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Arithmetic Hilbert Modular Functions II

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Introduction

The purpose of this paper, which is a continuation of [2, 3], is to prove further results about arithmetic modular forms and functions. In particular we shall demonstrate here a q-expansion principle which will be useful in proving a reciprocity law for special values of arithmetic Hilbert modular functions, of which the classical results on complex multiplication are a special case. The main feature of our treatment is, perhaps, its independence of the theory of abelian varieties. In that respect these developments may be considered as an extension of Hecke's thesis [13] and Habilitationsschrift [14]. We should also mention a contribution of Sugawara [34]. More recently Karel has shown how to apply such ideas to the classical case of elliptic modular functions in an adelic setting [16].

To date the furthest reaching results in this area, beyond those in the classical case, belong to a long list of distinguished contributors who have freely used the known facts about elliptic functions, elliptic curves, and abelian varieties, notably Hasse [11,12], Deuring [10], Shimura-Taniyama [32], Shimura [28, 29, 30] (and many others), Taniyama [35], Shih [33], Miyake [25], Milne-Shih [22, 23, 24], Deligne [8, 9], Borovoi [7], and Milne [21]. The last mentioned work, which uses the preceding ones together with results of Kazhdan [17, 18], contains very general results. It has recently been complemented by a work of Milne, still in unpublished form, written to put the results of [18] on a firmer basis.

However, our purpose is to develop a theory independent of abelian varieties based on the properties of the modular functions themselves and thereby also, it is hoped, to learn more about these functions and their own intrinsic arithmetic properties. The inspiration for this approach comes from the paper *Der Hilbertsche Klassenkörper eines imaginärquadratischen Zahlkörpers,* Math. Zeitschr. 64 (1956), by M. Eichler, and it is to Prof. Eichler that we wish to dedicate this article. In it we have relied most heavily on the work of Hecke, the second paper of Hasse, the papers [27, 28] of Shimura (for facts about CM-fields and reduction of algebraic varieties modulo a prime), Deligne [8] (especially for topological properties of the adelic double coset space), and Karel [16] (as will be explained later). We hope these efforts, to be continued in subsequent publications, may be of some interest to mathematicians in this field.

1. The Adelic Space

In this paper we generally follow the notation and conventions of [3]. For convenience we give some of the most frequently used notation. Let k be a totally real number field with ring of integers $\mathfrak{o}$ and $[k\colon Q] = n > 1$, let $\mathbf{A}$ resp. $\mathbf{A}(k)$ denote the adele rings of Q resp. of k, and let $\mathbf{I}(k)$ be the ideles or group of units of $\mathbf{A}(k)$, each supplied with its usual topology. The subscripts ∞ and f will denote the projections of an adelic object to its archimedean and non-archimedean components respectively and the subscript $+$ will indicate adelic objects with non-negative archimedean components. $\hat{\mathbb{Z}}$ resp. $\hat{\mathfrak{o}}$ will be the maximal compact subrings of $\mathbf{A}_f$ and of $\mathbf{A}(k)_f$ respectively. We denote by $\mathfrak{H}$ the upper half complex plane $Im z > 0$, by $\mathfrak{H}^n$ its n-th Cartesian power, and by i.e., the point $(i, \ldots, i) \in \mathfrak{H}^n$.

Moreover, G' will denote the algebraic group GL_2 defined over k and G will be the group $R_{k/\mathbf{Q}}G'$ defined over $\mathbf{Q}$. There is a canonical isomorphism ϕ of $G'(\mathbf{A}(k))$ onto $G(\mathbf{A})$ and of $G'(k)$ onto $G(\mathbf{Q})$ such that if the integral structures on G' and on G are those associated to $G'(\hat{\mathfrak{o}}) = GL_2(\hat{\mathfrak{o}})$ and to $G(\hat{\mathbb{Z}})$ (with respect to suitable bases of the vector spaces on which G' and G act), then $\phi(G'(\hat{\mathfrak{o}})) = G(\hat{\mathbb{Z}})$ (cf. [5]).

Z' is the center of G' and $Z = R_{k/\mathbf{Q}}Z'$, that of G, and $G_+(\mathbb{R})/Z(\mathbb{R})$ acts effectively on $\mathfrak{H}^n$. If $\mathbb{K}_\infty$ is the isotropy group of i.e in $G_+(\mathbb{R})$ and $\mathbb{K}'_\infty = \mathbb{K}_\infty \cap G_+^{\mathrm{der}}(\mathbb{R})$, one has

$$G_+(\mathbf{A}) = G_+(\mathbb{R})G(\mathbf{A}_f) = P_+(\mathbf{A})\mathbb{K}'_\infty G(\hat{\mathbb{Z}}) \tag{1}$$

(the corrected form of §1.2(2) of [3]), where $P = R_{k/\mathbf{Q}}P'$ and P' is the group of upper triangular matrices in G'.

Let $\mathbb{K}$ be an open compact subgroup of $G(\hat{\mathbb{Z}})$ and denote by Γ or $\Gamma(\mathbb{K})$ the arithmetic subgroup $G(\mathbf{Q}) \cap G_+(\mathbb{R}) \cdot \mathbb{K}$ of $G_+(\mathbf{Q})$, whose projection into $G_+(\mathbb{R})$ we also denote by Γ or $\Gamma(\mathbb{K})$.

The space of left cosets of $\mathbb{K}\mathbb{K}_\infty$ in $G_+(\mathbf{A})$, $X_{\mathbb{K}} = G_+(\mathbf{A})/\mathbb{K}\mathbb{K}_\infty$, is the union of countably many connected components, each one of the form

$$X_\omega = \omega G_+(\mathbb{R})\mathbb{K}/\mathbb{K}\mathbb{K}_\infty, \quad \omega \in G(\mathbf{A}_f), \tag{2}$$

X_ω being complex analytically isomorphic to $\mathfrak{H}^n$. The group $G_+(\mathbf{Q})$ permutes these components under left translation in $G_+(\mathbf{A})$ and has finitely many orbits among them, the stabilizer in $G_+(\mathbf{Q})$ of X_ω being $\Gamma_\omega = \Gamma({}^\omega\mathbb{K})$. If we let $\mathfrak{C}_{\mathbb{K}}(\omega) = G_+(\mathbf{Q})\omega G_+(\mathbb{R})\mathbb{K}$, then $V_\omega = \Gamma_\omega \backslash X_\omega$ may be identified with

$$G_+(\mathbf{Q})\backslash\mathfrak{C}_{\mathbb{K}}(\omega)/\mathbb{K}\mathbb{K}_\infty,$$

and the collection of double cosets $\mathfrak{C}_{\mathbb{K}}(\omega)$ or components V_ω is in natural one-to-one correspondence with the set of elements of the group (cf. [8], Variante 2.5)

$$\mathfrak{g}_+[\mathbb{K}] = I_+(k)/k^\times k_{\infty+} \det(\mathbb{K}) \simeq I(k)_f/k_f^\times \det(\mathbb{K}),$$

where $k_\infty = \bigoplus_{v|\infty} k_v$, k_v being the completion of k at the archimedean place v, and $\det(\mathbb{K})$ is the group of $\det(\mathbf{k})$, $\mathbf{k} \in \mathbb{K}$.

Define the double coset space $V_{\mathbb{K}} = G_+(\mathbf{Q})\backslash X_{\mathbb{K}}$; this is a union

$$V_{\mathbb{K}} = \cup_{\omega\in\Omega} V_\omega, \qquad V_\omega = \Gamma_\omega\backslash X_\omega \simeq \Gamma_\omega\backslash\mathfrak{H}^n, \tag{3}$$

where Ω is a finite set of indexing representatives of the orbits of $G_+(\mathbf{Q})$ among the components X_ω. (Cf. [3].)

In this paper we consider properties of $V_{\mathbb{K}}$ in connection with arithmetic automorphic forms and functions on $G_+(\mathbf{A})$ with respect to $\mathbb{K}$ and study the arithmetic properties of such functions by means of arithmetically defined Eisenstein series on the components X_ω. We follow the ideas and program of [3] and [16], to which must be added a certain q-expansion principle and other ideas related to [13, 14] and [11, 12], as well as properties of CM-types to be found in [32, 28].

We generally adhere to the notation $\mathfrak{A}(\mathbb{K})$, $\mathfrak{A}(\mathbb{K}, w)$, etc., of §1 of [3] for the graded algebra of modular forms with respect to $\mathbb{K}$, the forms of weight w, etc.

2. Special Points and Idelic Action

We follow here the pattern of §§1.2-1.3 of [16], taking account of differences needed to accomodate the more general situation discussed in [3]. Therefore our discussion will be abbreviated by making suitable references to [3] and [16].

The group $G'_+(k)$ acts on $\mathfrak{H}^n$ in a manner analogous to that in which $GL_{2+}(\mathbb{Q})$ acts on $\mathfrak{H}$ by linear fractional transformations. If $\Sigma = (\sigma_1, \ldots, \sigma_n)$ is the set of isomorphisms of k onto subfields of $\mathbb{R}$, if $S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G'_+(k)$, and $(z) = (z_1, \ldots, z_n) \in \mathfrak{H}^n$, then $S(z) = (S^{\sigma_1} \cdot z_1, \ldots, S^{\sigma_n} \cdot z_n)$, where

$$S^{\sigma_j} \cdot z_j = (\alpha^{\sigma_j} z_j + \beta^{\sigma_j})/(\gamma^{\sigma_j} z_j + \delta^{\sigma_j}).$$

We denote by K a purely imaginary quadratic extension of k. Then $S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G'(k)$ acts on $K - k$ by linear fractional transformations, $S \cdot \tau = (\alpha\tau + \beta)/(\gamma\tau + \delta)$, $\tau \in K$. Consider an imbedding

$$q: K \hookrightarrow M_2(k): 2 \times 2 \quad \text{matrices over } k,$$

of K as a k-algebra such that $q(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. By the Skolem-Noether theorem, the representation q of K as a k-algebra is equivalent to the regular representation of K on itself and $\det(q(x)) = x\bar{x} = N_{K/k}(x) > 0$. As a subgroup of $G'_+(k)$, $q(K^\times)$ has precisely two fixed points $\tau, \bar{\tau} \in K - k$, where $\bar{\tau}$ is the complex conjugate of τ. Conversely, if $\tau \in K - k$, then by taking τ, 1, as a k-basis of K for the regular representation, we see that each $\tau \in K - k$ defines such an imbedding $q = q_\tau$, that its complex conjugate $\bar{\tau}$ defines the conjugate imbedding, and that $q(K^\times)$ as a subgroup of $G'(k)$ has precisely the two fixed points $\tau, \bar{\tau}$. Thus we have a one-to-one correspondence between conjugate pairs of imbeddings q of K in $M_2(k)$ and complex conjugate pairs $\tau, \bar{\tau}$ of elements of $K - k$.

