



The angular derivative problem for petals of one-parameter semigroups in the unit disk

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Abstract. We study the angular derivative problem for petals of one-parameter semigroups of holomorphic self-maps of the unit disk. For hyperbolic petals, we prove a necessary and sufficient condition for the conformality of the petal in terms of the intrinsic hyperbolic geometry of the petal and the backward dynamics of the semigroup. For parabolic petals, we characterize conformality of the petal in terms of the asymptotic behaviour of the Koenigs function at the Denjoy–Wolff point.

Dedicated to the memory of Professor Larry Zalcman

1. Introduction

One-parameter semigroups of holomorphic self-maps of the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ have been studied for more than a century, but many key properties regarding their boundary and asymptotic behaviour have been established only in the last two decades. Apart from being interesting in their own right, one-parameter semigroups in the unit disk act as pivotal role models for other complex dynamical systems involving holomorphic functions of one or several complex variables – in the continuous as well as the discrete setting. A comprehensive overview of one-parameter semigroups in the unit disk spanning from the basic theory to the numerous recent achievements can be found in the monograph [14].

Traditionally, understanding the forward dynamics of the orbits of a dynamical system has been of particular interest. However, in recent years the study of the *backward* dynamics of one-parameter semigroups, see e.g. [15, 26, 32], as well of discrete complex dynamical systems (in one and several complex variables), see e.g. [2, 5, 12, 35], has become another focal point of research.

One of the most striking results about the forward dynamics of one-parameter semigroups of the unit disk \mathbb{D} is the continuous version of the celebrated Denjoy–Wolff theorem. It guarantees that all forward orbits of the semigroup converge to the same point $\tau \in \mathbb{D} \cup \partial\mathbb{D}$,

the Denjoy–Wolff point of the semigroup, or *DW-point*, for short. In contrast, the backward flow of a one-parameter semigroup is defined only on a proper subset \mathcal{W} of \mathbb{D} , the so-called *backward invariant set*. The connected components of the interior of \mathcal{W} are simply connected domains, and are referred to as the *petals* of the semigroup. Each petal Δ is associated to a specific boundary point $\sigma \in \partial\mathbb{D}$. This boundary point σ plays a similar role for the backward dynamics of the semigroup as the DW-point plays for the forward dynamics: σ is the common limit of all backward orbits starting inside the petal Δ . Bearing in mind the conventional terminology in dynamical systems (see, e.g., p. 225 in Section 17, Chapter IV, of [4]), we call σ the α -point of the petal Δ . We refer to Section 2 for the precise definition of these concepts, and in particular to the pioneering papers [15, 18, 26].

The goal of this paper is to study the angular derivative problem for the petals Δ of one-parameter semigroups at their α -points σ . The corresponding angular derivative problem related to the forward dynamics has recently been studied in [8, 10, 22, 31].

The angular derivative problem for general simply connected domains G is concerned with the existence of a finite non-zero angular derivative of a Riemann map f from the unit disk \mathbb{D} onto G at a given boundary point $\xi \in \partial\mathbb{D}$. It can be shown that the existence of the angular derivative of f does not depend on the choice of the conformal map f but only on the domain G itself. Accordingly, when this angular derivative condition is satisfied at a boundary point ξ , we say that the domain G is conformal at $f(\xi)$. In his thesis [3], Ahlfors raised the question¹ of finding necessary and sufficient geometric conditions on G near the boundary point ξ , so that G is conformal at $f(\xi)$. This is the angular derivative problem. It has attracted much interest, and continues to do so. Seminal contributions have been given by Rodin and Warschawski [40, 41], Jenkins and Oikawa [30], followed by many others, e.g., Burdzy [16], Carroll [17], as well as very recently by Betsakos and Karamanlis [11].

We can now state our main result. It is concerned with *hyperbolic* petals, i.e., petals Δ for which the α -point σ is different from the Denjoy–Wolff point. The simpler case of *parabolic* petals, i.e., petals for which the α -point coincides with the Denjoy–Wolff point, will be discussed in Section 5. We denote the density of the hyperbolic metric of a domain G by λ_G . We adopt the convention from [6, 33], so the hyperbolic metric of a domain G is the unique complete conformal metric on G with constant negative curvature -1 . In particular, the density of the hyperbolic metric of the unit disk \mathbb{D} is

$$\lambda_{\mathbb{D}}(z) = \frac{2}{1 - |z|^2}.$$

Theorem 1.1. *Let Δ be a hyperbolic petal of a non-trivial one-parameter semigroup (ϕ_t) in the unit disk, and let $z_0 \in \Delta$. Then the petal Δ is conformal at its α -point σ if and only if*

$$(1.1) \quad \int_{-\infty}^0 \log \left(\frac{\lambda_{\Delta}(z_0)}{\lambda_{\mathbb{D}}(\phi_t(z_0)) |\phi_t'(z_0)|} \right) dt < +\infty.$$

In this case, the integral (1.1) converges for every $z_0 \in \Delta$; in fact, locally uniformly in Δ .

¹“Welche geometrischen Eigenschaften sollen das Gebiet G kennzeichnen, um dass die so bestimmte Funktion im Punkt ξ eine von Null und Unendlich verschiedene Winkelableitung besitze?”, page 47 in [3].

We note that the second named author and Zarvalis showed recently, see Theorem 1.3 in [32], that as $t \rightarrow -\infty$,

$$(1.2) \quad \lambda_{\mathbb{D}}(\phi_t(z)) |\phi_t'(z)| \nearrow \lambda_{\Delta}(z) \quad \text{locally uniformly in } \Delta.$$

So Theorem 1.1 relates the rate of convergence in (1.2) with the conformality of Δ at σ .

Remark 1.2. It is easy to construct examples of hyperbolic petals Δ such that $\partial\Delta$ coincides in a neighbourhood of its α -point σ with $\partial\mathbb{D}$, see e.g. Examples 7.4 and 7.8 in [15]; clearly, in such a case, Δ is conformal at σ . An example of a *non-conformal* hyperbolic petal based on certain subtle properties of the hyperbolic distance was given in Section 8 of [15]. In Remark 2.22, we will describe a simple device which allows a painless construction of numerous examples of conformal as well as non-conformal hyperbolic petals.

The proof of Theorem 1.1 is long and is therefore divided into several steps. We shall require various tools from the general theory of one-parameter semigroups of the unit disk, in particular the Berkson–Porta theory, holomorphic models and pre-models and basic properties of petals. These tools are collected and explained in a preliminary Section 2. This section also thoroughly introduces the angular derivative problem. In Section 3, we state and prove two technical, but crucial auxiliary results: an integral criterion for conformality of domains which are starlike at infinity, and a lemma on convergence of conformal mappings on the boundary. The proof of Theorem 1.1 is given in Section 4, and is divided into several steps. In Subsection 4.1, we state a conformality criterion for the Koenigs domain Ω of the semigroup (Theorem 4.1), and show how it implies Theorem 1.1. In Subsection 4.2, we prove the if-part of Theorem 4.1 by establishing in Theorem 4.2 a pointwise lower bound, given in Euclidean terms, for the quotient of the hyperbolic densities of a domain Ω which is starlike at infinity and a maximal strip contained in Ω . The proof of this lower bound uses a mixture of tools from geometric function theory such as monotonicity of hyperbolic densities, Green’s function, harmonic measure and kernel convergence. The only-if part of Theorem 4.1 is proved in Subsection 4.3 by comparing the Koenigs domain Ω of (ϕ_t) with carefully chosen slit domains and using potential-theoretic tools. The proof of Theorem 4.1 is finished in Subsection 4.4, where we show that the previously obtained pointwise estimates in fact hold locally uniformly. In Section 5, we state and prove a conformality criterion for the case of a parabolic petal, see Theorem 5.1. In the concluding Section 6, we discuss how the results of this paper are related to several other recent results, in particular the conformality conditions obtained by Betsakos and Karamanlis [11]. In addition, we indicate some potential alternative approaches to the conformality problem for petals, and raise several questions that remain open.

2. Preliminaries

2.1. One-parameter semigroups in the unit disk

Here we briefly recall the main definitions and basic facts concerning one-parameter semigroups of holomorphic functions. For more details and proofs of the statements cited in this section, we refer interested readers to the monographs [14, 25, 42], to Chapter 4 in [1], and to Chapter 4 in [24].

For a domain $D \subset \mathbb{C}$ and a set $E \subset \mathbb{C}$, we denote by $\text{Hol}(D, E)$ the set of all holomorphic functions in D with values in E . As usual, we endow $\text{Hol}(D, E)$ with the topology of locally uniform convergence. Then $\text{Hol}(D, D)$ becomes a topological semigroup with respect to the composition operation $(\phi, \psi) \mapsto \phi \circ \psi$.

Definition 2.1. A *one-parameter semigroup* in the unit disk \mathbb{D} is a continuous semigroup homomorphism $[0, +\infty) \ni t \mapsto \phi_t \in \text{Hol}(\mathbb{D}, \mathbb{D})$ from the semigroup $([0, +\infty), +)$ with the Euclidean topology to the semigroup $\text{Hol}(\mathbb{D}, \mathbb{D})$.

Equivalently, a family $(\phi_t)_{t \geq 0} \subset \text{Hol}(\mathbb{D}, \mathbb{D})$ is a one-parameter semigroup if and only if it satisfies the following three conditions: (i) $\phi_0 = \text{id}_{\mathbb{D}}$; (ii) $\phi_s \circ \phi_t = \phi_{s+t}$ for any $s, t \geq 0$; (iii) $\phi_t \rightarrow \text{id}_{\mathbb{D}}$ in $\text{Hol}(\mathbb{D}, \mathbb{D})$ as $t \rightarrow 0^+$. Thanks to Montel's normality criterion, see e.g. Theorem 1 in Section II.7 of [27], the continuity condition (iii) is equivalent to the pointwise convergence: $\phi_t(z) \rightarrow z$ as $t \rightarrow 0^+$ for each $z \in \mathbb{D}$. At the same time, in the presence of (i) and (ii), condition (iii) is equivalent to a much stronger property: the map $(z, t) \mapsto \phi_t(z)$ is jointly real-analytic in $\mathbb{D} \times [0, +\infty)$. Moreover, every one-parameter semigroup in \mathbb{D} represents the semiflow of a (uniquely defined) holomorphic vector field $G: \mathbb{D} \rightarrow \mathbb{C}$, known as the *infinitesimal generator* of (ϕ_t) . This means that for each fixed $z \in \mathbb{D}$, the function $t \mapsto \phi_t(z)$ is the unique solution to the initial value problem

$$(2.1) \quad \frac{d}{dt} \phi_t(z) = G(\phi_t(z)), \quad t \geq 0; \quad \phi_0(z) = z.$$

Using conditions (i) and (ii), one can easily deduce from (2.1) the following PDE:

$$(2.2) \quad \frac{\partial \phi_t}{\partial t} = G(z) \phi_t'(z), \quad t \geq 0, \quad z \in \mathbb{D}.$$

Remark 2.2. It follows from the standard uniqueness results for solutions of ODEs that every element of a one-parameter semigroup is an injective map.

Remark 2.3. The definition of a one-parameter semigroup can be literally extended to an arbitrary domain $D \subset \mathbb{C}$. However, this yields an interesting class of objects only if D is conformally equivalent to \mathbb{D} or to $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$, with the latter case easily reduced to the former one; see e.g. [14], Section 8.4. Clearly, given that D admits a conformal mapping f onto \mathbb{D} , a family $(\phi_t)_{t \geq 0} \subset \text{Hol}(D, D)$ is a one-parameter semigroup in D if and only if the mappings $f \circ \phi_t \circ f^{-1}$ form a one-parameter semigroup in \mathbb{D} .

Another way to modify the definition of a one-parameter semigroup is to allow negative values of the parameter t . In such a case, we have $\phi_t \circ \phi_{-t} = \text{id}_{\mathbb{D}}$ for any $t \in \mathbb{R}$, and hence we end up with a *one-parameter group of automorphisms* $(\phi_t)_{t \in \mathbb{R}}$.

The classical representation formula due to Berkson and Porta [7] characterizes infinitesimal generators in \mathbb{D} as functions of the form

$$(2.3) \quad G(z) = (\tau - z)(1 - \bar{\tau}z)p(z), \quad z \in \mathbb{D},$$

where p is a holomorphic function in \mathbb{D} with $\text{Re } p \geq 0$ and τ is a point in the closure of \mathbb{D} . The function p is uniquely determined by G . The same concerns τ unless $G \equiv 0$.

In order to exclude from consideration certain degenerate cases, we accept the following assumption.

Assumption. The one-parameter semigroups (ϕ_t) we consider in this paper do not extend to one-parameter groups of automorphisms.

This assumption is equivalent to requiring that there exists no $t > 0$ such that ϕ_t is an automorphism.

The distinguished point τ in the Berkson–Porta representation formula (2.3) has a very clear dynamic meaning: $\phi_t(z) \rightarrow \tau$ locally uniformly in \mathbb{D} as $t \rightarrow +\infty$. If $\tau \in \mathbb{D}$, then

$$(2.4) \quad \phi_t(\tau) = \tau, \quad \phi'_t(\tau) = e^{\lambda t}, \quad \lambda := G'(\tau), \quad \text{for all } t \geq 0,$$

with $\operatorname{Re} \lambda < 0$.

If $\tau \in \partial\mathbb{D}$, then (2.4) holds in the sense of angular limits. Here and in what follows, given a holomorphic function $f: \mathbb{D} \rightarrow \mathbb{C}$ and a point $\zeta \in \partial\mathbb{D}$, by $f(\zeta)$ we denote the angular limit $\angle \lim_{z \rightarrow \zeta} f(z) \in \overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. Similarly, if $f(\zeta)$ does exist finitely, then by $f'(\zeta)$ we denote the angular derivative

$$f'(\zeta) := \angle \lim_{z \rightarrow \zeta} \frac{f(z) - f(\zeta)}{z - \zeta}.$$

Remark 2.4. One special important case, in which the existence of the angular derivative is guaranteed, is when $f \in \operatorname{Hol}(\mathbb{D}, \mathbb{D})$ and $f(\zeta)$ exists and belongs to $\partial\mathbb{D}$, see e.g. Proposition 4.13 on p. 82 of [37]. In this case, ζ is called a *contact point* for the self-map f ; the angular derivative $f'(\zeta)$ at a contact point does not vanish, but it can be infinite.

A *boundary fixed point* of $f \in \operatorname{Hol}(\mathbb{D}, \mathbb{D})$ is a contact point ζ such that $f(\zeta) = \zeta$. A boundary fixed point (or more generally, a contact point) ζ is said to be *regular* if $f'(\zeta) \neq \infty$. Boundary fixed points which are not regular are also called *super-repulsive* (or *super-repelling*) fixed points of f . The angular derivative $f'(\zeta)$ at a boundary regular fixed point ζ is a positive real number, and further, two subcases are distinguished: the boundary fixed point ζ is *repulsive* (or *repelling*) if $f'(\zeta) > 1$, while for $f'(\zeta) \in (0, 1]$, it is called *attracting*.

Remark 2.5. It is worth mentioning that for elements of one-parameter semigroups, the angular limit $\phi_t(\zeta)$ exists at *every* point $\zeta \in \partial\mathbb{D}$, see [20, 28]. Moreover, the orbit $t \mapsto \phi_t(\zeta)$ is continuous for each $\zeta \in \partial\mathbb{D}$. At the same time, the extensions of the holomorphic maps $\phi_t(\cdot)$ to $\partial\mathbb{D}$ by angular limits are not necessarily continuous on $\partial\mathbb{D}$.

