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New slope inequalities for families of complete intersections

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Abstract. We prove f-positivity of $\mathcal{O}_X(1)$ for arbitrary dimensional fibrations over curves $f\colon X \to B$ whose general fibre is a complete intersection. In the special case where the family is a global complete intersection, we prove numerical sufficient and necessary conditions for f-positivity of powers of $\mathcal{O}_X(1)$ and for the relative canonical sheaf. From these results we also derive a Chow instability condition for the fibres of relative complete intersections in the projective bundle of a μ-unstable bundle.

1. Introduction

Let $f\colon X \to B$ be a surjective morphism with connected fibres from an n-dimensional smooth projective variety X to a smooth projective curve B, with general fibre F. Let L be a divisor on X. We say that L *is f-positive* if the following inequality holds:

$$e(L) := L^n - n\,\frac{L_{|F}^{n-1}}{h^0(F, L_{|F})}\,\deg f_*\mathcal{O}_X(L) \geq 0. \tag{1.1}$$

The quantity $e(L)$ is a numerical invariant of the data, originally introduced by Cornalba and Harris in [9], and its positivity has a profound relation with Chow and Hilbert stability of the pair $(F, L_{|F})$, and it is also related with sheaf stability properties of the vector bundle $f_*\mathcal{O}_X(L)$: see [5] for a detailed discussion; some of these connections will also emerge from the results of the present article.

The case when L is the relative canonical divisor $K_f = K_X - f^*K_B$ of a relatively minimal fibration is of particular interest: inequality (1.1) is called in this case *the canonical slope inequality*:

$$K_f^n \geq n\,\frac{K_F^{n-1}}{p_g(F)}\,\deg f_*\mathcal{O}_X(K_f). \tag{1.2}$$

The case $n = 2$ was the first case studied. Cornalba and Harris [9] (see also [29]) and Xiao [30] proved with two different methods the canonical slope inequality for relatively minimal fibred surfaces with fibres of genus greater than or equal to 2. The importance of

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the slope inequality can hardly be underestimated: in particular, it has consequences both in the classification of surfaces ([28,30]) and in the study of the structure of the ample and nef cones of the moduli space of stable curves ([1,16]).

For the case of an arbitrary L in dimension 2, we proved in [5] that f-positivity holds in case that L is a nef line bundle such that the restriction $L_{|F}$ is linearly stable (see Definition 2.16 in [25]), and that this linear stability on the fibres is the assumption needed in both the Cornalba–Harris and the Xiao methods, and also in a third method due to Moriwaki. It is worth remarking though that (differently from Cornalba–Harris and Moriwaki's ones) Xiao's method works also with weaker assumptions than linear stability for $L_{|F}$, producing a (weaker) inequality for the ratio between L^2 and $\deg f_*\mathcal{O}_X(L)$.

For the case $n = 3$, many inequalities between K_f^3 and $\deg f_*\mathcal{O}_X(K_f)$ are proved in [2] and [27] and more recently in [19]; still, a slope inequality or more generally the f-positivity of some class of line bundles is in general not known.

For higher values of n, only few results are known. The most important one is in the original paper of Cornalba and Harris [9]: an inequality is proved in any dimension under a Hilbert stability condition on the map induced by the line bundle on the fibres (Theorem 2.1). A slightly improved result (assuming the Chow stability of maps induced on the general fibres) has been proved by Bost in arbitrary characteristic [7]. Unfortunately, the Hilbert or Chow stability on the fibres in dimension ≥ 2 is not known. In particular, for the case of $L = K_f$ and general fibre F of general type, such a condition is known only asymptotically, i.e., for high enough powers of the canonical sheaf.

Very recently, the first named author has proved a bound in any dimension for irregular fibrations (see [4] and [3]); similar results have also been proved by Hu and Zhang in [20]. Even more recently Codogni, Tasin and Viviani in [8] applied Xiao's method to families of K-semistable and KSB-semistable varieties and their moduli.

In [6], we gave a complete treatment of the particular case where X is a relative hypersurface in a projective bundle $\pi\colon \mathbb{P}_B(\mathcal{E}) \to B$, where $\mathcal{E}$ is a vector bundle over B, and $f\colon X \to B$ is the morphism induced by π. In that case, f-positivity of $\mathcal{O}_X(h)$ for any $h \geq 1$ is equivalent to the canonical slope inequality, and both are equivalent to a numerical relation between the class of X in the Néron–Severi space $\mathrm{N}^1(\mathbb{P})$ and the slope $\mu(\mathcal{E}) := \deg\mathcal{E}/\mathrm{rank}\mathcal{E}$ of $\mathcal{E}$. From this we could deduce instability and singularity conditions for the fibres and also for the total space X.

In this paper, we study fibrations whose general fibre is a complete intersection of arbitrary codimension. The main results of the present paper are three.

Firstly we prove a very general slope inequality for these fibrations, with mild conditions on L and on the general fibre F.

Theorem 1.1 (Theorem 2.4). *Let X be an n-dimensional variety with a surjective morphism with connected fibres $f\colon X \to B$ over a smooth curve B. Let L be a line bundle over X which is relatively ample with respect to f. Suppose that the general fibre F is embedded in $\mathbb{P}^{h^0(F,L_{|F})-1} = \mathbb{P}^{r-1}$ by $|L_{|F}|$ as the complete intersection of $r-n$ hypersurfaces Y_i of degree d_i, such that for any $i = 1,\dots,r-n$,*

$$\mathrm{lct}(\mathbb{P}^{r-1}, Y_i) \geq \frac{r}{d_i},$$

where lct *is the log canonical threshold of the pair $(\mathbb{P}^{r-1}, Y_i)$. Then L is f-positive.

Remark 1.2. In particular, the result holds if the general fibre is the (not necessarily smooth) complete intersection of smooth hypersurfaces.

This is to our knowledge the most general result holding in any dimension. Its proof is however a simple application of Cornalba–Harris and Bost's results, combined with:

(1) a result of Lee relating the Chow stability of a projective variety with the log canonical threshold of its Chow form [22];

(2) the simple but remarkable fact that the proper intersection of stable varieties is stable (Proposition 2.2), that follows from a result of Ferretti [13].

In the second (and longer) part of the paper, we study thoroughly the case of a codimension c complete intersection X in a relative projective bundle $\mathbb{P} := \mathbb{P}_B(\mathcal{E})$ over a curve B, where $\mathcal{E}$ is a rank $r \geq 3$ vector bundle over B, together with the morphism $f\colon X \to B$ induced by π. Here not only we study the f-positivity of $\mathcal{O}_X(1)$, but also the positivity of its powers, and of the relative canonical sheaf $\omega_f = \mathcal{O}_X(K_f)$. It turns out that in many cases there is a numerical inequality governing the f-positivity of these sheaves: inequality (1.3) below. The main results are the following.

Theorem 1.3 (Theorem 3.9, Theorem 3.15, Proposition 3.26). *Let X be a complete intersection of c hypersurfaces*

$$X_i \equiv k_i H - y_i \Sigma, \quad \text{with } k_i \geq 2 \text{ for } i = 1, \ldots, c.$$

Consider the inequality

$$c\mu(\mathcal{E}) \geq \sum_{i=1}^{c} \frac{y_i}{k_i}. \tag{1.3}$$

The following three statements are equivalent:

(A1) *the sheaf $\mathcal{O}_X(h)$ is f-positive for any $h < \min\{k_i\}$*;

(A2) *there exists $h < \min\{k_i\}$ such that $\mathcal{O}_X(h)$ is f-positive*;

(A3) *inequality (1.3) holds.*

Suppose moreover that X is balanced (i.e., $k_i = k$ for any $i = 1, \ldots, c$). The following statements hold:

(B1) *if $\mathcal{O}_X(h)$ is f-positive for $h \gg 0$, then (1.3) holds*;

(B2) *if (1.3) holds with strict inequality, then $\mathcal{O}_X(h)$ is strictly f-positive for $h \gg 0$.*

Suppose that X is balanced and that $r < ck$ (i.e., the fibres of f are of general type). Then we have the following:

(C1) *if $(c-1)k < r$, then ω_f is f-positive if and only if (1.3) holds*;

(C2) *if $k \gg 0$ and $c = 2, 3, 4$, then ω_f is f-positive if and only if (1.3) holds*;

(C3) *if c is fixed and $r \gg 0$, then ω_f is f-positive if and only if (1.3) holds.*

Remark 1.4. This is a wide generalization of Theorem 2.8 of [6], where we proved that condition (1.3) is equivalent to the f-positivity of ω_f and of *any* power of $\mathcal{O}_X(1)$ (see Remark 3.17).

Remark 1.5. In the case $c = r - 2$ with X balanced, Enokizono made in [12] an explicit computation of the invariants K_f^2 and $\deg f_*\omega_f$, proving an equality of the form $K_f^2 = \lambda(r,k) \deg f_*\omega_f$. This equality in particular implies our last results (C1) in case $c = r-2$, as we discuss in detail in Remark 3.28.

Remark 1.6. Theorem 1.1 is an extremely general result of f-positivity, its assumption being on the general fibres. However, notice that in the case of a global complete intersection in $\mathbb{P}$, it does not even imply items (A1) and (A2) of Theorem 1.3. Indeed, first of all we do not ask any condition on the singularities of the fibres in Theorem 1.3; moreover, in (A1) and (A2) we obtain a sufficient and *necessary* condition for f-positivity. This, combined with Theorem 1.1, can be used to obtain the following strong theorem in the case of relative complete intersections

Theorem 1.7 (Theorem 3.18). *Let X be a complete intersection of c hypersurfaces*

$$X_i \equiv k_i H - y_i \Sigma, \quad \text{with } k_i \geq 2 \ \text{ for } i = 1, \dots, c.$$

Suppose that for the general fibre $\Sigma \cong \mathbb{P}^{r-1}$ we have that

$$\operatorname{lct}(\Sigma, X_i \cdot \Sigma) \geq \frac{r}{k_i} \quad \text{for any } i = 1, \dots, c. \tag{1.4}$$

Then the following statements hold:

(1) *the sheaf $\mathcal{O}_X(h)$ is f-positive for any $h < \min_i\{k_i\}$;*

(2) *inequality (1.3) holds.*

In the third and last part of the paper, following the spirit of our work on hypersurfaces [6], we use the results obtained to study the cones in the Néron–Severi space of cycles $\mathrm{N}^c(\mathbb{P})$. Indeed, equation (1.3) tells us something about the position of the class of X inside $\mathrm{N}^c(\mathbb{P})$; we define a cone $\mathbb{B}$ in the 2-dimensional space $\mathrm{N}^c(\mathbb{P})$ as follows (Definition 4.3):

$$\mathbb{B} := \mathbb{R}^+[H^{c-1}\Sigma] \oplus \mathbb{R}^+[H^c - c\mu(\mathcal{E})H^{c-1}\Sigma],$$

where $\mu(\mathcal{E}) = \deg \mathcal{E}/\mathrm{rank}\mathcal{E}$ is the slope of the vector bundle $\mathcal{E}$.

