e^{nu}\, dw}\Big) = 0, \quad \xi \in \mathbb{S}^n,$$

for all $n \geq 2$ by the same method. Here,

$$P_n = \begin{cases} \prod_{k=0}^{(n-2)/2} (-\Delta + k(n-k-1)), & \text{for } n \text{ even}, \\ (-\Delta + ((n-1)/2)^2)^{1/2} \prod_{k=0}^{(n-3)/2} (-\Delta + k(n-k-1)), & \text{for } n \text{ odd}. \end{cases}$$

Note that Theorem 1.3 can be proved by the integral identity (2.16). Here we provide a more essential proof using Pohozaev-type identity.

Now we focus on axially symmetric solutions to (4.1). It is readily checked that u satisfies

$$\alpha(1-x^2)[(1-x^2)^2u']''' + 6(1-x^2)(1-e^{4u}) = 0, \quad x \in (-1,1). \tag{4.4}$$

Multiplying (4.4) by $F_2 = \frac{1}{4}(5x^2-1)$ and integrating, we get

$$\alpha \int_{-1}^{1} [(1-x^2)^2 u']'''(1-x^2)F_2 - 6\int_{-1}^{1}(1-x^2)e^{4u}F_2 = 0,$$

or

$$\alpha \int_{\mathbb{S}^4} (P_4 u) F_2 \, dw - 6\int_{\mathbb{S}^4} e^{4u} F_2 \, dw = 0.$$

Note that

$$\alpha \int_{\mathbb{S}^4} (P_4 u) F_2 \, dw = \alpha \int_{\mathbb{S}^4} u P_4 F_2 \, dw = \alpha\lambda_2(\lambda_2+2)\int_{\mathbb{S}^4} u F_2 \, dw = 120\alpha \int_{\mathbb{S}^4} u F_2 \, dw.$$

Therefore,

$$\int_{\mathbb{S}^4} u F_2 \, dw = \frac{1}{20\alpha}\int_{\mathbb{S}^4} e^{4u} F_2 \, dw. \tag{4.5}$$

Multiply (4.1) by $\langle \nabla u, \nabla F_2\rangle$ and integrate,

$$\begin{aligned}
\alpha \int_{\mathbb{S}^4} (P_4 u)\langle \nabla u, \nabla F_2\rangle \, dw &= 6\int_{\mathbb{S}^4} (e^{4u}-1)\langle \nabla u, \nabla F_2\rangle \, dw \\
&= 6\int_{\mathbb{S}^4} \frac{1}{4}\langle \nabla e^{4u}, \nabla F_2\rangle - 6\int_{\mathbb{S}^4}\langle \nabla u, \nabla F_2\rangle \, dw \\
&= 6\int_{\mathbb{S}^4}\Big(-\frac{1}{4}e^{4u}\Delta F_2 + u\Delta F_2\Big)\, dw = 3\int_{\mathbb{S}^4}\Big(5-\frac{1}{\alpha}\Big)e^{4u}F_2 \, dw.
\end{aligned} \tag{4.6}$$

Direct computations show that in the spherical coordinate

$$\nabla u = ((1-x^2)u', 0, 0, 0), \quad \nabla F_2 = ((1-x^2)F_2', 0, 0, 0),$$
$$\langle \nabla u, \nabla F_2\rangle = g_{11}(1-x^2)^2 u' F_2' = (1-x^2)u' F_2',$$

which together with (4.6) imply that

$$\int_{\mathbb{S}^4} (P_4 u)\langle \nabla u, \nabla F_2\rangle \, dw = \frac{5}{2}\int_{-1}^{1}[(1-x^2)^2u']''' x(1-x^2)^2 u'. \tag{4.7}$$

Applying integration by parts to (4.7), we get

$$\begin{aligned}
\int_{\mathbb{S}^4} (P_4 u)\langle \nabla u, \nabla F_2\rangle \, dw &= -\frac{15}{8}\int_{-1}^{1}[(1-x^2)^2u']''[(1-x^2)^2u' + x((1-x^2)^2u')'] \\
&= \frac{45}{16}\int_{-1}^{1}|((1-x^2)^2u')'|^2.
\end{aligned}$$

Hence we obtain the following Pohozaev-type inequality for solutions to (4.4):

$$\int_{-1}^{1}|((1-x^2)^2u')'|^2 = \frac{4}{5\alpha}\Big(5-\frac{1}{\alpha}\Big)\int_{-1}^{1}(1-x^2)e^{4u}F_2. \tag{4.8}$$

This is equivalent to (2.16). When $\alpha = 1/5$, it follows that

$$\int_{-1}^{1} |((1-x^2)^2 u')'|^2 = 0.$$

We further have

$$((1-x^2)^2 u')' \equiv 0, \quad x \in (1-, 1).$$

Therefore,

$$(1-x^2)^2 u' \equiv C, \quad x \in (1-, 1)$$

for some constants C. Since the term $(1-x^2)u'$ is bounded on $(-1, 1)$, we have $C = 0$.

