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Regularity of the Boundary of a Capillary Drop on an Inhomogeneous Plane and Related Variational Problems

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1. Introduction

Consider the functional

$$J(E) = \int_{\Omega_0} |D\phi_E| + g \int_{\Omega_0} x_{n+1}\phi_E \, dx - \int_{\partial\Omega_0} \lambda(x')\phi_E \, dx' \tag{1.1}$$

where $\Omega_0 = \{x = (x', x_{n+1}), x_{n+1} > 0\}$, $x' = (x_1, \ldots, x_n)$ varies in $\mathbb{R}^n$, g is a positive constant, $\lambda(x')$ is a given function, $|\lambda(x')| < 1$, ϕ_E is the characteristic function of a set E, and E varies in the class

$$\mathcal{A} = \{E \subset \bar{\Omega}_0; E \text{ has finite perimeter } \int |D\phi_E|\}. \tag{1.2}$$

For a given positive number V set

$$\mathcal{A}_V = \{E \in \mathcal{A}, H^{n+1}(E) = V\}. \tag{1.3}$$

Problem $(\mathcal{A}_V)$: Find E in $\mathcal{A}_V$ such that

$$J(E) = \min_{F \in \mathcal{A}_V} J(F).$$

For $n = 2$ this is precisely the sessile drop problem, i.e., the problem of a capillary drop occupying the set E and sitting on in inhomogeneous plane $\{x_{n+1} = 0\}$. The first term in $J(E)$ is the energy due to surface tension, the second term is the gravitational energy, and the last term is the wetting energy with contact angle $\theta(x')$ given by

$$\cos\theta(x') = \lambda(x'), \qquad 0 < \theta(x') < \pi.$$

In (the homogeneous) case $\lambda \equiv$ const., a minimizer E can be found having the form

$$E = \{x; |x'| < \rho(x_{n+1})\}$$

(see [11] [12]). In case $\lambda \not\equiv$ const., existence of a minimizer E was recently established by Caffarelli and Spruck [4] under some mild assumptions on $\lambda(x')$. For a strictly curved bottom $\partial\Omega_0$, existence was proved by Giusti [10].

We shall restrict λ to satisfy

$$0 < \lambda < 1; \tag{1.4}$$

then one can show that E is an x_{n+1}-subgraph that is

$$E = \{x; 0 \geqslant x_{n+1} < u(x'), x' \in S\} \tag{1.5}$$

for some function u with support S. In this paper we are interested in studying the boundary ∂S of S; ∂S may be conceived as the free boundary for the sessile drop problem. We prove that, for the case $n = 2$, ∂S is regular; more precisely,

$$\begin{aligned} &\text{if} \quad \lambda \in C^{m+\alpha} \quad \text{then} \quad \partial S \in C^{m+1+\alpha}; \\ &\text{if} \quad \lambda \quad \text{is analytic then} \quad \partial S \quad \text{is analytic;} \end{aligned} \tag{1.6}$$

the same holds for $3 \leqslant n \leqslant 6$ under some «flatness condition.»

Our method is based on extensions of the results of [1] and [2] to the minimal surface operator. To explain this connection, consider the functional

$$J(v) = \int_{\Omega\cap\{v>0\}} f(x, v, \nabla v)\, dx \tag{1.7}$$

where Ω is, say, a bounded domain in $\mathbb{R}^n$ and v varies in the class of $H^{1,2}(\Omega)$ functions satisfying a boundary condition $v \doteq u^0$ on a portion $\partial_0\Omega$ of $\partial\Omega$; $u^0 \geqslant 0$. If u is a minimizer, then (see [1]) formally

$$\nabla \cdot f_p(x, u, \nabla u) - f_z(x, u, \nabla u) = 0 \quad \text{in} \quad \Omega\cap\{u > 0\} \tag{1.8}$$

where $z = u$, $p = \nabla u$, and $u = 0$,

$$f_p(x, u, \nabla u) \cdot \nabla u - f(x, u, \nabla u) = 0 \text{ on the free boundary } \Omega \cap \partial\{u > 0\}. \quad (1.9)$$

Taking in particular

$$f(x, u, \nabla u) = \sqrt{1 + |\nabla u|^2} + \frac{g}{2}u^2 - \mu u - \lambda(x) \qquad (\mu \geqslant 0), \tag{1.10}$$

we get

$$\operatorname{div}\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} - gu = -\mu \quad \text{in} \quad \Omega \cap \{u > 0\}, \tag{1.11}$$

$$u = 0, \qquad \frac{1}{1 + |\nabla u|^2} = \lambda^2 \quad \text{on} \quad \Omega \cap \partial\{u > 0\}. \tag{1.12}$$

(This incidentally shows that for a regular free boundary to exist one must assume that $|\lambda| < 1$.)

Observe now that the functional (1.1) for E a subgraph (as in (1.5)) reduces to the functional of the form (1.7) with f given by (1.10) and $\mu = 0$. The sessile drop problem, however, includes a volume constraint $H^{n+1}(E) = V$, which can actually be replaced by adding a "penalty" term $f_{\epsilon 0}(V_E)$ into the functional, where $V_E = H^{n+1}(E)$ for any $E \in \mathcal{Q}$. More precisely, Caffarelli and Spruck [4] introduce the functional

$$\tilde{J}(F) = J(F) + f_{\epsilon 0}(V_F) \qquad (F \in \mathcal{Q})$$

where $f_{\epsilon 0}(t) = (V - t)/\epsilon_0$ if $t < V$, $f_{\epsilon 0}(t) = 0$ if $t \geqslant V$ and prove that if ϵ_0 is positive and small enough then a minimizer E exists, $V_E = V$ and E is a solution of problem $(\mathcal{Q}_V)$.

The methods of the present paper apply also to the modified functional $\tilde{J}$. For the sake of clarity we shall first establish the regularity result (1.6) of the free boundary for the variational problem involving (1.7), (1.10) and then consider the sessile drop problem, indicating the minor changes in the proof.

The regularity of the free boundary for the variational problem for (1.7) was established by Alt and Caffarelli [1] in case $f(x, z, p) = |p|^2$ (corresponding to the Laplace operator), and by Alt, Caffarelli and Friedman [2] in the case of general $f(p) = F(|p|^2)$ corresponding to quasi-linear uniformly elliptic operator; the case $f = |p|^2 - Q(x)z$ with $Q > 0$ was considered by Friedman [5]. The main novelty of the present paper stems from the fact that the quasi-linear elliptic operator corresponding to (1.10) is not uniformly elliptic. Thus the crucial step is the proof that any minimizer u is Lipschitz continuous.

In §§2-4 we study the variational problem corresponding to (1.7), (1.10) and establish regularity of the minimizer and of the free boundary. In §5 we shall apply the results to the sessile drop problem as well as to other related capillary problems.

We always assume in this paper that $n \leqslant 6$; this ensures the regularity of the boundary of any perimeter minimizing set.

ADDED IN PROOF. Jean Taylor («Boundary regularity for solutions to various capillarity and free boundary problems,» *Comm. P.D.E.,* **2** (1977), 323-257) proved regularity of the free boundary surface in $\mathbb{R}^3$, using Almgren's approach.