By a lifting of Σ, or "type" for the given CM-field K/k, we mean a set $\tilde{\Sigma} = \Phi$ of extensions $(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n)$ of Σ to a set of n imbeddings of K into $\mathbb{C}$ such that $\tilde{\sigma}_j | k = \sigma_j$, $j = 1, \ldots, n$. If $\tau \in K - k$, there is a unique lifting $\tilde{\Sigma} = \tilde{\Sigma}(\tau)$ defined by the requirement $\mathrm{Im}(\tilde{\sigma}_j(\tau)) > 0$, $j = 1, \ldots, n$. Conversely, given any lifting $\tilde{\Sigma}$, define the set $K_{\tilde{\Sigma}} = \{\tau \in K - k \mid \tilde{\Sigma}(\tau) = \tilde{\Sigma}\}$.

If $z = (\tau_1, \ldots, \tau_n) \in \mathfrak{H}^n$ is the fixed point of a non-central element of $G'_+(k) = G_+(\mathbf{Q})$, then [3] there is a uniquely determined purely imaginary quadratic extension $K = K_z$ of k and $\tau \in K - k$ such that if $\tilde{\Sigma}(\tau) = (\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_n)$, then $\tau_j = \tau^{\tilde{\sigma}_j}$, $j = 1, \ldots, n$. Moreover, from our previous discussion it follows that there is an imbedding $q = q_\tau$ of K into $M_2(k)$ such that $z = (\tau)$ is the unique fixed point of $q(K^\times) \subset G'_+(k)$ in $\mathfrak{H}^n$. In fact, the isotropy subgroup of (τ) in $G'_+(k)$ has to be a torus; since $q(K)$ is a maximal commutative subalgebra of $M_2(k)$, it follows that $q(K^\times)$ is the full isotropy group of (τ) in $G'_+(k)$.

In the future we view any point $\tau \in K - k$ as being imbedded into $\mathfrak{H}^n$ by means of $\tilde{\Sigma}(\tau)$, and write (τ) for its image there. Having fixed the CM-extension K, we refer to the set K_Σ of points (τ) for $\tau \in K - k$ as the set of special points of $\mathfrak{H}^n$ relative to the extension K/k.

If $\mathfrak{v}$ is a fractional ideal of k and $\tau \in K - k$, then $\mathfrak{J}_{\mathfrak{v},\tau} = \mathfrak{v}\tau + \mathfrak{o}$ is an $\mathfrak{o}$-module in K of rank two. Let $\mathfrak{R} = \mathfrak{R}_{\mathfrak{v}}(\tau)$ be its order, $\mathfrak{R} = \{x \in K \mid x \cdot \mathfrak{J}_{\mathfrak{v},\tau} \subset \subset \mathfrak{J}_{\mathfrak{v},\tau}\}$, so that $\mathfrak{J}_{\mathfrak{v},\tau}$ is a proper $\mathfrak{R}$-ideal in K. Let $\Xi_{\mathfrak{v}}(\mathfrak{R}) = \{\tau \in K - k \mid \mathfrak{R}_{\mathfrak{v}}(\tau) = = \mathfrak{R}\}$, and $\Xi_{\mathfrak{v}}(\mathfrak{R}, \tilde{\Sigma}) = \{\tau \in \Xi_{\mathfrak{v}}(\mathfrak{R}) \mid \tilde{\Sigma}(\tau) = \tilde{\Sigma}\}$.

Returning to the double coset decomposition

$$G_+(\mathbf{A}) = \bigcup_{\omega \in \Omega} G_+(\mathbf{Q})\omega G_+(\mathbf{R})\mathbb{K} \tag{4}$$

associated to the decomposition of the double coset space

$$V_{\mathbb{K}} = G_+(\mathbf{Q})\backslash G_+(\mathbf{A})/\mathbb{K}\mathbb{K}_\infty = \bigcup_{\omega \in \Omega} V_\omega \tag{5}$$

into its component varieties as in the preceding section, we recall from [3] that the double coset representatives ω may be chosen in diagonal form $\omega = \begin{pmatrix} \omega' & 0 \\ 0 & 1 \end{pmatrix}$, $\omega' \in I(k)_f$, let $\mathfrak{v} = \mathrm{id}(\omega')^{-1}$, and $\mathfrak{v}_\omega = \mathfrak{v}$, $x_\omega = x_{\mathfrak{v}}$. In particular when $\mathbb{K} = G(\hat{\mathbb{Z}})$, let $\Theta = \Omega$ and $\omega = \theta \in \Theta$, $\theta = \begin{pmatrix} \theta' & 0 \\ 0 & 1 \end{pmatrix}$, $\mathfrak{v} = \mathrm{id}(\theta')^{-1}$ and $\Theta' = = \{\theta' \mid \theta \in \Theta\}$. Then

$$G(\mathbf{A})_f = \bigcup_{\theta \in \Theta} G_+(\mathbf{Q})_f \theta G(\hat{\mathbb{Z}}). \tag{6}$$

For a fixed order $\mathfrak{R}$ in K such that $\mathfrak{R}$ contains the ring of integers of k, and for any $\theta \in \Theta$, we define the set of special points on X_θ to be the set

$$\Xi_\theta(\mathfrak{R}) = \{\theta \cdot (\tau) \in \theta G_+(\mathbb{R})G(\hat{\mathbb{Z}})/\mathbb{K}_\infty G(\hat{\mathbb{Z}}) \mid \tau \in \Xi_{\mathfrak{v}_\theta}(\mathfrak{R})\}. \tag{7}$$

In other words, identifying $\mathfrak{H}^n$ with $X_\theta = \theta \cdot \mathfrak{H}^n$, $\Xi_\theta(\mathfrak{R})$ is the set of special points of $\mathfrak{H}^n$ coming from elements $\tau \in \Xi_{\mathfrak{v}_\theta}(\mathfrak{R}) \subset K - k$. Then define $\Xi_{\theta,\infty}(\mathfrak{R})$ to be the subset of $g \in G_+(\mathbb{R})$ such that $g(\text{i.e.}) \in \Xi_\theta(\mathfrak{R})$ and define the sets

$$\Xi_A(\mathfrak{R}) = G_+(\mathbf{Q})\Xi_\infty(\mathfrak{R})G(\hat{\mathbb{Z}}), \quad \text{where} \quad \Xi_\infty(\mathfrak{R}) = \bigcup_{\theta \in \Theta} \Xi_{\theta,\infty}(\mathfrak{R}) \cdot \theta.$$

We also let

$$\Xi_\theta(\mathfrak{R}, \tilde{\Sigma}) = \{\theta \cdot (\tau) \in \Xi_\theta(\mathfrak{R}) \mid \tilde{\Sigma}(\tau) = \tilde{\Sigma}\}$$

and define $\Xi_A(\mathfrak{R}, \tilde{\Sigma})$ and $\Xi_\infty(\mathfrak{R}, \tilde{\Sigma})$ analogously. If $\mathbb{K}$ is an open compact subgroup of $G(\hat{\mathbb{Z}})$, let $\Xi_{\mathbb{K}}(\mathfrak{R})$ be the image, via the canonical projection to double cosets, of $\Xi_A(\mathfrak{R})$ in $V_{\mathbb{K}}$. In particular, if $\mathfrak{n}$ is an integral ideal in $\mathfrak{o}$ and if $\mathbb{K} = \mathbb{K}(\mathfrak{n})$ is the principal congruence subgroup of those $k \in G'(\hat{\mathfrak{o}}) = G(\hat{\mathbb{Z}}) \subset \subset M_2(\hat{\mathfrak{o}})$ such that $k - I_2 \in \mathfrak{n} \cdot M_2(\hat{\mathfrak{o}})$ (where $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$), denote $V_{\mathbb{K}}$ by $V_{\mathfrak{n}}$, and $\Xi_{\mathbb{K}}(\mathfrak{R})$ by $\Xi_{\mathfrak{n}}(\mathbb{R})$. Let $\mathfrak{U}_{\mathfrak{n}}(\hat{\mathfrak{R}})$ be the principal congruence subgroup mod $\mathfrak{n}$ of the group $\mathfrak{U}(\hat{\mathfrak{R}}) = \hat{\mathfrak{R}}^{\times}$ of unit ideles, where $\hat{\mathfrak{R}}$ is the closure of $\mathfrak{R}$ in $\mathbf{A}(K)_f$, and let $C_{\mathfrak{n}}(\mathfrak{R})$ be the group

$$C_{\mathfrak{n}}(\mathfrak{R}) = \mathbf{A}(K)^{\times}/K^{\times} \cdot \mathbf{A}(K)_{\infty}^{\times} \cdot \mathfrak{U}_{\mathfrak{n}}(\hat{\mathfrak{R}}) \tag{8}$$

of ray classes modulo $\mathfrak{n}$ of proper $\mathfrak{R}$-ideals. If $\theta \in \Theta$ and $\tau \in \Xi_{\mathfrak{d}_\theta}(\mathfrak{R})$, and if q_τ is the imbedding associated to τ of K into $M_2(k)$, then we have

$$q_\tau(\mathfrak{U}_{\mathfrak{n}}(\hat{\mathfrak{R}}) \cap (K_f)^{\times}) \subset {}^{\theta}\mathbb{K}(\mathfrak{n}). \tag{9}$$

Let $C_{\mathfrak{R}}(\mathfrak{R})$ be the group of classes of proper fractional ideals of $\mathfrak{R}$ in K. Suppose $\gamma \in \Gamma_\theta = \Gamma({}^{\theta}G(\hat{\mathbb{Z}}))$ and $\tau \in \Xi_{\mathfrak{d}_\theta}(\mathfrak{R})$, so that $\mathfrak{A}_\tau = \mathfrak{d}\tau + \mathfrak{o}$ is a proper fractional $\mathfrak{R}$-ideal of K and so that if (as in [2]) we put $R_{\mathfrak{d}} = \begin{pmatrix} \mathfrak{o} & \mathfrak{d}^{-1} \\ \mathfrak{d} & \mathfrak{o} \end{pmatrix}$, then $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R_{\mathfrak{d}+}^{\times}$. Then if $\tau' = \gamma \cdot \tau$, a direct calculation shows that $\mathfrak{A}_{\tau'} = \mathfrak{d}\tau' + \mathfrak{o} = (c\tau + d)^{-1}\mathfrak{A}_\tau$, therefore τ' also belongs to $\Xi_{\mathfrak{d}_\theta}(\mathfrak{R})$. Consequently the $\mathfrak{R}$-ideal class of $\mathfrak{d}\tau + \mathfrak{o}$ is constant along the orbits of Γ_θ, and similarly if $\mathfrak{A}_\tau \sim \mathfrak{A}_{\tau'}$, then $\tau' \in R_{\mathfrak{d}}^{\times} \cdot \tau$ (linear fractional operation on the right-hand side), so that Γ_θ has only finitely many orbits in $\Xi_{\mathfrak{d}_\theta}(\mathfrak{R})$.