Finally, $u' \equiv 0$, and so $u \equiv$ constant, $x \in (1-, 1)$. Theorem 1.3 has been proven.

5. Bifurcation

In this section we shall obtain results on bifurcation curves to (1.6) in general for $\alpha > 0$ and in particular for $\alpha \in (1/5, 1/2)$. We shall first apply the standard bifurcation theory to analyze the local bifurcation diagram. Let us recall the following general theorem.

Theorem 5.1 (Theorem 1.7 in [7]). *Let X, Y be Hilbert spaces, V a neighborhood of* 0 *in X, and $F\colon (-1,1) \times V \to Y$ a map with the following properties*:

(1) $F(t, 0) = 0$ *for any* t;

(2) $\partial_t F, \partial_x F$ *and* $\partial^2_{t,x} F$ *exist and are continuous*;

(3) $\ker(\partial_x F(0,0)) = \operatorname{span}\{w_0\}$ *and* $Y/\mathcal{R}(\partial_x F(0,0))$ *are one-dimensional*;

(4) $\partial^2_{t,x} F(0,0) w_0 \notin \mathcal{R}(\partial_x F(0,0))$.

If Z is any complement of $\ker(\partial_x F(0,0))$ *in X, then there exist $\varepsilon_0 > 0$, a neighborhood of* $(0,0)$ *in $U \subset (-1,1) \times X$, and continuously differentiable maps $\eta\colon (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ and $z\colon (-\varepsilon_0, \varepsilon_0) \to Z$ such that $\eta(0) = 0$, $z(0) = 0$ and*

$$F^{-1}(0) \cap U \setminus ((-1,1) \times \{0\}) = \{(\eta(\varepsilon), \varepsilon w_0 + \varepsilon z(\varepsilon)) \mid \varepsilon \in (-\varepsilon_0, \varepsilon_0)\}.$$

Recall that the shape of the above local bifurcating branch can be further described by the following theorem (see, e.g., Section I.6 in [20]):

Theorem 5.2. *In the setting of Theorem 5.1, let $\psi \neq 0 \in Y^{-1}$ satisfy*

$$\mathcal{R}(\partial_x F(0,0)) = \{y \in Y \mid \langle \psi, y \rangle = 0\},$$

where Y^{-1} is the dual space of Y. Then we have

$$\eta'(0) = -\frac{\langle \partial^2_{x,x} F(0,0)[w_0, w_0], \psi \rangle}{2\|w_0\| \langle \partial^2_{t,x} F(0,0) w_0, \psi \rangle}.$$

Furthermore, the bifurcation is transcritical provided that $\eta'(0) \neq 0$.

Note that critical points of $I_\alpha(u)$ satisfy the following, with $\rho = 6/\alpha$:

$$(1-x^2)[(1-x^2)^2 u']''' + \rho(1-x^2)\Big(1 - \frac{4}{3} \frac{e^{4u}}{\int_{-1}^{1} (1-x^2) e^{4u}}\Big) = 0, \quad x \in (-1, 1). \tag{5.1}$$

Let

$$\mathcal{V} = \Big\{u \in H^4(\mathbb{S}^4) : u = u(x),\ \int_{\mathbb{S}^4} u\,dw = 0\Big\},$$
$$\mathcal{W} = \Big\{u \in L^2(\mathbb{S}^4) : u = u(x),\ \int_{\mathbb{S}^4} u\,dw = 0\Big\}.$$

To apply Theorem 5.1, we define a nonlinear operator $\mathcal{T} : \mathbb{R} \times \mathcal{V} \to \mathcal{W}$ as

$$\mathcal{T}(\rho, u) = P_4 u + \rho\Big(1 - \frac{e^{4u}}{\int_{\mathbb{S}^4} e^{4u}\,dw}\Big).$$

Obviously, the operator $\mathcal{T}$ is well defined. After direct computations, one has

$$\partial_u \mathcal{T}(\rho, 0)\phi = P_4\phi - 4\rho\phi.$$

Note that $P_4 : \mathcal{V} \to \mathcal{W}$ is surjective and the kernel of P_4 is trivial. Thus, P_4 is invertible. On the other hand, in view of $P_4 u = [(1 - x^2)^2 u']'''$, letting $u(x) = \sum_{k=1}^{\infty} c_k F_k(x)$, we readily check that

$$P_4 u = \sum_{k=1}^{\infty} \bar{\lambda}_k c_k F_k(x), \quad P_4^{-1} u = \sum_{k=1}^{\infty} \bar{\lambda}_k^{-1} c_k F_k(x),$$

where

$$\bar{\lambda}_k = \lambda_k(\lambda_k + 2), \quad \text{with} \quad \lambda_k = k(k + 3).$$