According to the Denjoy–Wolff theorem, a self-map $f \in \operatorname{Hol}(\mathbb{D}, \mathbb{D}) \setminus \{\operatorname{id}_{\mathbb{D}}\}$ either has an attracting fixed point $\tau \in \partial\mathbb{D}$ and no fixed points in \mathbb{D} , or it has a fixed point $\tau \in \mathbb{D}$ and no attracting fixed points on $\partial\mathbb{D}$. In both cases, τ is unique and it is called the *Denjoy–Wolff point* (or *DW-point* for short) of the self-map f .

From (2.4), it is clear that τ in the Berkson–Porta representation formula (2.3) is the DW-point for each ϕ_t with $t > 0$. If $\tau \in \mathbb{D}$, then (ϕ_t) is said to be *elliptic*. If $\tau \in \partial\mathbb{D}$, then $\lambda \leq 0$, and depending on whether $\lambda < 0$ or $\lambda = 0$, the one-parameter semigroup (ϕ_t) is said to be *hyperbolic* or *parabolic*, respectively. By the continuous version of the Denjoy–Wolff theorem, $\phi_t(z) \rightarrow \tau$ locally uniformly in \mathbb{D} as $t \rightarrow +\infty$.

Similarly to the DW-point, repulsive (and super-repulsive) fixed points are common for all elements of a one-parameter semigroup. More precisely, $\sigma \in \partial\mathbb{D}$ is a repulsive (or

super-repulsive) fixed point of ϕ_t for some $t > 0$ if and only if it is a repulsive (respectively, super-repulsive) fixed point of ϕ_t for all $t > 0$; see e.g. [20].

Fixed points of a one-parameter semigroup can be characterized in terms of the infinitesimal generator. It is known [19] that $\sigma \in \partial\mathbb{D}$ is a boundary regular fixed point of (ϕ_t) if and only if $G(\sigma) = 0$ and $\lambda := G'(\sigma)$ exists finitely; see also Section 12.2 in [14]. In such a case, $\lambda \in \mathbb{R}$ and $\phi'_t(\sigma) = e^{\lambda t}$ for all $t \geq 0$. Clearly, if $\lambda > 0$, then σ is a repulsive fixed point; otherwise, i.e., if $\lambda \leq 0$, then σ is the DW-point of (ϕ_t) .

The following remark contains a useful construction indicating that every elliptic one-parameter semigroup, which is not a group, is correlated with a unique non-elliptic one parameter-semigroup.

Remark 2.6. Suppose that (ϕ_t) is an elliptic semigroup with the DW-point $\tau \in \mathbb{D}$. Then (ϕ_t) can be regarded as a one-parameter semigroup in $\mathbb{D} \setminus \{\tau\}$. Consider the covering map $T \circ C: \mathbb{D} \rightarrow \mathbb{D} \setminus \{\tau\}$, where $C(z) := \exp(-\frac{1+z}{1-z})$ and $T(w) := (w + \tau)/(1 + \bar{\tau}w)$. It is known, see Section 2 in [28], that there is a (unique) one-parameter semigroup (φ_t) which is a lifting of (ϕ_t) with respect to $C \circ T$, i.e., such that $\phi_t \circ T \circ C = T \circ C \circ \varphi_t$ for all $t \geq 0$. Further details and application of the above construction follow in the proof of Theorem 1.1.

2.2. Holomorphic models and Koenigs function

It is known, see e.g. Section 9.2 of [14] or [13], that any one-parameter semigroup admits a *holomorphic model* (Ω_0, h, L_t) . This means that $\Omega_0 \subset \mathbb{C}$ is a simply connected domain, referred to as the *base space*, $h: \mathbb{D} \rightarrow \Omega_0$ is an injective holomorphic map, and (L_t) is a one-parameter group of holomorphic automorphisms of Ω_0 with the following two properties:

$$(2.5) \quad h \circ \phi_t = L_t \circ h, \quad \text{for all } t \geq 0;$$

$$(2.6) \quad \bigcup_{t \leq 0} L_t(\Omega) = \Omega_0, \quad \text{where } \Omega := h(\mathbb{D}).$$

Up to a naturally defined isomorphism, a holomorphic model for a given one-parameter semigroup is unique.

The theory which shall be presented in the current and the upcoming sections can also be generalized (with appropriate modifications) to the case of an elliptic one-parameter semigroup, which is not an elliptic group. Taking Remark 2.6 into consideration, from this point onward, we can safely consider only non-elliptic one-parameter semigroups (i.e., hyperbolic or parabolic).

Non-elliptic one-parameter semigroups admit holomorphic models for which $L_t(z) = z + t$, $t \in \mathbb{R}$, and Ω_0 is the whole \mathbb{C} , or a half-plane, or a strip with $\partial\Omega_0$ composed of one or two lines parallel to \mathbb{R} . For such holomorphic models, equation (2.5) becomes Abel's functional equation

$$(2.7) \quad h(\phi_t(z)) = h(z) + t \quad \text{for all } z \in \mathbb{D} \text{ and all } t \geq 0.$$

The function h is called the *Koenigs function* of (ϕ_t) , and it is unique up to an additive constant. The set $\Omega := h(\mathbb{D}) \subset \Omega_0$ is called the *Koenigs* (or sometimes, *planar*) *domain*

of (ϕ_t) . Abel's equation implies an important property of Ω : for every of its points w , the Koenigs domain contains the ray $\{w + t : t \geq 0\}$. Such domains are said to be *starlike at infinity*.

Many dynamical properties of (ϕ_t) are encoded in the geometry of the corresponding Koenigs domain Ω . Moreover, any starlike-at-infinity domain Ω different from the whole plane is the Koenigs domain of a non-elliptic one-parameter semigroup. This is often used to construct examples of one-parameter semigroups with given behaviour, see e.g. Remark 2.22 below.

Definition 2.7. We denote by \mathbb{S} the “standard” horizontal strip $\{z : |\operatorname{Im} z| < \pi/2\}$, and more generally we denote $\mathbb{S}(a, b) := \{z : a < \operatorname{Im} z < b\}$ for $a, b \in \mathbb{R}$ with $a < b$. Let Ω be the Koenigs domain of a non-elliptic one-parameter semigroup (ϕ_t) . A strip $\mathbb{S}(a, b)$ contained in Ω is said to be a *maximal strip* for (ϕ_t) if $\mathbb{S}(a, b) \subset \mathbb{S}(a', b') \subset \Omega$ holds only for $(a', b') = (a, b)$.

It is easy to see that the maximal strips defined above are connected components of the interior of $\bigcap_{t \geq 0} \Omega + t$.

Remark 2.8. It is known [18] that there exists a one-to-one correspondence between the repulsive fixed points $\sigma \in \partial\mathbb{D}$ of (ϕ_t) and the maximal strips in the Koenigs domain of (ϕ_t) . If S is a maximal strip for (ϕ_t) and $w \in S$, then $h^{-1}(w + t)$ tends, as $t \rightarrow -\infty$, to the corresponding repulsive fixed point σ . Moreover, the width $\nu(S)$ of the maximal strip S is related to the angular derivative at σ : namely, $\nu(S) G'(\sigma) = \pi$.

2.3. Backward orbits, invariant petals, and pre-models

In this section, we follow the terminology from [15]. For the proofs of statements quoted below, we refer the reader to the same source. Let us denote by $d_{\mathbb{D}}$ the hyperbolic distance in a hyperbolic domain D .

Definition 2.9 (Definition 3.1 in [15]). A continuous curve $\gamma : [0, +\infty)$ is called a *backward orbit* of a one-parameter semigroup (ϕ_t) if for any $t > 0$ and any $s \in (0, t)$, we have $\phi_s(\gamma(t)) = \gamma(t - s)$. A backward orbit γ is said to be *regular* if

$$\limsup_{t \rightarrow +\infty} d_{\mathbb{D}}(\gamma(t), \gamma(t + 1)) < +\infty.$$

Remark 2.10. Let (ϕ_t) be a non-elliptic one-parameter semigroup in \mathbb{D} . Fix $z \in \mathbb{D}$. It is easy to see that the following three conditions are equivalent:

- (i) there exists a backward orbit γ with $\gamma(0) = z$;
- (ii) $z \in \mathcal{W} := \bigcap_{t \geq 0} \phi_t(\mathbb{D})$;
- (iii) the line $\{h(z) + t : t \in \mathbb{R}\}$, where h is the Koenigs function of (ϕ_t) , is contained in the Koenigs domain Ω of (ϕ_t) .

If the above conditions are satisfied, then the backward orbit γ in (i) is unique and it is given by $\gamma(t) := \phi_t^{-1}(z) = h^{-1}(h(z) - t)$ for all $t \geq 0$. Moreover, this backward orbit γ is regular if and only if $z \in \mathcal{W}^\circ$.

Remark 2.11. The *negative iterates* $\phi_{-t} := \phi_t^{-1}$, $t > 0$, are well-defined and holomorphic in \mathcal{W}° . Thus, for $z \in \mathcal{W}^\circ$, the differential equations (2.1) and (2.2) are valid for all $t < 0$.

Definition 2.12. The set \mathcal{W} in Remark 2.10 is called the *backward invariant set* of (ϕ_t) . Each non-empty connected component of \mathcal{W}° is called a *petal*.

Every petal Δ is a simply connected domain and $(\phi_t|_\Delta)_{t \in \mathbb{R}}$ is a one-parameter group of automorphisms of Δ . The boundary of Δ contains the DW-point τ of (ϕ_t) . All regular backward orbits that lie in a petal Δ converge to the same boundary fixed point of (ϕ_t) , which lies on the boundary of the petal. We call this unique limit point *the α -point of the petal* Δ . The following dichotomy holds:

- (P) either the α -point of the petal Δ coincides with τ and it is the only fixed point of (ϕ_t) contained in $\partial\Delta$;
- (H) or $\partial\Delta$ contains exactly two fixed points of the semigroup: the DW-point τ and a repulsive fixed point σ , which is the α -point of the petal Δ .

The case (P) arises only if the one-parameter semigroup is parabolic. In this case, the image $h(\Delta)$ of the petal Δ with respect to the Koenigs function h is a half-plane bounded by a line parallel to \mathbb{R} ; it is maximal in the sense that there exist no half-plane $H \neq h(\Delta)$ such that $h(\Delta) \subset H \subset \Omega$.

In case (H), the petal Δ is said to be *hyperbolic* and $h(\Delta)$ coincides with the maximal strip corresponding to the repulsive fixed point σ in the sense of Remark 2.8. Moreover, there is a one-to-one correspondence² between the repulsive fixed points and the hyperbolic petals, as the pre-image $h^{-1}(S)$ of any maximal strip S is a hyperbolic petal. In what follows, the hyperbolic petal corresponding to a given repulsive fixed point σ will be denoted by $\Delta(\sigma)$.

The Koenigs function can be regarded as a global change of variables reducing the dynamics of (ϕ_t) to the canonical form $w \mapsto w + t$. When studying dynamics of the one-parameter semigroup in a petal Δ , instead of the holomorphic model, it is more convenient to work with the so-called pre-model. This notion has been introduced for discrete iteration in [34]. The definition below is a slight modification of that from Definition 3.8 in [15] combined with Remark 3.9 in [15]. We denote by \mathbb{H} the right half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

Definition 2.13. Let $\sigma \in \partial\mathbb{D}$ be a repulsive fixed point of a one-parameter semigroup (ϕ_t) with associated infinitesimal generator G . The triple (\mathbb{H}, ψ, Q_t) is called a *pre-model* for (ϕ_t) at σ if the following conditions are met:

- (i) for each $t \geq 0$, Q_t is the automorphism of \mathbb{H} given by $Q_t(z) := e^{\lambda t} z$, where $\lambda := G'(\sigma)$;
- (ii) the map $\psi: \mathbb{H} \rightarrow \mathbb{D}$ is holomorphic and injective, $\angle \lim_{w \rightarrow 0} \psi(w) = \sigma$, and ψ is isogonal at 0, i.e.,

$$(2.8) \quad \angle \lim_{w \rightarrow 0} \operatorname{Arg} \frac{1 - \bar{\sigma} \psi(w)}{w} = 0;$$

- (iii) $\psi \circ Q_t = \phi_t \circ \psi$ for all $t \geq 0$.

²In the case of non-elliptic one-parameter semigroups, this one-to-one correspondence was discovered by Contreras and Díaz-Madrígal [18]. Similar results for elliptic and hyperbolic one-parameter semigroups were independently established by Elin, Shoikhet and Zalcman in [26]. A proof covering all the cases and more details can be found in Section 4 of [15].

Remark 2.14. It is known (Theorem 3.10 in [15]) that every one-parameter semigroup, at each repulsive fixed point σ , admits a pre-model unique up to the transformation $\psi(w) \mapsto \psi(cw)$, where c is an arbitrary positive constant. Moreover, $\psi(\mathbb{H})$ is the hyperbolic petal $\Delta(\sigma)$ with α -point σ . The map ψ can be expressed via the Koenigs function h of (ϕ_t) . Namely, if the strip $h(\Delta(\sigma)) = \mathbb{S}(a, b)$, then the map ψ in the pre-model for (ϕ_t) at σ is given by

$$\psi(w) := h^{-1}\left(\frac{b-a}{2\pi} \log w + \frac{b+a}{2} i + s\right), \quad w \in \mathbb{H},$$

where s is an arbitrary real constant.

Remark 2.15. One important consequence of the facts mentioned in Remark 2.14 is that every backward orbit γ starting from a point z in a hyperbolic petal $\Delta(\sigma)$ converges to σ non-tangentially and with a definite slope, i.e., there exists the limit

$$\theta(z) := \lim_{t \rightarrow +\infty} \text{Arg}(1 - \bar{\sigma}\gamma(t)),$$

with $\theta(z) \in (-\pi/2, \pi/2)$.

2.4. Conformality of a domain at a boundary point

The geometry of Koenigs domains of non-elliptic one-parameter semigroups is strongly affiliated to the notion of conformality in the “strip normalization” studied in detail, e.g., in [41].

Definition 2.16. Let $S := \mathbb{S}(a, b)$ be a maximal strip contained in a domain Ω , and let g be a conformal mapping of Ω onto $\mathbb{S}(a, b)$ such that

$$(2.9) \quad \text{Re } g(t + iy_0) \rightarrow -\infty, \quad \text{as } t \rightarrow -\infty,$$

for some and hence all $y_0 \in (a, b)$. The domain Ω is said to *have an angular derivative* or to be *conformal at $-\infty$ with respect to S* if for any $\varepsilon \in (0, b - a)$ there exists the finite real limit

$$(2.10) \quad \lim_{D(\varepsilon) \ni z \rightarrow -\infty} g(z) - z, \quad \text{where } D(\varepsilon) := \{z \in S : \text{Re } z \leq 0, \text{dist}(z, \partial S) \geq \varepsilon/2\}.$$

Remark 2.17. Clearly, the map g above is not uniquely defined. However, it is easy to see that if condition (2.10) holds for one conformal map g of Ω onto $\mathbb{S}(a, b)$ satisfying (2.9), then (2.10) holds for all such mappings g .

In a different geometric setting, it is natural to consider a similar and closely related notion of conformality with respect to the unit disk \mathbb{D} . In this case, we restrict ourselves to subdomains of \mathbb{D} .

Definition 2.18. A simply connected domain $U \subset \mathbb{D}$ is said to be *conformal at a point* $\sigma \in \partial U \cap \partial \mathbb{D}$ with respect to \mathbb{D} if there exists a conformal mapping φ of \mathbb{D} onto U such that $\varphi(1) = \sigma$ in the sense of angular limits and the angular derivative $\varphi'(1)$ is finite.

Note that the condition $\varphi(1) = \sigma$ in the above definition means that $\zeta = 1$ is a contact point of φ ; hence, the angular derivative $\varphi'(1)$ exists and does not vanish, but in general, can be infinite; see Remark 2.4. To simplify the terminology in the case when U is a petal, we make the following definition.