Let $g \in G_+(\mathbf{A})$ and $\tau = g_\infty(\text{i.e})$. Denote by $j(g)$ the double coset

$$G_+(\mathbf{Q})gG_+(\mathbb{R})G(\hat{\mathbb{Z}})$$

to which g belongs. Let $\mathfrak{A}$ be a proper fractional $\mathfrak{R}$-ideal of K. Since $\mathfrak{A}$ is proper and its order contains $\mathfrak{o}$, $\mathfrak{A}$ is at every finite place $\mathfrak{P}$ of K locally principal as an $\mathfrak{R}_{\mathfrak{P}}$-ideal; hence, there is a finite idele $\alpha \in I(K)_f$ such that $\mathfrak{A} = K \cap \mathbf{A}(K)_\infty \alpha \hat{\mathfrak{R}}$. If $\hat{\mathfrak{A}}$ is the $\mathfrak{o}$-ideal generated by $\mathfrak{A}$, as in [3], let $\hat{N}\mathfrak{A} = N_{K/k}\hat{\mathfrak{A}}$. Suppose $g \in \Xi_{\mathbf{A}}(\mathfrak{R})$ and that $g = \xi\theta$, $\xi \in \Xi_{\theta,\infty}(\mathfrak{R})$, $\theta \in \Theta$, and let $(\tau) = \xi(\text{i.e})$ and $\mathfrak{A}_\tau = \mathfrak{d}\tau + \mathfrak{o}$ with $\mathfrak{d} = \text{id}(\det(\theta))^{-1}$. Let $\gamma \in G_+(\mathbf{Q})$ and write $g' = \gamma g$. Then $\mathfrak{A}\mathfrak{A}_\tau$ is in the same proper $\mathfrak{R}$-ideal class as $\mathfrak{A}_{\tau_1} = \mathfrak{d}_1\tau_1 + \mathfrak{o}$ with $\mathfrak{d}_1 = \mathfrak{d}_{\theta_1}$ and $\tau_1 \in \Xi_{\mathfrak{d}_1}(\mathfrak{R})$ for some $\theta_1 \in \Theta$, with $\mathfrak{d}_1$ in the same narrow ideal class as $\mathfrak{d} \cdot \hat{N}\mathfrak{A}$, and $(\tau_1) = \xi_1(\text{i.e})$ for some $\xi_1 \in \Xi_{\theta_1,\infty}(\mathfrak{R})$.

Lemma 1. *With the notation just introduced, we may choose* $(\tau_1) = \xi_1(\text{i.e})$ *such that* $\tilde{\Sigma}(\tau_1) = \tilde{\Sigma}(\tau)$, *and then we have* $j(\theta_1\xi_1) = j(\gamma q_\tau(\alpha^{-1})\theta\xi)$.

PROOF. Of course it suffices to prove that

$$j(\theta_1\xi_1) = j(q_\tau(\alpha^{-1})\theta\xi).$$

Let $q = q_\tau$. If $\mathfrak{p}$ is a prime ideal of $\mathfrak{o}$, let $\mathfrak{R}_{\mathfrak{p}} = \mathfrak{R} \otimes_{\mathfrak{o}} \mathfrak{o}_{\mathfrak{p}}$ and define an $\mathfrak{R}_{\mathfrak{p}}$-module $\mathfrak{B}_{\mathfrak{p}}$ by $\mathfrak{B}_{\mathfrak{p}} = \alpha_{\mathfrak{p}}(\mathfrak{d}_{\mathfrak{p}}\tau + \mathfrak{o}_{\mathfrak{p}})$. Clearly $(\mathfrak{A}\mathfrak{A}_\tau)_{\mathfrak{p}} = \mathfrak{B}_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ of $\mathfrak{o}$, thus $\mathfrak{B} = \mathfrak{A}\mathfrak{A}_\tau$ is the proper fractional $\mathfrak{R}$-ideal with localization $\mathfrak{B}_{\mathfrak{p}}$ for every $\mathfrak{p}$, and $\hat{N}\mathfrak{B}$, as defined above, is in the same narrow ideal class as $\mathfrak{d}_1$ and

$\mathfrak{v} \cdot \hat{N}\mathfrak{A}$. Then $\mathfrak{B}$ is in the same proper $\mathfrak{R}$-ideal class as $\mathfrak{A}_{\tau_1} = \mathfrak{v}_1\tau_1 + \mathfrak{o}$, with $\mathfrak{v}_1 = \mathfrak{v}_{\theta_1}$ and $\tau_1 \in \Xi_{\mathfrak{v}_1}(\mathfrak{R})$ for some $\theta_1 \in \Theta$. Then by the calculations of §3.1 of [3], we have

$$\mathfrak{v}_1 = (\Delta(\tau_1, 1)/\Delta(\tau, 1)) \cdot \hat{N}\mathfrak{A} \cdot \mathfrak{v}. \tag{10}$$

Since $\Delta(\tau, 1) = 2\,\mathrm{Im}(\tau)$, this says the principal fractional ideals $(\mathrm{Im}(\tau))$ and $(\mathrm{Im}(\tau_1))$ of k are in the same narrow ideal class, and therefore there exists a unit $\eta \in \mathfrak{o}$ such that

$$\tilde{\Sigma}(\eta\tau_1) = \tilde{\Sigma}(\tau),$$

or, if we replace τ_1 by $\eta\tau_1$, which does not change the ideal $\mathfrak{v}_1$, we may assume that $\tilde{\Sigma}(\tau_1) = \tilde{\Sigma}(\tau)$. Thus we may write

$$\mathfrak{B} = \mathfrak{v}_1 B + \mathfrak{o}B',$$

with $\tilde{\Sigma}(B/B') = \tilde{\Sigma}(\tau_1) = \tilde{\Sigma}(\tau)$. Therefore there exists $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G'_+(k)$ such that

$$P\begin{pmatrix} \tau \\ l \end{pmatrix} = \begin{pmatrix} B \\ B' \end{pmatrix}. \tag{$'$}$$

For each prime $\mathfrak{p}$ of $\mathfrak{o}$, we have

$$\mathfrak{B}_\mathfrak{p} = \mathfrak{v}_{1\mathfrak{p}}B + \mathfrak{o}_\mathfrak{p}B'$$

and also

$$\mathfrak{B}_\mathfrak{p} = \mathfrak{v}_\mathfrak{p}\alpha_\mathfrak{p}\tau + \mathfrak{o}_\mathfrak{p}\alpha_\mathfrak{p};$$

therefore, there exists $\omega_\mathfrak{p} \in \theta_\mathfrak{p}G'(\mathfrak{o}_\mathfrak{p})\theta_{1\mathfrak{p}}^{-1}$, say $\omega_\mathfrak{p} = \theta_\mathfrak{p}\gamma_\mathfrak{p}\theta_{1\mathfrak{p}}^{-1}$, such that

$$\omega_p\begin{pmatrix} B \\ B' \end{pmatrix} = \begin{pmatrix} \alpha_\mathfrak{p}\tau \\ \alpha_\mathfrak{p} \end{pmatrix},$$

i.e., such that

$$\omega_\mathfrak{p}P\begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_\mathfrak{p}\tau \\ \alpha_\mathfrak{p} \end{pmatrix} \quad \text{for each prime } \mathfrak{p}.$$

Then $q(\alpha)_\mathfrak{p} = \omega_\mathfrak{p}P$ for each prime $\mathfrak{p}$. Therefore, $q(\alpha^{-1})_\mathfrak{p} = P^{-1}\theta_{1\mathfrak{p}}\gamma_\mathfrak{p}^{-1}\theta_\mathfrak{p}^{-1}$, or since $\alpha \in I(K)_f$,

$$q(\alpha^{-1}) = P_f^{-1}\theta_1\gamma\theta^{-1}, \qquad \gamma \in G(\hat{\mathbb{Z}}),$$

hence $q(\alpha^{-1})\theta\xi = P_f^{-1}\theta_1\gamma\xi = P^{-1}P_\infty\xi\theta_1\gamma$. Now by ('), $P_\infty\xi(\text{i.e}) = \xi_1(\text{i.e})$, so that

$$j(q(\alpha^{-1})\theta\xi) = j(\xi_1\theta_1) = j(\theta_1\xi_1).$$

This proves the lemma. (Cf. [16: §1.2.4]).

Now $\Xi_\infty(\mathfrak{R}) = \cup_{\theta\in\Theta}\Xi_{\theta,\infty}(\mathfrak{R})\theta = \cup_\theta \Xi_\theta(\mathfrak{R})$. Then, following Karel [16], we may introduce an action of the group $I(K)_f$ on the set $\Xi_{\mathbf{A}}(\mathbb{R})$ as follows: Define a map

$$F: I(K)_f \times G_+(\mathbf{Q}) \times \Xi_\infty(\mathfrak{R}) \times G(\hat{\mathbb{Z}}) \to G_+(\mathbf{A}) \tag{11}$$

by $F(y, \lambda, \xi\theta, \omega) = \lambda q_{\xi(\text{i.e})}(y^{-1})\xi\theta\omega$, where $y \in I(K)_f$, $\lambda \in G_+(\mathbf{Q})$, $\xi \in \Xi_{\theta,\infty}(\mathfrak{R})$, and $\omega \in G(\hat{\mathbb{Z}})$. We first verify that $F(y, \lambda, \xi\theta, \omega)$ depends only on y and on the product $\lambda\xi\theta\omega \in \Xi_{\mathbf{A}}(\mathfrak{R})$. Suppose that $\lambda\xi\theta\omega = \lambda'\xi'\theta'\omega'$ with analogous meanings for the primed elements. By looking at the non-archimedean components, we see that

$$\det(\theta\theta'^{-1}) \in \det(G'(\hat{\mathfrak{o}}))\det(G'_+(k)_f),$$

so that θ and θ' represent the same double coset in (6) and therefore $\theta = \theta'$. Consequently

$$\lambda^{-1}\lambda' = \xi\xi'^{-1}\theta\omega\omega'^{-1}\theta^{-1} \in \Gamma({}^\theta G(\hat{\mathbb{Z}})),$$

the arithmetic group acting on X_θ. Put $\gamma = \lambda'^{-1}\lambda$. Then $\xi' = \gamma_\infty\xi$ so that $\theta\xi(\text{i.e}) = \theta\gamma_\infty\xi'(\text{i.e})$. The representation $q_\tau: K \hookrightarrow M_2(k)$ is defined by