Define

$$\mathcal{F}(\rho, u) = u + \rho P_4^{-1}\Big(1 - \frac{e^{4u}}{\int_{\mathbb{S}^4} e^{4u}\,dw}\Big) \quad \text{and} \quad \mathcal{G}(u) = P_4^{-1}\Big(1 - \frac{e^{4u}}{\int_{\mathbb{S}^4} e^{4u}\,dw}\Big).$$

Let $\mathcal{S}$ denote the closure of the set of nontrivial solutions of

$$\mathcal{F}(\rho, u) = 0. \tag{5.2}$$

Lemma 5.3. *Let $\rho_k = \frac{(k+3)!}{4(k-1)!}$ for $k = 1, 2, 3, \ldots$ Then the kernel of $\partial_u \mathcal{T}(\rho_k, 0)$ is 1-dimensional, and*

$$\ker(\partial_u \mathcal{T}(\rho_k, 0)) = \operatorname{span}\{F_k\}. \tag{5.3}$$

Moreover, the range of the operator $\partial_u \mathcal{T}(\rho_k, 0))$ is given by

$$\mathcal{R}(\partial_u \mathcal{T}(\rho_k, 0)) = \Big\{\varphi \in L^2(-1, 1) : \int_{-1}^{1} (1 - x^2)\varphi F_k = 0\Big\}, \tag{5.4}$$

and it has co-dimension 1. In addition, we have

$$\partial^2_{\rho,u} \mathcal{T}(\rho_k, 0) F_k \notin \mathcal{R}(\partial_u \mathcal{T}(\rho_k, 0)). \tag{5.5}$$

Proof. We can choose

$$X = \mathcal{V} \text{ and } Y = \mathcal{W}.$$

It is easy to compute that

$$\partial_u \mathcal{T}(\rho_k, 0)\phi = P_4\phi - 4\rho_k\phi, \quad \partial^2_{uu} \mathcal{T}(\rho_k, 0)(\phi, \phi) = -16\rho_k\phi^2 + 16\rho_k \int_{-1}^{1} (1 - x^2)\phi^2.$$

Then (5.3) follows from (2.1). From the orthogonal property (2.2), we deduce that

$$\mathcal{R}(\partial_u \mathcal{T}(\rho_k, 0)) \text{ coincides with the orthogonal of } \ker(\partial_u \mathcal{T}(\rho_k, 0)).$$

Note $\ker(\partial_u \mathcal{F}) = \ker(\partial_u \mathcal{T})$. Differentiating $\partial_u \mathcal{T}$ with respect to ρ at the point $(\rho_k, 0)$, we get

$$\partial^2_{\rho,u} \mathcal{T}(\rho_k, 0)\phi = -4\phi,$$

which, combined with the fact that $\int_{-1}^{1}(1-x^2)\, F_k^2 \neq 0$ gives (5.5). ■

For $k \in \mathbb{N}^+$, the following local bifurcation result is an immediate consequence of Theorem 5.1 and Lemma 5.3.

Theorem 5.4. *Let $\rho_k = \frac{(k+3)!}{4\cdot(k-1)!}$ for $k = 1, 2, 3, \ldots$ Then the points $(\rho_k, 0)$ are bifurcation points for the curve of solutions $(\rho, 0)$. In particular, there exist $\varepsilon_0 > 0$ and continuously differentiable functions $\rho_k \colon (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ and $\psi_k \colon (-\varepsilon_0, \varepsilon_0) \to \{F_k\}^\perp$ such that $\rho_k(0) = \rho_k$, $\psi_k(0) = 0$, and every nontrivial solution of (5.1) in a small neighborhood of $(\rho_k, 0)$ is of the form*

$$(\rho_k(\varepsilon), \varepsilon F_k + \varepsilon \psi_k(\varepsilon)).$$

In particular, when $k = 2$, the bifurcation point $(\rho_2, 0) = (30, 0)$ is a transcritical bifurcation point. Indeed, we have

$$\rho_2'(0) = -60\, \frac{\int_{-1}^{1}(1-x^2)\, F_2^3}{\int_{-1}^{1}(1-x^2)\, F_2^2} = -20. \tag{5.6}$$

Corollary 5.5. *Let $\alpha_k = \frac{24\cdot(k-1)!}{(k+3)!}$ for $k = 1, 2, 3, \ldots$ Then the points $(\alpha_k, 0)$ are bifurcation points for the curve of solutions $(\alpha, 0)$ of (1.6). Moreover, when $k = 2$, the bifurcation point $(1/5, 0)$ is a transcritical bifurcation point.*

Remark 5.1. When $k = 1$, the bifurcation leads to the family of solutions $u = -\ln(1-ax)$, $a \in (-1, 1)$, and $\rho = 6$. It is clear that $(\rho_k, 0)$ is not a transcritical bifurcation point for k odd since F_k is an odd function and $\rho'(0) = 0$ in this case. It should be true that $(\rho_k, 0)$ is a transcritical bifurcation point for k even, we only need to check if $\int_{-1}^{1}(1-x^2)\, F_k^3 \neq 0$ in this case, which can be confirmed for small k numerically. However, in this paper we only need to use the transcriticality of $(\rho_2, 0)$.