Definition 2.19. Let $\Delta(\sigma)$ be a hyperbolic petal of a one-parameter semigroup (ϕ_t) with α -point σ . We say that $\Delta(\sigma)$ is *conformal* if $\Delta(\sigma)$ is conformal at σ with respect to \mathbb{D} .

Remark 2.20. Note that, in general, there can exist two conformal mappings φ_k , $k = 1, 2$, of \mathbb{D} onto the same domain $U \subset \mathbb{D}$ such that $\varphi_k(1) = \sigma$, $k = 1, 2$, for some $\sigma \in \partial\mathbb{D}$, but with $\varphi'_1(1) = \infty$ while $\varphi'_2(1)$ is finite. This phenomenon may happen if the geometric point σ corresponds to at least two different accessible boundary points of U . However, this is never the case for petals, see Proposition 4.15 in [15]. Therefore, in order to determine whether a petal $\Delta(\sigma)$ is conformal it is sufficient to construct *just one* conformal map φ of \mathbb{D} onto $\Delta(\sigma)$ with $\varphi(1) = \sigma$ in the sense of angular limits and check whether the angular derivative $\varphi'(1)$ is finite. According to Remark 2.14, every one-parameter semigroup admits a pre-model (\mathbb{H}, ψ, Q_t) at each repulsive fixed point σ . A conformal map of \mathbb{D} onto $\Delta(\sigma)$ taking 1 to σ is given by $\varphi(z) := \psi\left(\frac{1-z}{1+z}\right)$. Therefore, a hyperbolic petal $\Delta(\sigma)$ is conformal if and only if the pre-model (\mathbb{H}, ψ, Q_t) is *regular* in the sense that the angular derivative $\psi'(0)$ is finite. Note that this condition is stronger than the isogonality condition (2.8), which is also sometimes called *semi-conformality*; see e.g. Section 4.3 in [37]. Condition (2.8) is satisfied in our context by the very definition of a pre-model.

For a non-elliptic one-parameter semigroup (ϕ_t) , the two versions of the angular derivative problem introduced above turn out to be equivalent.

Proposition 2.21. *In the above notation, a hyperbolic petal $\Delta(\sigma)$ is conformal if and only if the Koenigs domain Ω is conformal at $-\infty$ w.r.t. the maximal strip $S(\sigma) := h(\Delta(\sigma))$.*

Proof. Using conformal automorphisms of \mathbb{D} , we may assume that $\sigma = -1$ and $\tau = 1$. Denote $S := S(\sigma)$, $a := \inf_{z \in S} \operatorname{Im} z$ and $b := \sup_{z \in S} \operatorname{Im} z$. Let

$$q(z) := \frac{e^{L(z)} - 1}{e^{L(z)} + 1}, \quad \text{where } L(z) := \pi \frac{z - (b+a)i/2}{b-a}.$$

The function q maps S conformally onto the disk \mathbb{D} in such a way that $q(z) \rightarrow \tau = 1$ as $\operatorname{Re} z \rightarrow +\infty$, and $q(z) \rightarrow \sigma = -1$ as $\operatorname{Re} z \rightarrow -\infty$.

Then $g := (h \circ q)^{-1}$ is a conformal mapping of Ω onto S . Moreover, by Remarks 2.8 and 2.10, for any $y_0 \in (a, b)$, the curve $[0, +\infty) \ni t \mapsto h^{-1}(-t + iy_0)$ is a backward orbit in $\Delta(\sigma)$ and hence it converges to σ as $t \rightarrow +\infty$. Therefore, g satisfies the normalization (2.9). By the very definition, it follows that Ω is conformal at $-\infty$ with respect to S if and only if g satisfies condition (2.10).

It is elementary to see that (2.10) is in turn equivalent to the existence of finite angular derivative at $\sigma = -1$ for the holomorphic self-map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ defined by $\varphi := q \circ g \circ q^{-1}$. At the same time, we have

$$\varphi = q \circ (h \circ q)^{-1} \circ q^{-1} = h^{-1} \circ q^{-1}.$$

Hence φ maps \mathbb{D} conformally onto the hyperbolic petal $\Delta(\sigma)$. Using again Remark 2.8, we see that the radial limit of φ at $\sigma = -1$ equals σ . By Lindelöf's theorem (see e.g. Theorem 1.5.7 on p. 27 of [14]), the latter means that $\varphi(\sigma) = \sigma$ in the sense of angular limits. According to Remark 2.20, the existence of the finite angular derivative of $z \mapsto \varphi(-z)$ at $z = 1$ implies that $\Delta(\sigma) = \varphi(\mathbb{D})$ is conformal at σ with respect to \mathbb{D} , and hence we obtain the desired result. ■

Remark 2.22 (Examples of non-conformal hyperbolic petals). With some efforts, an example of a non-conformal hyperbolic petal was constructed in Section 8 of [15]. We briefly indicate how one can easily obtain many other examples of conformal and non-conformal hyperbolic petals. Lemma 3.2, which we prove in the next section, allows one to construct starlike-at-infinity domains Ω containing a maximal strip S with respect to which Ω is conformal (or non-conformal) at $-\infty$. Let h be a conformal mapping of \mathbb{D} onto Ω . Then $\Delta := h^{-1}(S)$ is a hyperbolic petal for the non-elliptic semigroup (ϕ_t) given by $\phi_t := h^{-1} \circ (h + t)$, $t \geq 0$. By Proposition 2.21, the petal Δ is conformal (respectively, non-conformal). Note that, using the trick described in Remark 2.6, this technique can be extended to elliptic semigroups as well.

3. Auxiliary results

3.1. Strong Markov property for the Green function

Let $D \subsetneq \mathbb{C}$ be a simply connected domain. Let G_D and ω_D denote the (positive) Green function and the harmonic measure for the domain D , respectively; see, e.g., Chapter 4 in [39]. In the course of the proofs, we make use of the following remarkable property of the Green function G_D , see [38], p. 111.

Lemma 3.1 (Strong Markov property for the Green function). *Let D_1 and D_2 be two simply connected domains with $D_1 \subset D_2 \subsetneq \mathbb{C}$. Then for all $z, w \in D_1$, $z \neq w$,*

$$(3.1) \quad G_{D_2}(z, w) - G_{D_1}(z, w) = \int_A G_{D_2}(\alpha, z) \omega_{D_1}(w, d\alpha),$$

where $A := D_2 \cap \partial D_1$.

In the proof of the main results, we occasionally replace the hyperbolic density with the conformal radius; the reader may refer to Section 2.1 in [23]. For simply connected domains $D \subsetneq \mathbb{C}$, the conformal radius $\mathcal{R}(z_0, D)$ of D with respect to the point $z_0 \in D$ is just the reciprocal of the hyperbolic density $\lambda_D(z_0)$; namely, $\mathcal{R}(z_0, D) = 2/\lambda_D(z_0)$.

It is further known that $G_D(z, w) + \log |w - z| \rightarrow \log \mathcal{R}(w, D)$ as $z \rightarrow w \in D$. Therefore, (3.1) implies

$$(3.2) \quad \log \frac{\mathcal{R}(w, D_2)}{\mathcal{R}(w, D_1)} = \int_A G_{D_2}(\alpha, w) \omega_{D_1}(w, d\alpha), \quad w \in D_1.$$

3.2. An integral criterion for conformality of domains starlike at infinity

Let Ω be a domain which is starlike at infinity, and let us denote by $S = \mathbb{S}(a, b) = \{x + iy : a < y < b\}$ a maximal horizontal strip contained in Ω .

One of the ingredients of the proof of Theorem 4.1 is a characterization of conformality of Ω at the boundary point $-\infty$ with respect to S in Euclidean terms. Such a characterization can easily be deduced from the work of Rodin and Warschawski [41] and Jenkins and Oikawa [30] as follows.

Fix a point $w_0 = iy_0 \in S$, $y_0 \in (a, b)$, and denote

$$\delta_{\Omega,1}(t) := a - \inf I(t) \quad \text{and} \quad \delta_{\Omega,2}(t) := \sup I(t) - b,$$

where $I(t)$ is the connected component of $\{y: t + iy \in \Omega\}$ containing y_0 . It is clear that $\delta_{\Omega,j}(t) \geq 0$ for all $t \in \mathbb{R}$, $j = 1, 2$, because $S \subset \Omega$. We denote

$$(3.3) \quad \delta_{\Omega}(t) := \max \{ \delta_{\Omega,1}(t), \delta_{\Omega,2}(t) \}.$$

Note that $\delta_{\Omega,1}(t)$, $\delta_{\Omega,2}(t)$ and $\delta_{\Omega}(t)$ do not depend on the choice of the base point w_0 . We should also note that there might exist $t \in \mathbb{R}$ such that $\delta_{\Omega}(t) = +\infty$. Due to maximality of the strip S inside Ω , there always exists a $t_0 \in \mathbb{R}$ such that $\delta_{\Omega}(t) < +\infty$ for all $t \leq t_0$. In fact, starlikeness of Ω at infinity assures that δ_{Ω} is monotonically non-decreasing in $(-\infty, t_0]$ and that $\delta_{\Omega}(t) \rightarrow 0$ as $t \rightarrow -\infty$.

Since the Koenigs function h is defined modulo an additive constant, by shifting the domain Ω along the real axis, in our proofs we may assume without loss of generality that $t_0 = 1$. The following lemma relates the conformality of the starlike-at-infinity domain Ω at $-\infty$ with respect to the strip S with the integrability of the Euclidean quantity δ_{Ω} .

Lemma 3.2. *In the above notation, Ω is conformal at $-\infty$ w.r.t. the strip S if and only if*

$$(3.4) \quad \int_{-\infty}^0 \delta_{\Omega}(t) dt < +\infty.$$

Proof. Suppose that (3.4) holds. Let $(Q_n)_{n \geq 0}$ be a sequence of pairwise disjoint squares contained in $(\Omega \setminus S) \cap \{z: \operatorname{Re} z \leq 0\}$ and such that for each Q_n one of the sides is contained on ∂S . From (3.4), it follows easily that $\sum_{n \geq 0} \operatorname{area}(Q_n) < +\infty$. Hence, by Theorem 2 in [41], Ω is conformal at $-\infty$ with respect to S .

Now suppose that Ω is conformal at $-\infty$ with respect to S . Since $\Omega + x \subset \Omega$ for any $x \geq 0$, the functions $\delta_j := \delta_{\Omega,j}$, $j = 1, 2$, are monotonically non-decreasing. Therefore, by the implication (i) \Rightarrow (iii) of Theorem 2 in [41], there is an increasing sequence $0 = u_0 < u_1 < \dots < u_n < \dots$, tending to $+\infty$, such that

$$\sum_{n=0}^{\infty} (u_{n+1} - u_n)^2 < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} \delta_j(-u_{n+1})^2 < +\infty, \quad j = 1, 2.$$

With the help of the Cauchy–Schwarz–Bunyakovsky inequality, it follows that

$$\int_{-\infty}^0 \delta_j(t) dt = \sum_{n=0}^{\infty} \int_{-u_{n+1}}^{-u_n} \delta_j(t) dt \leq \sum_{n=0}^{\infty} (u_{n+1} - u_n) \cdot \delta_j(-u_{n+1}) < +\infty,$$

by the monotonicity of δ_j . ■

3.3. A lemma on convergence of Riemann mappings on the boundary

Let (D_n) be a sequence of simply connected domains in \mathbb{C} and \mathcal{B} an open subarc of $\partial\mathbb{D}$ with the following properties: $\mathbb{D} \subset D_n$ and $\mathcal{B} \subset \partial D_n$ for each $n \in \mathbb{N}$. Denote by σ_k , $k = 1, 2$, the end-points of the arc \mathcal{B} .

Lemma 3.3. *In the above notation, suppose additionally that each D_n is a Jordan domain. For each $n \in \mathbb{N}$, let f_n denote the conformal map of \mathbb{D} onto D_n normalized by $f_n(0) = 0$, $f_n'(0) > 0$ and extended by continuity to a homeomorphism between the closures. If (D_n) converges to \mathbb{D} with respect to 0 in the sense of kernel convergence, then $f_n^{-1}(\sigma_k) \rightarrow \sigma_k$, $k = 1, 2$, as $n \rightarrow +\infty$.*

Remark 3.4. Requiring that each of the domains D_n is a Jordan domain is not really essential in the above lemma, but this condition is satisfied in the setting for which we will apply Lemma 3.3. For more general domains, both the statement of the lemma and its proof would become more technical. On the other hand, the assumption that $\mathbb{D} \subset D_n$ for all $n \in \mathbb{N}$ seems to play an important role. Again, in our setting this assumption holds, but without it the conclusion of Lemma 3.3 may fail as the following example demonstrates. The sequence of Jordan domains

$$D_n := \left\{ z \in \mathbb{C} : |z| < 1 - \frac{1}{n} \right\} \cup \left\{ z \in \mathbb{D} : \left| \arg z \right| < \frac{\pi}{2n} \right\} \cup \left\{ z \in \mathbb{D} : \operatorname{Re} z > 0, |z| > 1 - \frac{1}{2n} \right\} \subset \mathbb{D}$$

converges to \mathbb{D} with respect to 0 in the sense of kernel convergence. Although the arc $\mathcal{B} := \{z \in \partial\mathbb{D} : \operatorname{Re} z > 0\}$ lies on the boundary of D_n for all $n \in \mathbb{N}$, and although the functions f_n defined as above converge locally uniformly in \mathbb{D} to the identity map, the pre-images $f_n^{-1}(\mathcal{B})$ shrink as $n \rightarrow +\infty$ to the point $\sigma_0 = 1$ rather than converge to \mathcal{B} .

Proof of Lemma 3.3. Denote $g_n := f_n^{-1}|_{\overline{\mathbb{D}}}$, where $\overline{\mathbb{D}}$ denotes the closure of \mathbb{D} . By Carathéodory's kernel convergence theorem (Theorem 1.8 in [37]), (f_n) and (g_n) converge locally uniformly in \mathbb{D} to the identity mapping. Moreover, the restrictions $g_n|_{\mathbb{D}}$ can be extended by the Schwarz reflection principle to conformal mappings g_n^* of $\mathbb{D}_{\mathcal{B}} := \{z \in \mathbb{C} : |z| \neq 1\} \cup \mathcal{B}$ into \mathbb{C} . Recall that $g_n(0) = 0$ for all $n \in \mathbb{N}$. Moreover, by the locally uniform convergence of (g_n) in \mathbb{D} , the sequence $|g_n'(0)|$ is bounded. It follows that the extended functions g_n^* form a normal family in $\mathbb{D}_{\mathcal{B}}$ and hence $g_n^* \rightarrow \operatorname{id}_{\mathbb{D}_{\mathcal{B}}}$ locally uniformly in $\mathbb{D}_{\mathcal{B}}$. This fact, however, does not imply on its own the conclusion of the lemma, because $\sigma_k \notin \mathbb{D}_{\mathcal{B}}$. On the other hand, $g_n(w) = g_n^*(w)$ for all $w \in \mathcal{B}$ and all $n \in \mathbb{N}$, and hence we may conclude that (g_n) converges uniformly on any closed subarc of \mathcal{B} to the identity mapping.