2. The variational problem

A Borel function $v(x)$ defined in an open set $A \subset \mathbb{R}^m$ is said to be of bounded variation (BV) if

$$\int_A |Dv| \equiv \sup\Big\{\int_A v \operatorname{div} G;\ G = (G_1, \dots, G_m) \in C_0^1(A),\quad |G(x)|^2 = \sum_{i=1}^{m} G_i^2(x) \leqslant 1\Big\}$$

is a finite number. A Borel set $E \subset \mathbb{R}^m$ is said to have a finite perimeter in an open set $\Omega \subset \mathbb{R}^m$ if

$$\int_\Omega |D\phi_E| < \infty$$

where ϕ_E is the characteristic function of E.

We denote by E^* the one-sided Steiner symmetrization of a set E in $\mathbb{R}^m$ with respect to the plane $\Pi = \{x_m = 0\}$; more precisely, E^* lies in $\{x_m \geqslant 0\}$, $E^* \cap \{x' = x_0'\}$ consists of a single interval $0 \leqslant x_m \leqslant x_m'$ (for any $x_0' = (x_1^0, \dots, x_{m-1}^0)$), where $x = (x', x_m)$, $x' = (x_1, \dots, x_{m-1})$, and

$$H^1(E^* \cap \{x' = x_0'\}) = H^1(E \cap \{x' = x_0'\}).$$

We recall [14] that if $E \subset \{x_m \geqslant 0\}$ then

$$\int_{\{x_m > 0\}} |D\phi_{E^*}| \leqslant \int_{\{x_m > 0\}} |D\phi_E|. \tag{2.1}$$

Consider a set

$$E = \{x; x_m < u(x'), x' \in S\}$$

where $x = (x', x_m) \in \mathbb{R}^m$, $x' = (x_1, \dots, x_{m-1})$, and S is an open set in $\mathbb{R}^{m-1}$.

For any open set $A \subset S$ one defines (see [14])

$$\int_A \sqrt{1+|Du|^2}\,dx' = \sup\Big\{\int_A (u \operatorname{div} G + G_m)\,dx';$$
$$G = (G_1, \ldots, G_{m-1}),\, G_i \in C_0^1(A),\, \sum_{i=1}^m G_i^2(x) \leqslant 1\Big\}, \tag{2.2}$$

and then there holds ([14; Prop. 1.9 and (1.5)])

$$\int_{A\times\mathbb{R}} |D\phi_E| = \int_A \sqrt{1+|Du|^2}\,dx'. \tag{2.3}$$

In particular, E has a finite perimeter if and only if u is a BV function.

Let Ω be a bounded domain in $\mathbb{R}^n$ whose boundary is locally a Lipschitz graph and let u^0 be a nonnegative Lipschitz continuous function defined on $\partial\Omega$. Let

$$S_0 = \{(x, x_{n+1});\, x \in \partial\Omega,\, 0 \leqslant x_{n+1} < u^0(x)\}$$

and denote by K_0 the class of sets in $\bar{\Omega} \times [0, \infty)$ with finite perimeter in $\Omega \times (0, \infty)$, which coincide with S_0 on $\partial\Omega \times [0, \infty)$. For $E \in K_0$, let

$$J_0(E) = \int_{\Omega\times(0,\infty)} |D\phi_E| + \int_{\Omega\times(0,\infty)} (gx_{n+1} - \mu)\phi_E - \int_{\Omega\times\{x_{n+1}=0\}} \lambda\phi_E \tag{2.4}$$

and consider the problem: Find E such that

$$E \in K_0, \qquad J_0(E) = \lim_{G\in K_0} J_0(G). \tag{2.5}$$

We assume that

$$g > 0, \qquad \mu \geqslant 0 \qquad (g, \mu \quad \text{are constants}) \tag{2.6}$$

and

$$\begin{gathered} \lambda(x) \quad \text{is Lipschitz continuous,} \\ 0 < \lambda(x) < 1 \quad \text{in} \quad \bar{\Omega}. \end{gathered} \tag{2.7}$$

Theorem 2.1. *There exists a solution E of problem* (2.5), *and E is a bounded set.*

PROOF. Since J_0 is clearly bounded from below, to prove existence for (2.5) it suffices to prove that J_0 is lower semicontinuous, or just that

$$\int_{\Omega\times(0,\infty)} |D\phi_E| - \int_{\Omega\times\{x_{n+1}=0\}} \lambda\phi_E$$

is lower semicontinuous; but this can be established as in [10; Th. 1.2]. The proof that a minimizer E is a bounded set is the same as in [10; Th. 2.3].

Notice that if E is in K_0 then, by (2.1), its onesided Steiner symmetrization E^* decreases the perimeter and strictly decreases the remaining part of J_0, unless $E^* = E$ a.e. Thus for a minimizer E we must have that

$$E^* = E. \tag{2.8}$$

We shall henceforth normalize E (as in [8; §3.1]) so that

$$0 < |B_\rho(X) \cap (\Omega \times (0, \infty))| < |B_\rho(X)| \qquad \forall X \in \partial E.$$

If $X^0 \in \partial E$, $X^0 \in \Omega \times (0, \infty)$ then take a small ball $B = B_r(X^0)$ contained in $\Omega \times (0, \infty)$. Clearly E is then a minimizer of

$$\tilde{J}_0(G) = \int_B |D\phi_G| + \int_B (g x_{n+1} - \mu)\phi_G$$

in the class of sets which coincide with E on ∂B. Hence, by Massari [13] (recall that $n \leqslant 6$), ∂E is in $C^{2+\alpha}$ in B and, in fact, since μ, g are constants,

$$\partial E \quad \text{is analytic in } B. \tag{2.9}$$

In view of (2.8) we can write

$$E = \{(x, x_{n+1}); 0 \leqslant x_{n+1} < u(x), x \in \Omega\} \tag{2.10}$$

for some fuction $u(x)$. In view of (2.3), $u \in BV(\Omega)$. Since E is a bounded set, we also have that

$$u(x) < C \quad \text{for all} \quad x \in \Omega \qquad (C \text{ constant}). \tag{2.11}$$

Lemma 2.2. *The function $u(x)$ is continuous in Ω.*

PROOF. Suppose $u(x^0) > 0$. If $u(x)$ is not continuous at x^0, then from (2.10) it follows that ∂E contains a vertical line segment. In view of the analyticity of ∂E, ∂E must then contain the entire interval $\{x = x^0, x_{n+1} > 0\}$, a contradiction to the boundedness of E. For the same reason, if $u(x^0) = 0$, then $u(x) \to 0$ if $x \to x^0$.

Lemma 2.3. *For any $v \in BV(\Omega) \cap C^0(\Omega)$, $v \geqslant 0$, or $v \in H^{1,2}(\Omega)$, $v \geqslant 0$,*

$$\int_\Omega \sqrt{1 + |Dv|^2}\, I_{\{v>0\}} = \int_\Omega (\sqrt{1 + |Dv|^2} - 1) + \int_\Omega I_{\{v>0\}} \tag{2.12}$$

where I_A denotes the characteristic function of a set A.