In order to analyze the global bifurcation diagram, we employ a global bifurcation theorem via degree arguments (see [20,26]) and also exploit special properties of solutions to (5.1).

First, we recall a global bifurcation result (see Theorem II.5.8 in [20]).

Proposition 5.6. *In Theorem 5.4, the bifurcation at $(\rho_k, 0)$ is global and satisfies the Rabinowitz alternative, i.e., a global continuum of solutions to (5.1) either goes to infinity in $R \times \mathcal{W}$ or meets the trivial solution curve at $(\rho_m, 0)$ for some $m \geq 1$ and $m \neq k$.*

Next we state and prove the following more specific global bifurcation result regarding (5.1).

Theorem 5.7. (1) *For $k \geq 2$, there exists a global continuum of solutions $\mathcal{B}_k^+ \subset \mathcal{S} \setminus \{(\rho, 0), \rho \in \mathbb{R}\}$ of (5.1) which coincides in a small neighborhood of $(\rho_k, 0)$ with*

$$\{(\rho_k(\varepsilon), \varepsilon F_k + \varepsilon \psi_k(\varepsilon)), \varepsilon < 0\}.$$

The set $\mathcal{B}_k^+$ is contained in $\mathcal{N}_2 := \{(\rho, u) : \rho > 30,\ u \in L^2(-1,1)\}$ and is uniformly bounded in $L^2(-1,1)$ for ρ in any fixed finite interval $[\rho_m, \rho_M] \subset (30, \infty)$. Furthermore, $\mathcal{B}_k^+$ satisfies the improved Rabinowitz alternative, i.e., either $\mathcal{B}_k^+$ extends in ρ to infinity, or meets the trivial solution curve at $(\rho_m, 0)$ for some $m \geq 2$.

(2) *Similarly, for $k \geq 2$, there exists a global continuum of solutions $\mathcal{B}_k^-$ which coincides in a small neighborhood of $(\rho_k, 0)$ with $\{(\rho_k(\varepsilon), \varepsilon F_k + \varepsilon \psi_k(\varepsilon)), \varepsilon > 0\}$. When $k \geq 3$, the set $\mathcal{B}_k^-$ is contained in $\mathcal{N}_2$ and satisfies the boundedness for ρ in any fixed finite interval $[\rho_m, \rho_M] \subset (12, \infty)$. Furthermore, the improved Rabinowitz alternative holds.*

(3) *Moreover, $\mathcal{B}_k^+ = \{u : u(x) = v(-x), v \in \mathcal{B}_k^-\}$ when k is odd.*

(4) *The global continuum of solutions $\mathcal{B}_2^-$ of (5.1) must be contained in the set*

$$\mathcal{N}_1 := \left\{(\rho, u) : \rho \in \left(\frac{6 \times 1800}{473 + \sqrt{209329}}, 30\right) \supset (12, 30),\ u \in L^2(-1,1)\right\}.$$

Furthermore, the set $\mathcal{B}_2^-$ is unbounded in $L^\infty([-1,1])$, and there exists a sequence $(\rho^{(k)}, u^{(k)}) \in \mathcal{B}_2^-$, $k = 1, 2, \ldots$, such that $\rho^{(k)} \to 30$ and $\|u^{(k)}\|_{L^\infty([-1,1])} \to \infty$. As an immediate consequence, there is a nontrivial solution to (5.1) for any $\rho \in (12, 30)$.

Proof. To prove (1) and (2), we only need to first apply the general global bifurcation theory and then use Theorem 1.3 to show $\mathcal{B}_k^+$ and $\mathcal{B}_k^-$ are contained in $\mathcal{N}_2 := \{(\rho, u) : \rho > 30,\ u \in L^2(-1,1)\}$.

A general compactness result (see Theorem 1.1 in [24]) says that the solutions to (5.1) can only blow up in $L^\infty([-1,1])$ at $\rho = 6k$ for an positive integer k when (5.1) is considered as an fourth order Q-curvature type equation on $\mathbb{S}^4$, and k is the number of blowup points. (See also Theorem 4.3 in [16] from a view point of constrained inequalities.) Since an axially symmetric solution can blow up at most two points at a finite parameter ρ, we must have $k = 1, 2$. Therefore, this leads to the boundedness of $\mathcal{B}_k^+$, for $k \geq 2$, and of $\mathcal{B}_k^-$, for $k \geq 3$, for ρ in any fixed finite interval $[\rho_m, \rho_M] \subset (30, \infty)$.