Consider the sequence $h_n(w) := \sigma_0 g_n(w) / g_n(\sigma_0)$, $w \in \overline{\mathbb{D}}$, where σ_0 is the midpoint of the arc \mathcal{B} . Note that $h_n(\mathbb{D}) \subset \mathbb{D}$, $h_n(\mathcal{B}) \subset \partial\mathbb{D}$, and $h_n(\sigma_0) = \sigma_0$. Therefore, by Loewner's lemma (Proposition 4.15 in [37]), $\mathcal{B} \subset h_n(\mathcal{B})$. Since $g_n(\sigma_0) \rightarrow \sigma_0$, it is enough to show that $h_n(\sigma_k) \rightarrow \sigma_k$ as $n \rightarrow +\infty$, $k = 1, 2$. Suppose this is not the case. Then, passing if necessary to a subsequence, we may assume that there exists an open arc \mathcal{C} on $\partial\mathbb{D}$ such that $\mathcal{C} \not\subset \mathcal{B}$, and

$$(3.5) \quad \mathcal{C} \subset h_n(\mathcal{B}) \quad \text{for all } n \in \mathbb{N}.$$

In particular, $h_n^{-1}(\mathcal{C}) \subset \partial\mathbb{D}$ for any $n \in \mathbb{N}$. The functions h_n^{-1} are restrictions of the maps $z \mapsto f_n(z g_n(\sigma_0) / \sigma_0)$ to $\overline{\mathbb{D}}$. Therefore, arguing as above, we see that $h_n^{-1} \rightarrow \operatorname{id}_{\mathcal{C}}$ on \mathcal{C} . Since $\mathcal{C} \not\subset \mathcal{B}$, it follows that $h_n^{-1}(\mathcal{C}) \not\subset \mathcal{B}$ for $n \in \mathbb{N}$ large enough. To complete the proof, it remains to notice that the latter conclusion contradicts (3.5). ■

4. Proof of the main result

4.1. Reformulation of the problem

In this section, we reduce Theorem 1.1 to showing that if a domain Ω is starlike at infinity, then its conformality at $-\infty$ with respect to a maximal strip $\mathbb{S}(a, b) \subset \Omega$ is equivalent to a certain condition on how fast λ_{Ω} approaches $\lambda_{\mathbb{S}(a, b)}$ along a horizontal ray $\{t + iy_0 : t \leq 0\} \subset \mathbb{S}(a, b)$ as $t \rightarrow -\infty$. The precise statement of this result is as follows.

Theorem 4.1. *Let $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, be a domain starlike at infinity and let $S := \mathbb{S}(a, b)$ be a maximal strip contained in Ω . Fix a point $w_0 \in S$. Then Ω is conformal at $-\infty$ with respect to S if and only if*

$$(4.1) \quad \int_{-\infty}^0 \log \frac{\lambda_S(t + w_0)}{\lambda_\Omega(t + w_0)} dt < +\infty.$$

In this case, the integral (4.1) converges for every $w_0 \in S$; in fact, locally uniformly in S .

In this subsection, we show that Theorem 4.1 implies Theorem 1.1. This is not difficult for non-elliptic semigroups (ϕ_t) in \mathbb{D} , but less obvious in the elliptic case. The proof of Theorem 4.1 will be given in Subsections 4.2 and 4.3.

Proof of Theorem 1.1. First we prove Theorem 1.1 for the case of a non-elliptic semigroup (ϕ_t) in \mathbb{D} . Differentiating Abel's equation (2.7) with respect to z , we obtain

$$(4.2) \quad h'(\phi_t(z)) \phi_t'(z) = h'(z)$$

for all $z \in \mathbb{D}$, $t \geq 0$. Moreover, in view of Remark 2.11, the identity (4.2) holds also for all $t < 0$ if $z \in \mathcal{W}^\circ$.

Denote by Ω the Koenigs domain of (ϕ_t) and by $S = \mathbb{S}(a, b)$ the maximal strip in Ω associated to the α -point σ of a hyperbolic petal $\Delta = \Delta(\sigma)$, see Remark 2.8. Then Ω is starlike at infinity and the Koenigs function h maps \mathbb{D} conformally onto Ω and $\Delta(\sigma)$ onto S , see Theorem 13.5.5 in [14]. Fix $z_0 \in \Delta(\sigma)$. Taking into account that the hyperbolic metric is invariant under conformal mappings and using equality (4.2), we see that the integrand in (1.1) can be written as

$$\log \frac{\lambda_\Delta(z_0)}{\lambda_{\mathbb{D}}(\phi_t(z_0)) |\phi_t'(z_0)|} = \log \frac{|h'(z_0)| \lambda_S(h(z_0))}{|h'(\phi_t(z_0))| \lambda_\Omega(h(\phi_t(z_0))) |\phi_t'(z_0)|} = \log \frac{\lambda_S(h(z_0))}{\lambda_\Omega(h(\phi_t(z_0)))},$$

which is equal for any $t \in \mathbb{R}$, according to Abel's equation (2.7) and Remark 2.11, to

$$\log \frac{\lambda_S(h(z_0))}{\lambda_\Omega(h(z_0) + t)}.$$

Furthermore, obviously $\lambda_S(h(z_0)) = \lambda_S(h(z_0) + t)$ for any $t \in \mathbb{R}$. Consequently, for $w_0 := h(z_0)$, the integral in (1.1) is identical to the integral in (4.1). Thus, for non-elliptic semigroups, Theorem 1.1 follows from Theorem 4.1 and Proposition 2.21.

Now we show how the elliptic case can be reduced to the non-elliptic case. Consider a semigroup (ϕ_t) in \mathbb{D} with DW-point $\tau \in \mathbb{D}$. According to Remark 2.6, there exists a parabolic semigroup (φ_t) in \mathbb{D} such that for all $t \geq 0$,

$$(4.3) \quad \phi_t \circ F = F \circ \varphi_t, \quad \text{where } F := T \circ C, \quad C(z) := \exp\left(-\frac{1+z}{1-z}\right), \quad T(w) := \frac{w+\tau}{1+\bar{\tau}w}.$$

Note that $F(\zeta_1) = F(\zeta_2)$ if and only if $L^{on}(\zeta_1) = \zeta_2$ for some $n \in \mathbb{Z}$, where L is the automorphism of \mathbb{D} given by $L(\zeta) := H^{-1}(H(\zeta) + 2\pi i)$, $H(z) := (1+z)/(1-z)$, and L^{on} denotes the n -th iterate of L . The semigroup (φ_t) satisfies for each $t \geq 0$ the functional equation $\varphi_t \circ L = L \circ \varphi_t$. This is clear from the construction given in Section 2

of [28]. Therefore, if $\zeta \in \varphi_t(\mathbb{D})$ for some $t \geq 0$, then $\varphi_t(\mathbb{D})$ contains all the points $\zeta' \in \mathbb{D}$ satisfying $F(\zeta') = F(\zeta)$. With the notation

$$\mathcal{W} := \bigcap_{t \geq 0} \phi_t(\mathbb{D}), \quad \mathcal{U} := \bigcap_{t \geq 0} \varphi_t(\mathbb{D}),$$

it follows that

$$F(\mathcal{U}) = \bigcap_{t \geq 0} F(\varphi_t(\mathbb{D})) = \bigcap_{t \geq 0} \phi_t(F(\mathbb{D})) = \bigcap_{t \geq 0} \phi_t(\mathbb{D} \setminus \{\tau\}) = \mathcal{W} \setminus \{\tau\}.$$

Note that $\Delta(\sigma)$ is a simply connected domain and $\tau \in \partial\Delta(\sigma)$; see Proposition 13.4.2 in [14]. Moreover, by the definition of a petal, $\Delta(\sigma)$ is a connected component of the interior of \mathcal{W} . Since $F: \mathbb{D} \rightarrow \mathbb{D} \setminus \{\tau\}$ is a covering map, it follows that there exists a connected component D of the interior of \mathcal{U} such that F maps D conformally (and in particular, injectively) onto $\Delta(\sigma)$. By the very definition, D is a petal for (φ_t) .

Moreover, by Proposition 13.4.9 in [14], D is a Jordan domain and $\partial\Delta(\sigma)$ is locally connected. It follows, see e.g. Section 2.2 in [37], that $F_* := F|_D$ extends continuously to ∂D and that $F_*(\partial D) = \partial\Delta(\sigma)$. Recall that $\sigma \in \partial\Delta(\sigma)$. Therefore, there exists a point $\zeta \in \partial D$ such that $F_*(\zeta) = \sigma$. Since F is continuous with $|F| < 1$ in \mathbb{D} and since $\sigma \in \partial\mathbb{D}$, we have $\zeta \in \partial\mathbb{D}$. We claim that, $\zeta \neq 1$. Indeed, let Γ be a Jordan arc in $D \cup \{1\}$ with one of the end-points at 1. Since F_* is continuous in the closure of D , we have $F(z) \rightarrow F_*(1)$ as $\Gamma \ni z \rightarrow 1$. Taking into account that the only asymptotic value of $\mathbb{H} \ni \zeta \mapsto e^{-\zeta}$ at ∞ is 0, it follows that $F_*(1) = \tau \neq \sigma$.

Note that F extends holomorphically to any point of $\partial\mathbb{D} \setminus \{1\}$. Hence $F(\zeta) = \sigma$. As a consequence, using Remark 2.5, it is easy to see that ζ is a repulsive fixed point of (φ_t) .

Thus we have constructed a hyperbolic petal D for (φ_t) with α -point ζ such that F maps D conformally onto $\Delta(\sigma)$, with $F(\zeta) = \sigma$. Since F is holomorphic at ζ and $F'(\zeta) \neq 0$, the petal D is conformal if and only if the petal $\Delta(\sigma)$ is conformal.

It remains to show that condition (1.1) for (ϕ_t) and a point $z_0 \in \Delta(\sigma)$ is equivalent to (1.1) with (ϕ_t) , $\Delta(\sigma)$, and z_0 replaced by (φ_t) , D , and $\zeta_0 := F_*^{-1}(z_0)$, respectively.

First of all since F maps D conformally onto $\Delta(\sigma)$, with $F(\zeta_0) = z_0$, we have $\lambda_D(\zeta_0) = |F'(\zeta_0)|\lambda_{\Delta(\sigma)}(z_0)$. Furthermore, by (4.3),

$$\phi_t(z_0) = F(\varphi_t(\zeta_0)) \quad \text{and} \quad \varphi_t'(z_0) = \phi_t'(\zeta_0) \frac{F'(\zeta_0)}{F'(\varphi_t(\zeta_0))}$$

for all $t \in \mathbb{R}$. Using these relations, we obtain

$$\begin{aligned} \frac{\lambda_D(\zeta_0)}{\lambda_{\mathbb{D}}(\varphi_t(\zeta_0)) |\varphi_t'(z_0)|} &= \frac{|F'(\zeta_0)| \lambda_{\Delta(\sigma)}(z_0)}{\lambda_{\mathbb{D}}(\varphi_t(\zeta_0))} \frac{|F'(\varphi_t(\zeta_0))|}{|\phi_t'(\zeta_0) F'(\zeta_0)|} \\ (4.4) \qquad &= \frac{\lambda_{\Delta(\sigma)}(z_0)}{\lambda_{\mathbb{D}}(\phi_t(z_0)) |\phi_t'(z_0)|} \frac{|F'(\varphi_t(\zeta_0))| \lambda_{\mathbb{D}}(\phi_t(z_0))}{\lambda_{\mathbb{D}}(\varphi_t(\zeta_0))} \\ &= \frac{\lambda_{\Delta(\sigma)}(z_0)}{\lambda_{\mathbb{D}}(\phi_t(z_0)) |\phi_t'(z_0)|} \frac{|F'(\varphi_t(\zeta_0))| \lambda_{\mathbb{D}}(F(\varphi_t(\zeta_0)))}{\lambda_{\mathbb{D}}(\varphi_t(\zeta_0))} \end{aligned}$$

for all $t \in \mathbb{R}$. The backward orbit $\gamma(t) := \varphi_{-t}(\zeta_0)$ of (φ_t) converges to ζ at an exponential rate, that is,

$$(4.5) \qquad \lim_{t \rightarrow -\infty} \frac{1}{t} \log(1 - \bar{\zeta} \varphi_t(\zeta_0)) = \lambda \in (0, +\infty),$$

see Proposition 13.4.14 in [14], and non-tangentially, see Remark 2.15. Moreover, for any $w \in \mathbb{D}$,

$$0 \geq \log \frac{|F'(w)| \lambda_{\mathbb{D}}(F(w))}{\lambda_{\mathbb{D}}(w)} = \log \frac{|C'(w)| \lambda_{\mathbb{D}}(C(w))}{\lambda_{\mathbb{D}}(w)} = \log \frac{\mu(w)}{\sinh \mu(w)} \geq -\frac{\mu(w)^2}{6},$$

where $\mu(w) := (1 - |w|^2)/(1 - |w|)$. The first equality holds because $F = T \circ C$ with $T \in \text{Aut}(\mathbb{D})$, the second one comes from (4.3) by direct calculation, and the inequality sign can be established by comparing the Maclaurin expansion of $\sinh(x)/x$ with that of $e^{x^2/6}$. Note that $1 - |\varphi_t(\zeta_0)|^2 < 2|1 - \bar{\zeta}\varphi_t(\zeta_0)|$ for all $t < 0$, and that $|1 - \varphi_t(\zeta_0)| \rightarrow |1 - \zeta| \neq 0$ as $t \rightarrow -\infty$. Hence, it follows from (4.5) that $\int_{-\infty}^0 \mu(\varphi_t(\zeta_0))^2 dt < +\infty$, and so we can conclude that the integral

$$\int_{-\infty}^0 \log \frac{|F'(\varphi_t(\zeta_0))| \lambda_{\mathbb{D}}(F(\varphi_t(\zeta_0)))}{\lambda_{\mathbb{D}}(\varphi_t(\zeta_0))} dt$$

converges.

In fact, the above integral converges locally uniformly with respect to $\zeta_0 \in D$ because the limit (4.5) is attained locally uniformly in D , which in turn follows from the fact that the values of the holomorphic functions

$$D \ni \zeta \mapsto \frac{1}{t} \log(1 - \bar{\zeta}\varphi_t(\zeta)), \quad t < -1,$$

lie in the strip \mathbb{S} , and hence these functions form a normal family in D . In view of (4.4), the proof that Theorem 4.1 implies Theorem 1.1 for elliptic semigroups is therefore reduced to the previous case of non-elliptic semigroups. ■

4.2. Proof of the if-part of Theorem 4.1: Condition (4.1) implies conformality

As a matter of taste, we prefer to work with the conformal radius instead of the density of the hyperbolic metric. Therefore, condition (4.1) in Theorem 4.1 can be rewritten as

$$(4.6) \quad \int_{-\infty}^0 \log \frac{\mathcal{R}(t + w_0, \Omega)}{\mathcal{R}(t + w_0, S)} dt < +\infty, \quad w_0 \in S.$$

In order to simplify notation, throughout the current and the following section we restrict ourselves to the case $\text{Re } w_0 = 0$. The following theorem is the key result of this section. It shows that the integrand in (4.6) can be estimated from below by the Euclidean quantity $\delta_{\Omega}(t)$ introduced in (3.3).

Theorem 4.2. *Let Ω be a proper subdomain of \mathbb{C} which is starlike at infinity and such that the standard strip $\mathbb{S} = \{z \in \mathbb{C} : |\text{Im } z| < \pi/2\}$ is a maximal horizontal strip contained in Ω . Then for any compact set $K \subset (-\pi/2, \pi/2)$, there are constants $c > 0$ and $T \leq 0$ such that for any $y \in K$,*

$$(4.7) \quad \log \frac{\mathcal{R}(t + iy, \Omega)}{\mathcal{R}(t + iy, \mathbb{S})} \geq c \delta_{\Omega}(t) \quad \text{for all } t \leq T.$$

It is clear from our previous considerations that Theorem 4.2 implies the “if-part” of Theorem 4.1. Indeed, if condition (4.1) holds for some $w_0 = iy_0 \in S$, $y_0 \in (-\pi/2, \pi/2)$, then Theorem 4.2 and Lemma 3.2 imply that Ω is conformal at $-\infty$.

The proof of Theorem 4.2 is quite long and will be broken into several steps. The idea of the proof is as follows.