By reformulating in a suitable way a result of Fulger [14] (using the so-called virtual slopes of the vector bundle $\mathcal{E}$), we see that this cone $\mathbb{B}$ is always intermediate between the pseudoeffective and the nef cones in $\mathrm{N}^c(\mathbb{P})$. Then we can reformulate the results obtained, for instance by saying that the class of X lies in the interior of $\mathbb{B}$ if and only if $\mathcal{O}_X(1)$ is f-positive (Proposition 4.4).

Eventually, using the same reasoning as in [6], we combine the result of Cornalba–Harris and Bost with Theorem 1.3, and obtain an instability condition for the fibres of relative complete intersections.

Theorem 1.8 (Corollary 4.5). *Let $X \subset \mathbb{P}$ be a relative complete intersection in the projective bundle $\mathbb{P} = \mathbb{P}_B(\mathcal{E})$ satisfying Assumptions 3.1. If $\sum_{i=1}^c y_i/k_i > c\mu$ (equivalently, if $[X] \notin \mathbb{B}$), then*:

(i) *the fibres of f are Chow unstable with the restriction of $\mathcal{O}_{\mathbb{P}^{r-1}}(h)$ for any $h < \min\{k_i\}$.*

(ii) *Assume moreover that X is balanced. Then the fibres of f are Chow unstable with the restriction of $\mathcal{O}_{\mathbb{P}^{r-1}}(h)$ for $h \gg 0$.*

(iii) *Assume moreover that X is balanced, $r < kc$, and* (1), (2) *or* (3) *in Proposition* 3.26 *holds. Then the fibres of f are unstable with respect to their dualizing sheaf.*

This leads to an example of unstable complete intersections of general type with only one point as singularity (Proposition 4.7). Note that such varieties need to be singular (Remark 4.8), so these examples have the smaller possible singularity (set-theoretically).

2. f-positivity of families of complete intersections

We work over the complex field $\mathbb{C}$. In this section, we derive some results on the f-positivity and the Chow stability of fibres of a fibration whose general fibres are complete intersections. For the definition of Chow stability of a projective variety, see [13] and the references therein. From now on, anytime we say (semi)stable we mean Chow (semi)stable. The main result relating this conditions is due to Bost, see Theorem 3.3 in [7], and to Cornalba–Harris [9, 29] (see [5] for references to similar results).

Theorem 2.1 (Bost, Cornalba–Harris). *Let X be an n-dimensional variety with a surjective morphism $f: X \to B$ over a smooth curve B. Let L be a divisor over X such that*

(i) *for a general fibre F, $L_{|F}$ is very ample*;

(ii) *L is relatively nef with respect to f.*

If the general fibre of f is Chow semistable with respect to the immersion induced by $L_{|F}$, then L is f-positive.

We now ask ourselves: what can we say about the stability of fibres which are complete intersections? We now state a very natural stability result, which derives from a formula of R. G. Ferretti (Theorem 1.5 in [13]). This application of Ferretti's result was suggested to the second author by Yongnam Lee.

Let Y and Z be two irreducible subvarieties of $\mathbb{P}^n = \mathbb{P}(V^\vee)$ whose intersection is proper. Let $Y \cdot Z$ be the intersection cycle of Y and Z.

Proposition 2.2. *If Y and Z are semistable, then $Y \cdot Z$ is semistable. If, moreover, at least one among Y and Z is stable, then the intersection $Y \cdot Z$ is stable.*

Proof. We use the Hilbert–Mumford criterion for stability. Let us consider a 1-parameter subgroup of $\mathrm{GL}(V)$ and let F be the associated weighted filtration of V, with weights r_i. Then, for any subvariety $X \subset \mathbb{P}^n$, there is a well defined integer $e_F(X)$, which in the notation of [13] is called *degree of contact*. The Hilbert–Mumford criterion says that an irreducible subvariety $T \subset \mathbb{P}(V^\vee)$ is semistable (respectively, stable) if and only if for any weighted filtration F of V, the following inequality holds:

$$\frac{e_F(T)}{(\dim T + 1)(\deg T)} \leq \frac{1}{n+1} \sum_{i=0}^{n} r_i \qquad \text{(respectively, } <\text{).} \tag{2.1}$$

Choosing Y and Z properly intersecting in $\mathbb{P}(V^\vee)$, Ferretti proves in Theorem 1.5 of [13] the following "Bézout type" formula for the degree of contact of the cycle intersection $Y \cdot Z$:

$$e_F(Y \cdot Z) = \deg(Y)\, e_F(Z) + \deg(Z)\, e_F(Y) - \deg(Y)\deg(Z) \sum_{i=0}^{n} r_i . \tag{2.2}$$

Let us now suppose that Y and Z are semistable. From the Hilbert–Mumford criterion, we have that

$$\frac{e_F(Y)}{(\dim Y + 1)(\deg Y)} \leq \frac{1}{n+1}\sum_{i=0}^{n} r_i \quad \text{and} \quad \frac{e_F(Z)}{(\dim Z + 1)(\deg Z)} \leq \frac{1}{n+1}\sum_{i=0}^{n} r_i .$$

Call y and z the dimensions of Y and Z, respectively. By the properness assumption, we have that

$$\dim(Y \cdot Z) = y + z - n .$$

We thus have the following chain of inequalities:

$$\begin{aligned}
&\frac{e_F(Y \cdot Z)}{(\dim(Y \cdot Z) + 1)(\deg(Y \cdot Z))} \\
&\quad = \frac{e_F(Y)}{(y+z-n+1)\deg(Y)} + \frac{e_F(Z)}{(y+z-n+1)\deg(Z)} - \frac{\sum_{i=0}^{n} r_i}{y+z-n+1} \\
&\quad \leq \frac{(z+1)+(y+1)}{(y+z-n+1)(n+1)} \sum_{i=0}^{n} r_i - \frac{\sum_{i=0}^{n} r_i}{y+z-n+1} \\
&\quad = \frac{1}{n+1}\sum_{i=0}^{n} r_i ,
\end{aligned}$$

as wanted. The first equality is Ferretti's formula (2.2), while the inequality derives from the condition of semistability of Y and Z. The result with strict stability follows by substituting strict inequality for (at least) one of the varieties. ■

Let us now recall the definition of log canonical threshold (see [21] for reference). Let (Y, Δ) be a pair, with Y a normal $\mathbb{Q}$-Gorenstein variety and Δ a $\mathbb{Q}$-Cartier, $\mathbb{Q}$-divisor on Y. Given any birational morphism $\varphi\colon T \to Y$ with T normal, we have

$$K_T + \varphi_*^{-1}\Delta \equiv \varphi^*(K_Y + \Delta) + \sum a(E_i, Y, \Delta) E_i ,$$

where $\varphi_*^{-1}\Delta$ is the strict transform of Δ and the E_i's are the exceptional irreducible divisors associated to φ. Then we define the *discrepancy* $\operatorname{discrep}(Y, \Delta)$ of the pair (Y, Δ) to be the infimum of the $a(E, Y, \Delta)$, taken for any birational morphism φ and any exceptional irreducible divisor. The pair (Y, Δ) is said to be *log canonical* (l.c.) if $\operatorname{discrep}(Y, \Delta) \geq -1$. The log canonical threshold of (Y, Δ) is

$$\operatorname{lct}(Y, \Delta) := \sup\{t > 0 \mid (Y, t\Delta) \text{ is log canonical}\} \in \mathbb{Q} \cap (0, 1].$$

The log canonical threshold is a measure of the singularities of the pair (Y, Δ): for instance, if Y is smooth and Δ is reduced and normal crossing, then $\mathrm{lct}(Y, \Delta) = 1$.

In [22], Lee proved the following beautiful condition for a variety to be Chow semistable in terms of the log canonical threshold of its Chow form.

Theorem 2.3 (Lee). *Let Y be an s-dimensional variety together with a non-degenerate degree d immersion in $\mathbb{P}^n$. Let $Z(Y) \subset G$ be the corresponding Chow variety in the Grassmanian $G := \mathrm{Gr}(n-s-1, \mathbb{P}^n)$. Suppose that the following inequality holds*:

$$\mathrm{lct}(G, Z(Y)) \geq \frac{n+1}{d} \quad (\textit{respectively, } >). \tag{2.3}$$

Then $Y \subset \mathbb{P}^n$ is Chow semistable (*respectively, Chow stable*).

We are now ready to state the main theorem of this section.

Theorem 2.4. *Let X be an n-dimensional variety with a surjective morphism $f\colon X \to B$ over a smooth curve B. Let L be a line bundle over X which is relatively nef with respect to f. Suppose that for the general fibre F the line bundle $L_{|F}$ is very ample, and call $r := h^0(F, L_{|F})$. Suppose moreover that one of the following conditions holds*:

(1) *F is embedded in $\mathbb{P}^{r-1}$ by $|L_{|F}|$ as the complete intersection of $r-n$ hypersurfaces Y_i of degree d_i such that, for any $i = 1, \dots, r-n$,*

$$\mathrm{lct}(\mathbb{P}^{r-1}, Y_i) \geq \frac{r}{d_i};$$

(2) *F is embedded in $\mathbb{P}^{r-1}$ by $|L_{|F}|$ as a degree d variety such that, with notations as in Theorem 2.3,*

$$\mathrm{lct}(G, Z(Y)) \geq \frac{n+1}{d}.$$

Then L is f-positive.