To prove (3), we note that $u(x) = v(-x)$ is a solution to (5.1) if so is $v(x)$, and $u(x)$ is not an even function for $u \in \{u : u(x) = v(-x), v \in \mathcal{B}_k^-\}$ near the bifurcation point $\{(\rho_k, 0)\}$. Therefore, by the local bifurcation result Theorem 5.4, we know $\{u : u(x) = v(-x), v \in \mathcal{B}_k^-\}$ is different from $\mathcal{B}_k^-$ and hence coincides with $\mathcal{B}_k^+$ near $\{(\rho_k, 0)\}$. Then (3) follows immediately.

To prove (4), we first use the transcriticality (5.6) to get $\mathcal{B}_2^- \cap \mathcal{N}_1 \neq \emptyset$. By Theorems 1.1 and 1.3, we conclude that $\mathcal{B}_2^- \subset \mathcal{N}_1$. Since there is no other bifurcation points for ρ between $\frac{6\times 1800}{473+\sqrt{209329}} > 6$ and 30 , and $\rho = 12$ is the only blowup point, we conclude that $\mathcal{B}_2^-$ must go to infinity in $\mathcal{W}$ and in $L^\infty([-1,1])$ at $\rho = 12$.

This completes the proof. ∎

Remark 5.2. The above theorem implies that $\mathcal{B}_2^-$ does not coincide with other bifurcation branches. It would be interesting to see whether the solution branches bifurcating from different points $(\rho_k, 0)$ coincide with each other or not, i.e., whether $\mathcal{B}_k^+ = \mathcal{B}_m^+$ or $\mathcal{B}_k^+ = \mathcal{B}_m^-$ in general for any $m \neq k$, or particularly for $m \equiv k+1 \pmod 2$. Also it is not clear whether $\mathcal{B}_k^+ = \mathcal{B}_k^-$ for some $k \geq 3$.

Proof of Theorem 1.4. Theorem 1.4 follows immediately from Theorem 5.7. This leads to the existence of a nontrivial solution to (1.6) for $\alpha \in (1/5, 1/2)$. ■

6. Discussion

In this section, we shall discuss some ideas to close the gap $\alpha \in (\frac{1}{2}, \frac{473+\sqrt{209329}}{1800})$. Note that Gui and Wei [15] used an induction method to show

$$\frac{1}{\alpha} - \beta \leq \frac{4}{\lambda_n} \quad \text{for all } n \geq 4,$$

with the sequence $\lambda_n \to \infty$. So it follows $1/\alpha - \beta \to 0$ as $n \to \infty$, which leads to a contradiction. Following the arguments in [15], we divide (3.28) by λ_{n+1} to get

$$\begin{aligned} \frac{5\beta}{3\lambda_{n+1}} \Big[-\frac{8}{\alpha^2} + (15\lambda_{n+1} + 136)\frac{1}{\alpha} - 180 - 30\lambda_{n+1} \Big] \\ \geq \Big(\frac{1}{\alpha} - \beta\Big)\Big[5\beta\Big(3 - \frac{12}{\lambda_{n+1}} + \frac{32}{5\alpha\lambda_{n+1}}\Big) - \frac{3\bar{c}_n}{\lambda_{n+1}}\Big(\frac{1}{\alpha} - \beta\Big)\Big]. \end{aligned} \tag{6.1}$$

A direct calculation shows that

$$\text{LHS of (6.1)} \leq \frac{100\beta}{\lambda_{n+1}}, \tag{6.2}$$

which is the basic ingredient for the induction procedure in [15]. Next, the major task is to find an appropriate d so that

$$\frac{1}{\alpha} - \beta \leq \frac{d}{\lambda_n} \quad \text{for all } n \geq n_0.$$

However, we do not know what the initial value n_0 should be, which is dependent on the choice of d. We assume, by induction, that

$$\frac{1}{\alpha} - \beta \leq \frac{d}{\lambda_n} \tag{6.3}$$

for some $n \geq n_0$. Then

$$\frac{1}{\alpha} - \beta \leq \frac{d}{\lambda_{n+1}} \tag{6.4}$$

must hold. It follows from (6.3) that

$$\text{RHS of (6.1)} \geq \Big(\frac{1}{\alpha} - \beta\Big)\Big[5\beta\Big(3 - \frac{12}{\lambda_{n+1}} + \frac{32}{5\alpha\lambda_{n+1}}\Big) - \frac{3\bar{c}_n}{\lambda_{n+1}}\frac{d}{\lambda_n}\Big].$$

To ensure (6.4), it suffices from (6.2) to prove that

$$5\beta\Big(3-\frac{12}{\lambda_{n+1}}+\frac{32}{5\alpha\lambda_{n+1}}\Big)-\frac{3\bar{c}_n}{\lambda_{n+1}}\,\frac{d}{\lambda_n}\geq\frac{100}{d}\,\beta. \tag{6.5}$$

For n large, this requires

$$\frac{15d-100}{\alpha}\geq\frac{3d^2}{2} \tag{6.6}$$

for $\alpha\in(\frac{1}{2},\frac{473+\sqrt{209329}}{1800})$. However, there does not exist such a constant d for (6.6) to hold. Hence, this method does not seem to yield the optimal constant $\alpha=1/2$ for this problem.