Remark 4.3 (Idea of proof of Theorem 4.2). Recall that $\mathbb{H} = \{z : \operatorname{Re} z > 0\}$ denotes the right half-plane. For $\delta > 0$, we consider the enlarged strip

$$\mathbb{S}_\delta := \left\{ z : -\frac{\pi}{2} < \operatorname{Im} z < \frac{\pi}{2} + \delta \right\}$$

and the “half-widened” standard strip

$$\mathbb{S}_\delta^* := \mathbb{S} \cup (\mathbb{S}_\delta \cap \mathbb{H}).$$

Fix a point $iy_0 \in \Omega$, $y_0 \in (-\pi/2, \pi/2)$. By definition of $\delta_2(t) := \delta_{\Omega,2}(t)$, see Subsection 3.2, and since Ω is starlike at infinity, it follows at once that

$$\mathbb{S}_{\delta_2(t)}^* + t \subset \Omega.$$

Thus, by domain monotonicity of the conformal radius,

$$(4.8) \quad \log \frac{\mathcal{R}(t + iy_0, \Omega)}{\mathcal{R}(t + iy_0, \mathbb{S})} \geq \log \frac{\mathcal{R}(t + iy_0, \mathbb{S}_{\delta_2(t)}^* + t)}{\mathcal{R}(t + iy_0, \mathbb{S})} = \log \frac{\mathcal{R}(iy_0, \mathbb{S}_{\delta_2(t)}^*)}{\mathcal{R}(iy_0, \mathbb{S})}, \quad t \leq 0.$$

The crux of the proof is to show that the expression on the right-hand side of (4.8) is bounded below by

$$(4.9) \quad \log \frac{\mathcal{R}(iy_0, \mathbb{S}_{\delta_2(t)})}{\mathcal{R}(iy_0, \mathbb{S})},$$

at least up to a positive multiplicative constant which depends only on y_0 but in a fairly controllable way. The quantity occurring in (4.9) is explicitly computable in terms of $\delta_2(t)$ (and y_0), and as we shall see, in fact comparable to $\delta_2(t)$ locally uniformly with respect to y_0 . This provides a lower bound also for the left-hand side in (4.8) in terms of $\delta_2(t) = \delta_{\Omega,2}(t)$ and then, by symmetry, in terms of $\delta_\Omega(t) = \max\{\delta_{\Omega,1}(t), \delta_{\Omega,2}(t)\}$, and (4.7) follows.

Proposition 4.4. For $\delta > 0$, let $\Phi_\delta: (-\pi/2, \pi/2) \rightarrow (0, +\infty)$ be defined by

$$(4.10) \quad \Phi_\delta(y) := \log \frac{\mathcal{R}(iy, \mathbb{S}_\delta^*)}{\mathcal{R}(iy, \mathbb{S})} \left(\log \frac{\mathcal{R}(iy, \mathbb{S}_\delta)}{\mathcal{R}(iy, \mathbb{S})} \right)^{-1}.$$

Then

$$(4.11) \quad \lim_{\delta \rightarrow 0^+} \Phi_\delta = \frac{1}{2}$$

locally uniformly on $(-\pi/2, \pi/2)$.

Remark 4.5. The two quantities $\mathcal{R}(\cdot, \mathbb{S})$ and $\mathcal{R}(\cdot, \mathbb{S}_\delta)$ occurring in (4.10) do have simple explicit expressions. In principle, the third quantity $\mathcal{R}(\cdot, \mathbb{S}_\delta^*)$ does also have an explicit expression, since the conformal map of the unit disk \mathbb{D} onto \mathbb{S}_δ^* is “explicitly” known, see formula (5.2.15) in p. 272 of [43]. However, this explicit formula is fairly involved and seems quite unsuitable for obtaining precise information about $\mathcal{R}(\cdot, \mathbb{S}_\delta^*)$, which is needed for proving Proposition 4.4. The proof of Proposition 4.4 below circumvents this difficulty by making use of the strong Markov property for the Green function, see Section 3.1.

In order to prove Proposition 4.4, we further need several auxiliary lemmas.

Lemma 4.6. Fix some $\beta \in (-\pi/2, \pi/2)$, some positive $\alpha_0 \leq \frac{1}{2}(\pi/2 - \beta)$, and some $\alpha \in (0, \alpha_0)$, and let

$$h(x) := G_{\mathbb{S}}(i\beta, x + i(\frac{\pi}{2} - \alpha)), \quad x \in \mathbb{R}.$$

Then

$$(4.12) \quad h(0) \geq h(x) \geq h(0) \frac{1 - \cos \alpha_0}{\cosh x - \cos \alpha_0}$$

for all $x \in \mathbb{R}$.

Proof. Let $z := x + i(\pi/2 - \alpha)$. Using the formula for the Green function of \mathbb{S} (see [39], p. 109), we find

$$h(x) = G_{\mathbb{S}}(i\beta, z) = \log \left| \frac{e^z + e^{-i\beta}}{e^z - e^{i\beta}} \right| = \frac{1}{2} \log q(\cosh x, \alpha), \quad q(u, \alpha) := \frac{u - \sin(\beta - \alpha)}{u - \sin(\beta + \alpha)}.$$

The proof of (4.12) is now elementary. For convenience, we provide the main steps. As $q(1, \alpha) \geq q(u, \alpha) > 1$ for any $u \geq 1$, we see that the left inequality in (4.12) holds. Since $u \mapsto \varrho(u) := (u - \sin(\beta + \alpha)) \log q(u, \alpha)$ is concave with $\lim_{u \rightarrow \infty} \varrho'(u) = 0$, the function $\varrho(u)$ is increasing and we have

$$\begin{aligned} \log q(u, \alpha) &\geq \frac{1 - \sin(\beta + \alpha)}{u - \sin(\beta + \alpha)} \log q(1, \alpha) > \frac{1 - \sin(\beta + \alpha_0)}{u - \sin(\beta + \alpha_0)} \log q(1, \alpha) \\ &\geq \frac{1 - \cos \alpha_0}{u - \cos \alpha_0} \log q(1, \alpha). \end{aligned}$$

The right inequality in (4.12) follows easily. ■

Lemma 4.7. Let $\mathcal{B} := \{e^{i\theta} : \alpha < \theta < \beta\}$ and $\mathcal{B}' := \{e^{i\theta} : \alpha' < \theta < \beta'\}$. Let $z, z' \in \mathbb{D}$. Then

$$(4.13) \quad |\omega_{\mathbb{D}}(z, \mathcal{B}) - \omega_{\mathbb{D}}(z', \mathcal{B}')| \leq \frac{|\alpha' - \alpha| + |\beta' - \beta| + 2|z' - z|}{\pi(1 - r)},$$

where $r := \max\{|z|, |z'|\}$.

Proof. Recall that $2\pi\omega_{\mathbb{D}}(0, \cdot)$ coincides with one-dimensional Lebesgue measure on the unit circle $\partial\mathbb{D}$. Hence

$$(4.14) \quad |\omega_{\mathbb{D}}(0, \mathcal{B}) - \omega_{\mathbb{D}}(0, \mathcal{B}')| \leq \frac{|\alpha' - \alpha| + |\beta' - \beta|}{2\pi}.$$

To prove (4.13) in the case $z' = z \neq 0$, it is sufficient to apply (4.14) to the arcs $T(\mathcal{B}, z)$ and $T(\mathcal{B}', z)$, where $T(\sigma, z) := (\sigma - z)/(1 - \sigma\bar{z})$, and take into account that

$$\left| \frac{\partial T(\sigma, z)}{\partial \sigma} \right| \leq \frac{1 + |z|}{1 - |z|} < \frac{2}{1 - r} \quad \text{for any } \sigma \in \partial\mathbb{D}.$$

Finally, to prove (4.13) in the general case, we notice that for any $\sigma \in \partial\mathbb{D}$ and $\theta \in \mathbb{R}$,

$$\left| \frac{\partial T(\sigma, z + te^{i\theta})}{\partial t} \right|_{t=0} \leq \left| \frac{\partial T(\sigma, z)}{\partial z} \right| + \left| \frac{\partial T(\sigma, z)}{\partial \bar{z}} \right| \leq \frac{2}{1 - r}.$$

Hence, $|\arg T(\sigma, z')/T(\sigma, z)| \leq 2|z' - z|/(1 - r)$ for any $\sigma \in \partial\mathbb{D}$. As a consequence,

$$2\pi |\omega_{\mathbb{D}}(z', \mathcal{B}') - \omega_{\mathbb{D}}(z, \mathcal{B}')| \leq 4|z' - z|/(1 - r),$$

and the general case for the estimate (4.13) follows immediately. \blacksquare

Proof of Proposition 4.4. Applying formula (3.2), we immediately get

$$(4.15) \quad \log \frac{\mathcal{R}(iy, \mathbb{S}_\delta)}{\mathcal{R}(iy, \mathbb{S})} = \int_A G_{\mathbb{S}_\delta}(iy, z) \omega_{\mathbb{S}}(iy, dz),$$

where $A := \{z : \operatorname{Im} z = \pi/2\}$, and

$$(4.16) \quad \log \frac{\mathcal{R}(iy, \mathbb{S}_\delta)}{\mathcal{R}(iy, \mathbb{S}_\delta^*)} = \int_{B \cup C_\delta} G_{\mathbb{S}_\delta}(iy, z) \omega_{\mathbb{S}_\delta^*}(iy, dz),$$

where $B := \{z \in A : \operatorname{Re} z < 0\}$ and $C_\delta := \{iy : \pi/2 \leq y < \pi/2 + \delta\}$.

Clearly, we may restrict consideration to small $\delta > 0$; namely, we will suppose that

$$\delta \in (0, \delta_0), \quad \text{where } \delta_0 := \min_{y \in K} \frac{1}{2} \left(\frac{\pi}{2} - y \right).$$

Fix temporarily $y \in K$ and $\delta \in (0, \delta_0)$. By Lemma 4.6 applied with

$$\beta := \left(y - \frac{\delta}{2} \right) \frac{\pi}{\pi + \delta}, \quad \alpha_0 := \delta_0, \quad \alpha := \delta \frac{\pi}{\pi + \delta},$$

we have that

$$h(x) = h(x; y, \delta) := G_{\mathbb{S}_\delta}(iy, x + i\pi/2)$$

satisfies inequality (4.12). On the one hand, in combination with the explicit formula from Table 4.1 in p. 100 of [39]

$$\omega_{\mathbb{S}}(iy, dz) = (2\pi)^{-1} \cos y (\cosh x - \sin y)^{-1} dx, \quad z = x + i\pi/2 \in A,$$

this allows us to estimate the right-hand side of (4.15) from below:

$$(4.17) \quad \int_A G_{\mathbb{S}_\delta}(iy, z) \omega_{\mathbb{S}}(iy, dz) \geq \frac{h(0; y, \delta)}{2\pi} \int_{\mathbb{R}} \frac{(1 - \cos \delta_0) \sin 2\delta_0}{(\cosh x - \cos \delta_0)(\cosh x + 1)} dx \\ = m(\delta_0) h(0; y, \delta),$$

for all $\delta \in (0, \delta_0)$ and all $y \in K$, where $m(\delta_0)$ is a positive constant depending only on δ_0 . On the other hand, in view of (4.12), we have

$$(4.18) \quad \begin{aligned} \int_A G_{\mathbb{S}_\delta}(iy, z) \omega_{\mathbb{S}}(iy, dz) &\leq h(0; y, \delta) \int_A \omega_{\mathbb{S}}(iy, dz) = h(0; y, \delta) \left(\frac{y}{\pi} + \frac{1}{2} \right) \\ &\leq h(0; y, \delta) \left(1 - \frac{2\delta_0}{\pi} \right). \end{aligned}$$

The main ingredient of the proof is the following:

Claim 1. As $\delta \rightarrow 0^+$,

$$(4.19) \quad \sup_{y \in K} \frac{1}{h(0; y, \delta)} \left| \int_B G_{\mathbb{S}_\delta}(iy, z) \omega_{\mathbb{S}_\delta^*}(iy, dz) - \frac{1}{2} \int_A G_{\mathbb{S}_\delta}(iy, z) \omega_{\mathbb{S}}(iy, dz) \right| \rightarrow 0.$$

To prove this claim we first notice that, due to symmetry, one can remove the factor $1/2$ in (4.19) if the set A in the second integral is replaced by B . Moreover, by the monotonicity property of harmonic measure, see e.g. Corollary 4.3.9 on p. 102 of [39],

$$\mu_{y, \delta} := \omega_{\mathbb{S}_\delta^*}(iy, \cdot)|_B - \omega_{\mathbb{S}}(iy, \cdot)|_B$$

is a non-negative bounded measure on B . Recall also that $0 < h(x; y, \delta) \leq h(0; y, \delta)$ for all $x \in \mathbb{R}$, see (4.12). Therefore, in order to prove (4.19), it is sufficient to show that

$$(4.20) \quad \sup_{y \in K} \mu_{y, \delta}(B) \rightarrow 0 \quad \text{as } \delta \rightarrow 0^+.$$

To this end, we will take advantage of conformal invariance of harmonic measure. Denote by F_δ , for $\delta \in (0, \delta_0)$, and by F the conformal mappings of \mathbb{S}_δ^* and \mathbb{S} , respectively, onto \mathbb{D} with the normalization $F(0) = F_\delta(0) = 0$, $F'(0) > 0$, and $F'_\delta(0) > 0$. Clearly, the map $F(z) = (e^z - 1)/(e^z + 1)$ extends holomorphically and injectively to the wider strip $|\operatorname{Im} z| < \pi$, and hence we can write $F_\delta = f_\delta^{-1} \circ F$, where f_δ is the conformal mapping of \mathbb{D} onto the Jordan domain $D_\delta := F(\mathbb{S}_\delta^*)$ with $f_\delta(0) = 0$ and $f'_\delta(0) > 0$.

Denote $\mathcal{B} := F(B) = \{z \in \partial\mathbb{D} : \pi/2 < \arg z < \pi\}$. By conformal invariance of harmonic measure, $\omega_{\mathbb{S}}(iy, B) = \omega_{\mathbb{D}}(F(iy), \mathcal{B})$ and $\omega_{\mathbb{S}_\delta^*}(iy, B) = \omega_{\mathbb{D}}(f_\delta^{-1}(F(iy)), f_\delta^{-1}(\mathcal{B}))$.

Note that $\mathbb{D} \subset D_\delta$ for any $\delta \in (0, \delta_0)$ and that $D_\delta \rightarrow \mathbb{D}$ with respect to 0 in the sense of kernel convergence when $\delta \rightarrow 0^+$. Denote $z = z(y) := F(iy)$ and $z' = z'(y, \delta) := F_\delta(iy) = f_\delta^{-1}(z(y))$. As $\delta \rightarrow 0^+$, by the Carathéodory kernel convergence theorem, we have that $z'(y, \delta) \rightarrow z(y)$ uniformly with respect to $y \in K$. Note also that by the Schwarz lemma, for any $y \in K$ and any $\delta \in (0, \delta_0)$ the estimate $|z'(y, \delta)| \leq |z(y)| \leq r_0 := \max_{y \in K} |F(iy)| < 1$ holds. Therefore, (4.20) and hence (4.19) follow from Lemmas 3.3 and 4.7.