$$q_\tau(b)\begin{pmatrix}\tau\\1\end{pmatrix} = \begin{pmatrix}b\tau\\b\end{pmatrix}, \qquad \tau \in K - k, \qquad b \in K, \tag{12}$$

and q_τ may be extended to a representation of $\mathbf{A}(K)$ into $M_2(\mathbf{A}(k))$, in particular to a representation of $\mathbb{C} = K \otimes \mathbb{R}$ into $M_2(\mathbb{R})$. Therefore, if $S = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in GL_2^+(k)$ and $S\cdot\tau = (a\tau + b)(c\tau + d)^{-1}$, $\tau \in K - k$, we have

$$q_{S\cdot\tau} = Sq_\tau S^{-1}. \tag{13}$$

Applying this with $S = \gamma$ we obtain

$$\lambda' q_{\xi'(\text{i.e})}(y^{-1})\xi'\theta'\omega' = \lambda\gamma^{-1}q_{\xi'(\text{i.e})}(y^{-1})\gamma_\infty\xi\theta\omega' = \lambda\gamma_f^{-1}q_{\xi'(\text{i.e})}(y^{-1})\xi\theta\omega' =$$

$$= \lambda q_{\gamma_\infty^{-1}\xi'(\text{i.e})}(y^{-1})\gamma_f^{-1}\theta\xi\omega' = \lambda q_{\xi(\text{i.e})}(y^{-1})\xi\gamma_f^{-1}\theta\omega' = \lambda q_{\xi(\text{i.e})}(y^{-1})\xi\theta\omega,$$

since $\gamma_f^{-1} = \theta\omega\omega'^{-1}\theta^{-1}$. This says, as claimed, that $F(y, \lambda, \xi\theta, \omega)$ depends only on y and on the product $\lambda\xi\theta\omega$.

We can also see that $F(y, \lambda, \xi\theta, \omega)$ belongs to $\Xi_{\mathbf{A}}(\mathfrak{R})$. For this it suffices to prove that

$$q_{\xi(\text{i.e})}(y^{-1})\theta\xi \in \Xi_{\mathbf{A}}(\mathfrak{R}) \tag{14}$$

(noting that $\theta\xi = \xi\theta$), and this is immediate from Lemma 1 since (in the notation of that lemma) $\theta_1\xi_1 \in \Xi_{\mathbf{A}}(\mathfrak{R})$.

Thus one has, in analogy with §1.2.5 of [16], a map

$$I(K)_f \times \Xi_{\mathbf{A}}(\mathfrak{R}) \to \Xi_{\mathbf{A}}(\mathfrak{R}), \tag{15}$$

written as $(y, \xi) \to y_*\xi$ for $y \in I(K)_f$, $\xi \in \Xi_{\mathbf{A}}(\mathfrak{R})$. One extends this to an action of $I(K)$ by defining the action of $I(K)_\infty$ to be trivial (which is appropriate since $q_{\xi(\mathrm{i.e})}(\alpha).\ \xi(\mathrm{i.e}) = \xi(\mathrm{i.e})$ for every $\alpha \in I(K)_\infty$). In this way one obtains an action of the group $I(K)/I(K)_\infty \cdot K^\times$ of idele classes, modulo the connected component of the identity, on $G_+(\mathbf{Q})\Xi_{\mathbf{A}}(\mathfrak{R})/\mathbb{K}_\infty$ and hence on the image of $\Xi_{\mathbf{A}}(\mathfrak{R})$ in each of the double coset spaces $V_{\mathbb{K}}$ in (5). Now $V_{\mathbb{K}} = \cup_{\omega\in\Omega} V_\omega$ and it is easy to verify that

$$\det(q_{\xi(\mathrm{i.e})}(y^{-1})) = N_{K/k}(y^{-1}), \qquad y \in I(K). \tag{16}$$

Hence, $\xi \to y * \xi$ moves a special point of the component X_ω to one of $X_{\omega'}$, where $\omega' \in \Omega$ is defined uniquely by

$$\det(\omega') \in N(y^{-1})\det(\omega)(k_+^\times)_f \det(\mathbb{K}). \tag{17}$$

Also, for $x \in \mathfrak{U}_{\mathfrak{n}}(\hat{\mathfrak{R}})$, $q_\tau(x)$ belongs to the group of units of $\hat{R}_{\mathfrak{v}+}$ which are $\equiv 1 \bmod \mathfrak{n}$, i.e., $q_\tau(x) \in {}^{\omega}\mathbb{K}(\mathfrak{n})$, where τ belongs to $\mathfrak{H}^n$ and $\mathfrak{R}$ is the order of $\tau\mathfrak{v} + \mathfrak{o}$, and $\mathfrak{v} = \mathrm{id}.(\det(\omega)^{-1})$. Therefore if $\tau = \xi(\mathrm{i.e})$ and $\zeta = \lambda\xi\omega\eta$ represents a point of a component V_ω of $V_{\mathbb{K}}$, where $\mathbb{K} = \mathbb{K}(\mathfrak{n})$, $\lambda \in G_+(\mathbf{Q})$, and $\eta \in \mathbb{K}(\mathfrak{n})$, then $q_\tau(x^{-1}) = {}^{\omega}\eta'$ for some $\eta' \in \mathbb{K}(\mathfrak{n})$ and therefore $x * \zeta = \lambda q_\tau(x^{-1})\xi\omega\eta = \lambda\xi\omega\eta'\eta$, which represents the same point of $V_{\mathbb{K}}$. This means that, as a subgroup of $I(K)$, the principal congruence subgroup $\mathfrak{U}_{\mathfrak{n}}(\hat{\mathfrak{R}})$ of $I(K)_f$ acts trivially on the projection $\Xi_{\mathfrak{n}}(\mathfrak{R})$ of $\Xi_{\mathbf{A}}(\mathfrak{R})$ into $V_{\mathfrak{n}}$. In other words, the ray class group

$$C_{\mathfrak{n}}(\mathfrak{R}) = R_{\mathfrak{n}}(K, \mathfrak{R}) = I(K)/I(K)_\infty \cdot K^\times \mathfrak{U}_{\mathfrak{n}}(\hat{\mathfrak{R}})$$

of K with respect to $\mathfrak{R}$ modulo $\mathfrak{n}$ acts on $\Xi_{\mathfrak{n}}(\mathfrak{R})$. It is known already from [3:§3.1] that

$$\Xi(\mathfrak{R}) = \Xi_1(\mathfrak{R}) = \cup_{\mathfrak{v}} \Xi_{\mathfrak{v}}(\mathfrak{R})/\Gamma_{\mathfrak{v}}^+$$

is finite, where $\mathfrak{v}$ runs over a set of representatives of narrow ideal classes of k. Since the canonical projection of $V_{\mathfrak{n}}$ onto $V = V_1$ has only finite fibers, it follows that $\Xi_{\mathfrak{n}}(\mathfrak{R})$ is a finite set in which $R_{\mathfrak{n}}(K, \mathfrak{R})$, of course, has only finitely many orbits.

At the same time, the relation $\tilde{\Sigma}(\tau_1) = \tilde{\Sigma}(\tau)$ from Lemma 1 implies that if $\xi \in \Xi_{\mathbf{A}}(\mathfrak{R}, \tilde{\Sigma})$ and $y \in I(K)$, then $y * \xi \in \Xi_{\mathbf{A}}(\mathfrak{R}, \tilde{\Sigma})$. Thus, the action of $I(K)$ on $\Xi_{\mathbf{A}}(\mathfrak{R})$ is also an action of $I(K)$ on $\Xi_{\mathbf{A}}(\mathfrak{R}, \tilde{\Sigma})$ for each lifting $\tilde{\Sigma}$.

The action described in [16], as well as that defined here, of the idele group on the special points of the adelic double coset space, is, of course, closely related to the action given in [31] of the idele group on the special values of arithmetic modular functions.

3. Conjugation

We now formulate a generalization of results of §1.3 of [16] to cover our present situation. The proofs, which parallel closely those of *loc. cit.*, will be mostly omitted. The purpose of these results is to effect, for each $x = \xi\theta \in \Xi_\infty(\mathfrak{R})$, an extension of the k-algebra homomorphism $q_{\xi(\mathrm{i.e})}$ to an isomorphism of a K-algebra

$$\tilde{K} = K + \iota K, \qquad \iota^2 = 1, \qquad \iota a = \bar{a}\iota \quad \text{for all} \quad a \in K,$$

with $M_2(k)$.

Let $x = \xi\theta \in \Xi_{\infty,\theta}(\mathfrak{R})$ and $z = \xi(\mathrm{i.e})$. Then $z = (\tau^{\tilde{\sigma}_1}, \ldots, \tau^{\tilde{\sigma}_n})$ for some $\tau \in K$ where $\tilde{\Sigma}(\tau) = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n)$. For $a \in K$, $q(a) = q_\tau(a)$ is defined by (12) of §2. We define

$$q(\iota) = \begin{pmatrix} -1 & \tau + \bar{\tau} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & x \\ 0 & 1 \end{pmatrix} \in G'_-(k),$$

where $x = 2Re\tau \in k$ and $G'_-(k) = \{M \in G'(k) \mid \det(M) \leqslant 0\}$. Then using (12) we get

$$q(\iota)q(a) = q(\bar{a})q(\iota). \tag{18}$$

Hence q, so extended to $\tilde{K}$, becomes a faithful k-algebra isomorphism of $\tilde{K}$ with $M_2(k)$. Then if $x \in M_2(k)$ satisfies $q(a)x = xq(\bar{a})$ for all $a \in K$ it follows that $xq(\iota)$ centralizes $q(K) \subset M_2(k)$ and therefore $xq(\iota) = q(b)$ for some $b \in K$, i.e., $x = q(b\iota)$.