PROOF. Suppose first that $v \in BV(\Omega) \cap C^0(\Omega)$. We can approximate v by mollifiers v_m such that (see [8] [14])

$$\int_\Omega \sqrt{1 + |Dv_m|^2}\, I_A \to \int_\Omega \sqrt{1 + |Dv|^2}\, I_A$$

for $A = A_k \equiv \left\{ v > \frac{1}{k} \right\}$ $(k = 1, 2, \dots)$ and for $A = \Omega$ (notice that A_k is open). Since v_m is smooth,

$$\int_\Omega \sqrt{1 + |Dv_m|^2}\, I_{\{v_m > 0\}} = \int_\Omega (\sqrt{1 + |Dv_m|^2} - 1) + \int_\Omega I_{\{v_m > 0\}}. \tag{2.13}$$

Since v is continuous,

$$I_{\{v_m > 0\}} \geqslant I_{\{v > 1/k\}}$$

for any positive integer k, if $m \geqslant m_0(k)$. Hence

$$\int_\Omega (\sqrt{1 + |Dv_m|^2} - 1) I_{\{v_m > 0\}} \geqslant \int_\Omega (\sqrt{1 + |Dv_m|^2} - 1) I_{\{v > 1/k\}} \to$$
$$\to \int_\Omega (\sqrt{1 + |Dv|^2} - 1) I_{\{v > 1/k\}}.$$

Since k is arbitrary,

$$\int_\Omega (\sqrt{1 + |Dv|^2} - 1) I_{\{v > 0\}} \leqslant \liminf_{m \to \infty} \int_\Omega (\sqrt{1 + |Dv_m|^2} - 1) I_{\{v_m > 0\}}.$$

Using this in (2.13) we obtain

$$\int_\Omega \sqrt{1 + |Dv|^2}\, I_{\{v > 0\}} \leqslant \int_\Omega (\sqrt{1 + |Dv|^2} - 1) + \int_\Omega I_{\{v > 0\}}. \tag{2.14}$$

To prove the reverse inequality we approximate $(v - 1/m)^+$ by mollifiers $v_{mj} (j \to \infty)$ such that

$$\int_\Omega \sqrt{1 + |Dv_{mj}|^2}\, I_A \to \int_\Omega \sqrt{1 + |D(v - 1/m)^+|^2}\, I_A$$

for $A = \{v > 0\}$ and for $A = \Omega$. Since $I_{\{v_{mj} > 0\}} \geqslant I_{\{v > 0\}}$ if j is large enough,

$$\limsup_{j \to \infty} \int_\Omega (\sqrt{1 + |Dv_{mj}|^2} - 1) I_{\{v_{mj} > 0\}} \leqslant \limsup_{j \to \infty} \int_\Omega (\sqrt{1 + |Dv_{mj}|^2} - 1) I_{\{v > 0\}} =$$
$$= \int_\Omega (\sqrt{1 + |D(v - 1/m)^+|^2} - 1) I_{\{v > 0\}}.$$

Noting that (2.13) holds for the smooth functions v_{mj} and taking a suitable sequence $j \to \infty$, we then obtain

$$\int_\Omega \sqrt{1 + |D(v - 1/m)^+|^2} I_{\{v > 0\}} \geqslant$$
$$\geqslant \int_\Omega (\sqrt{1 + |D(v - 1/m)^+|^2} - 1) + \int_\Omega I_{\{v > 0\}}. \tag{2.15}$$

If $F = \{(x, x_{n+1}); 0 \leqslant x_{n+1} < v(x), x \in \Omega\}$, then, by (2.3),

$$\int_\Omega \sqrt{1 + |D(v - 1/m)^+|^2} = \int_{\Omega \times (1/m, \infty)} |D\phi_F| \nearrow \int_{\Omega \times (0, \infty)} |D\phi_F|$$

as $m \to \infty$; the same holds with Ω replaced by $\Omega \cap \{v > 0\}$. Using these facts in (2.15) we obtain the reverse of (2.14), which completes the proof of (2.12).

If $v \in H^{1,2}(\Omega)$ then $Dv = 0$ a.e. on $\{v = 0\}$, so that

$$(\sqrt{1 + |Dv|^2} - 1)I_{\{v=0\}} = 0,$$

which immediately yields (2.12) if $v \geqslant 0$.

We introduce the functional

$$J_0(v) = \int_\Omega \sqrt{1 + |Dv|^2} I_{\{v>0\}}\, dx + \int_\Omega \left(\frac{g}{2} v^2 - \mu v\right) dx - \int_\Omega \lambda(x) I_{\{v>0\}}\, dx \tag{2.16}$$

and the admissible class

$$K = \{v \in BV(\Omega), v \geqslant 0 \quad \text{in} \quad \Omega, v = u^0 \quad \text{on} \quad \partial\Omega\}. \tag{2.17}$$

Consider the problem: Find u such that

$$u \in K \quad \text{and} \quad J_0(u) = \min_{v \in K} J_0(v). \tag{2.18}$$

We shall find it convenient to work also with the functional

$$J(v) = \int_\Omega (\sqrt{1 + |Dv|^2} - 1)\, dx + \int_\Omega \left(\frac{g}{2} v^2 - \mu v\right) dx + \\ + \int_\Omega (1 - \lambda) I_{\{v>0\}}\, dx. \tag{2.19}$$

Theorem 2.4. *Let u be defined by* (2.10) *where E is a solution of problem* (2.5). *Then u is a solution of problem* (2.18), $u \in C^0(\Omega)$ *and*

$$J(u) \leqslant J(v) \qquad \forall v \in K \cap C^0(\Omega). \tag{2.20}$$

PROOF. Since $J_0(u) = J_0(E)$ and

$$J_0(v) = J_0(G)$$

if $G = \{(x, x_{n+1}); 0 \leqslant x_{n+1} < v(x), x \in \Omega\}$, u is a solution of (2.18). The continuity of u was already established in Lemma 2.2, and (2.20) follows from the relation

$$J(v) = J_0(v) \quad \text{if} \quad v \in K \cap C^0(\Omega)$$

which is obtained using Lemma 2.3.

3. Lipschitz continuity

In this section we establish the Lipschitz continuity of the solution u asserted in Theorem 2.4.

We introduce the minimal surface operator

$$\mathcal{L}v = \operatorname{div}\frac{Dv}{\sqrt{1+|Dv|^2}}$$

and consider the Dirichlet problem

$$\begin{aligned} \mathcal{L}w - gw &= -\mu \quad \text{in } B_\epsilon(x_0), \\ w &= u \quad \text{on } \partial B_\epsilon(x_0) \end{aligned} \tag{3.1}$$

where $B_\epsilon(x_0) = \{x; |x-x_0| < \epsilon\}$ and $x_0 \in \Omega$. Set, for brevity, $B = B_\epsilon(x_0)$.

Lemma 3.1. *If ϵ is small enough then there exists a unique solution w in $C^{2+\alpha}(B) \cap C^0(\bar{B})$ of* (3.1).

By standard regularity results it follows that $w(x)$ is analytic.

PROOF. Uniqueness follows from the maximum principle. Existence is established in [7] in case $g = 0$; the proof in case $g > 0$ is similar and, for completeness, we briefly describe it. Denote the boundary values of u on ∂B by ϕ, and consider first the case where $\phi \in C^{2+\alpha}$. By comparing w (if existing) with $\pm M$ (M constant) we find that

$$\max_{\bar{B}} |w| < M \quad \text{if} \quad M > \sup|\phi|.$$

Hence $|gw| \leqslant gM$. We can now use [7; p. 285, Cor 13.5] (see (13.35), (13.36)) to deduce that

$$|Dw| \leqslant C_0 \quad \text{on} \quad \partial\Omega$$

if ϵ is small enough. By [7; p. 303, Th. 14.1] we then also get

$$|Dw| \leqslant C_0 \quad \text{in} \quad \Omega.$$

Thus $\mathcal{L}w$ is uniformly elliptic and then, by [7; p. 276, Th. 12.7],

$$[Dw]_{\alpha,\Omega} \leqslant C.$$

Having thus established an a priori $C^{1+\alpha}$ estimate on w, we can apply [7; p. 229, Th. 10.8] and deduce the existence of a $C^{2+\alpha}(\bar{B})$ solution of (3.1) in case $\phi \in C^{2+\alpha}$.