Taking into account (4.15)–(4.19), we see that it remains to estimate in (4.16) the part of the integral over C_δ . We claim that:

Claim 2. As $\delta \rightarrow 0^+$,

$$(4.21) \quad \sup_{y \in K} \frac{1}{h(0; y, \delta)} \left| \int_{C_\delta} G_{\mathbb{S}_\delta}(iy, z) \omega_{\mathbb{S}_\delta^*}(iy, dz) \right| \rightarrow 0.$$

Fix some $\delta \in (0, \delta_0)$ and $y \in K$. Clearly, $0 < G_{S_\delta}(iy, z) \leq h(0; y, \delta)$ for any $z \in C_\delta$. Therefore, it is sufficient to show that

$$(4.22) \quad \sup_{y \in K} \omega_{S_\delta^*}(iy, C_\delta) \rightarrow 0 \quad \text{as } \delta \rightarrow 0^+.$$

Consider again the mappings F_δ introduced above. As we have shown in the proof of Claim 1, $|F_\delta(iy)| \leq r_0$, where $r_0 \in [0, 1)$ depends only on the compact set K . Therefore, to prove (4.22) we only need to check that the linear Lebesgue measure of $F_\delta(C_\delta)$ tends to zero as $\delta \rightarrow 0^+$. Suppose this is not the case. Then there exist a sequence $(\delta_n) \subset (0, \delta_0)$ converging to 0 and a non-degenerate closed arc $\mathcal{C} \subset \partial\mathbb{D}$ such that for each $n \in \mathbb{N}$, $g_n := F_{\delta_n}^{-1}: \mathbb{D} \rightarrow S_{\delta_n}^*$ extends continuously to \mathcal{C} , with $g_n(\mathcal{C}) \subset C_{\delta_n}$.

Since g_n is continuous on the compact set $\{\sigma r : \sigma \in \mathcal{C}, r \in [0, 1]\}$, there exists $r_n \in (0, 1)$ with the property that $g_n(\mathcal{C}_n)$, where $\mathcal{C}_n := \{\sigma r_n : \sigma \in \mathcal{C}\}$, lies in the $1/n$ -neighbourhood of C_{δ_n} . Thus, $\text{diam } g_n(\mathcal{C}_n) \rightarrow 0$. Note that the functions g_n are “uniformly normal” in \mathbb{D} in the sense that

$$(4.23) \quad \sup_{z \in \mathbb{D}} \frac{|g_n'(z)|(1 - |z|^2)}{1 + |g_n(z)|^2} \leq M \quad \text{for all } n \in \mathbb{N}$$

and some constant $M > 0$ independent of n . Indeed, $S_\delta^* \subset S_{\delta_0}^*$. Therefore, $g_n = F_{\delta_0}^{-1} \circ \phi_n$, where ϕ_n is a holomorphic self-map of \mathbb{D} . Taking into account that $F_{\delta_0}^{-1}$ is univalent in \mathbb{D} and hence normal (see Lemma 9.3 on p. 262 of [36]), the desired conclusion (4.23) follows from the Schwarz–Pick lemma. Now by the no-Koebe-arcs theorem for sequences of holomorphic functions (see Theorem 9.2 and the subsequent remark on p. 265 of [36]), it follows that (g_n) converges locally uniformly in \mathbb{D} to a constant, which is impossible because $g_n \rightarrow F^{-1}$ by construction. This contradiction proves Claim 2.

The conclusion of the proposition follows easily from (4.15)–(4.19) and (4.21). \blacksquare

We are now in a position to prove Theorem 4.2.

Proof of Theorem 4.2. Let $\delta_j(t) := \delta_{\Omega, j}(t)$ and let K be a compact subset of the interval $(-\pi/2, \pi/2)$. It suffices to prove that there exist $T_j < 0$ and $c_j > 0$, $j = 1, 2$, such that

$$(4.24) \quad \log \frac{\mathcal{R}(t + iy, \Omega)}{\mathcal{R}(iy, \mathbb{S})} \geq c_j \delta_j(t) \quad \text{for all } t \leq T_j \text{ and all } y \in K.$$

Further, it clearly suffices to consider the case $j = 2$. Fix $t \leq 0$. Since $S_{\delta_2(t)}^* + t \subset \Omega$, we have

$$\mathcal{R}(t + iy, \Omega) \geq \mathcal{R}(t + iy, S_{\delta_2(t)}^* + t) = \mathcal{R}(iy, S_{\delta_2(t)}^*) \quad \text{for all } y \in (-\pi/2, \pi/2).$$

Hence Proposition 4.4 implies that

$$(4.25) \quad \log \left[\frac{\mathcal{R}(t + iy, \Omega)}{\mathcal{R}(iy, \mathbb{S})} \right] \geq \log \left[\frac{\mathcal{R}(iy, S_{\delta_2(t)}^*)}{\mathcal{R}(iy, \mathbb{S})} \right] \geq \Phi_{\delta_2(t)}(y) \cdot \log \left[\frac{\mathcal{R}(iy, S_{\delta_2(t)})}{\mathcal{R}(iy, \mathbb{S})} \right]$$

for any $t \leq 0$ and any $y \in (-\pi/2, \pi/2)$, where Φ_δ converges to $1/2$ uniformly on K as $\delta \rightarrow 0^+$. By employing the well-known explicit expression (see e.g. Example 7.9 in [6])

$$\mathcal{R}(iy, \mathbb{S}_\delta) = \frac{2}{\lambda_{\mathbb{S}_\delta}(iy)} = 4 \frac{\pi + \delta}{\pi} \cos\left(\frac{\pi}{2} \cdot \frac{2y - \delta}{\pi + \delta}\right)$$

for the conformal radius of the strip \mathbb{S}_δ , it is easily checked that

$$\lim_{\delta \rightarrow 0} \frac{1}{\delta} \log \left[\frac{\mathcal{R}(iy, \mathbb{S}_\delta)}{\mathcal{R}(iy, \mathbb{S})} \right] = \frac{1}{\pi} + \frac{(\pi + 2y) \tan y}{2\pi} \in (0, +\infty)$$

holds uniformly with respect to $y \in K$. Combining this fact with $\Phi_\delta \rightarrow 1/2$ uniformly on K as $\delta \rightarrow 0^+$ and $\lim_{t \rightarrow -\infty} \delta_2(t) = 0$ shows that inequality (4.25) implies the estimate (4.24) for $j = 2$. As noted above, this concludes the proof of Theorem 4.2. \blacksquare

4.3. Necessity of condition (4.1)

In this subsection, we prove the necessity of the condition (4.1). The proof relies on the fact that the integral in (4.6) can be estimated from above by the integral over the function δ_Ω . This is the content of the following theorem.

Theorem 4.8. *Let Ω be a proper subdomain of \mathbb{C} which is starlike at infinity and such that the standard strip $\mathbb{S} = \{z \in \mathbb{C} : |\operatorname{Im} z| < \pi/2\}$ is a maximal horizontal strip contained in Ω . Then for any compact set $K \subset (-\pi/2, \pi/2)$ there are constants $c > 0$ and $C > 0$ such that, for any $y \in K$,*

$$(4.26) \quad \int_{-\infty}^0 \log \frac{\mathcal{R}(t + iy, \Omega)}{\mathcal{R}(t + iy, \mathbb{S})} dt \leq C + 2c \int_{-\infty}^0 \delta_\Omega(t) dt.$$

Hence Theorem 4.8 in conjunction with Lemma 3.2 implies that the integral of the ratio of conformal radii in (4.26) converges, when the domain Ω is conformal at $-\infty$ with respect to \mathbb{S} .

In order to prove Theorem 4.8, we construct a family of two-slit domains $D_0(t)$ that all contain the Koenigs domain Ω . For the sake of brevity, denote

$$\delta := \delta_\Omega(1 + t/2), \quad \delta_j := \delta_{\Omega, j}(1 + t/2), \quad j = 1, 2, \quad t \leq 0.$$

Due to Ω being starlike at infinity, it is easy to see that

$$D_0(t) := \mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Re} z \leq 1 + t/2 \text{ and } \operatorname{Im} z \in \{-\pi/2 - \delta_1, \pi/2 + \delta_2\}\}, \quad t \leq 0,$$

indeed contains Ω . Let $y \in (-\pi/2, \pi/2)$. Then

$$(4.27) \quad \log \frac{\mathcal{R}(t + iy, \Omega)}{\mathcal{R}(t + iy, \mathbb{S})} \leq \log \frac{\mathcal{R}(t + iy, D_0(t))}{\mathcal{R}(t + iy, \mathbb{S})}.$$

Furthermore, let $S(t) := \mathbb{S}(-\pi/2 - \delta_1, \pi/2 + \delta_2)$, the maximal strip between the two slits of the domain $D_0(t)$, and let

$$D(t) := D_0(t) - (1 + t/2) = \mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Re} z \leq 0 \text{ and } \operatorname{Im} z \in \{-\pi/2 - \delta_1, \pi/2 + \delta_2\}\}.$$

Bearing in mind that the hyperbolic metric in a strip remains invariant under horizontal translations, it follows that

$$(4.28) \quad \log \frac{\mathcal{R}(t + iy, \Omega)}{\mathcal{R}(t + iy, \mathbb{S})} \leq \log \frac{\mathcal{R}(t/2 - 1 + iy, D(t))}{\mathcal{R}(t/2 - 1 + iy, S(t))} + \log \frac{\mathcal{R}(iy, S(t))}{\mathcal{R}(iy, \mathbb{S})}.$$

The idea for the proof of Theorem 4.8 is to analyze the asymptotic behavior of the summands in (4.28). This task is rather elementary for the second term, while the first term requires a bit more delicate work making use of an estimate of the Green function, which we establish in the following lemma. Denote by D_* the “standard” two-slit domain $\mathbb{C} \setminus \{w \in \mathbb{C} : \operatorname{Re} w \leq 0, |\operatorname{Im} w| = \pi/2\}$.

Lemma 4.9. *For all $w \in \mathbb{S}$ with $\operatorname{Re} w < 0$ and all $s \in D_* \cap \partial\mathbb{S}$, we have*

$$(4.29) \quad G_{D_*}(w, s) \leq \log \frac{1 + e^{\operatorname{Re} w}}{1 - e^{\operatorname{Re} w}}.$$

Proof. Using the explicit formula for the Green function of \mathbb{S} , we have

$$(4.30) \quad \begin{aligned} G_{\mathbb{S}}(z_1, z_2) &= \frac{1}{2} \log \frac{\cosh(x_2 - x_1) + \cos(y_2 + y_1)}{\cosh(x_2 - x_1) + \cos(y_2 - y_1)} \\ &\leq \log \frac{1 + e^{-|x_2 - x_1|}}{1 - e^{-|x_2 - x_1|}}, \quad x_k := \operatorname{Re} z_k, y_k := \operatorname{Im} z_k, k = 1, 2, \end{aligned}$$

for any pair of points $z_1, z_2 \in \mathbb{S}$.

In order to apply the above formula for estimating G_{D_*} , we notice that

$$g(z) := z + \frac{1 + e^{2z}}{2}$$

maps \mathbb{S} conformally onto D_* . Indeed,

$$f(\xi) := g(\log \xi) = \log(\xi) + \frac{1 + \xi^2}{2}$$

is the Schwarz–Christoffel map of the right-half plane onto D_* with the normalization $f(0) = \infty$, $f(\pm i) = \pm i\pi/2$, as follows from the equality

$$\frac{f''(\xi)}{f'(\xi)} = \frac{1}{\xi - i} + \frac{1}{\xi + i} - \frac{1}{\xi}, \quad \operatorname{Re} \xi > 0.$$

Let $w \in U := \{w \in \mathbb{S} : \operatorname{Re} w < 0\}$. It is easy to see that $g(U) \supset U$. Therefore, $z := g^{-1}(w) \in U$. It follows that

$$(4.31) \quad \operatorname{Re} g^{-1}(w) = \operatorname{Re} w - (1 + \operatorname{Re} e^{2z})/2 < \operatorname{Re} w \quad \text{for all } w \in U.$$

Moreover, it is easy to see that $g(\{w \in \mathbb{S} : \operatorname{Re} w > 0\})$ contains the set $A := D_* \cap \partial\mathbb{S}$. Hence,

$$(4.32) \quad \operatorname{Re} g^{-1}(s) > 0 \quad \text{for all } s \in A.$$

Combining now (4.30), (4.31) and (4.32), we obtain

$$G_{D_*}(w, s) = G_{\mathbb{S}}(g^{-1}(w), g^{-1}(s)) \leq \log \frac{1 + e^{\operatorname{Re} w}}{1 - e^{\operatorname{Re} w}}$$

for any $w \in U$ and any $s \in A$, as desired. ■

Lemma 4.10. *For any $w \in U := \{w \in \mathbb{S} : \operatorname{Re} w < 0\}$, we have*

$$(4.33) \quad \log \frac{\mathcal{R}(w, D_*)}{\mathcal{R}(w, \mathbb{S})} \leq \log \frac{1 + e^{\operatorname{Re} w}}{1 - e^{\operatorname{Re} w}}.$$

Proof. According to (3.2), for any $w \in \mathbb{S}$, the left-hand side of (4.33) equals

$$\int_A G_{D_*}(w, s) \omega_{\mathbb{S}}(w, ds), \quad A := D_* \cap \partial \mathbb{S},$$

and inequality (4.33) follows immediately from (4.29), taken into account that $\omega_{\mathbb{S}}(w, \cdot)$ is a probability measure. ■

Proof of Theorem 4.8. Using the explicit formula for the conformal radius of a strip, it is easy to see that there exists a constant $c > 0$, depending only on the compact set $K \subset (-\pi/2, \pi/2)$, such that

$$(4.34) \quad \log \frac{\mathcal{R}(iy, S(t))}{\mathcal{R}(iy, \mathbb{S})} \leq c\delta = c\delta_{\Omega}(1 + t/2) \quad \text{for all } t \leq 0 \text{ and all } y \in K.$$

Moreover, recall that the function δ_{Ω} is finite and monotonic on $(-\infty, 1]$. In particular, it is integrable on $[0, 1]$. Therefore, in view of inequality (4.28), it remains to show that

$$(4.35) \quad \sup_{y \in K} \int_{-\infty}^0 \log \frac{\mathcal{R}(t/2 - 1 + iy, D(t))}{\mathcal{R}(t/2 - 1 + iy, S(t))} dt < +\infty.$$

The linear function

$$F_t(z) := \frac{\pi}{\pi + \delta_1 + \delta_2} \left(z + i \frac{\delta_1 - \delta_2}{2} \right),$$

maps the strip $S(t)$ conformally onto the standard strip \mathbb{S} , and the two-slit domain $D(t)$ onto D_* . Therefore, the integrand in (4.35) equals

$$\log \frac{\mathcal{R}(w(t), D_*)}{\mathcal{R}(w(t), \mathbb{S})}, \quad \text{where } w(t) := F_t(t/2 - 1 + iy).$$

By the monotonicity of $\delta_{\Omega, j}$, $j = 1, 2$,

$$\operatorname{Re} w(t) \leq \frac{\pi}{\pi + \delta_{\Omega, 1}(1) + \delta_{\Omega, 2}(1)} \left(\frac{t}{2} - 1 \right) \quad \text{for any } t \leq 0,$$

and we are done because now (4.35) follows from Lemma 4.10. ■

Proof of necessity in Theorem 4.1. Suppose that Ω is conformal at $-\infty$ with respect to S . By Lemma 3.2, δ_{Ω} is integrable over $(-\infty, 0]$. Hence, Theorem 4.8 shows that condition (4.1) holds. ■

4.4. Completion of the proof of Theorem 4.1: Locally uniform convergence of (4.1)

We are left to prove that if (4.6) holds for one point $w_0 \in S$, then it holds for all $w_0 \in S$ and the integral in (4.6) converges locally uniformly with respect to $w_0 \in S$. Clearly, we may assume that $S = \mathbb{S}$.