We identify ι with the generator of $\mathrm{Gal}(K/k)$ and extend the action of $I(K)$ on

$$G_+(\mathbf{Q})\backslash G_+(\mathbf{A})/\mathbb{K}_\infty,$$

defined in §2, to an action of the semi-direct product $I(K)^\sim = I(K) \rtimes \mathrm{Gal}(K/ /k)$ defined by the exact sequence

$$\{1\} \to I(K) \to I(K)^\sim \to \mathrm{Gal}(K/k) \to \{1\},$$

in the following manner. If $x = \xi\theta$ as above, and if $\omega \in G(\hat{\mathbb{Z}}) = G'(\hat{\mathfrak{o}})$, let

$$\iota^*(G_+(\mathbf{Q})x\omega\mathbb{K}_\infty) = G_-(\mathbf{Q})q_\tau(\iota)_\infty x\omega\mathbb{K}_\infty,$$

where $G_-(\mathbf{Q}) = G'_-(k)$. It is easy to verify, just as in *loc. cit.*, that this definition is independent of the choice of representative $x\omega$ of the given double coset mod $G_+(\mathbf{Q})\backslash/\mathbb{K}_\infty$; that if v is such a double coset, then

$$b * \iota * v = \iota * \bar{b} * v, \qquad b \in I(K);$$

that for each prime ideal $\mathfrak{p}$ of the ring of integers $\mathfrak{o}$ of k and $\theta \in \Theta$, there exists $\delta_\mathfrak{p} \in {}^\theta G'(\mathfrak{o}_\mathfrak{p})$ such that $\delta_\mathfrak{p} q(a) \delta_\mathfrak{p}^{-1} = q(\bar{a})$ for all $a \in K_\mathfrak{p} = K \otimes_k k_\mathfrak{p}$; and then, in parallel to the proof of Lemma 1.3.2 of *loc. cit.*, that

$$\iota * (G_+(\mathbf{Q})\backslash \Xi_\mathbf{A}(\mathfrak{R})/\mathbb{K}_\infty) = G_+(\mathbf{Q})\backslash \Xi_\mathbf{A}(\mathfrak{R})/\mathbb{K}_\infty.$$

In this way, one constructs an action of $I(K)^\sim$ on the space of double cosets $G_+(\mathbf{Q})\backslash \Xi_\mathbf{A}(\mathfrak{R})/\mathbb{K}_\infty$. As in Section 2, this action commutes with right translation by elements of $G(\hat{\mathbb{Z}})$ and provides an action of

$$C_\mathfrak{n}(\mathfrak{R})^\sim = R_\mathfrak{n}(K, \mathfrak{R})^\sim = R_\mathfrak{n}(K, \mathfrak{R}) \rtimes \mathrm{Gal}(K/k)$$

on $\Xi_\mathfrak{n}(\mathfrak{R})$. (Note that complex conjugation permutes the ray classes modulo the ideal $\mathfrak{n}$ of $\mathfrak{o}$.)

At the same time, using the relations analogous to those described in the proof of Lemma 1.3.2 of [16], one sees that the action of $I(K)^\sim$ on the space of double cosets preserves each of the sets

$$G_+(\mathbf{Q})\backslash \Xi_\mathbf{A}(\mathfrak{R}, \tilde{\Sigma})/\mathbb{K}_\infty$$

and provides an action of $C_\mathfrak{n}(\mathfrak{R})^\sim$ on $\Xi_\mathfrak{n}(\mathfrak{R}, \tilde{\Sigma})$ for each lifting $\tilde{\Sigma}$.

4. Modular Forms and Eisenstein Series

4.1. If $\mathbb{K}$ is an open compact subgroup of $G(\mathbf{A}_f)$, we form the graded algebra [3: §1.2]

$$\mathfrak{A}(\mathbb{K}) = \oplus_{w \in \mathbb{Z}, w \geq 0} \mathfrak{A}(\mathbb{K}, w)$$

of modular forms with respect to $\mathbb{K}$ on $G_+(\mathbf{A})$. For given w, each element ϕ of $\mathfrak{A}(\mathbb{K}, w)$ induces on each component $X_\omega (\simeq \mathfrak{H}^n)$ a holomorphic modular form of weight w with respect to the arithmetic group Γ_ω and for the automorphy factor

$$j(g, z)^w = \mathrm{jac}(g, z)^w, \qquad g \in G_+(\mathbb{R}), \qquad z \in \mathfrak{H}^n,$$

where jac denotes the functional determinant. In general we adopt the notation of §1 of [3] for the graded algebras of modular forms, the graded k-subalgebras of those which are k-arithmetic, the graded algebra of homogeneous quotients of modular forms, and, in particular, for the ring of modular functions, respectively arithmetic modular functions, with respect to an open compact subgroup of $G(\mathbf{A}_f)$, or field of modular functions with respect to an arithmetic group acting on $\mathfrak{H}^n$. As remarked in *loc. cit.*, there is a standard procedure for lifting a modular form or function with respect to Γ_ω from $\mathfrak{H}^n$ $(= X_\omega)$ to one on $G_+(\mathbf{A})$ supported on the double coset $\mathfrak{C}_{\mathbb{K}}(\omega)$; we denote this lifting map by Λ_ω. There is also a standard process, given by a functor λ, for lowering a modular form ϕ on $G_+(\mathbf{A})$ of weight w with respect to $\mathbb{K}$ to a family $\{f_\omega\}_{\omega\in\Omega}$ of modular forms of weight w, where f_ω is a modular form on $\mathfrak{H}^n$ with respect to Γ_ω.

4.2. We now introduce Eisenstein series, following the constructions of [2, 3, 16], and record some basic facts about their Fourier expansions and behavior as transformed by elements of the Galois group $\mathfrak{G} = \mathrm{Gal}(\mathbf{Q}_{ab}/\mathbf{Q})$.

The Eisenstein series considered in [3] are constructed as follows: Let $\mathfrak{v}$ and $\mathfrak{b}$ be fractional ideals of k representing respectively a narrow ideal class $\mathfrak{K}$ and an ideal class $\mathfrak{K}$ of k. Let $\mathfrak{n} \subset \mathfrak{o}$ be an integral non-zero ideal of $\mathfrak{o}$, and denote by ρ_1 and ρ_2 respectively elements of $\mathfrak{b}\mathfrak{v}$ and of $\mathfrak{b}$ such that

$$g.c.d.(\mathfrak{v}^{-1}\mathfrak{b}^{-1}\rho_1, \mathfrak{b}^{-1}\rho_2, \mathfrak{n}) = (1).$$

Let $\mathfrak{a}_1 = \mathfrak{v}\mathfrak{b}$, $\mathfrak{a}_2 = \mathfrak{b}$. One forms the series

$$G_w(z; \rho_1; \rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n}) = \Sigma'_{(e)} \frac{(N(\mathfrak{a}_1\mathfrak{a}_2))^w}{N(\xi_1 z + \xi_2)^{2w}}, \qquad (e)$$

$$G_w^*(z; \rho_1; \rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n}) = \Sigma'_{(e)+(*)} \frac{(N(\mathfrak{a}_1\mathfrak{a}_2))^w}{N(\xi_1 z + \xi_2)^{2w}}, \qquad (e^*)$$

where in the first series the sum is over $\xi_i \equiv \rho_i \bmod \mathfrak{n}\mathfrak{a}_i$, $i = 1, 2$, modulo multiplication of the pair (ξ_1, ξ_2) by a totally positive unit $\eta \equiv 1 \bmod \mathfrak{n}$, and $(\xi_1, \xi_2) \neq (0, 0)$, while the conditions in the second summation are all these conditions plus the condition

$$(\mathfrak{v}^{-1}\xi_1, \xi_2) = \mathfrak{b}, \qquad (*)$$

where $(,)$ stands for *g.c.d.* Let $q = h(\mathfrak{n})$ be the order of the group of ray classes mod $\mathfrak{n}$ in k, let $C_1, \ldots, C_q$ be the distinct ray classes mod $\mathfrak{n}$, and for each $l = 1, \ldots, q$, let $\mathfrak{r}_l$ be an integral ideal in C_l and prime to $\mathfrak{n}$, and $\varkappa_l$ be an integer of k such that $\varkappa_l \equiv 0 \pmod{\mathfrak{r}_l}$ and $\varkappa_l \equiv 1 \pmod{\mathfrak{n}}$. Then the two sets of Eisenstein series

$$\{G_w(z; \varkappa_l\rho_1, \varkappa_l\rho_2; \mathfrak{b}\mathfrak{r}_l; \mathfrak{v}; \mathfrak{n}) \mid l = 1, \ldots, q\} \qquad (e_1)$$

and

$$\{ G_w^*(z; \varkappa_l\rho_1, \varkappa_l\rho_2; \mathfrak{b}\mathfrak{r}_l; \mathfrak{v}; \mathfrak{n}) \mid l = 1, \ldots, q \} \qquad (e_1{}^*)$$

are linearly independent and span the same vector-space of complex-valued functions on $\mathfrak{H}^n$. Each G_w is a linear combination of the function G_w^*, and vice versa, and the coefficients in these linear combinations are given explicitly as special values of certain Dirichlet series (cf. [19], §2.2) denoted $\theta_l^{*(w)}$ and $\theta_l^{(w)}$. Moreover, if one defines

$$E_w = (-2\pi i)^{-wn} |\sqrt{d}|^{-1} G_w,$$

then E_w has a Fourier expansion of the form

$$E_w(z; \rho_1, \rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n}) = b_0(\rho_1, \rho_2; \mathfrak{b}, \mathfrak{v}, \mathfrak{n}) + \\ + \frac{N\mathfrak{b}^{2w-1}}{((2w-1)!)^n N(\mathfrak{n})} \sum_{\substack{\xi_1 \equiv \rho_1(\mathfrak{n}\mathfrak{a}_1) \\ (\xi_1)_\mathfrak{n}^+}} \sum_{\substack{\mu \in (\mathfrak{n}\mathfrak{b}\mathfrak{d}_K)^{-1} \\ \xi_1\mu \gg 0}} sgn(N(\xi_1)) N\mu^{2w-1} e(\mu\rho_2) e(\xi_1 \mu z),$$

where $(\xi_1)_\mathfrak{n}^+$ denotes that the summation over ξ_1 is, again, modulo multiplication by totally positive units $\equiv 1 \bmod \mathfrak{n}$, and $e(\,) = e^{2\pi i \cdot tr(\,)}$. All the Fourier coefficients in this expansion lie in $\mathbf{Q}_{N(\mathfrak{n})}$. Moreover, the coefficients of the linear combinations by means of wich the E_w's are expressed in terms of the G_w^*'s, or vice versa, all lie in $\mathbf{Q}_{N(\mathfrak{n})}$, and if $\sigma \in \mathrm{Gal}(\mathbf{Q}_{N(\mathfrak{n})}/\mathbf{Q})$ is such that for every $N(\mathfrak{n})$-th root of unity ζ we have $\zeta^\sigma = \zeta^s$, $s \in \mathbb{Z}$, $(s, N(\mathfrak{n})) = (1)$, and if σ is applied to all the Fourier coefficients of E_w, the result is

$$E_w(z; \rho_1, \rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n})^\sigma = E_w(z; \rho_1, s\rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n}). \qquad (19)$$

By means of calculations based on Klingen's paper and in principle due to Karel, we may show that, as a consequence of (19),

$$G_w^*(z; \rho_1, \rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n})^\sigma = G_w^*(z; s^{-1}\rho_1, \rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n}). \qquad (20)$$

This equation, which will be proved later, has a convenient formal interpretation for which we now prepare.