Remark 6.1. We also intend to replace denominator in (6.3) and (6.4) by λ_n^t and λ_{n+1}^t, respectively, for some $t>0$. For this purpose, we only need to slightly modify the previous procedure. After some calculations, (6.5) becomes

$$5\beta\Big(3-\frac{12}{\lambda_{n+1}}+\frac{32}{5\alpha\lambda_{n+1}}\Big)-\frac{3\bar{c}_n}{\lambda_{n+1}}\,\frac{d}{\lambda_n^t}\geq\frac{100\lambda_n^{t-1}}{d}\,\beta.$$

Therefore, we need to show that for $\alpha\in(\frac{1}{2},\frac{473+\sqrt{209329}}{1800})$ and n large,

$$\frac{1}{\alpha}\,[15d\,\lambda_n^{t+1}-100\lambda_n^{2t}]\geq\frac{3d^2}{2}\lambda_{n+1}\,\lambda_n+o_n(1),$$

which suggests that $t=1$ is the best choice, since $\lambda_n\to\infty$ as $n\to\infty$.

It is worth pointing out that the inequality (3.17) ensured by Lemma 3.1 plays an important role in the proof of Theorem 1.1. In view of (3.8), we may want to estimate M more accurately by considering

$$M=\max_{x\in[-1,1]}G'(x)=\frac{1}{\alpha}\Big(1-\frac{4}{3\gamma}\,e^{4u}\Big). \tag{6.7}$$

To this purpose, first, we observe, due to $G'(x)\leq M\leq 1/\alpha$, that

$$-\frac{1}{\alpha}\,(1-x)\leq G(x)\leq\frac{1}{\alpha}\,(1+x),\quad x\in[-1,1]. \tag{6.8}$$

This leads to

$$-\frac{1}{\alpha(1+x)}\leq u'(x)\leq\frac{1}{\alpha(1-x)},\quad x\in(-1,1). \tag{6.9}$$

Assume, without loss of generality, $u(0)=0$. Then one has

$$u(x)\geq-\frac{1}{\alpha}\ln 2,\quad\text{that is,}\quad e^{4u}\geq 2^{-4/\alpha}. \tag{6.10}$$

We next estimate γ from above. Note that $u = -\ln(1-ax)$ is a solution when $\alpha = 1$. By some computations, we can see that (6.8)–(6.9) cannot be improved, and there is *no* a priori estimate for γ. However, if we assume

$$1 - \alpha\beta > 0, \tag{6.11}$$

then we can use (2.14) to estimate γ. Precisely, take a constant a such that

$$1 - a^2 = c(1-\alpha\beta), \quad c \in (0, 4/5). \tag{6.12}$$

Then

$$\frac{4}{5}(1-\alpha\beta) \le \frac{2}{\gamma}\int_0^a (1-x^2)(a^2-x^2)(1-x)^{-4/\alpha} + (1-a^2). \tag{6.13}$$

Since $\alpha \in (0.5, 0.517)$, one has

$$\begin{aligned} I &:= \int_0^a (1-x^2)(a^2-x^2)(1-x)^{-4/\alpha} \le \int_0^a (1+x)(a^2-x^2)(1-x)^{-7} \\ &= \frac{2a^2}{15}((1-a)^{-5}-1). \end{aligned}$$

It follows from (6.12) that $a = \sqrt{1-c(1-\alpha\beta)}$, and so

$$\frac{1}{1-a} = \frac{1+a}{c(1-\alpha\beta)} \le \frac{2}{c(1-\alpha\beta)}.$$

We further compute directly

$$I \le \frac{2}{15}(1-c(1-\alpha\beta))\left[\left(\frac{2}{c(1-\alpha\beta)}\right)^5 - 1\right] \le \frac{2^6[1-c(1-\alpha\beta)]}{15[c(1-\alpha\beta)]^5},$$

which, joint with (6.13), lead to

$$\gamma \le \left[\left(\frac{4}{5}-c\right)(1-\alpha\beta)\right]^{-1} \frac{2^7[1-c(1-\alpha\beta)]}{15[c(1-\alpha\beta)]^5}. \tag{6.14}$$

We see from (6.7), (6.10) and (6.14) that

$$M \le \frac{1}{\alpha}\left[1 - \frac{5c^5}{2^{13}}\left(\frac{4}{5}-c\right)\frac{(1-\alpha\beta)^6}{1-c(1-\alpha\beta)}\right] := \frac{1}{\alpha}[1 - B(\alpha,\beta)]. \tag{6.15}$$