Proof. If assumption (1) holds, then by Lee's result above, the embeddings $Y_i \hookrightarrow \mathbb{P}^{r-1}$ are Chow semistable for any $i = 1, \dots, r-n$, hence by Proposition 2.2 the complete intersection of the Y_i's is Chow semistable. We thus can apply Theorem 2.1 and conclude the proof. For the case of assumption (2), Theorem 2.3 directly proves Chow semistability of the intersection, and the rest follows as above. ∎

3. Relative complete intersections

3.1. Set-up and preliminaries

In the rest of the paper, we make the following assumptions. Let $\mathcal{E}$ be a vector bundle of rank $r \geq 3$ and degree $d = \deg(\det \mathcal{E})$ on a smooth projective curve B of genus b.

Let $\mathbb{P} := \mathbb{P}_B(\mathcal{E})$ be the relative projective bundle of duals, following Grothiendieck's notation. Let $\pi\colon \mathbb{P}_B(\mathcal{E}) \to B$ be the natural projection. Call Σ a general fibre of π (which is a $\mathbb{P}^{r-1}$).

Let $\mathcal{O}_{\mathbb{P}}(1)$ be the tautological sheaf on $\mathbb{P}$, and let H be an associated divisor, so that $\mathcal{O}_{\mathbb{P}}(1) \cong \mathcal{O}_{\mathbb{P}}(H)$. Let c be an integer between 1 and $r-2$. Let $X_1, \dots, X_c \subset \mathbb{P}$ be relative hypersurfaces of degree $k_i \geq 2$. As $\mathrm{Pic}(\mathbb{P}) = \mathbb{Z}[\mathcal{O}_{\mathbb{P}}(1)] \oplus \pi^*\mathrm{Pic}(B)$, each X_i is an effective divisor in a linear system of the form $|k_i H - \pi^* M_i|$, where M_i is a divisor on B, say of degree $y_i \in \mathbb{Z}$.

Let X be the scheme theoretic intersection of the X_i's, and let $f\colon X \to B$ be the induced fibration.

Assumption 3.1. *We assume that the intersection X is irreducible and proper, i.e., of dimension $r - c$.*

Definition 3.2. We shall call X *balanced* in case that the k_i's are all equal, $k_i = k$ for any $i \in \{1, \dots, c\}$.

Now we would like to use a celebrated result of Miyaoka–Nakayama [26] about the positive cones of divisors of $\mathbb{P}_B(\mathcal{E})$.

Before stating the result, let us recall the notion of the Harder–Narasimhan sequence of a vector bundle $\mathcal{E}$ (see [17]). It is the unique filtration of subbundles

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_l = \mathcal{E}$$

satisfying the following assumptions:

- for any $i = 0, \dots, l$, the sheaf $\mathcal{E}_i/\mathcal{E}_{i-1}$ is μ-semistable;
- if we set $\mu_i := \mu(\mathcal{E}_i/\mathcal{E}_{i-1})$, we have that $\mu_i > \mu_{i-1}$.

Note that $\mu_1 > \mu(\mathcal{E}) > \mu_l$, unless $\mathcal{E}$ is μ-semistable, in which case $1 = l$ and $\mu_1 = \mu(\mathcal{E})$. Note moreover that, setting $r_0 := 0$, we can express the degree of $\mathcal{E}$ as a combinations of the μ_i's and r_i's:

$$\deg(\mathcal{E}) = \sum_{i=1}^{l} \mu_i (r_i - r_{i-1}). \tag{3.1}$$

Theorem 3.3 (Miyaoka–Nakayama). *Given $k \in \mathbb{Z}^{>0}$, and M a divisor on B,*

(1) *the divisor $kH - \pi^* M$ is pseudoeffective* (*i.e., it is a limit of effective divisors*) *if and only if* $(\deg M)/k \leq \mu_1(\mathcal{E})$, *where $\mu_1(\mathcal{E})$ is the first slope of the Harder–Narasimhan sequence of $\mathcal{E}$.*

(2) *The divisor $kH - \pi^* M$ is nef if and only if* $(\deg M)/k \leq \mu_\ell(\mathcal{E})$, *where $\mu_\ell(\mathcal{E})$ is the last slope of the Harder–Narasimhan sequence of $\mathcal{E}$.*

Remark 3.4. Recall that a vector bundle $\mathcal{E}$ is called nef if the corresponding tautological sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is nef. From Theorem 3.3, we see that $\mathcal{E}$ is nef if and only if the smallest slope μ_l is greater or equal to 0. In this case, setting $\mu_{l+1} := 0$, we can reformulate equation (3.1) as follows:

$$\deg(\mathcal{E}) = \sum_{i=1}^{l} r_i (\mu_i - \mu_{i+1}).$$

Remark 3.5. Theorem 3.3 tells us that under our Assumptions 3.1, for any $i = 1, \dots, c$, being $X_i \in |k_i H - \pi^* M_i|$ effective, it necessarily holds the inequality $y_i/k_i \leq \mu_1$.

Notation 3.6. For a multi-index $I = \{i_1, i_2, \dots, i_l\} \subseteq \{1, 2, \dots, c\}$ of length $|I| = l$, we shall indicate by k_I and y_I the corresponding sums

$$k_I = \sum_{k=1}^{l} k_{i_k} \quad \text{and} \quad y_I = \sum_{k=1}^{l} y_{i_k}.$$

For $I = \emptyset$, we set $k_I = y_I = 0$.

Moreover, we call $J = \{1, \dots, c\}$ the full set of indices, so that $k_J = \sum_{i=1}^{c} k_i$ and $y_J = \sum_{i=1}^{c} y_i$.

Let H_X be the class of $\mathcal{O}_X(1)$, let $F = \Sigma \cap X$ be the class of a general fibre of f, and let H_F be the class of $\mathcal{O}_F(1)$. Consider the sheaf $\mathcal{O}_X(1) = j^*\mathcal{O}_{\mathbb{P}}(1)$ on X, where j is the natural inclusion $j\colon X \hookrightarrow \mathbb{P}$. We want to study the f-positivity of $\mathcal{O}_X(1)$ and of its powers.

We start by writing down the basic inequality that embodies the f-positivity of $\mathcal{O}_X(h)$:

$$(hH_X)^{r-c}\, h^0(F, hH_{|F}) - (r-c)(hH_{|F})^{r-c-1} \deg f_*\mathcal{O}_X(h) \geq 0. \tag{3.2}$$

We will use the following well-known result, whose proof we give for the reader's convenience.

Proposition 3.7. *Let $g\colon Y \to X$ be a morphism between varieties. Let*

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \cdots \longrightarrow \mathcal{F}_k \longrightarrow 0 \tag{3.3}$$

be an exact sequence of $\mathcal{O}_Y$-modules, with $k \geq 3$.

If $R^i g_* \mathcal{F}_j = 0$ for any $i > 0$ and j with $i + j \leq k - 1$, then there is an exact pushforward sequence:

$$0 \longrightarrow g_*\mathcal{F}_1 \longrightarrow g_*\mathcal{F}_2 \longrightarrow \cdots \longrightarrow g_*\mathcal{F}_k \longrightarrow 0.$$

Proof. For $i = 1, \dots, k-1$, let $\alpha_i\colon \mathcal{F}_i \longrightarrow \mathcal{F}_{i+1}$ be the maps in the exact sequence, and for $i = 1, \dots, k-2$ consider the short exact sequences

$$0 \longrightarrow \mathrm{im}\alpha_i \longrightarrow \mathcal{F}_{i+1} \longrightarrow \mathrm{im}\alpha_{i+1} \longrightarrow 0,$$

where $\mathrm{im}\alpha_1 = \mathcal{F}_1$ and $\mathrm{im}\alpha_{k-1} = \mathcal{F}_k$.

Observe that it is enough to prove that $R^1 g_* \mathrm{im}\alpha_i = 0$ for $i = 1, \dots, k-2$, which is a consequence of the following Claim.

Claim. *For any $i = 1, \dots, k-2$ and for any $j = 1, \dots, k-1-i$, we have $R^j g_* \mathrm{im}\alpha_i = 0$.*

Proof of Claim. We proceed by induction on i. For $i = 1$, we have $\mathrm{im}\alpha_1 = \mathcal{F}_1$, and so the statement holds by hypothesis. Assume now that the claim holds for $i < k-2$ and consider the short exact sequence

$$0 \longrightarrow \mathrm{im}\alpha_i \longrightarrow \mathcal{F}_{i+1} \longrightarrow \mathrm{im}\alpha_{i+1} \longrightarrow 0.$$

By induction hypothesis, we have that $R^j g_* \mathrm{im}\alpha_i = 0$ for $j = 1, \dots, k-1-i$, and by the hypothesis in the proposition, we have $R^j g_* \mathcal{F}_{i+1} = 0$ for $j = 1, \dots, k-2-i$. So considering the derived long exact sequence, we obtain $R^j g_* \mathrm{im}\alpha_{i+1} = 0$ for $j = 1, \dots, k-2-i = k-1-(i+1)$. ∎

Let us now compute the invariants associated to the f-positivity of $\mathcal{O}_X(h)$, following Notation 3.6.

Proposition 3.8. *With the above assumptions, let h be any integer ≥ 1. The following formulas hold*:

(1) $(H_X)^{r-c} = \big(\big(\prod_{i=1}^c k_i\big)d - \sum_{i=1}^c \big(\prod_{j\neq i} k_j\big)y_i\big)$;

(2) $(H_F)^{r-c-1} = \prod_{i=1}^c k_i$;

(3) $\operatorname{rank} f_*\mathcal{O}_X(h) = h^0(F, \mathcal{O}_F(h)) = \sum_{i=0}^c \big(\sum_{|I|=i} (-1)^i \binom{h-k_I+r-1}{r-1}\big)$;

(4) $\deg f_*\mathcal{O}_X(h) = \sum_{i=0}^c \big(\sum_{|I|=i} (-1)^i \binom{h-k_I+r-1}{r-1} \frac{(h-k_I)d+y_I r}{r}\big)$.

Note that here we use the standard convention that considers equal to zero a binomial of the form $\binom{n}{m}$ when $n < m$.