4.3. In [2] we calculated, in the special case of arithmetic groups commensurable with the Hilbert modular group, the explicit form of certain Eisenstein series considered in [15]. These Eisenstein series are associated to a quintuple $\mathfrak{C} = (G, P, \rho, \mathbb{K}, s)$, where G is an algebraic group, P, a parabolic subgroup, ρ, a one-dimensional character of P, all defined over $\mathbf{Q}$, $\mathbb{K}$ is an open compact subgroup of $G(\mathbf{A}_f)$, and s is a function on the double coset space

$$P_+(\mathbf{Q}) \backslash G(\mathbf{A}) / G_+(\mathbb{R}) \mathbb{K}; \qquad (21)$$

and these series are constructed by means of a certain function $\phi_{\mathfrak{S}}: G(\mathbf{A}) \to \mathbf{C}$. In this paper we extend the calculations of [2] to a more general situation and show how, in our case, such series are related to those in section §4.2. The construction of the series we consider also depends on a certain function

$$\phi_{\mathfrak{S}}: G_+(\mathbf{A}) \to \mathbf{C}$$

which satisfies

$$\phi_{\mathfrak{S}}(bg) = \phi_{\mathfrak{S}}(g), \qquad g \in G_+(\mathbf{A}), \qquad b \in P_+(\mathbf{Q}),$$

where P is defined in §1, and

$$\phi_{\mathfrak{S}}(gk) = \phi_{\mathfrak{S}}(g) J(k_\infty, \mathrm{i.e}), \qquad k_\infty \in \mathbb{K}_\infty,$$

where $J(g, z) = j(g, z)^w$ for some integer $w \geqslant 0$, and finally

$$\phi_{\mathfrak{S}}(gx) = \phi_{\mathfrak{S}}(g), \qquad x \in G'(\hat{\mathfrak{o}}),$$
$$\phi_{\mathfrak{S}}(b) = |\rho(b)|_{\mathbf{A}(k)}, \qquad b \in P_+(\mathbf{A}),$$

where ρ is the rational character, defined over k, on the group P' of §1, such that $\rho\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = (ad^{-1})^w$. Then one forms the series

$$\mathcal{E}_{\mathfrak{S}}(g) = \sum_{P_+(\mathbf{Q}) \diagdown G_+(\mathbf{Q}) \ni \gamma} s(\gamma g) \phi_{\mathfrak{S}}(\gamma g), \tag{22}$$

where s is *a* **Q**-valued function on the double coset space (21), and the series converges uniformly absolutely on compact subsets of $G_+(\mathbf{A})$ as long as $w \geqslant 2$. We have

$$G_+(\mathbf{A}) = \bigcup_{\omega \in \Omega} G_+(\mathbf{Q}) \omega G_+(\mathbb{R}) \mathbb{K}$$

and we may let $\Omega = \Theta \cdot H$, where H is a set of elements $\eta \in G(\mathbf{A}_f)$ such that id.$(\det(\eta))$ runs over representatives of the narrow ray classes mod $\mathfrak{n}$ contained in the principal narrow class of k, say $\eta = \begin{pmatrix} \eta' & 0 \\ 0 & 1 \end{pmatrix}$ for $\eta \in H$. (Here, for $x \in I(k)_f$, id.(x) is the ideal of k naturally associated to x.) Moreover, for each $\omega \in \Omega$, we have

$$G_+(\mathbf{Q}) = \bigcup_{\alpha \in \mathbf{A}(\omega)} P_+(\mathbf{Q}) \alpha \Gamma_\omega,$$
$$\Gamma_\omega = G_+(\mathbf{Q}) \cap G_+(\mathbb{R})^\omega \mathbb{K},$$
$$G_+(\mathbf{A}) = \bigcup_{\omega \in \Omega} \bigcup_{\alpha \in \mathbf{A}(\omega)} P_+(\mathbf{Q}) \alpha \omega G_+(\mathbb{R}) \mathbb{K}.$$

We may write

$$\alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix} \in G'_+(k) = G_+(\mathbf{Q}),$$

and let $s_{\omega,\alpha}$ be the characteristic function of the double coset

$$P_+(\mathbf{Q})\alpha\omega G_+(\mathbb{R})\mathbb{K}.$$

We assume $\mathbb{K}$ is the principal congruence subgroup $G'(\hat{\mathfrak{o}})(\mathfrak{n})$ of $G'(\hat{\mathfrak{o}})$ for some integral ideal $\mathfrak{n}$ of k, $\mathbb{K} = \mathbb{K}(\mathfrak{n})$. Then we may assume each $\theta \in \Theta$ is of the form $\left(\begin{smallmatrix}\theta' & 0\\ 0 & 1\end{smallmatrix}\right)$, where $\mathfrak{v} = \mathrm{id}.(\theta'^{-1})$ is an integral ideal prime to $\mathfrak{n}$. Moreover, a full set of representatives of the cosets C occurring in

$$I_+(k)/k_+^\times k_{\infty+}^\times \mathfrak{U}_\mathfrak{n}(\hat{\mathfrak{o}}) \qquad (\mathfrak{U}_\mathfrak{n}(\hat{\mathfrak{o}})\text{: units} \equiv 1 \bmod \mathfrak{n} \quad \text{of} \quad \hat{\mathfrak{o}})$$

for which the ideals of C belong to the narrow principal class, may be taken as units η' of the maximal compact subring $\hat{\mathfrak{o}}$ of $\mathbf{A}(k)_f$ (which are 1 at the archimedean places), and therefore each $\eta \in H$ may even be taken of the form

$$\eta = \begin{pmatrix}\eta' & 0\\ 0 & 1\end{pmatrix},$$

with $\eta' \in \hat{\mathfrak{o}}^\times$. We need the following

Lemma 2. *Let there be given two pairs (c, d) and (c', d') belonging to $(\mathfrak{v}\mathfrak{b}, \mathfrak{b})$ and such that*

$$g.c.d.(\mathfrak{v}^{-1}c, d) = g.c.d.(\mathfrak{v}^{-1}c', d') = \mathfrak{b},$$

where $\mathfrak{b}$ is an integral ideal prime to $\mathfrak{n}$. Assume $(c, d, \mathfrak{n}) = (c', d', \mathfrak{n}) = (1)$ and that $c' \equiv c$, $d' \equiv d \bmod \mathfrak{n}$. Then there exists $M'' \in R_{\mathfrak{v}+}^\times(\mathfrak{n})$ such that $\det(M'') = 1$ and $(c, d)M'' = (c', d')$ *(matrix multiplication on the right).*

PROOF. Let $\mathfrak{a}_1 = \mathfrak{b}\mathfrak{v}$, $\mathfrak{a}_2 = \mathfrak{b}$; these are integral ideals prime to $\mathfrak{n}$. We know ([3], §2.3) that there exist $a, a' \in \mathfrak{a}_2^{-1}$, $b, b' \in \mathfrak{a}_1^{-1}$ such that if

$$S = \begin{pmatrix}a & b\\ c & d\end{pmatrix}, \qquad S' = \begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}, \qquad \text{then } \det(S) = \det(S') = 1.$$

Clearly $M = S'^{-1}S$ belongs to $R_\mathfrak{v}^\times$. We have $S = S'M$. Since $(c, d, \mathfrak{n}) = (1)$, there is (by strong approximation) a non-singular matrix T of determinant 1 in $R_\mathfrak{v}^\times$ such that modulo $\mathfrak{n}$ we have $(c_1, d_1) = (c, d)T \equiv (c^*, 0)$, $(c^*, \mathfrak{n}) = (1)$. Since $S = S'M$, we have $(c, d) = (c', d')M$ and $(c_1, d_1) = (c, d)T = (c', d')MT =$ $= (c', d')TM' \equiv (c^*, 0) \bmod \mathfrak{n}$, where $M' = T^{-1}MT \in R_\mathfrak{v}^\times$. We have, since $c' \equiv c$, $d' \equiv d \bmod \mathfrak{n}$, $(c', d')T \equiv (c, d)T \equiv (c^*, 0) \bmod \mathfrak{n}$, hence $(c^*, 0)M' \equiv$ $\equiv (c^*, 0) \bmod \mathfrak{n}$, and since $(c^*, \mathfrak{n}) = 1$ and $\det(M') = 1$, we have

$$M' \equiv \begin{pmatrix}1 & 0\\ \gamma' & 1\end{pmatrix} \bmod \mathfrak{n} \quad \text{for some} \quad \gamma' \in \mathfrak{v}.$$

We want $M_2 \in R_{\mathfrak{v}}$ such that $(c_1, d_1)M_2 = (c_1, d_1)$ and

$$M_2 \equiv \begin{pmatrix} 1 & 0 \\ \gamma' & 1 \end{pmatrix} \bmod \mathfrak{n}.$$

The case $d_1 = 0$ being easily handled as a special case, we assume $c_1 d_1 \neq 0$. The matrices of the form

$$\begin{pmatrix} \alpha & c_1^{-1} d_1(\alpha - 1) \\ -d_1^{-1} c_1(\alpha - 1) & 2 - \alpha \end{pmatrix}, \qquad \alpha \in k,$$

all fix (c_1, d_1), i.e., satisfy the second condition. Then the first and third conditions are expressed by

$$c_1^{-1} d_1 \beta^* \in \mathfrak{v}^{-1}, d_1^{-1} c_1 \beta^* \in \mathfrak{v}, \beta^* \in \mathfrak{n}, \beta^* \equiv -c_1^{-1} d_1 \gamma' \bmod c_1^{-1} d_1 \mathfrak{n},$$

where $\beta^* = \alpha - 1$. Since $\mathfrak{b}$ and $\mathfrak{v}$ are prime to $\mathfrak{n}$ and $\gamma' \in \mathfrak{v}$, it follows easily from the Chinese remainder theorem that such a $\beta^* \in k$ exists, hence M_2 satisfying the desired conditions also exists. Then $M' M_2^{-1} \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \bmod \mathfrak{n}$ in $R_{\mathfrak{v}}$ and we have $(c', d')TM'M_2^{-1}T^{-1} = (c, d)TM_2^{-1}T^{-1} = (c_1, d_1)M_2^{-1}T^{-1} = (c_1, d_1)T^{-1} = (c, d)$, while at the same time $M'M_2^{-1} \in R_{\mathfrak{v}}(\mathfrak{n})^{\times}$, therefore $M'' = TM'M_2^{-1}T^{-1} \in R_{\mathfrak{v}}(\mathfrak{n})$ and, actually, $\det(M'') = 1$. Therefore, $M'' \in \Gamma_{\mathfrak{v}}(\mathfrak{n})$. This proves the lemma.