We conclude from the above relations that the upper bound of M can be slightly improved in terms of $B(\alpha,\beta) > 0$ by choosing, e.g., $c = 2/5$, given $\beta \ne 0$ and $1-\alpha\beta > 0$. Thus, one has

$$B(\alpha,\beta) = \frac{1}{5^4 2^7}\frac{(1-\alpha\beta)^6}{3+2\alpha\beta}. \tag{6.16}$$

We next will use some notations from Section 3, and assume that

$$0.5165 \le \alpha < 0.51696. \tag{6.17}$$

Note that

$$\beta > h(0.5165) > 1.3375. \tag{6.18}$$

It follows from (3.33) and (6.11) that

$$1 > \alpha\beta > 0.5165\, h(0.5165) > 0.69057 \tag{6.19}$$

and

$$\frac{1}{\alpha} - \beta \le 0.255\,\beta. \tag{6.20}$$

Let $t = 1/\alpha - \beta$. We now estimate the lower bound of t. Noting that

$$f_n(\alpha) < f_6(0.5165) < -13.764, \quad \text{for } n \ge 6,$$

we derive from (3.28) and (6.18) that

$$\frac{5}{3} f_6(0.5165) > \frac{5}{3} f_6(\alpha) \ge A_6 t - \frac{B_6}{\beta} t^2 > A_6 t - \frac{B_6}{1.3375} t^2.$$

Noting that $t > 0$, a direct calculation indicates that

$$t > 0.253, \tag{6.21}$$

which joint with (6.19) and (6.20) suggests that

$$\begin{aligned} 7.85 \times 10^{-10} > \frac{0.255^6}{5^4\, 2^7 (3 + 2 \times 0.69057)} &\ge B(\alpha,\beta) = \frac{1}{5^4\, 2^7} \frac{(1-\alpha\beta)^6}{3 + 2\alpha\beta} \\ &> \frac{(0.5165 \times 0.253)^6}{5^5\, 2^7} > 1.132 \times 10^{-11}. \end{aligned} \tag{6.22}$$

On the other hand, we need to modify some inequalities in Section 3 by exploiting (6.15) instead of (3.1). First, inequality (3.15) becomes

$$\Big(\frac{24}{\alpha} - 12\Big) \int_{-1}^{1} (1-x^2) G^2 \ge \lfloor G \rfloor^2 + \Big(6 - \frac{4}{\alpha}(1-B)\Big) \int_{-1}^{1} (1-x^2)^2 (G')^2.$$

Here, B denotes $B(\alpha,\beta)$. Similarly, we have

$$\lfloor G \rfloor^2 \le \Big(\frac{4}{\alpha}(1-B) - 6\Big) \int_{-1}^{1} |[(1-x^2)G]'|^2 + \frac{16+8B}{\alpha} \int_{-1}^{1} (1-x^2) G^2$$

and

$$\begin{aligned} \bar{D} &:= \sum_{k=3}^{\infty} \Big[\lambda_k(\lambda_k+2) - \Big(10 - \frac{4+2B}{3\alpha}\Big)(\lambda_k+2) - \frac{16+8B}{\alpha}\Big] b_k^2 \\ &\le \Big(\frac{16-10B}{3\alpha} - 16\Big) \int_{-1}^{1} |[(1-x^2)G]'|^2 - \Big[6\Big(\frac{4+2B}{3\alpha} - 6\Big) - \frac{16+8B}{\alpha}\Big] \frac{4\beta^2}{15} \\ &\le \frac{16\beta}{15} \Big[\Big(\frac{16-10B}{3\alpha} - 16\Big)\Big(5 - \frac{1}{\alpha}\Big) + \Big(9 + \frac{2+B}{\alpha}\Big)\beta\Big]. \end{aligned} \tag{6.23}$$

From (2.16), (3.21) and (6.23), we derive that

$$\begin{aligned}&\sum_{k=3}^{n}\Big(\lambda_k-\lambda_{n+1}-\frac{4+2B}{15\alpha}\Big)(\lambda_k+2)\,b_k^2\\&+\Big(\lambda_{n+1}-10+\frac{4+2B}{5\alpha}\Big)\Big[\frac{16\beta}{15}\Big(5-\frac{1}{\alpha}\Big)-\frac{8}{5}\,\beta^2-\frac{8}{7}\,a_2^2\Big]\\&\qquad=\sum_{k=3}^{\infty}\Big[\lambda_k(\lambda_k+2)-\Big(10-\frac{4+2B}{3\alpha}\Big)(\lambda_k+2)-\frac{16+8B}{\alpha}\Big]\,b_k^2\\&\qquad\le\frac{16\beta}{15}\Big[\Big(9+\frac{2+B}{\alpha}\Big)\beta+\Big(\frac{16-10B}{3\alpha}-16\Big)\Big(5-\frac{1}{\alpha}\Big)\Big].\end{aligned}$$