Proof. The first two formulas are easily computed by standard intersection theory. Indeed,

$$\begin{aligned} H_X^{r-c} &= H^{r-c}X = H(k_1H - y_1\Sigma)\cdots(k_cH - y_c\Sigma) \\ &= \prod_{i=1}^c k_i H^r - \sum_{i=1}^c \Big(\prod_{j\neq i} k_j\Big) y_i H^{r-1}\Sigma, \end{aligned}$$

and

$$\begin{aligned} H_F^{r-c-1} &= H^{r-c-1}F = H^{r-c-1}X\Sigma \\ &= H^{r-c-1}(k_1H - y_1\Sigma)\cdots(k_cH - y_c\Sigma)\Sigma = \prod_{i=1}^c k_i H^{r-1}\Sigma, \end{aligned}$$

and now we just need to recall that $H^r = \deg \mathcal{E} = d$ and $H^{r-1}\Sigma = 1$.

For the last two formulas, let us consider the Koszul sequence that provides the resolution of $\mathcal{O}_X(h)$ (cf. for instance pp. 144–145 in Chapter III of [11]):

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathbb{P}}(-X_1 - \cdots - X_c + hH) \longrightarrow \cdots \longrightarrow \textstyle\bigoplus_{|I|=l} \mathcal{O}_{\mathbb{P}}(-X_I + hH) \longrightarrow \cdots \\ \cdots \longrightarrow \textstyle\bigoplus_{i=1}^c \mathcal{O}_{\mathbb{P}}(-X_i + hH) \longrightarrow \mathcal{O}_{\mathbb{P}}(h) \longrightarrow \mathcal{O}_X(h) \longrightarrow 0, \end{aligned}$$

where we use the notation X_I to indicate the divisor $\sum_{j\in I} X_j$. For $1 \leq i \leq r-2$, we have the vanishing of the higher direct image sheaves $R^i\pi_*\mathcal{O}(-X_I + hH) = 0$, so we are in the conditions of applying Proposition 3.7, and obtain by pushforward via π the long exact sequence

$$\begin{aligned} 0 \longrightarrow \pi_*\mathcal{O}_{\mathbb{P}}(-X_1 - \cdots - X_c + hH) \longrightarrow \cdots \longrightarrow \textstyle\bigoplus_{|I|=l} \pi_*\mathcal{O}_{\mathbb{P}}(-X_I + hH) \longrightarrow \cdots \\ \cdots \longrightarrow \textstyle\bigoplus_{i=1}^c \pi_*\mathcal{O}_{\mathbb{P}}(-X_i + hH) \longrightarrow \pi_*\mathcal{O}_{\mathbb{P}}(h) \longrightarrow f_*\mathcal{O}_X(h) \longrightarrow 0. \end{aligned}$$

It is then straightforward to compute $\operatorname{rank} f_*\mathcal{O}_X(h)$ obtaining (3).

In order to prove the last formula, observe that for any integer $a \geq 0$ and for any M divisor on B of degree y, the projection formula (Exercise 5.1 (d) in Chapter II of [18]) says that

$$\pi_* \mathcal{O}_{\mathbb{P}}(aH - \pi^* M) \cong \pi_* \mathcal{O}_{\mathbb{P}}(a) \otimes \mathcal{O}_B(-M) \cong \operatorname{Sym}^a \mathcal{E} \otimes \mathcal{O}_B(-M),$$

and that

$$\deg \pi_* \mathcal{O}_{\mathbb{P}}(aH - \pi^* M) = \deg \operatorname{Sym}^a \mathcal{E} \otimes \mathcal{O}_B(-M) = \binom{a+r-1}{r-1} \frac{ad - yr}{r}.$$

Notice moreover that for $a < 0$ we have $\pi_* \mathcal{O}_{\mathbb{P}}(a) = 0$, and so the formula above still holds. An easy computation now leads to formula (4) for $\deg f_* \mathcal{O}_X(h)$. ∎

3.2. f-positivity of $\mathcal{O}_X(h)$ for small enough h

We first prove a result about the case of small h.

Theorem 3.9. *Let $f\colon X \to B$ be a morphism as in Assumption 3.1. The following statements are equivalent:*

(1) $\sum_{i=1}^c y_i / k_i \leq c\mu(\mathcal{E})$;

(2) *the line bundle $\mathcal{O}_X(h)$ is f-positive for any $h < \min_i\{k_i\}$;*

(3) *there exists an $h < \min_i\{k_i\}$ such that $\mathcal{O}_X(h)$ is f-positive.*

Proof. Combining the formulas of Proposition 3.8, we see that the f-positivity of $\mathcal{O}_X(h)$ is equivalent to the following inequality:

$$h\Big(\prod_{i=1}^c k_i d - \sum_{i=1}^c \Big(\prod_{j \neq i} k_j\Big) y_i\Big) \sum_{i=0}^c \Big(\sum_{|I|=i} (-1)^i \binom{h - k_I + r - 1}{r-1}\Big)$$
$$\geq (r-c) \prod_{i=1}^c k_i \Big[\sum_{i=0}^c \Big(\sum_{|I|=i} (-1)^i \binom{h - k_I + r - 1}{r-1} \frac{(h - k_I)d + y_I r}{r}\Big)\Big].$$

Grouping terms, this inequality becomes

$$\begin{aligned} &\frac{h}{r}\Big[c\prod_{i=1}^c k_i d - \sum_{i=1}^c \Big(\prod_{j \neq i} k_j\Big) y_i r\Big] \sum_{i=0}^c \Big(\sum_{|I|=i} (-1)^i \binom{h - k_I + r - 1}{r-1}\Big) \\ &\quad + (r-c) \prod_{i=1}^c k_i \Big[\sum_{i=0}^c \Big(\sum_{|I|=i} (-1)^i \binom{h - k_I + r - 1}{r-1} \frac{k_I d - y_I r}{r}\Big)\Big] \geq 0. \end{aligned} \tag{3.4}$$

Now observe that for $h < \min_i\{k_i\}$ (hence in particular for $h = 1$), the second term in inequality (3.4) vanishes when $i > 0$ because all binomials are zero, and trivially vanishes when $i = 0$. Hence, in this case the inequality just becomes

$$\frac{h}{r}\binom{h+r-1}{r-1}\Big(c\prod_{i=1}^c k_i d - \sum_{i=1}^c \Big(\prod_{j \neq i} k_j\Big) y_i r\Big) \geq 0. \tag{3.5}$$

From these observations, we see that the condition for f-positivity for $h < \min_i\{k_i\}$ is precisely

$$c\prod_{i=1}^{c} k_i d - \sum_{i=1}^{c}\Big(\prod_{j\neq i} k_j\Big) y_i r \geq 0 \iff \sum_{i=1}^{c}\frac{y_i}{k_i} \leq c\,\frac{d}{r} = c\mu, \tag{3.6}$$

thus proving the equivalence between (1) and (2). Observe now that if condition (3) holds, then inequality (3.5) holds, so it holds (3.6), so (1) and (2) both hold. Note that the same statement holds for *strict* f-positivity with strict inequality in (3.4). ∎

Remark 3.10. This result in particular implies that strict f-positivity for relative complete intersections is stable under intersection. More precisely, consider two relative complete intersections X and X' (of codimension c and c' respectively, and with associated string of integers k_i, y_i and k'_i, y'_i respectively, and call f and f' the morphisms induced by π) in $\mathbb{P}$. Suppose that X and X' intersect properly, call $X'' := X \cdot X'$ and call $f''\colon X'' \to B$ the induced fibration. Suppose that the tautological bundles $\mathcal{O}_X(1)$ (respectively, $\mathcal{O}_{X'}(1)$) are strictly f-positive (respectively, f'-positive); then by the above theorem we have

$$\sum_{i=1}^{c}\frac{y_i}{k_i} + \sum_{i=1}^{c}\frac{y'_i}{k'_i} < c\mu(\mathcal{E}) + c'\mu(\mathcal{E}) = (c+c')\mu(\mathcal{E}),$$

so $\mathcal{O}_{X''}(1)$ is strictly f''-positive. This property should be compared with the analogue result holding for Chow stability (Proposition 2.2).

3.3. f-positivity of $\mathcal{O}_X(h)$ for big enough h

Remark 3.11. Let us consider the following number associated to our data:

$$\alpha := c\prod_{i=1}^{c} k_i d - \sum_{i=1}^{c}\Big(\prod_{j\neq i} k_j\Big) y_i r.$$

In the pervious section, we have seen that f-positivity of low powers of $\mathcal{O}_X(h)$ is related to the positivity of α.

Note moreover that if $y_i/k_i \leq \mu(\mathcal{E})$ (respectively, $<$) for any $i = 1,\dots,c$, then $\alpha \geq 0$ (respectively, $>$).

Let us start by recalling the asymptotic formula of Hirzebruch–Riemann–Roch, see Chapters 15 and 18 of [15].

Proposition 3.12 (Hirzebruch–Riemann–Roch). *Let L be an ample line bundle over a smooth n-dimensional variety Z. We have for $h \gg 0$,*

$$h^0(Z,\mathcal{O}_Z(hL)) = \frac{h^n}{n!}L^n - \frac{h^{n-1}}{2(n-1)!}L^{n-1}K_Z + \mathcal{O}(h^{n-2}). \tag{3.7}$$

The same formula holds true if Z is a (not necessarily smooth) local complete intersection in a projective space.