With $\eta \in H$ of the form $\left(\begin{smallmatrix} \eta' & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\eta' \in \hat{\mathfrak{o}}^{\times}$, η normalizes $\mathbb{K}(\mathfrak{n})$. Let $\omega = \theta\eta$ and let $\tilde{\mathcal{E}}_{\omega, \alpha, w}$ be the Eisenstein series $\tilde{\mathcal{E}}_{\mathfrak{C}}$ on $G_+(\mathbf{A})$ where $\mathfrak{C} = (G, P, \rho, \mathbb{K}(\mathfrak{n}), s_{\omega, \alpha})$. Since $s_{\omega, \alpha}$ is the characteristic function of

$$P_+(\mathbf{Q})\alpha\omega G_+(\mathbb{R})\mathbb{K}(\mathfrak{n}),$$

$\tilde{\mathcal{E}}_{\omega, \alpha, w}$ is supported on the union of translates by $G_+(\mathbf{Q})$ on the left of

$$G_{\omega, \mathfrak{n}}(\mathbf{A}) = \omega G_+(\mathbb{R})\mathbb{K}(\mathfrak{n}),$$

hence the corresponding holomorphic Eisenstein series $E_{\omega, \alpha, w}$ is supported on the union of translates by $G_+(\mathbf{Q})$ on the left of

$$X_\omega = \omega G_+(\mathbb{R})\mathbb{K}(\mathfrak{n})/\mathbb{K}_\infty \mathbb{K}(\mathfrak{n}),$$

which may be identified with $\mathfrak{H}^n$ on which $\Gamma_\omega(\mathfrak{n}) = \Gamma({}^{\omega}\mathbb{K}(\mathfrak{n})) = \Gamma({}^{\theta}\mathbb{K}(\mathfrak{n}))$ acts. If $\mathfrak{v} = \mathrm{id}.(\theta'^{-1})$, $\Gamma({}^{\theta}\mathbb{K}(\mathfrak{n})) = \Gamma_{\mathfrak{v}}(\mathfrak{n})$, the principal congruence subgroup of the group $\Gamma_{\mathfrak{v}}$ defined in [2]. The calculation in [2], §5.2, applied in similar fashion to the present case shows that if

$$\alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix} \in \begin{pmatrix} \mathfrak{a}_2^{-1} & \mathfrak{a}_1^{-1} \\ \mathfrak{a}_1 & \mathfrak{a}_2 \end{pmatrix}, \qquad \mathfrak{a}_1 = \mathfrak{v}\mathfrak{b}, \qquad \mathfrak{a}_2 = \mathfrak{b},$$

and $\det(\alpha) = 1$ (which implies $(\mathfrak{a}_1^{-1}c_\alpha, \mathfrak{a}_2^{-1}d_\alpha) = (1)$), then

$$E_{\omega,\alpha,w}(z) = \sum_{(\mathfrak{v}^{-1}c,d)=\mathfrak{b},\, c\equiv c_\alpha(\mathfrak{n}\mathfrak{a}_1),\, d\equiv d_\alpha(\mathfrak{n}\mathfrak{a}_2),\, (c,d)\mathfrak{n}^+} \frac{N(\mathfrak{a}_1\mathfrak{a}_2)^w}{N(\xi_1 z + \xi_2)^{2w}} = \tag{23}$$
$$= G_w^*(z; c_\alpha, d_\alpha; \mathfrak{b}; \mathfrak{v}; \mathfrak{n}),$$

because by Lemma 2, every pair (c, d) satisfying the conditions of summation is obtained by applying an element $\gamma \in \Gamma_{\mathfrak{v}}(\mathfrak{n})$ to the right of (c_α, d_α): $(c, d) = (c_\alpha, d_\alpha)\gamma$. Therefore on the component $X_\omega \simeq \mathfrak{H}^n$, $\tilde{\mathcal{E}}_{\omega,\alpha,w}$ induces a standard congruence Eisenstein series for the congruence group $\Gamma_{\mathfrak{v}}(\mathfrak{n})$, and induces the function $\equiv 0$ on any component not in the left translates of X_ω by $G_+(\mathbf{Q})$. Hence the coefficients of the Fourier expansion of this function on each component X_ω lie in the field of $N(\mathfrak{n})$-th roots of unity. Therefore, for $\sigma \in \mathrm{Gal}(\mathbf{Q}_{ab}/\mathbf{Q})$ such that $\zeta_{N(\mathfrak{n})}^\sigma = \zeta_{N(\mathfrak{n})}^s$, $(s, N(\mathfrak{n})) = 1$, we need to find the result of applying σ to all the Fourier coefficients of these expansions.

5. Galois Action and Transformation Theory

5.1. The calculations in this section are based on Klingen's paper [19] and were suggested by Karel's paper [16]. We keep the notation of §4.2 and, for the greater part where there is no conflict, also that of §3 of [19].

For each $l = 1, \ldots, h(\mathfrak{n})$, let $\mathfrak{g}_l$ be an integral ideal in the ray class C_l^{-1}. In our present notation, equation (20) on p. 186 of [19] reads (where Klingen's Vorzeichencharakter may be omitted since the weight is even)

$$G_w^*(z; \rho_1, \rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n}) = \sum_{l=1}^{h(\mathfrak{n})} \sum_{\mathfrak{g}\in C_l} \frac{\mu(\mathfrak{g})}{N\mathfrak{g}^{2w}} G_w(z; \rho_1, \rho_2; \mathfrak{b}\mathfrak{g}_l^{-1}; \mathfrak{v}; \mathfrak{n}),$$

and the Fourier coefficients of the Eisenstein series

$$\frac{\Delta^{1/2}}{(\pi i)^{2wn}} G_w(z; \rho_1, \rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n}) = G_w^{\#}(z; \rho_1, \rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n}) \tag{24}$$

lie in $\mathbf{Q}_{N(\mathfrak{n})}$. The constant term of the series (24) is zero unless $\rho_1 \equiv 0 \bmod \mathfrak{v}\mathfrak{b}\mathfrak{n}$, but if this congruence holds, then necessarily (according to our original assumptions) we have $(\mathfrak{b}^{-1}\rho_2, \mathfrak{n}) = (1)$, and in that case, the constant term is given as

$$a_0(\rho_1, \rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n}) = (\pi i)^{-2wn} N(\mathfrak{a}_1\mathfrak{a}_2)^w \Delta^{1/2} \sum_{m\equiv\rho_2(\mathfrak{n}\mathfrak{b}),\, (m)\mathfrak{n}^+} N(m)^{-2w},$$

which belongs to $\mathbf{Q}_{N(\mathfrak{n})}$ by Klingen's results. If $\sigma \in \mathfrak{G} = \mathrm{Gal}(\mathbf{Q}_{ab}/\mathbf{Q})$ is such that $\sigma(\zeta_{N(\mathfrak{n})}) = \zeta_{N(\mathfrak{n})}^s$, then, by §2.2 of [3],

$$a_0(\rho_1, \rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n})^\sigma = a_0(\rho_1, s\rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n}).$$

That is, for $\rho \in \mathfrak{b}$, and $(\mathfrak{b}^{-1}\rho, \mathfrak{n}) = (1)$, one has

$$((\pi i)^{-2wn} N(\mathfrak{a}_1\mathfrak{a}_2)^w \Delta^{1/2} \sum_{m \equiv \rho(\mathfrak{n}\mathfrak{b})}' Nm^{-2w})^\sigma =$$
$$= ((\pi i)^{2wn} N(\mathfrak{a}_1\mathfrak{a}_2)^w \Delta^{1/2} \sum_{\substack{m \equiv s\rho(\mathfrak{n}\mathfrak{b}) \\ (m)\mathfrak{n}^+}} Nm^{-2w}).$$

Let C be the ray class modulo $\mathfrak{n}$ to which $\mathfrak{b}^{-1}\rho$ belongs and define

$$\zeta_k(2w, C) = \sum_{\mathfrak{g} \in C} N\mathfrak{g}^{-2w}.$$

It is proved on p. 184 of [19] that the constant term may also be expressed by

$$a_0(\rho_1, \rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n}) = (\pi i)^{-2wn} e(\mathfrak{n}) \Delta^{1/2} N\mathfrak{v}^w \cdot \zeta_k(2w, C),$$

where, of course, a slight modification is necessary in Klingen's calculation to take account of the fractional ideal $\mathfrak{v}$, and where $e(\mathfrak{n})$ is a rational number depending on $\mathfrak{n}$ and on the structure of the units group of k. Define

$$\chi(C) = (\pi i)^{-2wn} e(\mathfrak{n}) \Delta^{1/2} N\mathfrak{v}^{2w} \cdot \zeta_k(2w, C) =$$
$$= (\pi i)^{-2wn} N(\mathfrak{a}_1\mathfrak{a}_2)^w \Delta^{1/2} \sum_{m \equiv \rho(\mathfrak{n}\mathfrak{b}),\, (m)\mathfrak{n}^+}' Nm^{-2w}.$$

Then $\chi(C)^\sigma = \chi(sC)$ and by [19] p. 186, line 3 from the bottom,

$$\sum_{t=1}^{h(\mathfrak{n})} \zeta_k(2w, CC_t^{-1}) \Big(\sum_{\mathfrak{g} \in C_t} \mu(\mathfrak{g}) N\mathfrak{g}^{-2w} \Big) = \begin{cases} 1 & \text{if } C \text{ is the principal ray mod } \mathfrak{n}, \\ 0 & \text{otherwise.} \end{cases}$$

Here μ is the ideal-theoretic Möbius function on integral ideals of k. This can be written as

$$\sum_{t=1}^{h(\mathfrak{n})} (\pi i)^{-2wn} \Delta^{1/2} \cdot \zeta_k(2w, CC_t^{-1}) \Big(\frac{(\pi i)^{2wn}}{\Delta^{1/2}} \sum_{\mathfrak{g} \in C_t} \mu(\mathfrak{g}) N\mathfrak{g}^{-2w} \Big) =$$
$$= \begin{cases} 1 & \text{if } C = H(\mathfrak{n}), \text{ the principal ray mod } \mathfrak{n}, \\ 0 & \text{if otherwise.} \end{cases}$$

Use $\theta(C)$ to denote the last factor in large parentheses on the left hand side of this statement. Then $\theta(C)$ belongs to $\mathbf{Q}_{N(\mathfrak{n})}$ and the last equations read

$$\sum_{t=1}^{h(\mathfrak{n})} \chi(CC_t^{-1}) \theta(C_t) = \begin{cases} 1 & \text{if } C = H(\mathfrak{n}), \\ 0 & \text{if otherwise.} \end{cases}$$

Applying $\sigma \in \mathfrak{G}$ with s defined as above:

$$\sum_{t=1}^{h(\mathfrak{n})} \chi(sCC_t^{-1}) \theta(C_t)^\sigma = \begin{cases} 1 & \text{if } C = H(\mathfrak{n}), \\ 0 & \text{if otherwise.} \end{cases}$$

Comparing these equations and using (21), p. 187 of [19] gives

$$\theta(C_l)^{\sigma} = \theta(s^{-1}C_l).$$

According to Klingen

$$G_w^*(z; \rho_1, \rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n}) = \sum_{l=1}^{h(\mathfrak{n})} \theta(C_l) G_w^{\#}(z; \rho_1, \rho_2; C_l\mathfrak{b}; \mathfrak{v}; \mathfrak{n}). \tag{25}$$

Applying σ to the Fourier coefficients we get

$$G_w^*(z; \rho_1, \rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n})^{\sigma} = \sum_{l=1}^{h(n)} \theta(s^{-1}C_l) G_w^{\#}(z; \rho_1, s\rho_2; C_l\mathfrak{b}; \mathfrak{v}; \mathfrak{n}),$$

which, by the equation preceding (20) on p. 186 of [19], is equal to

$$\sum_{l=1}^{h(\mathfrak{n})} \theta(s^{-1}C_l) G_w^{\#}(z; s^{-1}\rho_1, \rho_2; s^{-1}C_l\mathfrak{b}; \mathfrak{v}; \mathfrak{n}) = G_w^*(z; s^{-1}\rho_1, \rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n}).$$

This proves equation (20) of §4.2.