Consider next the case where ϕ is only assumed to be continuous, and approximate it uniformly by functions $\phi_m \in C^{2+\alpha}$. By the maximum principle, the corresponding solutions w_m satisfy:

$$|w_m - w_k|_{L^\infty(\Omega)} \leqslant |\phi_m - \phi_k|_{L^\infty(\partial\Omega)}.$$

Hence $w_m \to w$ uniformly in Ω.

By [7; p. 346, Cor. 15.6] (or [3]), for any $\Omega' \subset\subset \Omega$,

$$|Dw_m| \leqslant C \quad \text{in} \quad \Omega' \qquad (C = C(\Omega') \text{ constant})$$

provided $g = 0$; the proof extends with small changes to the case $g > 0$. But then also

$$[Dw_m]_{\alpha, \Omega'} \leqslant C$$

and consequently, for a subsequence, $w_m \to w$ in $C^{2+\alpha}(B) \cap C^0(\bar{B})$ where w is the desired solution.

Lemma 3.2. *The function u is an analytic solution of*

$$\mathcal{L}u - gu = -\mu \quad \text{in} \quad \{u > 0\}.$$

PROOF. Suppose $u(x_0) > 0$ and denote by w the solution of (3.1) where ϵ is sufficiently small so that $\overline{B_\epsilon(x_0)} \subset \{u > 0\}$. It suffices to show that $w \equiv u$. Consider the family of functions $w_M = w - M$ $(M > 0)$. If M is large enough then $w_M \leqslant u$ in $B = B_\epsilon(x_0)$. We decrease M until we arrive at the smallest value M_0 such that $w_{M_0} \leqslant u$. We claim that

$$M_0 = 0.$$

Indeed, if $M_0 > 0$ then there must exist a point $\bar{x} \in B$ such that $w_{M_0} = u$ at $\bar{x}$. Also,

$$\mathcal{L}w_{M_0} - gw_{m_0} = -\mu + gM_0 \quad \text{in} \quad B.$$

Recall that $(x, u(x))$ represents a smooth surface, by (2.9), and observe that the surfaces $(x, u(x))$, $(x, w_{M_0}(x))$ are tangent at $(\bar{x}, u(\bar{x}))$ and thus have a common normal $\bar{\nu}$. Using a coordinate system $(\tilde{x}, \tilde{x}_{n+1})$ in which $\bar{\nu}$ is in the direction of the $\tilde{x}_{n+1}$-axis, these surfaces can be represented in the form $\tilde{x}_{n+1} = U(\tilde{x})$ and $\tilde{x}_{n+1} = W(\tilde{x})$ respectively, and

$$\mathcal{L}U - \tilde{g}U = -\tilde{\mu}(\tilde{x}),$$
$$\mathcal{L}W - \tilde{g}W = -\tilde{\mu}(\tilde{x}) + \tilde{g}M_0$$

in a neighborhood N of $(\bar{x}, u(\bar{x}))$, where $\tilde{g}$, $\tilde{\mu}$ are the same functions in both equations and $\tilde{g} = \text{const.} > 0$. Indeed, if $\tilde{X} = (\tilde{x}, x_{n+1}) = TX$ where $X = (x, x_{n+1})$, T orthogonal matrix, $e_{n+1} = (0, 0, \ldots, 0, 1)$ and $\tilde{e} = Te_{n+1} = (b_1, \ldots, b_{n+1})$, then

$$\int gx_{n+1}\phi_G = \int gX \cdot e_{n+1}\phi_G = \int g\tilde{X} \cdot \tilde{e}\phi_G = \int gb_{n+1}\tilde{x}_{n+1}\phi_G + \int g \sum_{i=1}^{n} b_i\tilde{x}_i\phi_G$$

for any set G in K_0. Due to this change in the functional J_0 (or J) we find that

$$\tilde{g} = b_{n+1}g, \qquad \tilde{\mu}(\tilde{x}) = \mu - g \sum_{i=1}^{n} b_i\tilde{x}_i. \tag{3.2}$$

Notice that $b_{n+1} > 0$.

We now apply the maximum principle to $U - W$ and deduce that $M_0 = 0$ (and $U = M$ in N), a contradiction. We have thus proved that $M_0 = 0$ and $w \leqslant u$, and similarly $w \geqslant u$.

Later on we shall need to use radial solutions $s = s(r)$ of

$$\mathcal{L}s - \tilde{g}s = -\tilde{\mu} \quad \text{in a shell} \quad \rho < r < R, \tag{3.3}$$

$$s(R) = 0 \tag{3.4}$$

where $\tilde{g}$, $\tilde{\mu}$ are nonnegative constants and

$$\tilde{g} \ll \tilde{\mu}, \qquad \tilde{\mu} \ll 1 \tag{3.5}$$

Rewriting (3.3) in the form

$$\left(\frac{r^{n-1}s'}{\sqrt{1 + s'^2}}\right)' = (-\tilde{\mu} + \tilde{g}s)r^{n-1}$$

we find that

$$\frac{r^{n-1}s'}{\sqrt{1 + s'^2}} = \gamma - \frac{\hat{\mu}r^n}{r} \qquad (\gamma \text{ constant})$$

where $\hat{\mu} = \tilde{\mu}(1 + o(1))$ ($o(1) \to 0$ if $\tilde{g}/\tilde{\mu} \to 0$). Thus a solution is given by

$$s'(r) = -\frac{\gamma r^{1-n} - \hat{\mu}r/n}{[1 - (\gamma r^{1-n} - \hat{\mu}r/n)^2]^{1/2}}, \qquad s(R) = 0$$

provided γ is chosen so that

$$(\gamma r^{1-n} - \hat{\mu}r/n)^2 < 1.$$

Since $\hat{\mu}$ is small,

$$s'(r) \sim -\frac{\gamma r^{1-n}}{\sqrt{1-(\gamma r^{1-n})^2}} \quad \text{for} \quad \rho < r < R, \qquad s(R) = 0$$

$$\text{provided } (\gamma \rho^{1-n})^2 \leqslant 1/2. \qquad (3.6)$$

We now state the main result of this section.

Theorem 3.3. $u \in C^{0,1}(\Omega)$.

PROOF. Suppose the assertion is not true. Then we can find a sequence $X^m = (x^m, y^m)$ with $y^m = u(x^m) > 0$,

$$\rho_m = \operatorname{dist}(X^m, \text{free boundary}) \to 0$$

(the free boundary is the set $\partial\{u > 0\} \times \{x_{n+1} = 0\}$), and free boundary points $\bar{X}^m = (\bar{x}^m, 0)$ such that

$$|X^m - \bar{X}^m| = \rho_m, \qquad \frac{y^m}{|x^m - \bar{x}^m|} \to 0$$

and $\operatorname{dist}(x_m, \partial\Omega) \geqslant \text{const.} > 0$.

On the line segment $\overline{x^m \bar{x}^m}$ we can clearly find a point $\tilde{x}^m$ such that

$$(\tilde{x}^m, u(\tilde{x}^m)) \in B_{\rho_m/2}(X^m) \quad \text{and} \quad |\nabla u(\tilde{x}^m)| \to 0$$

if $m \to \infty$.