Let $S(t)$ and $D(t)$ be defined as in Subsection 4.3. For $t \leq 0$, $y \in (-\pi/2, \pi/2)$, and $k = 1, 2$ we denote

$$F_1(t, y) := \log \frac{\mathcal{R}(t/2 - 1 + iy, D(t))}{\mathcal{R}(t/2 - 1 + iy, S(t))}, \quad F_2(t, y) := \log \frac{\mathcal{R}(iy, S(t))}{\mathcal{R}(iy, \mathbb{S})},$$

$$J_k(y) := \int_{-\infty}^0 F_k(t, y) dt.$$

The function δ_Ω is monotonic on $(-\infty, 1]$, and hence integrable on any compact interval contained in $(-\infty, 1]$. Therefore, from Theorem 4.2 it follows that if (4.6) holds for at least one point $w_0 \in \mathbb{S}$, then the function δ_Ω is integrable on $(-\infty, 1]$.

Thanks to inequality (4.34), the integrability of δ_Ω on $(-\infty, 1]$ implies that the integral $J_2(y)$ above converges uniformly with respect to y on any compact interval $[y_1, y_2] \subset (-\pi/2, \pi/2)$. Moreover, the argument used in the proof of Theorem 4.8 shows that the integral $J_1(y)$ converges uniformly on the whole interval $(-\pi/2, \pi/2)$.

Now let $w_0 := x + iy \in \mathbb{S}$. By inequality (4.28), for all $t \leq \min\{-x, 0\}$ we have

$$0 \leq \log \frac{\mathcal{R}(t + w_0, \Omega)}{\mathcal{R}(t + w_0, \mathbb{S})} \leq F_1(t + x, y) + F_2(t + x, y).$$

Thus, for each rectangle $R := [x_1, x_2] \times [y_1, y_2]$ contained in \mathbb{S} , the integral in (4.6) converges uniformly with respect to $w_0 \in R$. This completes the proof. \blacksquare

5. The angular derivative problem for parabolic petals

In this section, we study conformality of parabolic petals at the Denjoy–Wolff point. An important role is played by the so-called hyperbolic step

$$q(z, s) := \lim_{t \rightarrow +\infty} d_{\mathbb{D}}(\phi_{t+s}(z), \phi_t(z)),$$

where $d_{\mathbb{D}}$ stands for the hyperbolic distance in \mathbb{D} . If $q(z, s) = 0$ for some $z \in \mathbb{D}$ and $s > 0$, then $q \equiv 0$ on $\mathbb{D} \times (0, +\infty)$, and the one-parameter semigroup (ϕ_t) is said to be of *zero hyperbolic step*. Otherwise, i.e., if $q(z, s) > 0$ for some (and hence all) $z \in \mathbb{D}$ and $s > 0$, we say that (ϕ_t) is of *positive hyperbolic step*.

Theorem 5.1. *Let Δ be a parabolic petal of a one-parameter semigroup (ϕ_t) with Denjoy–Wolff point τ and associated Koenigs function h . The following statements hold.*

(A) *The petal Δ is conformal at τ with respect to \mathbb{D} if and only if the angular limit*

$$L := \angle \lim_{z \rightarrow \tau} (z - \tau)h(z)$$

is finite.

(B) *If (ϕ_t) is of positive hyperbolic step, then Δ is conformal at τ with respect to \mathbb{D} (and hence $L \neq \infty$).*

Remark 5.2. The angular limit L in statement (A) above was previously considered by Contreras, Díaz-Madriral and Pommerenke in a more general context of discrete iteration, see Theorems 4.1 and 6.2 in [22]. Moreover, this limit is directly related to the conformality problem considered by Betsakos (Theorem 3 in [8]) and Karamanlis (Theorem 2 in [31]) for parabolic one-parameter semigroups of positive hyperbolic step. As we will see in the proof of Theorem 5.1, the presence of a parabolic petal ensures that the angular limit L exists and does not vanish, but it can be infinite. In fact, as illustrated by Example 5.4 given after the proof, for semigroups of zero hyperbolic step, both cases $L \in \mathbb{C} \setminus \{0\}$ and $L = \infty$ are possible.

Proof of Theorem 5.1. First of all, recall that the existence of a parabolic petal Δ implies that the one-parameter semigroup (ϕ_t) is parabolic. Moreover, the image of Δ with respect to the Koenigs function h of (ϕ_t) is a half-plane bounded by a line parallel to \mathbb{R} .

If (ϕ_t) is of positive hyperbolic step, then $\Omega := h(\mathbb{D})$ is contained in some half-plane U with ∂U parallel to \mathbb{R} , see Theorem 9.3.5 in [14]. Let T be a linear-fractional transformation of \mathbb{D} onto U with $T(1) = \infty$. Choose a number $c \in \mathbb{C}$ such that $w \mapsto w + c$ maps U onto $h(\Delta)$. Then

$$\varphi(z) := h^{-1}(T(z) + c), \quad z \in \mathbb{D},$$

is a conformal mapping of \mathbb{D} onto Δ with $\varphi(1) = \tau$ in the sense of the angular limit. Further, denote $\psi := T^{-1} \circ h : \mathbb{D} \rightarrow \mathbb{D}$. Since $\psi \circ \varphi = T^{-1} \circ (T + c)$ has a regular contact point at $\zeta = 1$, by the chain rule for angular derivatives, see e.g. Lemma 2 in [21] or Lemma 5.1 in [29], φ also has a regular contact point at $\zeta = 1$, which means that Δ is conformal at τ with respect to \mathbb{D} . This proves statement (B).

To prove (A), fix some $w_0 \in \mathbb{C} \setminus \Omega$ and denote $g := 1/(h - w_0)$. Further denote by D the image of the half-plane $h(\Delta)$ with respect to the map $w \mapsto 1/(w - w_0)$. Clearly, D is a disk contained in $g(\mathbb{D})$, with $0 \in \partial D$. Let f be the linear function mapping \mathbb{D} onto D and normalized by $f(\tau) = 0$. Then for a suitably chosen circular arc $\Gamma \subset \mathbb{D}$ with the end-point at τ , $g^{-1}(f(\Gamma))$ is the forward orbit of some point $z_0 \in \Delta$ with respect to the semigroup (ϕ_t) . In particular, $g^{-1}(f(z)) \rightarrow \tau$ as $\Gamma \ni z \rightarrow \tau$. Therefore, by the comparison theorem, see e.g. Theorem 10.6 on p. 307 of [36], there exists a finite angular derivative $g'(\tau)$. In turn, this implies that $L := \angle \lim_{z \rightarrow \tau} (z - \tau)h(z)$ exists, with $L = 1/g'(\tau) \in \overline{\mathbb{C}} \setminus \{0\}$.

Note that $\varphi(\zeta) := g^{-1}(f(\tau\zeta))$ is a conformal mapping of \mathbb{D} onto Δ , and that by Lindelöf's theorem, see e.g. Theorem 9.3 on p. 268 of [36], $\angle \lim_{\zeta \rightarrow \tau} \varphi(\zeta) = \tau$. Recall that since $\varphi \in \text{Hol}(\mathbb{D}, \mathbb{D})$ and since $\zeta = 1$ is a contact point for φ , the angular derivative

$$\varphi'(1) := \angle \lim_{\zeta \rightarrow 1} \frac{\varphi(\zeta) - \tau}{\zeta - 1}$$

does exist, finite or infinite, and $\varphi'(1) \neq 0$. The following simple calculation,

$$\tau f'(\tau) = (g \circ \varphi)'(1) = \lim_{(0,1) \ni x \rightarrow 1} \left(\frac{g(\varphi(x))}{\varphi(x) - \tau} \cdot \frac{\varphi(x) - \tau}{x - 1} \right),$$

shows that

$$R(z) := \frac{g(z)}{z - \tau}, \quad z \in \mathbb{D},$$

has at $z = \tau$ a finite asymptotic value $\tau f'(\tau)/\varphi'(1)$. Taking into account that g is univalent and does not vanish in \mathbb{D} , and arguing as in the proof of Theorem 10.5 on pp. 305–306 of [36], we see that $g'(\tau) = \angle \lim_{z \rightarrow \tau} R(z) = \tau f'(\tau)/\varphi'(1)$. It follows that $L = 1/g'(\tau) \neq \infty$ if and only if $\varphi'(1) \neq \infty$. This completes the proof of (A). ■

Remark 5.3. Note that the centre of the disk D considered in the above proof lies on the imaginary axis. It follows that under the hypothesis of Theorem 5.1, if $L \neq \infty$ then $\operatorname{Re}(\bar{\tau}L) = 0$.

Example 5.4. Let $h_1(z) := z/(1-z)^2$ be the classical Koebe function, and let

$$h_2(z) := w(z) - i\sqrt{w(z)}, \quad \text{where } w(z) := i(1+z)/(1-z), \quad z \in \mathbb{D},$$

and \sqrt{w} stands for the branch of the square root that maps the upper half-plane onto the first quadrant. It is not difficult to see that $\operatorname{Re}(1-z)^2 h'_k(z) > 0$ for all $z \in \mathbb{D}$, $k = 1, 2$. Therefore, see e.g. Theorem 9.4.11 in p. 257 of [14], the h_k 's are univalent in \mathbb{D} , and the formula

$$\phi_t^k := h_k^{-1} \circ (h_k + t), \quad t \geq 0, \quad k = 1, 2,$$

defines two parabolic one-parameter semigroups with the DW-point at $\tau = 1$. Note that $h_1(\mathbb{D}) = \mathbb{C} \setminus (-\infty, -1/4]$ and $h_2(\mathbb{D}) = \{\zeta : \operatorname{Re} \zeta > f(\operatorname{Im} \zeta)\}$, where $f(\eta) := -\infty$ if $\eta > 0$, $f(0) := 1/4$, and $f(\eta) := \eta^2$ if $\eta < 0$. Since the image domains $h_k(\mathbb{D})$ are not contained in any half-plane, both semigroups (ϕ_t^k) , $k = 1, 2$, are of zero hyperbolic step. Clearly, $\lim_{z \rightarrow 1} (z-1)h_1(z) = \infty$. The corresponding semigroup (ϕ_t^1) has two non-conformal parabolic petals $\{z \in \mathbb{D} : \pm \operatorname{Im} z > 0\}$. At the same time, $\angle \lim_{z \rightarrow 1} (z-1)h_2(z) = -2i \neq \infty$. Since $\{\zeta : \operatorname{Im} \zeta > 0\} \subset h_2(\mathbb{D})$, the semigroup (ϕ_t^2) has a parabolic petal. By Theorem 5.1 (A), this parabolic petal is conformal at τ with respect to \mathbb{D} .

6. Concluding remarks and open questions

6.1. Rate of convergence of regular backward orbits

If $\sigma \in \partial\mathbb{D}$ is a repulsive fixed point of a one-parameter semigroup of (ϕ_t) , then as it was proved in Proposition 4.20 of [15],

$$\lim_{t \rightarrow -\infty} \frac{1}{t} \log |\phi_t(z) - \sigma| = \lambda := G'(\sigma) \quad \text{for any } z \in \Delta(\sigma),$$

where G stands for the infinitesimal generator of (ϕ_t) . It is therefore natural to ask whether the limit

$$(6.1) \quad C(\sigma, z) := \lim_{t \rightarrow -\infty} e^{-\lambda t} |\phi_t(z) - \sigma|$$

does exist for $z \in \Delta(\sigma)$. The answer is immediate if we recall that (ϕ_t) admits at σ a pre-model $(\mathbb{H}, \psi, z \mapsto e^{\lambda t} z)$, where ψ is a conformal mapping of \mathbb{H} onto $\Delta(\sigma)$ with $\psi(0) = \sigma$; see Remark 2.14. We have $\phi_t(z) = \psi(e^{\lambda t} \psi^{-1}(z))$ for any $z \in \Delta(\sigma)$ and any $t \in \mathbb{R}$. Hence, for all $z \in \Delta(\sigma)$,

$$C(\sigma, z) = \lim_{t \rightarrow -\infty} e^{-\lambda t} |\psi(e^{\lambda t} \psi^{-1}(z)) - \sigma| = |\psi^{-1}(z) \psi'(0)|.$$

If the hyperbolic petal $\Delta(\sigma)$ is conformal, then by Remark 2.20, $\psi'(0) \in \mathbb{C} \setminus \{0\}$, and hence the limit (6.1) exists finitely and does not vanish for all points z in $\Delta(\sigma)$. If $\Delta(\sigma)$ is not conformal, then $\psi'(0) = \infty$ and hence $C(z, \sigma) = +\infty$ for all $z \in \Delta(\sigma)$.

Thus, for $z_0 \in \Delta(\sigma)$, condition (1.1) in Theorem 1.1 is equivalent to having that $C(z_0, \sigma) \in (0, +\infty)$. Since the backward orbits in $\Delta(\sigma)$ converge to σ non-tangentially, see Remark 2.15, $C(z_0, \sigma) \in (0, +\infty)$ if and only if $\frac{1}{2} \log(e^{-\lambda t}(1 - |\phi_t(z_0)|^2))$ tends to a finite limit as $t \rightarrow -\infty$. In turn, since by Remark 2.11, in $\Delta(\sigma)$ the ODE (2.1) holds for all $t \in \mathbb{R}$, the latter condition can be restated as the convergence of the integral

$$(6.2) \quad J(z_0) := \int_{-\infty}^0 \left(\frac{\lambda}{2} + \frac{\operatorname{Re}[G(\phi_t(z_0)) \overline{\phi_t(z_0)}]}{1 - |\phi_t(z_0)|^2} \right) dt.$$

At the same time, combining (2.1) and (2.2), we get $\phi_t'(z_0) = G(\phi_t(z_0))/G(z_0)$. It follows that condition (1.1) in Theorem 1.1 is equivalent to the convergence of

$$(6.3) \quad I(z_0) := \int_{-\infty}^0 \left(\frac{A(z_0)}{2} - \frac{|G(\phi_t(z_0))|}{1 - |\phi_t(z_0)|^2} \right) dt,$$

where $A(z_0) := \lim_{t \rightarrow -\infty} |G(\phi_t(z_0))| \lambda_{\mathbb{D}}(\phi_t(z_0)) = |G(z_0)| \lambda_{\Delta(\sigma)}(z_0)$.

6.2. Boundary behaviour of the Koenigs function

For a non-elliptic one-parameter semigroup (ϕ_t) , the conformality of hyperbolic petals is related to the boundary behaviour of the Koenigs function h . Let $\sigma \in \partial\mathbb{D}$ be a repulsive fixed point of (ϕ_t) . Since $S := h(\Delta(\sigma))$ is a maximal strip in $\Omega := h(\mathbb{D})$, by a result of Betsakos [9], the angular limit

$$v := \angle \lim_{z \rightarrow \sigma} (z - \sigma) h'(z) \in (0, +\infty)$$

exists and equals the width of the strip S divided by π . (Since $h' = 1/G$, the existence of the above limit follows also from [21, Theorem 1].)

For a suitable $b \in \mathbb{C}$, the function $\psi(\zeta) := h^{-1}(b + v \log \zeta)$ maps the right half-plane \mathbb{H} onto $\Delta(\sigma)$. Hence, the conformality of the hyperbolic petal $\Delta(\sigma)$ is equivalent to $\psi'(0) \neq \infty$. Thanks to the isogonality property (2.8), one can use the change of variables $\zeta := \psi^{-1}(z)$ to obtain

$$(6.4) \quad \angle \lim_{z \rightarrow \sigma} [h(z) - v \log(1 - \bar{\sigma}z)] = b - v \log(-\bar{\sigma}\psi'(0)).$$

The above limit exists, and is finite or infinite, because $\psi'(0)$ exists with $-\bar{\sigma}\psi'(0) \in (0, +\infty) \cup \{+\infty\}$. Thus, $\Delta(\sigma)$ is conformal if and only if the limit in the left-hand side of (6.4) is *finite*.