Theorem 3.13. *Let $f\colon X \to B$ be a morphism as in Assumption 3.1. The following statements hold*:

(1) *Assume that $y_i/k_i \le \mu(\mathcal{E})$ for any $i = 1, \dots, c$, and that for some $j \in \{1, \dots, c\}$ strict inequality holds; then the line bundle $\mathcal{O}_X(h)$ is strictly f-positive for $h \gg 0$.*

(2) *Assume that $y_i/k_i > \mu(\mathcal{E})$ for any $i = 1, \dots, c$; then for $h \gg 0$ the line bundle $\mathcal{O}_X(h)$ is not f-positive.*

Proof. Let us start from the equation of f-positivity (3.2):

$$h^{r-c} H_X^{r-c} \operatorname{rank} f_*\mathcal{O}_X(h) - (r-c) h^{r-c-1} H_F^{r-c-1} \deg f_*\mathcal{O}_X(h) \ge 0. \tag{3.8}$$

By using Hirzebruch–Riemann–Roch, and the computations in Proposition 3.8, we have that the left side term is for $h \gg 0$ a polynomial in h of the following form:

$$\begin{aligned} &h^{r-c} H_X^{r-c} \operatorname{rank} f_*\mathcal{O}_X(h) - (r-c) h^{r-c-1} H_F^{r-c-1} \deg f_*\mathcal{O}_X(h) \\ &\quad = h^{r-c-1} \big(h H_X^{r-c} h^0(F, \mathcal{O}_F(h)) - (r-c) H_F^{r-c-1} \deg f_*\mathcal{O}_X(h)\big) \\ &\quad = h^{r-c-1} \Big(h^{r-c} H_X^{r-c} \frac{H_F^{r-c-1}}{(r-c-1)!} - (r-c) h^{r-c-1} H_F^{r-c-1} \frac{H_X^{r-c}}{(r-c)!}\Big) + \mathcal{O}(h^{2r-2c-2}) \\ &\quad = h^{2r-2c-1} \Big(\frac{H_X^{r-c} H_F^{r-c-1}}{(r-c-1)!} - (r-c) \frac{H_X^{r-c} H_F^{r-c-1}}{(r-c)!}\Big) + \mathcal{O}(h^{2r-2c-2}). \end{aligned}$$

We see that the degree is at most $2r-2c-2$. We will compute the coefficient of $h^{2r-2c-2}$.

Using the third and fourth formulas of Proposition 3.8, grouping terms as in the proof of Theorem 3.9 above, we see that for any $h \in \mathbb{N}$, we have

$$\deg f_*\mathcal{O}_X(h) = \frac{dh}{r} \operatorname{rank} f_*\mathcal{O}_X(h) - \Big[\sum_{i=0}^{c} \Big(\sum_{|I|=i} (-1)^i \binom{h - k_I + r - 1}{r-1} \frac{k_I d - y_I r}{r}\Big)\Big].$$

We want to see the right-hand term as a polynomial in h and compute its leading term.

Lemma 3.14. *We have*

$$\begin{aligned} \beta(h) &:= \Big[\sum_{i=0}^{c} \Big(\sum_{|I|=i} (-1)^i \binom{h - k_I + r - 1}{r-1} \frac{k_I d - y_I r}{r}\Big)\Big] \\ &= -\frac{\alpha}{r} \frac{h^{r-c}}{(r-c)!} + \frac{h^{r-c-1}}{2(r-c-1)!} \Big(\frac{\alpha}{r}(k_J - r) + \prod_{i=1}^{c} k_i \sum_{i=1}^{c} \frac{k_i d - y_i r}{r}\Big) + \mathcal{O}(h^{r-c-2}). \end{aligned}$$

Proof. Let us compute the coefficient of $(k_1 d - y_1 r)/r$ in $\beta(h)$: it is immediate to see that this is

$$\begin{aligned} &\sum_{i=1}^{c} \Big(\sum_{1 \in I, |I|=i} (-1)^i \binom{h - k_I + r - 1}{r-1}\Big) \\ &\quad = -\sum_{j=0}^{c-1} \Big(\sum_{1 \notin J, |J|=j} (-1)^j \binom{h - k_1 - k_J + r - 1}{r-1}\Big), \end{aligned}$$

and this, by the very same computation as in Proposition 3.8, is precisely

$$-\operatorname{rank} f_{1*}\mathcal{O}_{Y_1}(h-k_1),$$

where Y_1 is the $(c-1)$-codimensional intersection of all the X_i's except for X_1, and $f_1\colon Y_1 \to B$ is the induced fibration. Let us call H_1 the tautological divisor on Y_1 and F_1 the class of a general fibre of f_1. Now, by the Hirzebruch–Riemann–Roch formula (3.7), this rank has the following asymptotic expansion for $h \gg 0$:

$$\begin{aligned}\operatorname{rank} f_{1*}\mathcal{O}_{Y_1}(h-k_1) &= h^0(F_1,(h-k_1)H_{F_1})\\ &= (h-k_1)^{r-c}\frac{(H_{|F_1})^{r-c}}{(r-c)!} - (h-k_1)^{r-c-1}\frac{(H_{F_1})^{r-c-1}K_{F_1}}{2(r-c-1)!} + \mathcal{O}(h^{r-c-2})\\ &= h^{r-c}\frac{(\prod_{i\neq 1}k_i)}{(r-c)!} - h^{r-c-1}\Big(\frac{k_1(r-c)H_{F_1}^{r-c}}{(r-c)!} + \frac{H_{F_1}^{r-c-1}K_{F_1}}{2(r-c-1)!}\Big) + \mathcal{O}(h^{r-c-2})\\ &= h^{r-c}\frac{(\prod_{i\neq 1}k_i)}{(r-c)!} - h^{r-c-1}\Big(\frac{2k_J + \prod_{i\neq 1}k_i(\sum_{i\neq 1}k_i - r)}{2(r-c-1)!}\Big) + \mathcal{O}(h^{r-c-2}),\end{aligned}$$

where we used that

- $H_{F_1}^{r-c} = \prod_{j\neq 1}k_j$;
- $H_{F_1}^{r-c-1}K_{F_1} = \prod_{j\neq 1}k_j(\sum_{j\neq 1}k_j - r)$,

because $K_{F_1} \equiv (\sum_{j\neq 1}k_j - r)H_F$. So, for $h \gg 0$ we obtain from the above computations a term of the form

$$-\Big(\prod_{i\neq 1}k_i\Big)\frac{k_1 d - y_1 r}{r} = -\frac{(\prod_{i=1}^{c}k_i)d - (\prod_{j\neq 1}k_j)y_1 r}{r}.$$

Now, recalling that

$$\beta(h) = \sum_{i=1}^{c}\Big[-\Big(\frac{k_i d - y_i r}{r}\Big)\operatorname{rank} f_{1*}\mathcal{O}_{Y_i}(h-k_i)\Big],$$

we obtain, summing up for any $i = 1,\dots,c$, as a degree $r-c$ term for $\beta(h)$,

$$-\frac{h^{r-c}}{(r-c)!}\frac{\alpha}{r}.$$

Let us compute now the term in degree $r-c-1$. We have

$$\begin{aligned}&\frac{h^{r-c-1}}{2(r-c-1)!}\Big[\sum_{i=1}^{c}\frac{k_i d - y_i r}{r}\Big(2\prod_{j=1}^{c}k_j + \prod_{j=1,j\neq i}^{c}k_j\Big(\sum_{j=1,j\neq i}^{c}k_j - r\Big)\Big)\Big]\\ &\qquad= \frac{h^{r-c-1}}{2(r-c-1)!}\Big[\sum_{i=1}^{c}\frac{k_i d - y_i r}{r}\prod_{j=1,j\neq i}^{c}k_j\Big(2k_i + \sum_{j=1,j\neq i}^{c}k_j - r\Big)\Big]\\ &\qquad= \frac{h^{r-c-1}}{2(r-c-1)!}\Big[(k_J - r)\frac{\alpha}{r} + \Big(\prod_{j=1}^{c}k_j\Big)\sum_{i=1}^{c}\frac{k_i d - y_i r}{r}\Big],\end{aligned}$$

as wanted. ■

Let us now resume the proof of Theorem 3.13. Consider again the f-positivity of $\mathcal{O}_X(h)$:

$$h^{r-c}\, H_X^{r-c}\, \operatorname{rank} f_*\mathcal{O}_X(h) - (r-c)\, h^{r-c-1}\, H_F^{r-c-1} \deg f_*\mathcal{O}_X(h) \geq 0.$$

Dividing by h^{r-c-1} and using the computations of Proposition 3.8, this is equivalent to

$$h\Big(H_X^{r-c} - \frac{d(r-c)}{r}\, {H_F}^{r-c-1}\Big) \operatorname{rank} f_*\mathcal{O}_X(h) + (r-c)\Big(\prod_{i=1}^{c} k_i\Big)\beta(h) \geq 0. \tag{3.9}$$

Now observe that, by Proposition 3.8 again,

$$\begin{aligned} &H_X^{r-c} - d\,\frac{r-c}{r}\, {H_F}^{r-c-1} \\ &\quad = \frac{r\big(\prod_{i=1}^{c} k_i\big)d - r\left(\sum_{i=1}^{c}\left(\prod_{j\neq i} k_j\right) y_i\right) - (r-c)\,d\,\left(\prod_{i=1}^{c} k_i\right)}{r} = \frac{\alpha}{r}. \end{aligned}$$

Let us recall now the formula (3.7) of Hirzebruch–Riemann–Roch for $\operatorname{rank} f_*\mathcal{O}_X(h) = h^0(F, \mathcal{O}_F(h))$ for $h \gg 0$:

$$\begin{aligned} \operatorname{rank} f_*\mathcal{O}_X(h) &= h^0(F, \mathcal{O}_F(h)) \\ &= \frac{h^{r-c-1}}{(r-c-1)!}\, H_F^{r-c-1} - \frac{h^{r-c-2}}{2(r-c-2)!}\, H_F^{r-c-1}\, K_F + \mathcal{O}(h^{r-c-3}) \\ &= \frac{h^{r-c-1}}{(r-c-1)!}\Big(\prod_{i=1}^{c} k_i\Big) - \frac{h^{r-c-2}}{2(r-c-2)!}\Big(\prod_{i=1}^{c} k_i\Big)(k_J - r) + \mathcal{O}(h^{r-c-3}), \end{aligned}$$

because $K_F \equiv (k_J - r)H_F$. Using the above expression and Lemma 3.14, we have that the term in degree $r-c$ in (3.9) is

$$\frac{h^{r-c}}{(r-c-1)!}\left(\prod_{i=1}^{c} k_i\right)\frac{\alpha}{r} + (r-c)\,\frac{h^{r-c}}{(r-c)!}\left(\prod_{i=1}^{c} k_i\right)\frac{(-\alpha)}{r} = 0,$$

according to the observation made at the beginning of the proof.