One further obtains that

$$\begin{aligned}0\le\ &\frac{16\beta}{15}\Big(5-\frac{1}{\alpha}\Big)\Big(\frac{68-56B}{15\alpha}-6-\lambda_{n+1}\Big)+\frac{8}{15}\,\beta^2\Big(3\lambda_{n+1}-12+\frac{32+16B}{5\alpha}\Big)\\&+\frac{8}{25}\Big(\frac{1}{\alpha}-\beta\Big)^2\Big[7\Big(\lambda_{n+1}-10+\frac{4+2B}{5\alpha}\Big)+c_{n,B}\Big],\end{aligned}\tag{6.24}$$

where

$$c_{n,B}=\frac{1}{2}\,\lambda_{n+1}^2+\Big(\frac{4+2B}{15\alpha}-14\Big)\lambda_{n+1}+90-(n+16)\,\frac{4+2B}{15\alpha}\cdot$$

Note that (6.24) is equivalent to

$$\begin{aligned}0\le\ &10\beta\Big(5-\frac{1}{\alpha}\Big)\Big(\frac{68-56B}{15\alpha}-6-\lambda_{n+1}\Big)+5\beta^2\Big(3\lambda_{n+1}-12+\frac{32+16B}{5\alpha}\Big)\\&+3\,\bar{c}_{n,B}\Big(\frac{1}{\alpha}-\beta\Big)^2\end{aligned}$$

with

$$\bar{c}_{n,B}=\frac{1}{2}\,\lambda_{n+1}^2-7\lambda_{n+1}+20+\frac{4+2B}{15\alpha}\,(\lambda_{n+1}+5-n).$$

Therefore,

$$\begin{aligned}&\frac{5\beta}{3}\Big[(32B-8)\frac{1}{\alpha^2}+(15\lambda_{n+1}+136-112B)\,\frac{1}{\alpha}-180-30\lambda_{n+1}\Big]\\&\qquad\ge\Big(\frac{1}{\alpha}-\beta\Big)\Big[5\beta\Big(3\lambda_{n+1}-12+\frac{32+16B}{5\alpha}\Big)-3\,\bar{c}_{n,B}\Big(\frac{1}{\alpha}-\beta\Big)\Big].\end{aligned}\tag{6.25}$$

When $n=3$, it follows from (6.20) that

$$\begin{aligned}\text{RHS of (6.25)}&=\Big(\frac{1}{\alpha}-\beta\Big)\beta\Big[360+\frac{16(2+B)}{\alpha}-0.255\Big(648+\frac{12(2+B)}{\alpha}\Big)\Big]\\&\ge\Big(\frac{1}{\alpha}-\beta\Big)\beta\Big(194.76+\frac{12.94(2+B)}{\alpha}\Big).\end{aligned}$$

Combining (6.22), (6.21) and (6.25), we conclude

$$
\begin{aligned}
0.253\beta\Big(194.76+\frac{12.94\,(2+1.132\times 10^{-11})}{\alpha}\Big) &< t\beta\Big(194.76+\frac{12.94\,(2+B)}{\alpha}\Big)\\
&\le \frac{5\beta}{3}\Big[\frac{32B-8}{\alpha^2}+\frac{556-112B}{\alpha}-1020\Big]\\
&\le \frac{5\beta}{3}\Big[\frac{32\times 7.85\times 10^{-10}-8}{\alpha^2}+\frac{556-112\times 1.132\times 10^{-11}}{\alpha}-1020\Big].
\end{aligned}
$$

Then one has

$$\alpha < 0.511.$$

This is a contraction with (6.17). Consequently,

$$\alpha < 0.5165.$$

We now assume that

$$0.516 \le \alpha < 0.5165. \tag{6.26}$$

Similarly, one has

$$\frac{1}{\alpha}-\beta < 0.2611\beta, \quad \beta > 1.3341 \quad \text{and} \quad t > 0.2492.$$

Furthermore,

$$9.05\times 10^{-10} > B(\alpha,\beta) > 1.13\times 10^{-11}.$$

Then we find

$$
\begin{aligned}
0 < {}&\frac{32\times 9.05\times 10^{-10}-8}{\alpha^2}+\frac{556-112\times 1.13\times 10^{-11}}{\alpha}\\
&-1020-0.6\times 0.2492\Big[190.8+\frac{12.8\,(2+1.13\times 10^{-11})}{\alpha}\Big].
\end{aligned}
$$

A direct calculation shows that

$$\alpha < 0.512.$$

This yields a contradiction.

Repeating previous arguments, we obtain a contradiction for the following ranges of α:

$$\alpha \in [0.5155, 0.516), \quad \alpha \in [0.515, 0.5155) \quad \text{and} \quad \alpha \in [0.5145, 0.515).$$

Hence, $\alpha < 0.5145$.

Unfortunately, it does not seem possible to improve the estimate of α significantly in this way and recursively, due to (6.22), let alone to obtain the possible optimal constant $\alpha = 1/2$.

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