The surface $y = U_m(x)$, where

$$U_m(x) = \frac{1}{\rho_m} u(x^m + \rho_m x),$$

will be denoted by S_m. By [13], $S_m \cap B_{1-\epsilon}$ are uniformly $C^{2+\alpha}$ surfaces, for any $\epsilon > 0$. Hence, for a subsequence,

$$S_m \cap B_{1-\epsilon} \to S \cap B_{1-\epsilon}$$

in $C^{2+\alpha}$ sense, for any $\epsilon > 0$. Since

$$\mathcal{L}u - gu = -\mu \quad \text{in} \quad \{u > 0\},$$

it is easily seen that

$$\mathcal{L}U_m - g_m U_m = -\mu_m \quad \text{in} \quad \tilde{S}_m$$

where $\tilde{S}_m$ is the projection of S_m on $\{x_{n+1} = 0\}$, and

$$g_m = g\rho_m^2, \qquad \mu_m = \mu\rho_m.$$

It follows that

$$S \cap B_1 \quad \text{is a minimal surface,} \tag{3.7}$$

and a graph $x_{n+1} = U(x)$.

Denote by $(z_m, U_m(z_m))$ the point corresponding to $(\tilde{x}^m, u(\tilde{x}^m))$. Then

$$|z_m| \leqslant \tfrac{1}{2}, \; |DU_m(z_m)| \to \infty \quad \text{if} \quad m \to \infty.$$
$$U_m(z_m) \to U(z_0).$$

We may assume that $z_m \to z_0$. Then the tangent to S at $Z^0 = (z_0, U(z_0))$ is vertical. Since S in an analytic surface, it is then given in a neighborhood W of Z^0 by

$$x_1 = w(x_2, \ldots, x_n, x_{n+1}) \tag{3.8}$$

with $\partial w/\partial x_{n+1} = 0$ at Z^0 (here we have made a suitable rotation of the axes $x_1, x_2, \ldots, x_n$). Denote by B_0 a ball such that $(x_1, w(x_2, \ldots, x_n)) \in W$ if $(x_2, \ldots, x_{n+1}) \in B_0$).

Since $S \cap B_1$ is x_{n+1}-graph, it follows from the representation (3.8) that $\partial w/\partial x_{n+1} \leqslant 0$ in B_0 if, say, $\{U > 0\} \cap W$ lies to the left of S.

Differentiating the minimal surface equation $\mathcal{L}w = 0$ with respect to x_{n+1} we find that $\partial w/\partial x_{n+1}$ satisfies a linear elliptic equation to which the strong maximum principle can be applied. Since $\partial w/\partial x_{n+1} \leqslant 0$ in B_0 whereas $\partial w/\partial x_{n+1} = 0$ at Z^0, it follows that

$$\frac{\partial w}{\partial x_{n+1}} \equiv 0 \quad \text{in} \quad B_0.$$

Consequently

$$x_1 = w(x_2, \ldots, x_n) \tag{3.9}$$

in B_0 and, by analytic continuation, the same holds throughout $S \cap B_1$. Thus $S \cap B_1$ is a cylinder whose generators are parallel to the x_{n+1}-axis. Further, since S is x_{n+1}-graph, given by $x_{n+1} = U(x)$,

$$U(x) = 0 \quad \text{in} \quad \{x_1 < w(x_2, \ldots, x_n)\}. \tag{3.10}$$

We shall now derive a contradiction to the fact that u is a minimizer. We can do it either (i) by working with u, or (ii) by working with U. It will be instructive to describe both methods.

Method (i). Since $S_m \cap B_{1-\epsilon} \to S \cap B_{1-\epsilon}$ in $C^{2+\alpha}$ (for any $\epsilon > 0$), it follows from (3.9), (3.10) that after rotating the x_{n+1}-axis by a small angle δ we have, in the new coordinate system which we again denote by $x_1, \ldots, x_n, x_{n+1}$,

$$u(x) > Md(x^m) \quad \text{in} \quad B_{\theta d(x^m)}(x^m) \qquad (0 < \theta < 1) \tag{3.11}$$

where $d(x^m) = \text{dist}(x^m, \partial\{u > 0\}), d(x^m) \to 0$ if $m \to \infty$; here $1 - \theta$ can be taken arbitrarily small and M can be taken arbitrary large provided δ is small enough and m sufficiently large.

We rescale by $d(x^m)$ so as to obtain a new function u such that

$$u(x) > M \quad \text{in} \quad B_\theta(y_0) \tag{3.12}$$

where y_0 corresponds to a particular point x^m with m large enough, $|y| = 1$, its nearest free boundary point is at the origin, and the corresponding g, μ (which are in fact $gd^2(x^m)$, $\mu d(x^m)$) satisfy (3.5), so that the radial solution s of (3.3) can be constructed as above.

Take a point z_0 in the internal $\overline{Oy_0}$ with $|z_0| \leqslant (1 - \theta)/2$ and consider the shell Σ with center z_0 and radii

$$r_1 = (1 - \theta)/2, \qquad r_2 = 1 - |z_0| + \epsilon \qquad (\epsilon > 0). \tag{3.13}$$

Introduce the function

$$w = \text{maw}\{u, s\} \tag{3.14}$$

where s is constructed as in (3.6) with $\rho = r_1$, $R = r_2$ and $r = |x - z_0|$.

In view of (3.12) and the smallness of $1 - \theta$, $1/M$, we can choose the constant γ in (3.6) such that

$$u \geqslant s \quad \text{on} \quad \{r = r_1\}$$

and

$$s'(r) \geqslant \sigma, \tag{3.15}$$

with σ large; in fact,

$$\sigma = \sigma(M, \theta) \to \infty \quad \text{if} \quad M \to \infty, \theta \to 1. \tag{3.16}$$

Notice also that

$$u \geqslant 0 = s \quad \text{on} \quad \{r = r_2\},$$

and consequently w is an admissible function. Denoting by J_Σ the part of J taken over the shell Σ, the minimality of u implies that

$$J_\Sigma(u) \leqslant J_\Sigma(w); \tag{3.17}$$

here we used Theorem 2.4 and, for simplicity, we work with the original J rather than with its scaled form.

We shall derive a contradiction to (3.17). For clarity let us first proceed in a formal way.

We have

$$J_\Sigma(u) - J_\Sigma(w) = \int_\Sigma (\sqrt{1 + |Du|^2} - \sqrt{1 + |Dw|^2})$$

$$+ \int_\Sigma \left[\left(\frac{g}{2} u^2 - \mu u \right) - \left(\frac{g}{2} w^2 - \mu w \right) \right] - \int_\Sigma (1 - \lambda) I_{\{u=0\}}. \tag{3.18}$$

For $\nabla u = b$, $\nabla w = a$ we shall use the identity

$$\sqrt{1 + |b|^2} - \sqrt{1 + |a|^2} - \frac{(b - a) \cdot a}{\sqrt{1 + |a|^2}} =$$

$$= \frac{(1 + |b|^2)(1 + |a|^2) - (1 + a \cdot b)^2}{\sqrt{1 + |a|^2}\,(\sqrt{1 + |a|^2}\sqrt{1 + |b|^2} + 1 + a \cdot b)}.$$