6.3. Semigroups with symmetry with respect to the real line

Consider a one-parameter semigroup (ϕ_t) in \mathbb{D} with a repulsive fixed point at $\sigma = -1$ and such that $\phi_t((-1, 1)) \subset (-1, 1)$ for all $t \geq 0$. Fix some $z_0 \in \Delta(-1) \cap \mathbb{R}$. In this rather special case, Theorem 1.1 (excluding the part concerning uniformity of convergence) admits a simple proof based on the following elementary lemma.

Lemma 6.1. *Let f be a function continuous on $[0, +\infty)$ and of class C^2 on $(0, +\infty)$. Suppose that*

- (i) $f(0) = 0$ and $f'(x) > 0$ for all $x \in (0, +\infty)$;
- (ii) $g(x) := f(x)/(xf'(x)) \geq 1$ for all $x \in (0, +\infty)$.

Fix $\xi_0 > 0$. If $f'(x) \rightarrow A$ as $x \rightarrow +\infty$ for some $A \in (0, +\infty)$, then

$$(6.5) \quad I := \int_0^{\xi_0} \frac{\log g(x)}{x} dx < +\infty.$$

If $f(x)/x \rightarrow +\infty$ as $x \rightarrow +\infty$, then $I = +\infty$.

Proof. Suppose that $f'(x) \rightarrow A \in (0, +\infty)$ as $x \rightarrow +\infty$. Then $f(x)/x \rightarrow A$ and $g(x) \rightarrow 1$ as $x \rightarrow +\infty$. Therefore, taking into account (ii), we see that the improper integral I converges because

$$\int_0^{\xi_0} \left(1 - \frac{1}{g(x)}\right) \frac{dx}{x} = \int_0^{\xi_0} \left(\frac{1}{x} - \frac{f'(x)}{f(x)}\right) dx = \lim_{a \rightarrow 0^+} \log \frac{x}{f(x)} \Big|_{x=a}^{x=\xi_0} \in \mathbb{R}.$$

Similarly, if $f(x)/x \rightarrow +\infty$ as $x \rightarrow +\infty$, then

$$I \geq \int_0^{\xi_0} \left(1 - \frac{1}{g(x)}\right) \frac{dx}{x} = \lim_{a \rightarrow 0^+} \log \frac{x}{f(x)} \Big|_{x=a}^{x=\xi_0} = +\infty. \quad \blacksquare$$

Proof of Theorem 1.1 in the symmetric case. Passing from the unit disk \mathbb{D} to \mathbb{H} with the help of the Cayley map

$$H(z) := \frac{1+z}{1-z},$$

we get a one-parameter semigroup (φ_t) in \mathbb{H} with a repulsive fixed point at 0 and such that $\varphi_t((0, +\infty)) \subset (0, +\infty)$ for all $t \geq 0$. It follows that the hyperbolic petal D of (φ_t) with the α -point at 0 is symmetric with respect to the real line. Therefore, for all $t \in \mathbb{R}$, all $z \in D$, and a suitable $\lambda > 0$, we have

$$\varphi_t(z) = f(e^{\lambda t} f^{-1}(z)),$$

where f is a conformal mapping of \mathbb{H} onto D with $f(0) = 0$ and $f((0, +\infty)) \subset (0, +\infty)$. As a result, for $z_0 \in \Delta(-1) \cap \mathbb{R}$, the integral in (1.1) equals

$$(6.6) \quad \int_{-\infty}^0 \log \frac{f(e^{\lambda t} \xi_0)}{e^{\lambda t} \xi_0 f'(e^{\lambda t} \xi_0)} dt = \frac{1}{\lambda} \int_0^{\xi_0} \left(\log \frac{f(x)}{x f'(x)} \right) \frac{dx}{x},$$

where $\xi_0 := f^{-1}(H(z_0))$.

Applying the Schwarz–Pick lemma, see e.g. Theorem 6.4 in [6], to $f: \mathbb{H} \rightarrow \mathbb{H}$, it is easy to see that the restriction of f to $[0, +\infty)$ satisfies the hypothesis of Lemma 6.1. If $\Delta(-1)$ is conformal, then $f'(x) \rightarrow f'(0) \in (0, +\infty)$ as $(0, +\infty) \ni x \rightarrow 0$, and hence by Lemma 6.1, the integral (6.6) converges.

Similarly, if $\Delta(-1)$ is not conformal, then $f(x)/x \rightarrow +\infty$ as $(0, +\infty) \ni x \rightarrow 0$. In this case, Lemma 6.1 guarantees that the integral (6.6) diverges. \blacksquare

The above proof of Theorem 1.1 for the symmetric case is based on the observation that $(\phi_t)_{t \in \mathbb{R}}$ is a one-parameter group of hyperbolic automorphisms of $\Delta(\sigma)$. In contrast to our proof for the general case, the fact that for $t \geq 0$, the ϕ_t 's are well-defined holomorphic functions in the whole unit disk is not essential for the proof in the symmetric case.

Attempting to adapt the method used in this section to the general case, instead of the integral (6.5), one would need to consider

$$(6.7) \quad I_1(\theta) := \int_0^{\rho_0} \left(\log \frac{\operatorname{Re} f(\rho e^{i\theta})}{|f'(\rho e^{i\theta})| \rho \cos \theta} \right) \frac{d\rho}{\rho}, \quad \theta \in (-\pi/2, \pi/2),$$

where $\rho_0 > 0$ is fixed. Since f is isogonal at 0, $\operatorname{Re} f(\rho e^{i\theta})/|f(\rho e^{i\theta})| \rightarrow \cos \theta$ as $\rho \rightarrow 0^+$. Hence it would be also reasonable to consider

$$(6.8) \quad I_2(\theta) := \int_0^{\xi_0} \log \left| \frac{f(\rho e^{i\theta})}{\rho f'(\rho e^{i\theta})} \right| \frac{d\rho}{\rho}.$$

Note that the argument of $\log |\cdot|$ in (6.8) tends to 1 as $\rho \rightarrow 0^+$, because by Proposition 4.11 on p. 81 of [37], the conformal map $f \circ H_1$, with $H_1(z) := (1 - z)/(1 + z)$, satisfies at $\zeta = 1$ the Visser–Ostrowski condition.

6.4. Hyperbolic length of backward orbits

For a hyperbolic domain $D \subset \mathbb{C}$ and a rectifiable curve $\gamma: [0, T] \rightarrow D$, we denote by

$$\ell_D(\gamma) := \int_\gamma \lambda_D(z) |dz|$$

the hyperbolic length of γ . It is easily checked that Theorem 1.1 can be restated as follows: *a hyperbolic petal Δ of a one-parameter semigroup (ϕ_t) in \mathbb{D} is conformal if and only if*

$$(6.9) \quad \lim_{T \rightarrow +\infty} [\ell_\Delta(\gamma_z|_{[0, T]}) - \ell_{\mathbb{D}}(\gamma_z|_{[0, T]})] < +\infty$$

for any backward orbit $\gamma_z(t) := \phi_{-t}(z)$, $z \in \Delta$, in the petal Δ . Note that the above limit always exists, finite or infinite, because $\Delta \subset \mathbb{D}$ and hence $\lambda_\Delta(z) \geq \lambda_{\mathbb{D}}(z)$ for all $z \in \Delta$.

6.5. The angular derivative problem and hyperbolic length

Theorem 1.1 is closely related to a recent conformality condition due to Betsakos and Karamanlis [11]. We briefly discuss this issue. The conformality conditions obtained in [11] are valid for any simply connected domain $\Omega \subsetneq \mathbb{C}$ containing the real line \mathbb{R} . In order to compare the results of [11] with our Theorem 4.1, we additionally assume that Ω contains the standard strip \mathbb{S} . Taking into account Ostrowski's characterization of semi-conformality, see e.g. Theorem A in [11], we see that in this case, Theorem 1 in [11] says that Ω is conformal at $-\infty$ with respect to \mathbb{S} if and only if

$$(6.10) \quad \operatorname{dist}(z, \partial\Omega) := \inf_{w \in \partial\Omega} |z - w| \rightarrow 0 \quad \text{as } \operatorname{Re} z \rightarrow -\infty, z \in \partial\mathbb{S},$$

and

$$(6.11) \quad d_{\mathbb{S}}(iy + a, iy + b) - d_{\Omega}(iy + a, iy + b) \rightarrow 0 \quad \text{as } a, b \rightarrow -\infty, a, b \in \mathbb{R},$$

for some and hence any $y \in (-\pi/2, \pi/2)$. In fact, in [11] the conformality condition (6.11) is stated only for $y = 0$, but in our case we additionally have $\mathbb{S} \subset \Omega$, and the proof in [11] also works for $y \in (-\pi/2, \pi/2)$.

If Ω is the Koenigs domain of a one-parameter semigroup (ϕ_t) and \mathbb{S} is a *maximal* strip contained in Ω , then condition (6.10) is automatically satisfied.

On the other hand, it is easy to see that Theorem 4.1 says that Ω is conformal at $-\infty$ with respect to \mathbb{S} if and only if for some and hence all $y \in (-\pi/2, \pi/2)$,

$$(6.12) \quad \int_a^b (\lambda_{\mathbb{S}}(iy + x) - \lambda_{\Omega}(iy + x)) dx \rightarrow 0 \quad \text{as } a, b \rightarrow -\infty, \quad a, b \in \mathbb{R}.$$

Let us compare conditions (6.11) and (6.12).

Since $\int_a^b \lambda_{\mathbb{S}}(x) dx = b - a = d_{\mathbb{S}}(a, b)$ and $\int_a^b \lambda_{\Omega}(x) dx \geq d_{\Omega}(a, b)$ for any $a < b$, $a, b \in \mathbb{R}$, condition (6.11) for $y = 0$ implies condition (6.12) for $y = 0$. Thus, if Ω is conformal at $-\infty$ with respect to \mathbb{S} and if $\mathbb{S} \subset \Omega$, then regardless of whether Ω is starlike at infinity or not, our condition (6.12) holds for $y = 0$.

A similar remark applies to the relation between (6.12) and the following necessary and sufficient condition for conformality established by Bracci et al. (see equation (8.2) in [15]):

$$(6.13) \quad \limsup_{x \rightarrow -\infty} [d_{\mathbb{S}}(iy, iy + x) - d_{\Omega}(iy, iy + x)] < +\infty$$

for some and hence all $y \in (-\pi/2, \pi/2)$. Arguing as above, one can see that (6.13) implies (6.12), but apparently, only for $y = 0$. Although Section 8 in [15] addresses the angular derivative problem in the context of one-parameter semigroups, the proof of (6.13) does not depend on the fact that Ω is starlike at infinity; it actually works for any simply connected domain $\Omega \subsetneq \mathbb{C}$ containing \mathbb{S} .

6.6. Open questions

To conclude, we state several open questions. Let (ϕ_t) be a non-elliptic one-parameter semigroup with associated infinitesimal generator G , Koenigs function h , and Koenigs domain $\Omega := h(\mathbb{D})$. Further, let $\Delta(\sigma)$ be a hyperbolic petal of (ϕ_t) with α -point σ . As above, for simplicity we suppose that $h(\Delta(\sigma)) = \mathbb{S}$.

Question 1. Similarly to Theorem 1 in [11] by Betsakos and Karamanlis, our condition (1.1), as well as its restatements (4.1) and (6.12), uses hyperbolic geometry. However, in the proof we make use of Euclidean quantities related to the Koenigs domain Ω . Is it possible to prove one (or even both) of the implications in Theorem 1.1 *without* employing criteria for conformality in terms of Euclidean geometry?

One possible way to answer the above question would be to study in detail the relations between the convergence of the integrals $I(z_0)$ and $J(z_0)$ introduced in Section 6.1. An alternative direction is indicated in the next question.

Question 2. If Ω is starlike at infinity, then the conformality condition (6.11) due to Bet-sakos and Karamanlis and our condition (6.12) are equivalent in terms of Theorem 4.1 and Theorem 1 in [11]. Is there a more direct way to prove the equivalence (6.11) and (6.12) for such domains? How are conditions (6.11) and (6.12) related, when Ω is semi-conformal at $-\infty$ in the sense of Ostrowski, see e.g. Theorem A in [11], but not necessarily starlike at infinity?

In the elementary proof of Theorem 1.1 for the symmetric case, given in Section 6.3, we consider a generic injective holomorphic self-map isogonal at a contact point. In contrast to the general (non-symmetric) case, the argument does not depend on the fact that the image of the self-map is a hyperbolic petal of a one-parameter semigroup in \mathbb{D} .

On the one hand, the argument in Section 6.3 works only for the backward orbits contained in $(-1, \tau)$. This is similar to the situation in Section 6.5, where restricting to the symmetry line $y = 0$ apparently becomes necessary at some point.

On the other hand, motivated by the symmetric case, it is natural to ask whether even in the general case, Theorem 1.1 remains valid for a semigroup of hyperbolic automorphisms $(\phi_t)_{t \geq 0} \subset \text{Aut}(\Delta)$ of a simply connected domain $\Delta \subset \mathbb{D}$, provided that all the backward orbits converge to the point $\sigma \in \partial\mathbb{D}$ at which Δ is isogonal, but we do not assume that (ϕ_t) extends to a one-parameter semigroup of holomorphic self-maps of the whole unit disk \mathbb{D} . This leads to the following problem.

Question 3. Let $f: \mathbb{H} \rightarrow \mathbb{H}$ be an injective holomorphic self-map with a boundary fixed point at $\zeta = 0$. Suppose that f is isogonal at $\zeta = 0$. Is there any relation between the convergence of the integrals (6.7) and (6.8) introduced in Section 6.3 and the finiteness of the angular derivative $f'(0)$?

A closely related question is as follows. Let φ be a conformal mapping of \mathbb{D} onto the hyperbolic petal $\Delta(\sigma)$ satisfying $\varphi(-1) = \sigma$ and $\varphi(1) = \tau$. According to Theorems 1 and 3 in [26], the mapping φ is a solution to the non-linear ODE

$$(6.14) \quad G'(\sigma)(1 - z^2) \varphi'(z) = 2G(\varphi(z)), \quad z \in \mathbb{D}.$$

Question 4. What kind of non-trivial conclusion about the infinitesimal generator G can be drawn from the equation (6.14), if we suppose that the petal $\Delta(\sigma)$ is conformal?

The next open question concerns the geometry of hyperbolic petals near the Denjoy–Wolff point τ of (ϕ_t) .

Question 5. Recall that $\tau \in \partial\Delta(\sigma)$. Let $\tilde{\varphi}$ be a conformal mapping of \mathbb{D} onto $\Delta(\sigma)$ with $\tilde{\varphi}(1) = \tau$. What is the asymptotic behaviour of $\tilde{\varphi}(z)$ as $z \rightarrow 1$ within a Stolz angle? In particular, if (ϕ_t) is hyperbolic, then using results of Contreras and Díaz-Madrigal [18], it is possible to show that $\partial\Delta(\sigma)$ has a corner of opening $\alpha\pi$ at τ , with $\alpha := |G'(\tau)|/G'(\sigma)$. Is it always true (and if not always, then under which conditions) that the function

$$f(z) := (\tilde{\varphi}(z) - \tau)^{1/\alpha}, \quad z \in \mathbb{D},$$

is conformal at $z = 1$, i.e., f has angular derivative $f'(1) \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$?

We conclude the paper with a question on parabolic petals. Theorem 5.1, in case of semigroups of zero hyperbolic step, reduces the angular derivative problem for parabolic

petals to another problem of similar nature for the Koenigs function h . Although the limit relation (1.2) holds for parabolic petals as well, our Theorem 1.1 does not seem to extend to this case. So it is natural to raise the following question.

Question 6. Is it possible to characterize conformality of *parabolic* petals in terms of the intrinsic hyperbolic geometry of the petal and the backward (or forward) dynamics of the semigroup, without involving the Koenigs function of the semigroup?

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