Now, we compute the degree $r-c-1$ term:

$$\begin{aligned} &\frac{h^{r-c-1}}{2(r-c-1)!}\Big(\prod_{i=1}^{c} k_i\Big)\frac{\alpha}{r}\,(k_J - r)\,(-(r-c-1) + (r-c)) \\ &\qquad + \Big(\prod_{i=1}^{c} k_i\Big)(r-c)\prod_{j=1}^{c} k_j \sum_{i=1}^{c} \frac{k_i d - y_i r}{r} \\ &\quad = \frac{h^{r-c-1}}{2(r-c-1)!}\left(\prod_{i=1}^{c} k_i\right)\Big((k_J - r)\,\frac{\alpha}{r} + (r-c)\,\frac{\gamma}{r}\Big), \end{aligned} \tag{3.10}$$

where we set

$$\gamma = \prod_{j=1}^{c} k_j \sum_{i=1}^{c} \frac{k_i d - y_i r}{r}.$$

So we see that, if we assume that $k_J - r > 0$ (i.e., F of general type), this coefficient is a strictly positive combination of α and γ. It is now immediate to check (see also Remark 3.11) that in the assumption of (1) both α and γ are strictly positive. However, we now will see that the assumption $k_J - r > 0$ is not needed. Let us rearrange the terms in (3.10) as follows:

$$\begin{aligned}\Big((k_J - r)\frac{\alpha}{r} + (r-c)\frac{\gamma}{r}\Big) \\ &= \Big(\prod_{j=1}^{c} k_j\Big)\Big((k_J - r)\sum_{i=1}^{c}\frac{k_i d - y_i r}{k_i r} + (r-c)\sum_{i=1}^{c}\frac{k_i d - y_i r}{r}\Big) \\ &= \Big(\prod_{j=1}^{c} k_j\Big)\sum_{i=1}^{c}\frac{k_i d - y_i r}{k_i r}\,(k_J - r + (k_i(r-c)) \\ &= \Big(\prod_{j=1}^{c} k_j\Big)\sum_{i=1}^{c}\frac{k_i d - y_i r}{k_i r}\,(k_J - c + (k_i - 1)(r-c))\,.\end{aligned}$$

So it is clear that if the assumptions of (1) hold, then the coefficient of h^{r-c-1} is strictly positive, and so $\mathcal{O}(h)$ is f-positive for $h \gg 0$.

If on the contrary all the terms $k_i d - y_i r$ are strictly smaller than 0, for $i = 1, \dots, c$, then so is the coefficient of h^{r-c-1} in (3.10), and proposition (2) is thus proved. ∎

We now see that in the balanced case, we can trace back the asymptotic f-positivity of $\mathcal{O}_C(h)$ to the positivity of α.

Theorem 3.15. *Let $f: X \to B$ be a morphism as in Assumption 3.1. Assume moreover that f is balanced. The following implications hold.*

- *If $y_J/k = \sum_{i=1}^{c} y_i/k < c\mu(\mathcal{E})$, then the line bundle $\mathcal{O}_X(h)$ is strictly f-positive for $h \gg 0$.*
- *Conversely, if $\mathcal{O}_X(h)$ is f-positive for $h \gg 0$, then $y_J/k = \sum_{i=1}^{c} y_i/k \le c\mu(\mathcal{E})$.*

Proof. The proof of this result is straightforward by computing the coefficient of h^{r-c-1} in (3.10) in the balanced case: we obtain

$$k^c \sum_{i=1}^{c}\frac{kd - y_i r}{kr}\,(ck - r + k(r-c)) = (k-1)\alpha.$$

So both implications are clear. ∎

Remark 3.16. Clearly in case of equality $\alpha = 0$, in order to check f-positivity, one should investigate the non-negativity of the coefficient in degree $(2r - 2c - 3)$ in the polynomial (3.2).

Remark 3.17. In case $c = 1$, both terms in inequality (3.4) are multiple of the number $(dk_1 - ry_1)$, and one can see – as we do in [6] – that the multiplying term is positive, so that in Theorem 3.13 the condition of $\mathcal{O}_X(h)$ being f-positive for $h \gg 0$ is equivalent to $\mathcal{O}_X(h)$ being f-positive for *any* fixed value $h \ge 1$.

Let us now put together the results of Section 2 with the ones in this section and obtain the following.

Theorem 3.18. *Let $f\colon X \to B$ be a morphism as in Assumptions 3.1. Suppose that the relative hypersurfaces X_i's are such that for the general fibre $\Sigma \cong \mathbb{P}^{r-1}$ of $\pi\colon \mathbb{P}(\mathcal{E}) \to B$, we have that*

$$\mathrm{lct}(\Sigma, X_i \cdot \Sigma) \geq \frac{r}{k_i}\cdot$$

Then

(1) *the sheaf $\mathcal{O}_X(h)$ is f-positive for any $h < \min_i\{k_i\}$;*

(2) $\sum_{i=1}^{c} y_i/k_i \leq c\mu(\mathcal{E})$.

Proof. The assumption on the log canonical threshold of the pair $(\Sigma, X_i \cdot \Sigma)$ implies by Theorem 2.4, that $\mathcal{O}_X(1)$ is f-positive. We can apply Theorem 3.9 to deduce that (1) and (2) hold. ■

3.4. The slope inequality

We shall now address the problem of the slope inequality (1.2), i.e., of the f-positivity of the relative canonical sheaf ω_f. Let us first establish some preparatory results.

Firstly, we calculate the numerical class of K_f in our setting. Recall [24] that the relative canonical bundle of $\pi\colon \mathbb{P} \to B$ is

$$K_\pi = -rH + \pi^* \det(\mathcal{E}) \equiv -rH + d\Sigma.$$

Hence, by the adjunction theorem, we have that

$$\begin{aligned} K_f &= \Big(K_\pi + \sum X_i\Big)_{|X} \equiv \Big(\sum k_i - r\Big) H_X - \Big(\sum y_i - d\Big) F \\ &= (k_J - r)H_X - (y_J - d)F, \end{aligned} \tag{3.11}$$

following Notation 3.6 setting $J = \{1, \ldots, c\}$, the whole set of indexes.

Now we recall that f-positivity is stable under sum of pullbacks of divisors on the base curve, see Remark 1 in Section 2 of [5]. It is worth recalling here the proof.

Lemma 3.19. *A divisor D on X is f-positive if and only if the divisor $D + f^*M$ is f-positive for any divisor M on B.*

Proof. The f-positivity of D is

$$h^0(F, D_{|F})D^n - n(D_{|F})^{n-1} \deg f_*\mathcal{O}_X(D) \geq 0.$$

The f-positivity of $D + f^*M$ is

$$\begin{aligned} &h^0(F, (D+f^*M)_{|F})(D + f^*M)^n \\ &\qquad - n((D+f^*M)_{|F})^{n-1} \deg f_*\mathcal{O}_X(D+f^*M) \geq 0. \end{aligned} \tag{3.12}$$

Let us analyze this last inequality. We have:

- $(D + f^*M)_{|F} \cong D_{|F}$;
- $(D + f^*M)^n = D^n + nD_{|F}^{n-1}(\deg M)$;
- $\deg f_*\mathcal{O}_X(D + f^*M) = \deg f_*\mathcal{O}_X(D) + (\deg M)\,\mathrm{rank} f_*\mathcal{O}_X(D)$
  $= \deg f_*\mathcal{O}_X(D) + (\deg M) h^0(F, D_{|F})$.

Hence, we have that (3.12) is indeed

$$\begin{aligned}0 &\leq h^0(F, D_{|F})(D^n + nD_{|F}^{n-1}(\deg M)) - n(D_{|F})^{n-1} \\ &\quad \cdot \left[\deg f_*\mathcal{O}_X(D) + (\deg M) h^0(F, D_{|F})\right] \\ &= h^0(F, D_{|F}) D^n - n(D_{|F})^{n-1} \deg f_*\mathcal{O}_X(D),\end{aligned}$$

so, precisely the f-positivity of D. ■

Corollary 3.20. *The slope inequality* (1.2) *for a morphism $f: X \to B$ that satisfies Assumptions* 3.1 *is equivalent to the f-positivity of $\mathcal{O}_X(k_J - r)$. If the morphism is balanced, then the slope inequality is equivalent to the f-positivity of $\mathcal{O}_X(ck - r)$.*

Theorem 3.21. *With the notations above, suppose that X is balanced, and that $k > 1$ and that $r < ck$.*

Then the following are equivalent:

(1) $K_f^{r-c} \geq 0$;

(2) $\mu(\mathcal{E}) \geq \frac{\sum y_i}{\sum k_i} = \frac{\sum y_i}{ck} = \frac{y_J}{ck}$.

Moreover, if condition (1) (*or* (2)) *holds, the slope inequality is equivalent to the following inequality*:

$$(ck - r) h^0(F, K_F) \geq k(r - c) h^0(F_1, K_{F_1}), \tag{3.13}$$

where $f_1: Y_1 \to B$ is the morphism induced by π on the intersection Y_1 of all X_i's except for X_1, and F_1 is a general fibre of f_1.

Proof. As in the proof of Theorem 3.13, let us fix the notation

$$\alpha := c \prod_{i=1}^{c} k_i d - \sum_{i=1}^{c} \Big(\prod_{j \neq i} k_j\Big) y_i r = cdk^c - ry_J k^{c-1}.$$

Firstly we prove that K_f^{r-c} is a strict positive multiple of α. This of course tells us that (1) $\iff$ (2). From (3.11), we get

$$K_f^{r-c} = (k_J - r)^{r-c-1}\big((k_J - r) H_X^{r-c} - (r - c)(y_J - d) H_F^{r-c-1}\big). \tag{3.14}$$

Now using the formulas of Lemma 3.8 and dividing by $(k_J - r)^{r-c-1}$ (which is by assumption strictly greater than 0), we get

$$\frac{K_f^{r-c}}{(k_J - r)^{r-c-1}} = d \prod_{i=1}^{c} k_i (k_J - c) - (k_J - r)\Big(\sum_{i=1}^{c} \prod_{j \neq i} k_j\Big) y_i - (r - c) y_J \prod_{i=1}^{c} k_i.$$

Now, using the assumption that X is balanced, we have

$$\prod_{i=1}^{c} k_i = k^c, \quad \prod_{j \neq i} k_j = k^{c-1} \quad \text{and} \quad k_J = ck,$$

and we obtain

$$\begin{aligned}\frac{K_f^{r-c}}{(ck-r)^{r-c-1}} &= dk^c(ck-c) - (ck-r)(k^{c-1}y_J) - (r-c)k^c y_J \\ &= k^{c-1}(k-1)(cdk - ry_J) = (k-1)\alpha.\end{aligned}$$

Let us now turn our attention on the f-positivity of ω_f. As noted in Corollary 3.20, we have that this is equivalent to the f-positivity of $\mathcal{O}_X(ck-r)$.