By convexity, the right hand side is always $\geqslant 0$; however on the set $\{u = 0\}$ (where, formally, $\nabla u = 0$), we have the stronger inequality

$$\geqslant \frac{|a|^2}{\sqrt{1 + |a|^2}\,(\sqrt{1 + |a|^2} + 1)}.$$

Thus we obtain from (3.18)

$$J_\Sigma(u) - J_\Sigma(w) \geqslant \int_\Sigma \frac{\nabla(u - w) \cdot \nabla w}{\sqrt{1 + |\nabla w|^2}}$$

$$+ \int_{\Sigma \cap \{u=0\}} \frac{s'^2}{\sqrt{1 + s'^2}(\sqrt{1 + s'^2} + 1)}$$

$$+ \int_\Sigma \left[\left(\frac{g}{2} u^2 - \mu u \right) - \left(\frac{g}{2} w^2 - \mu w \right) \right]$$

$$- \int_\Sigma (1 - \lambda) I_{\{u=0\}}. \tag{3.19}$$

Since, formally,

$$\int_\Sigma \frac{\nabla(u - w) \cdot \nabla w}{\sqrt{1 + |\nabla w|^2}} = \int_\Sigma (u - w) \mathcal{L} s = \int_\Sigma (u - w)(-gs - \mu)$$

$$= \int_\Sigma (u - w)(-gw - \mu) \tag{3.20}$$

and

$$(u-w)(-gw)+\frac{g}{2}u^2-\frac{g}{2}w^2=\frac{g}{2}(u-w)^2\geqslant 0,$$

we obtain

$$J_\Sigma(u)-J_\Sigma(w)\geqslant\int_{\Sigma\cap\{u=0\}}\left[\frac{s'^2}{\sqrt{1+s'^2}(\sqrt{1+s'^2}+1)}-(1-\lambda)\right]$$

$$>0 \qquad \text{by (3.15), (3.16)}$$

(provided M is chosen large enough and $1-\theta$ is chosen small enough, depending on λ) which is a contradiction to (3.17).

In order to carry out the preceding argument rigorously, we approximate u first by $(u-\epsilon_j)^+$ and then by mollifiers $v_{j,k}\equiv(u-\epsilon_j)^+*\eta_k$ in a neighborhood of $\bar{\Sigma}$. Note that

$$\int_\Sigma\sqrt{1+|D(u-\epsilon_j)^+|^2}\leqslant\int_\Sigma\sqrt{1+|Du|^2}$$

and

$$\int_\Sigma\sqrt{1+|Dv_{j,k}|^2}\to\int_\Sigma\sqrt{1+|D(u-\epsilon_j)^+|^2}$$

if

$$\int_{\partial\Sigma}\sqrt{1+|D(u-\epsilon_j)^+|^2}=0$$

(which we may assume to be the case, by slightly changing the radii of Σ). Since also

$$I_{\{v_{j,k}>0\}}<I_{\{u>0\}}$$

if k is large enough (depending on ϵ_j), we may choose a sequence $u_j=v_{j,k(j)}$ such that

$$I_{\{u_j>0\}}<I_{\{u>0\}} \tag{3.21}$$

and

$$\int_\Sigma\sqrt{1+|Du_j|^2}<\int_\Sigma\sqrt{1+|Du|^2}+\eta_j,\qquad \eta_j\to 0. \tag{3.22}$$

Setting $w_j=\max(s,u_j)$ and observing that $w_j=u_j$ on ∂S, we can proceed as before (but this time rigorously) to establish that

$$J_\Sigma(u_j)-J_\Sigma(w_j)\geqslant\int_{\Sigma\cap\{u_j=0\}}\left[\frac{s'^2}{\sqrt{1+s'^2}(\sqrt{1+s'^2}+1)}-(1-\lambda)\right]\geqslant$$

$$\geqslant c\int_\Sigma I_{\{u=0\}},\qquad c>0. \tag{3.23}$$

On the other hand, by (3.21), (3.22),

$$J_\Sigma(u) \geqslant \limsup_{j\to\infty} J_\Sigma(u_j)$$

and by the lower semicontinuity of the perimeter

$$J_\Sigma(w) \leqslant \liminf_{j\to\infty} J_\Sigma(w_j).$$

Thus, taking $j \to \infty$ in (3.23) we find that

$$J_\Sigma(u) - J_\Sigma(w) > 0, \tag{3.24}$$

a contradiction to (3.17). This completes the proof of the Lipschitz continuity by method (i).

Method (ii). Here we work directly with the blow up limit U. First we must establish that the subgraph E_U of U is a minimizer. The proof is similar to the proof of [2; Lemma 3.3] which asserts that the blow up limit of u_m with respect to $B_{\rho_m}(x^m)$ is a minimizer. However, in that lemma it is given that $|\nabla u_m| \leqslant C$, which is not the case here. In our case one can easily show that the perimeter of the subgraphs of u_m and of $v + (1-\eta)(u_m - u_0)$ are uniformly small outside the set $\{\eta = 1\}$ and then proceed as in [2], using the lower semicontinuity of the perimeter.

Suppose now that E_U lies in $\{x_1 < w(x_2, \ldots, x_n)\}$. Since w is analytic, the set $S \cap \{x_{n+1} \geqslant 0\}$ is regular and we can therefore repeat the proof of (3.24) working directly with the set S and with the set T_s, the subgraph of s (where s is now a radial solution of the minimal surface equation). The calculations are in fact simpler as well as rigorous (i.e., there is no need to justify the formal calculations by approximation).

The blow up limit E_U may have, however, another portion E_2 in $\{x_1 > w\}$ (we denote the portion in $\{x_1 < w\}$ by E_1). By what was said in the preceding paragraph, we have, analogously to (3.24),

$$J_\Sigma(E_1 \cup T_s) < J_\Sigma(E_1). \tag{3.25}$$

In order to derive a contradiction to the minimality of the set E_U it suffices to show that

$$J_\Sigma(E_1 \cup E_2 \cup T_s) < J_\Sigma(E_1 \cup E_2).$$

But this follows from the well known inequality

$$\mathrm{Per}(E_2 \cup T_s) \leqslant \mathrm{Per}(E_2) + \mathrm{Per}(T_s).$$

4. Regularity of the free boundary

Let Ω' be a subdomain of Ω and let M be a positive number larger than the Lipschitz coefficient of u in Ω'. Define

$$F(t) = \begin{cases} \sqrt{1+t} - 1 & \text{if} \quad 0 \leqslant t \leqslant M, \\ \sqrt{1+t} - 1 + \epsilon \dfrac{(t-M)^2}{1+(t-M)} & \text{if} \quad t > M; \end{cases}$$

if ϵ is positive then the function $f(p) = F(|p|^2)$ satisfies, for some $\beta > 0$ and all $p \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$,

$$\beta|\xi|^2 \leqslant \sum_{i,j=1}^{n} \frac{\partial^2 f(p)}{\partial p_i \partial p_j} \leqslant \beta^{-1}|\xi|^2,$$

$$\beta|p|^2 \leqslant f_p(p)p, \qquad |f_p(p)| \leqslant \beta^{-1}|p|,$$

$$\beta|p|^2 \leqslant f(p) \leqslant \beta^{-1}|p|^2, \tag{4.1}$$

and these are precisely the conditions which are needed for the results in [2].

Consider the functional

$$J'(v) = \int_{\Omega'} F(|\nabla v|^2)\, dx + \int_{\Omega'} \left(\frac{g}{2} v^2 - \mu v\right) dx + \int_{\Omega'} (1-\lambda) I_{\{v>0\}}\, dx$$

and the admissible class

$$K_{\Omega'} = \{v \in H^{1,2}(\Omega'), v \geqslant 0, v = u \quad \text{on} \quad \partial\Omega'\}.$$

Noting that

$$\sqrt{1 + |\nabla v|^2} \leqslant F(|\nabla v|^2) \quad \text{if} \quad v \in K_{\Omega'},$$

$$\sqrt{1 + |\nabla u|^2} = F(|\nabla u|^2) \quad \text{in} \quad \Omega',$$

it is clear that

$$J'(u) \leqslant J'(v) \qquad \forall v \in K_{\Omega'}. \tag{4.2}$$

Set

$$P(x, v) = \frac{g}{2} v^2 - \mu. \tag{4.3}$$

For any ball $B = B_r(x_0)$ in Ω', let v be the solution of

$$-\text{div} f_P(Dv) + P_v(x, v) = 0 \quad \text{in} \quad B, \qquad v = u \quad \text{on} \quad \partial B,$$

where $f(p) = F(|p|^2)$. Since $P_v = gv$, we can apply the maximum principle to conclude that $v \geqslant 0$. It follows that v (when extended by u into $\Omega' \setminus B$) is in $K_{\Omega'}$.

Using the above remark we can now extend all the results of [2] to the present problem (4.2). (Note that in [2] f satisfies the conditions in (4.1) and $P \equiv 0$; see also [5] where $F(p) = |p|$ and $P \not\equiv 0$). In particular, the following theorems are valid:

Theorem 4.1. *If* $n = 2$ *then* (i) *if* $\lambda \in C^{k,\alpha}$ *then the free boundary is in* $C^{k+1,\alpha}$; (ii) *if* λ *is analytic then the free boundary is analytic.*

Theorem 4.2. *Let* $3 \leqslant n \leqslant 6$ *and let* D *be a domain with* $\bar{D} \subset \Omega$. *There exist positive constants* α, β, σ_0, τ, C *such that for any free boundary point* x^0 *in* D *the following is true:*

If in some coordinate system

$$u(x) = 0 \quad \text{in} \quad B_\rho(x_0) \cap \{x_n - x_n^0 > \sigma\rho\} \tag{4.4}$$

where $x^0 = (x_1^0, \ldots, x_n^0)$, $\sigma \leqslant \sigma_0$, $v \leqslant \tau_0 \sigma^{2/\beta}$, *then* $\partial\{u > 0\} \cap B_{\rho/4}(x_0)$ *has the form* $x_n = g(x')$ $(x' = (x_1, \ldots, x_{n-1}))$ *with* $g \in C^{1,\alpha}$ *and*

$$|Dg(x') - Dg(\bar{x}')| \leqslant C \left| \frac{x' - \bar{x}'}{\rho} \right|^\alpha .$$

Further, $g \in C^{k+1,\gamma}$ *if* $\lambda \in C^{k,\gamma}$ *and* g *is analytic if* λ *is analytic.*

The condition (4.4) is called the flatness condition. In general, not assuming flatness, one can assert for the set S of singularities of the free boundary that $H^{n-1}(S) = 0$.

5. Applications

Consider a capillary drop on a horizontal inhomogeneous plane $\Omega_0 = \mathbb{R}^n$; the contact angle $\theta(x)$ is non-constant in general. To study this problem we introduce the functional

$$J_{\epsilon_0}(G) = \int_{Q_0} |D\phi_G| + \int_{Q_0} g x_{n+1} \phi_G - \int_{\partial Q_0} \lambda \phi_G + f_{\epsilon_0}(V_G) \tag{5.1}$$

where $Q_0 = \Omega_0 \times (0, \infty)$, $\lambda = \cos\theta$, $V_G = H^{n+1}(G)$ and

$$f_{\epsilon_0}(t) = \begin{cases} -\dfrac{1}{\epsilon_0}(t - V) & \text{if} \quad t < V \\ 0 & \text{if} \quad t > V \qquad (\epsilon_0 > 0) \end{cases}$$

where V is prescribed positive number (the volume of the drop). Caffarelli and Spruck [4] proved that there exists a set $E \subset \bar{Q}_0$ such that

$$J_{\epsilon 0}(E) = \min_G J_{\epsilon 0}(G), \qquad G \subset \bar{Q}_0; \tag{5.2}$$

furthermore, E is a bounded set and

$$V_E = V$$

provided ϵ_0 is small enough. (Notice that since Ω_0 is unbounded, Theorem 2.1 is not applicable to this situation.) As in §2, E is a subgraph of a function $x_{n+1} = u(x)$ with support S, say.

We may consider E as a minimizer in a smaller class K_Ω:

$$\begin{aligned} G \in K_\Omega \quad \text{if} \quad & G \subset \Omega \times [0, \infty) \quad \text{and} \\ & G \quad \text{coincides with} \quad \partial\Omega \times \{0\} \quad \text{on} \quad \partial\Omega \times [0, \infty), \end{aligned} \tag{5.3}$$

where Ω is any bounded domain which contains the set S; the integral $\int_{\partial Q_0}$ in (5.1) is replaced by $\int_{\Omega \cap \{x_{n+1}=0\}}$.

Because of the presence of the term $f_{\epsilon 0}(V_G)$, we cannot apply the results of §§2-4 directly to the present problem. However, going over the various arguments we discover that all the results remain valid with some modifications, as we shall now explain.

The fact that $\partial_0 E \equiv \partial E \cap \{x_{n+1} > 0\}$ is in $C^{2+\alpha}$ can be established by the method of Massari [13] (see also [12] for regularity of $\partial_0 E$ when the volume constraint is imposed as a side condition); the analyticity of $\partial_0 E$ follows from the existence of multipliers (see [6] [9]). We can now establish the continuity of $u(x)$ as before.

In any open set $S \subset \{u > 0\}$ there exists a point x_S such that the tangent to ∂E at $(x_S, u(x_S))$ is not vertical; thus $u(x)$ is analytic in some ball B_S with center x_S.

Take a smooth nonnegative function $u_S(x)$ with support in B_S such that $\int u_S(x)\,dx = 1$. For any $\zeta \in C_0^1(B_S)$ and for any real ϵ, $|\epsilon|$ small enough, the function $u + \epsilon\zeta - \epsilon\left(\int \zeta\right)u_S$ is an admissible function having the same volume V as u. From the inequality

$$J_{\epsilon 0}\left(u + \epsilon\zeta - \epsilon\left(\int \zeta\right)u_S\right) \geqslant J_{\epsilon 0}(u)$$

we then obtain

$$\int_{B_S} \left(\frac{\nabla u \cdot \nabla \zeta}{\sqrt{1 + |\nabla u|^2}} + g u \zeta - \mu_S \zeta \right) dx = 0 \tag{5.4}$$

where

$$\mu_S = \int_{B_S} \left(\frac{\nabla u \cdot \nabla u_S}{\sqrt{1 + |\nabla u|^2}} + g u u_S \right) dx.$$

Taking $u + \epsilon u_S - \epsilon u_{S'} - \epsilon u_{S'}$ as an admissible function with S' another open set with its corresponding $u_{S'}$ and $\mu_{S'}$, and ϵ any small real number, we find that $\mu_{S'} = \mu_S$. Further from $J_{\epsilon 0}(u + \epsilon u_S) \geqslant J_{\epsilon 0}(u)$ $(\epsilon > 0)$ we find that $\mu_S \geqslant 0$. Thus

$$\mu = \mu_S \quad \text{is independent of } S \text{, and} \quad \mu \geqslant 0. \tag{5.5}$$

From (5.4), (5.5) we deduce that

$$\mathfrak{L}u - gu = -\mu \quad \text{in} \quad B_S. \tag{5.6}$$

By using local coordinates we can actually obtain a «parametric» form of (5.6) valid throughout $\partial_0 E$, whereby g, μ are to be replaced by $\tilde{g}$, $\tilde{\mu}$ (cf. (3.2)); this however will not be needed.