Let us first establish a formula for the f-positivity of $\mathcal{O}_X(h)$ for any $h \geq 1$ in the balanced case.

Lemma 3.22. *Let h be any integer greater or equal to* 1. *Under the assumptions of Theorem* 3.21, *the f-positivity of $\mathcal{O}_X(h)$ is equivalent to*

$$\begin{aligned}\frac{\alpha}{r}\Bigg(h\sum_{i=0}^{c}(-1)^i\binom{h-ik+r-1}{r-1}\binom{c}{i} \\ + k(r-c)\sum_{i=1}^{c}(-1)^i\binom{h-ik+r-1}{r-1}\binom{c-1}{i-1}\Bigg) \geq 0.\end{aligned} \tag{3.15}$$

Proof of Lemma 3.22. Let us start from equation (3.4); the f-positivity of $\mathcal{O}_X(h)$ is equivalent to the non-negativity of

$$\begin{aligned}\alpha\,\frac{h}{r}\Bigg(\sum_{i=0}^{c}(-1)^i\sum_{|I|=i}\binom{h-ik+r-1}{r-1}\Bigg) \\ + (r-c)k^c\Bigg(\sum_{i=0}^{c}(-1)^i\sum_{|I|=i}\binom{h-ik+r-1}{r-1}\frac{ikd-y_I r}{r}\Bigg).\end{aligned} \tag{3.16}$$

Let us now observe that the second summand of (3.16) can be taken starting from $i=1$, because for $i=0$ we have that $(ikd-y_I r)/r = 0$. Observe moreover that the first term can be simplified via the following identity:

$$\sum_{|I|=i}\binom{h-ik+r-1}{r-1} = \binom{c}{i}\binom{h-ik+r-1}{r-1}.$$

We now would like to rearrange the second term in equation (3.16) in a similar manner to what we have done in Lemma 3.14. The key observation is that

$$\sum_{i=1}^{c}(-1)^i\sum_{|I|=i}\binom{h-ik+r-1}{r-1}y_I = \Bigg(\sum_{i=1}^{c}(-1)^i\binom{h-ik+r-1}{r-1}\binom{c-1}{i-1}\Bigg)y_J,$$

where it is worth recalling that $y_J = \sum_{i=1}^{c} y_i$. Using these identities, we have that the second term in (3.16) becomes

$$\begin{aligned}
&\sum_{i=1}^{c}(-1)^i \sum_{|I|=i} \binom{h-ik+r-1}{r-1}\frac{ikd-y_I r}{r} \\
&\qquad = \sum_{i=1}^{c}\frac{(-1)^i}{r}\binom{h-ik+r-1}{r-1}\left(\binom{c}{i}ikd-\binom{c-1}{i-1}y_J r\right) \\
&\qquad = \sum_{i=1}^{c}(-1)^i\binom{h-ik+r-1}{r-1}\binom{c-1}{i-1}\frac{ckd-y_J r}{r} \\
&\qquad = \frac{\alpha}{rk^{c-1}}\sum_{i=1}^{c}(-1)^i\binom{h-ik+r-1}{r-1}\binom{c-1}{i-1},
\end{aligned}$$

and the proof is thus concluded. ■

Let us now complete the proof of Theorem 3.21. Set $h = ck - r$. The expression multiplying α/r in equation (3.15) is

$$\begin{aligned}
&(ck-r)\sum_{i=0}^{c}(-1)^i\binom{ck-r-ik+r-1}{r-1}\binom{c}{i} \\
&\qquad + k(r-c)\sum_{i=1}^{c}(-1)^i\binom{ck-r-ik+r-1}{r-1}\binom{c-1}{i-1}.
\end{aligned}$$

Now observe that

$$\begin{aligned}
&\sum_{i=1}^{c}(-1)^i\binom{ck-r-ik+r-1}{r-1}\binom{c-1}{i-1} \\
&\qquad = -\sum_{j=0}^{c-1}(-1)^j\binom{(c-1)k-r-jk+r-1}{r-1}\binom{c-1}{j}.
\end{aligned}$$

Now, let us call Y_1 be the intersection in $\mathbb{P}$ of $X_2, \dots, X_c$, and let $f_1\colon Y_1 \to B$ be the morphism induced. We see from the very computation made in Proposition 3.8 that the above expression is $-\operatorname{rank}(f_1)_*\mathcal{O}_{Y_1}((c-1)k-r)$. So, the slope inequality is equivalent to the non-negativity of

$$\begin{aligned}
&\frac{\alpha}{r}\left[(ck-r)\operatorname{rank} f_*\mathcal{O}_X(ck-r) - k(r-c)\operatorname{rank}(f_1)_*\mathcal{O}_{Y_1}((c-1)k-r)\right] \\
&\qquad = \frac{\alpha}{r}\left[(ck-r)h^0(F,K_F) - k(r-c)h^0(F_1,K_{F_1})\right].
\end{aligned}$$ ■

Remark 3.23. Note that condition $r < ck$ in Theorem 3.21 implies that the canonical sheaf on the fibres of f is very ample, i.e., that K_f is relatively very ample, so the fibres are of general type.

Remark 3.24. In the case $c = 1$, we have that $F_1 = \mathbb{P}^{r-1}$, and so inequality (3.13) is trivially satisfied, so Theorem 1.2 (1) in [6] is implied by this result.

Remark 3.25. In the non-balanced case, one could still obtain an inequality, but more involved.

In general, it does not seem clear to understand the positivity of expression (3.13). However, we can prove the slope inequality in some cases, as follows.

Proposition 3.26. *Under the assumptions of Theorem 3.21, suppose that one of the following conditions holds*:

(1) $(c-1)k < r$ (*this is equivalent to asking that F_1 is not of general type*).

(2) $c = 2, 3, 4$ *and* $k \gg 0$.

(3) c *is fixed and* $r \gg 0$.

Then the slope inequality is equivalent to $\mu(\mathcal{E}) \geq y_J/(ck)$.

Proof. In case (1), $h^0(F_1, K_{F_1})$ is trivially zero, so by (3.13) the slope inequality is equivalent to the non-negativity of α, as wanted.

As for case (2), let us compute (3.13) as a polynomial in k, and see its leading coefficient:

$$
\begin{aligned}
&(ck-r)\sum_{i=0}^{c}(-1)^i\binom{ck-r-ik+r-1}{r-1}\binom{c}{i}\\
&\qquad + k(r-c)\sum_{i=1}^{c}(-1)^i\binom{ck-r-ik+r-1}{r-1}\binom{c-1}{i-1}\\
&\qquad = (ck-r)\binom{ck-r+r-1}{r-1}\\
&\qquad + \sum_{i=1}^{c}(-1)^i\binom{ck-r-ik+r-1}{r-1}\left((ck-r)\binom{c}{i} + k(r-c)\binom{c-1}{i-1}\right)\\
&\qquad = \frac{k^r}{(r-1)!}\left[c^r + \sum_{i=1}^{c}(-1)^i(c-i)^{r-1}\left(\frac{c^2}{i}+r\right)\binom{c-1}{i-1}\right] + \mathcal{O}(k^{r-1}).
\end{aligned}
$$

Now, if we compute the coefficient for small c, we can see that it is non-negative:

- For $c = 2$, we have $2^r - (r+4)$, which is always strictly greater than 0 as soon as $r \geq 2$ (indeed, in our case r need to be greater than 4).
- For $c = 3$, we obtain $3^r - 2^{r-1}(r+9) + 2(r+9/4)$, which is strictly greater than 0 for $r \geq 5$.
- For $c = 4$, we obtain $4^r - 3^{r-1}(r+16) + 2^{r-1}3(r+4) - 3(r+16/9)$, which is greater than 0 for $r \geq 6$

As for the third case, we just need to observe that the leading coefficient as a polynomial in k^r in the above expression is positive if c is fixed and $r \gg 0$. ∎

Remark 3.27. Note that the result (2) for $k \gg 0$ cannot be derived from the f-positivity of $\mathcal{O}_X(h)$ for $h \gg 0$, because changing k does change also X.

Remark 3.28. In Lemma 1.1 of [12], Enokizono proved that, in the balanced case with $c = r - 2$ (so X is a surface), the following equalities hold:

$$K_f^2 = ((k-1)r - 2k)(k-1)\alpha;$$
$$\deg f_*\omega_f = \frac{1}{24}((3r-5)k - 3r + 1))(k-1)\alpha.$$

So, we see that the non-negativity of both these invariants correspond to the non-negativity of α, coherently with our results. It is easy to see that these equalities imply the slope inequality in case $\alpha \geq 0$. In particular, by Enokizono's result, there is always an equality, regardless of the sign of α:

$$K_f^2 = \frac{24((r-2)k - r)}{(3r-5)k - 3r + 1} \deg f_*\omega_f.$$

4. The cones of cycles of $\mathbb{P}_B(\mathcal{E})$ and an instability condition

4.1. The cone of "f-positive complete intersections" in $N^c(\mathbb{P})$

In this section, we interpret some of the results obtained in terms of cones in the real Néron–Severi space of codimension c cycles of X. We see that inequality (1.3) defines a meaningful cone, which is intermediate with respect to the nef and the pseudoeffective ones. For stating this, we rewrite in a more compact form a result of Fulger [14], that describes completely this last two cones.

Recall first that given any vector bundle $\mathcal{E}$ over a curve B, the real Néron–Severi space of codimension c cycles of $\mathbb{P} = \mathbb{P}_B(\mathcal{E})$,

$$N^c(\mathbb{P}) := \frac{\{\text{real span of classes of } c\text{-dimensional subvarieties of } \mathbb{P}\}}{\text{numerical equivalence}},$$

is 2-dimensional, generated by the classes H^c and $H^{c-1}\Sigma$, where $H = [\mathcal{O}_{\mathbb{P}}(1)]$ and Σ is the class of a fibre of π.