We shall now extend Lemma 3.2. Take any point $X^0 = (x^0, u(x^0))$ with $u(x^0) > 0$ and let $B_\delta(x^0)$ be any ball such that $u(x) > 0$ if $x \in B_\delta(x^0)$. We shall prove that u is a smooth solution of

$$\mathfrak{L}u - gu = -\mu \quad \text{in} \quad B_\delta(x^0). \tag{5.7}$$

Introducing the analytic solution w of

$$\begin{aligned} \mathfrak{L}w - gw &= -\mu \quad \text{in} \quad B_\delta(x^0), \\ w &= u \quad\ \ \text{on} \quad \partial B_\delta(x^0), \end{aligned} \tag{5.8}$$

it suffices to show that $w \equiv u$. Proceeding as in the proof of Lemma 3.2, we perform, at the same point in the same argument as before an orthogonal transformation $(\tilde{x}, \tilde{x}_{n+1}) = T(x, x_{n+1})$. The surfaces $x_{n+1} = w(x)$ and $x_{n+1} = u(x)$ become $\tilde{x}_{n+1} = W(\tilde{x})$ and $\tilde{x}_{n+1} = U(\tilde{x})$ respectively, $((\tilde{x}, \tilde{x}_{n+1}) = (0, 0)$ corresponds to point $(x, x_{n+1}) = (\bar{x}, u(\bar{x})))$ and it remains to show that the analytic functions W, U in some ball $B_\rho(\tilde{O})$ with center $\tilde{O} = (0, 0)$ satisfy

$$\mathfrak{L}W - \tilde{g}W = -\tilde{\mu}(\tilde{x}), \tag{5.9}$$

$$\mathfrak{L}U - \tilde{g}U = -\tilde{\mu}(\tilde{x}) \tag{5.10}$$

where $\tilde{\mu}$, $\tilde{g}$ are given by (3.2).

By the manner by which the transformation T changes the functional J (see the paragraph containing (3.2)) it is clear how the corresponding Euler equation changes, namely, (5.8) changes into (5.9). Similarly by choosing S a small ball about $\bar{x}$, (5.6) yields

$$\mathfrak{L}U - \tilde{g}U = -\tilde{\mu}(\tilde{x})$$

in a small ball $B_{\tilde{\rho}}(\tilde{x}_*)$ contained in $B_\rho(\tilde{O})$; by analytic continuation it then follows that (5.10) holds throughout $B_\rho(\tilde{O})$.

Having proved (5.9), (5.10), we can now complete the proof of Lemma 3.2 as before.

We next proceed to establish the Lipschitz continuity of u, as in §2. If we use Method (i) then the proof is the same since the terms $f_{\epsilon 0}(V_E)$ cancel out when we compare $J_{\epsilon 0}(u)$ with $J_{\epsilon 0}(w)$. On the other hand, if we use Method (ii), then $f_{\epsilon 0}(t)$ must be replaced by

$$\tilde{f}_{\epsilon 0}(t) = \lim_{m\to\infty} f_{\epsilon 0}\left(\frac{t}{\rho_m^n} + V_E - V_{E\cap B_{\rho_m}(X^m)}\right).$$

Having proved Lipschitz continuity in compact subsets of Ω, we next trucate $\sqrt{1+t}$ as in §4 and consider the functional

$$J'_{\epsilon 0}(v) = \int_\Omega F(|\nabla v|^2)\,dx + \int_\Omega \frac{g}{2} v^2\,dx + \int_\Omega (1-\lambda) I_{\{v>0\}}\,dx + f_{\epsilon 0}(V_v) \quad (5.11)$$

where

$$V_v = H^{n+1}\{(x, x_{n+1}); 0 < x_{n+1} \leqslant v(x), x \in \Omega'\}.$$

and proceed as in [2].

The proof of non-degeneracy remains the same and so do all the results of [2]. However, in checking the various details one must pay attention to the term $f_{\epsilon 0}(V_v)$. If $V_v \geqslant V_u$ then $f_{\epsilon 0}(V_u) = f_{\epsilon 0}(V_v)$ and these two terms cancel out. If however $V_v < V_u$ then

$$f_{\epsilon 0}(V_v) = f_{\epsilon 0}(V_u) + 0(V_u - V_v). \quad (5.12)$$

This causes some changes in the proofs, usually trivial ones. The only slightly significant difference occurs in Theorem 4.3 of [2] where one takes $v = u -$ $- \min(u, \epsilon\zeta)$. The error in (5.12) must now be controlled by $o(r^2)$. We recall that

$$\int_{B_\rho} \zeta = O\left(\rho^2/\log\frac{\rho}{r}\right)$$

so that

$$|V_u - V_v| \leqslant C\left[\frac{\rho^2}{\log\frac{\rho}{r}} + r^2\right]\epsilon, \quad \text{and} \quad \epsilon = Cr;$$

taking $\rho = r^{\theta}$ with $\theta < 1$, $1 - \theta$ small, we get

$$\frac{1}{r^2}|\tilde{J}_{\epsilon 0}(V_u) - \tilde{J}_{\epsilon 0}(V_v)| \leqslant \frac{C}{r^{\beta}} \quad \text{for some} \quad \beta > 0.$$

We can now proceed as in [2] and obtain the extensions of Theorem 4.3, namely,

$$\frac{1}{r^2}\int_{B_r \cap \{u>0\}} ((\lambda^*)^2 - |\nabla u|^2)^+ \leqslant C/\log\frac{1}{r}$$

where

$$(\lambda^*)^2 = \frac{1}{\lambda^2} - 1.$$

The rest of the proof of theorems 4.1, 4.2 is the same as in [2]. We can therefore state, for $n = 2$:

Theorem 5.1. *The free boundary of the sessile drop problem is in $C^{k+1,\alpha}$ if $\lambda \in C^{k,\alpha}$, and it is analytic if λ is analytic.*

Remark 5.1. Consider the minimization problem for the functional ($\Omega_0 \subset \mathbb{R}^n$, $Q_0 = \Omega_0 \times (0, \infty)$)

$$\hat{J}_{\epsilon 0}(G) = \int_{Q_0} |D\phi_G| + \int_{Q_0} g x_{n+1}\phi_G - \\ - \int_{\Omega_0 \times \{0\}} \lambda \phi_G - \int_{\partial\Omega_0 \times (0,\infty)} \hat{\lambda}\phi_G + f_{\epsilon 0}(G) \qquad (5.13)$$

with $G \subset \bar{Q}_0$, $\lambda = \cos\theta$, $\hat{\lambda} = \cos\hat{\theta}$. This functional is similar to (5.1); the additional term $\int \hat{\lambda}\phi_G$ represents the wetting energy on the lateral boundary of the tube Q_0. The minimization problem models capillary fluid in the tube Q_0 with a given volume V. If V is small enough then a portion of the bottom will remain dry. Theorem 5.1 immediately extends to the present case (with $n = 2$) showing that the boundary of the dry portion of the bottom is analytic.

The results of this paper also extend to functionals in which $\frac{1}{2}gv^2$ is replaced by more general functions $P(x, v)$, provided $P_v \geqslant 0$.

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