Let us consider the Harder–Narasimhan filtration of $\mathcal{E}$,

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_\ell = \mathcal{E},$$

and call

$$\mu_i := \mu(\mathcal{E}_i/\mathcal{E}_{i-1}) \quad \text{and} \quad \mu := \mu(\mathcal{E}).$$

Recall that $r_i := \operatorname{rank} \mathcal{E}_i$, and that $\operatorname{rank} \mathcal{E} = r = r_\ell$. Recall in particular that

$$\mu_\ell < \mu_{\ell-1} < \cdots < \mu_1. \tag{4.1}$$

It is useful to introduce the following notation.

Definition 4.1. With the above notations, we define the *virtual slopes* $\widetilde{\mu}_1 \geq \widetilde{\mu}_2 \geq \cdots \geq \widetilde{\mu}_r$ of $\mathcal{E}$ as follows. Let r_j be the rank of $\mathcal{E}_j$, and let $i \in \{1, \ldots, r-1\}$. If $r_j \leq i < r_{j+1}$, then we define $\widetilde{\mu}_i = \mu_j$. We define coherently $\widetilde{\mu}_r = \mu_\ell$.

Observe that

$$d = \deg \mathcal{E} = r\mu = \sum_{i=1}^{r} \widetilde{\mu}_i .$$

With this notation, we can restate Fulger's result as follows.

Theorem 4.2 (Fulger [14]). *With the notations defined above, we have*

$$\operatorname{Pseff}^c(\mathbb{P}) = \langle H^{c-1}\Sigma\,,\ H^c - (\widetilde{\mu}_1 + \cdots + \widetilde{\mu}_c)\, H^{c-1}\Sigma\rangle; \tag{4.2}$$

$$\operatorname{Nef}^c(\mathbb{P}) = \langle H^{c-1}\Sigma\,,\ H^c - (\widetilde{\mu}_{r-c+1} + \cdots + \widetilde{\mu}_r)\, H^{c-1}\Sigma\rangle. \tag{4.3}$$

Proof. The first formula is just the content of Theorem 1.1 in [14], with the notation suitably adapted. Observe that in [14] the indexes of slopes are reversed with respect to our definition. With this in mind, and following the construction given by Figure 2 in [14] and the notation therein, one immediately obtains that $\nu^{(i)} = -(\widetilde{\mu}_1 + \cdots + \widetilde{\mu}_c)$.

For the second formula, one just needs to use the fact that $\operatorname{Nef}^c(\mathbb{P})$ is the subset of $\operatorname{Pseff}^c(\mathbb{P})$ defined by positivity product with

$$\operatorname{Pseff}_c(\mathbb{P}) = \operatorname{Pseff}^{r-c}(\mathbb{P}) = \langle H^{r-c-1}, H^{r-c} - (\widetilde{\mu}_1 + \cdots + \widetilde{\mu}_{r-c})\Sigma\rangle,$$

and so it is the two dimensional cone determined by $H^{c-1}\Sigma$ and $H^c - aH^{c-1}\Sigma$ such that

$$\begin{aligned} 0 = (H^c - aH^{c-1}\Sigma)(H^{r-c} - (\widetilde{\mu}_1 + \cdots + \widetilde{\mu}_{r-c})\Sigma) &= H^r - (a + \widetilde{\mu}_1 + \cdots + \widetilde{\mu}_{r-c}) \\ = d - (a + \widetilde{\mu}_1 + \cdots + \widetilde{\mu}_{r-c}) = \widetilde{\mu}_{r-c+1} + \cdots + \widetilde{\mu}_r - a. \end{aligned}$$ ∎

We can now reformulate the results of Section 3 using the language of cones.

Definition 4.3. Let $\mathbb{B}$ be the cone in $\operatorname{Nef}^c(\mathbb{P})$ generated by the classes $[H^{c-1}\Sigma]$ and $[H^c - c\mu H^{c-1}\Sigma]$.

Note that for any $c \in \{1, \dots, r-1\}$, we have that

- $\sum_{i=1}^{c} \widetilde{\mu}_i > c\mu$;
- $\sum_{i=1}^{c} \widetilde{\mu}_{r-i+1} < c\mu$.

This means that the cone $\mathbb{B}$ is indeed contained in the pseudoeffective cone and contains the nef cone:

$$\operatorname{Nef}^c(\mathbb{P}) \subseteq \mathbb{B} \subseteq \operatorname{Pseff}^c(\mathbb{P}).$$

Note that by Theorem 3.3, the inclusions are strict unless $\mathcal{E}$ is semistable (in which case the cones all coincide). Theorems 3.9, 3.13 and 3.21 tell us the following.

Proposition 4.4. *In the above notations, let $X \subset \mathbb{P}$ be a codimension c cycle which is a complete intersection of c relative hypersurfaces $X_1, \dots, X_c$ in $\mathbb{P}$ of degree at least 2.*

1. *The numerical class of X is contained in $\mathbb{B}$ if and only if there exists $h < \min\{k_i\}$ such that $\mathcal{O}_X(h)$ is f-positive.*
2. *The numerical class of X is contained in $\mathbb{B}$ if and only if $\mathcal{O}_X(h)$ is f-positive for any $h < \min\{k_i\}$.*

(3) *If X is balanced and the numerical class of X lies in the interior of $\mathbb{B}$, then $\mathcal{O}_X(h)$ is strictly f-positive for $h \gg 0$.*

(4) *If X is balanced and $\mathcal{O}_X(h)$ is f-positive for $h \gg 0$, then the numerical class of X is contained in $\mathbb{B}$.*

(5) *If X is balanced, $r < kc$ and (1) or (2) in Proposition 3.26 hold, then the numerical class of X is contained in $\mathbb{B}$ if and only if the slope inequality holds.*

4.2. An instability condition for of the fibres

We can use the results of Section 3 vice versa as in [6], and prove the following *instability* condition for the fibres of a global relative complete intersection. As usual, let $\mathcal{E}$ be a rank $r \geq 3$ vector bundle over a curve B, and let $\pi\colon \mathbb{P} = \mathbb{P}_B(\mathcal{E}) \to B$ be the projective bundle.

Corollary 4.5. *Let $X \subset \mathbb{P}$ be a relative complete intersection in the projective bundle $\mathbb{P} = \mathbb{P}_B(\mathcal{E})$ satisfying Assumptions 3.1. If $\sum_{i=1}^{c} y_i/k_i > c\mu$ (equivalently, if $[X] \notin \mathbb{B}$), then*

(i) *the fibres of f are Chow unstable with the restriction of $\mathcal{O}_{\mathbb{P}^{r-1}}(h)$ for any $h < \min\{k_i\}$.*

(ii) *Assume moreover that X is balanced. Then the fibres of f are Chow unstable with the restriction of $\mathcal{O}_{\mathbb{P}^{r-1}}(h)$ for $h \gg 0$.*

(iii) *Assume moreover that X is balanced, $r < kc$ and (1), (2) or (3) in Proposition 3.26 holds. Then the fibres of f are unstable with respect to their dualizing sheaf.*

Proof. Immediate from Theorem 1.3 and Theorem 2.1. ∎

Remark 4.6. In [6], in the codimension one case, we proved a more general instability condition, and this led us to a singularity condition (a bound on the log canonical threshold of the fibres of f via Lee's result). In the general codimension case, of course from Lee's result we could obtain a singularity condition on the Chow form of the fibres; but as for the fibres themselves, it is not so easy to get geometric information from an instability condition.

As an application of these last results, together with the detailed study of the hypersurface case made in [6], we use Corollary 4.5 to construct families of complete intersections whose general fibre is of general type, asymptotically unstable and has only one (very) singular point, as follows.

Proposition 4.7. *Let $a \geq 1$ and $r \geq 3$ be integers. Consider the rank $r \geq 3$ vector bundle over $\mathbb{P}^1$*

$$\mathcal{E} := \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus r-1} \oplus \mathcal{O}_{\mathbb{P}^1}(a-1).$$

Let $0 < c < r-1$ be an integer. Then there exists an $m \in \mathbb{N}$ big enough, such that calling X the complete intersection of c general members of the linear system

$$X_i \in |m((r+1)a-1)H - m(r+1)\Sigma|$$

on $\mathbb{P}_B(\mathcal{E})$ and $f\colon X \to B$ the induced morphism, we have that

(1) *the general fibre F of f is a complete intersection of general type with only one singular point*;
(2) *the pair $(F, hH_{|F})$ is Chow unstable for any $h \in \mathbb{N}^{>0}$.*

Proof. The Harder–Narasimhan sequence of $\mathcal{E}$ is simply

$$\begin{array}{ccccc} \mathcal{E}_0 & \hookrightarrow & \mathcal{E}_1 & \hookrightarrow & \mathcal{E}_2 \\ \| & & \| & & \| \\ 0 & \hookrightarrow & \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus r-1} & \hookrightarrow & \mathcal{E}, \end{array}$$

and

$$\mu_1 = a > \mu(\mathcal{E}) = \frac{(r-1)a + a - 1}{r} = a - \frac{1}{r} > \mu_2 = a - 1.$$

We have that

$$\frac{m[((r+1)a-1)]}{m(r+1)} = \frac{(r+1)a-1}{r+1} = a - \frac{1}{r+1},$$

so, as this ratio is between μ_1 and μ_2,

$$\mu_1 = a > a - \frac{1}{r+1} > a - 1 = \mu_2,$$

we are in the conditions of applying Theorem 1.7 in [6], that states that for $m \gg 0$, the general member of $|m((r+1)a-1)H - m(r+1)\Sigma|$ has $\mathbb{P}(\mathcal{E}/\mathcal{E}_1)$ as base locus, so if we consider c general members of this linear system, they will intersect in a variety X smooth outside the section $\mathbb{P}(\mathcal{E}/\mathcal{E}_1)$. Now we just need to observe that

$$a - \frac{1}{r+1} > \mu(\mathcal{E}),$$

so by Corollary 4.5 we obtain the statement. ∎

Remark 4.8. Note that an asymptotically Chow unstable complete intersection of general type need to be singular. Indeed, let X be such a variety. By [23], Chow-stability for $h \gg 0$ is equivalent to Hilbert-stability for $h \gg 0$. By Corollary 4 in [10], this is implied by the existence of a Kähler–Einstein metric on X (because the automorphism group of X is finite). Hence, if a smooth complete intersection is not Chow-stable for $h \gg 0$, then it cannot carry a Kähler-Einstein metric. When K_X is ample this cannot happen by the Aubin–Yau theorem.

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