Now by applying the above together with §2.3 of [3], it is easy to see that if we apply $\sigma \in \mathfrak{G}$ to the Fourier coefficients of the expansion of the holomorphic modular form induced on each component by $\tilde{\mathcal{E}}_{\omega,\alpha,w}$, we get the collection of holomorphic modular forms induced on each component by $\tilde{\mathcal{E}}_{\omega,{}^s\alpha,w}$, where ${}^s\alpha$ is an element of determinant one and congruent modulo $N_{k/\mathbf{Q}}(\mathfrak{n}\mathfrak{b}\mathfrak{v})^2$ to

$$\begin{pmatrix} a_\alpha & sb_\alpha \\ s^*c_\alpha & d_\alpha \end{pmatrix}, \qquad s^*s \equiv 1 \bmod N_{k/\mathbf{Q}}(\mathfrak{n}\mathfrak{b}\mathfrak{v})^2.$$

If we let $\hat{\sigma} \in \hat{\mathbb{Z}}^{\times} \simeq \mathfrak{G}$, and if for $\mathfrak{C} = (G, P, \rho_w, \mathbb{K}, s)$ we define $\hat{\sigma}\mathfrak{C} = (G, P, \rho_w, {}^{\mu(\hat{\sigma})}\mathbb{K}, \hat{\sigma}\cdot s)$, where $\hat{\sigma}\cdot s(x) = s(\mu(\hat{\sigma})^{-1}x\mu(\hat{\sigma}))$, then linearization of the result we have just obtained shows that if we apply $\hat{\sigma}$ to the Fourier coefficients of the expansions of $\lambda(\tilde{\mathcal{E}}_{\mathfrak{C}})$ on all components X_ω, the result is $\lambda(\tilde{\mathcal{E}}_{\hat{\sigma}\mathfrak{C}})$. This is analogous to Karel's result in §2.2.1 of [16].

5.2. One may then proceed precisely as in [16] to establish an operation

$$\hat{\sigma}\colon \tilde{\mathcal{E}}_{\mathfrak{C}} \to \beta(\hat{\sigma})\tilde{\mathcal{E}}_{\mathfrak{C}} = R(\mu(\hat{\sigma})^{-1})\tilde{\mathcal{E}}_{\hat{\sigma}\mathfrak{C}}$$

or $\hat{\mathbb{Z}}^{\times} = \mathfrak{G}$ on the Eisenstein series for $\mathbb{K}$ on $G_+(A)$. (In this equation, μ stands for a homomorphism of $\hat{\mathbb{Z}}^{\times}$ into the adele group as defined by

$$\mu(\hat{\sigma}) = \begin{pmatrix} \tilde{\sigma} & 0 \\ 0 & 1 \end{pmatrix}$$

in [16], and $R(x)$ is the right regular representation of a group on the functions on it.) Define $\tilde{\mathcal{E}}_{\mathfrak{C}}$ and $\lambda(\tilde{\mathcal{E}}_{\mathfrak{C}})$ to be F-rational, F being a subfield of $\mathbf{Q}_{ab}$, if $\beta(\tilde{\sigma})\tilde{\mathcal{E}}_{\mathfrak{C}} = \tilde{\mathcal{E}}_{\mathfrak{C}}$ for all $\tilde{\sigma} \in \mathrm{Gal}(\mathbf{Q}_{ab}/F)$.

With a fixed ideal $\mathfrak{n}$ and $\mathbb{K} = \mathbb{K}(\mathfrak{n})$, let $\tilde{\mathcal{E}}(\mathfrak{n})$ be the graded subalgebra of the algebra $\mathfrak{A}(\mathbb{K}(\mathfrak{n}))$ of modular forms with respect to $\mathbb{K}(\mathfrak{n})$ generated by all the Eisenstein series $\tilde{\mathcal{E}}_{\mathfrak{C}}$ of weights $w > 1$, where $\mathfrak{C}$ is defined as above. $\tilde{\mathcal{E}}_{\mathfrak{C}}$ is a linear combination with (arbitrary) rational coefficients (if s is $\mathbf{Q}$-valued) of the Eisenstein series $\tilde{\mathcal{E}}_{\alpha,\omega,w}$ discussed above. We let ϕ be an element of the ring of homogeneous quotients (with respect to its non-zero divisors) of $\tilde{\mathcal{E}}(\mathfrak{n})$, of degree zero, and form the transformation polynomial

$$T_{\phi,\mathbb{K},s_0}(g)(X) = X^N + \sum_{\nu < N} \alpha_\nu(g)X^\nu, \tag{26}$$

as in [3], §2.1(23), where $S_0 \in G(\mathbf{A}_f) \cap \hat{R}_1$, $\alpha_\nu \in \mathfrak{M}(\mathbb{K}, 0, \{\varkappa\})$. By a straightforward generalization of Karel's results ([16], §2.2.4) we have

$$\beta(\tilde{\sigma})R(M)\phi = R(M)\beta(\tilde{\sigma})\phi, \qquad \tilde{\sigma} \in \mathfrak{G}, \qquad M \in G(\mathbf{A}_f). \tag{27}$$

Since α_ν is a symmetric function of functions $R(S_j)\phi$ and the relation (27) holds with M replaced by MS_j, it follows that also

$$\beta(\tilde{\sigma})R(M)\alpha_\nu = R(M)\beta(\tilde{\sigma})\alpha_\nu. \tag{28}$$

Using this relation in conjunction with Proposition 3 and its Corollary of [3], we see that we have

Proposition 1. *Let $\mathfrak{A}(\mathbb{K}, Q_{ab})$ be the graded algebra of arithmetic modular forms on $G_+(\mathbf{A})$ for $\mathbb{K}$ and let the operators $\beta(\tilde{\sigma})$ and $R(M)$ (for $M \in G(\mathbf{A}_f)$) be defined on $\mathfrak{A}(\mathbb{K}, Q_{ab})$ by the obvious extension. Then for every $\psi \in$ $\in \mathfrak{A}(\mathbb{K}, \mathbf{Q}_{ab})$ one has*

$$R(M)\beta(\tilde{\sigma})\psi = \beta(\tilde{\sigma})R(M)\psi. \tag{29}$$

PROOF. For from the foregoing considerations this relation holds for a set of generators of $\mathfrak{A}(\mathbb{K}, \mathbf{Q}_{ab})$.

Corallary. *Relation* (29) *holds for an arbitrary element ψ of the ring of homogeneous quotients of elements of $\mathfrak{A}(\mathbb{K}, \mathbf{Q}_{ab})$.*

Now the results of [16], §§2.3-2.3.4 have straightforward generalizations as expressed in the following propositions.

Proposition 2. *$\mathfrak{A}(\mathbb{K}(\mathfrak{n}))$ is the integral closure of $\tilde{\mathcal{E}}(\mathfrak{n})$ in its graded ring of homogeneous quotients and we have*

$$\mathfrak{A}(\mathbb{K}(\mathfrak{n})) = \mathfrak{A}(\mathbb{K}(\mathfrak{n}), \mathbf{Q}_{N(\mathfrak{n})}) \otimes \mathbf{C}.$$

Remark. The last statement follows since each Eisenstein series $\tilde{\mathcal{E}}_{\omega,\alpha,w}$ has all Fourier coefficients in $\mathbf{Q}_{N(\mathfrak{n})}$.

If F is any subfield of $\mathbf{Q}_{ab}$, define $\psi \in \mathfrak{A}(\mathbb{K}, \mathbf{Q}_{ab})$ to be F-rational if $\beta(\tilde{\sigma})\psi = \psi$ for every $\tilde{\sigma} \in \mathrm{Gal}(\mathbf{Q}_{ab}/F)$. Denote the graded F-algebra of such modular forms by $\mathfrak{A}(\mathbb{K})^F$. Then we have:

Proposition 3. *Let* $\mathbb{K} = \mathbb{K}(\mathfrak{n})$. *As a graded algebra over* $\mathbf{Q}_{ab}$, $\mathfrak{A}(\mathbb{K}, \mathbf{Q}_{ab})$ *is generated by* $\mathfrak{A}(\mathbb{K})^{\mathbf{Q}}$.

Since for $\theta \in \mathbf{Q}_{ab}$, $\tilde{\sigma} \in \mathfrak{G}$, we have $\beta(\tilde{\sigma})(\theta\psi) = \theta^{\tilde{\sigma}}\beta(\tilde{\sigma})\psi$, this follows, as remarked in [16], from §14 of [6, AG]. Therefore

$$V_{\mathbb{K}}^{*} = \bigcup_{\omega \in \Omega} V_{\omega}^{*},$$

where V_{ω}^{*} is the Satake compactification of V_{ω}, is an algebraic variety defined over $\mathbf{Q}$ (but not irreducible over $\mathbf{Q}_{ab}$, of course), and by the result of Borel and Narasimhan referred to in §2.5 of [3], the set of cusps $V_{\mathbb{K}}^{*} - V_{\mathbb{K}}$ is defined over $\mathbf{Q}$ as well, while each component V_{ω}^{*} is defined over $\mathbf{Q}_{N(\mathfrak{n})}$, and V_{ω} is a $\mathbf{Q}_{N(\mathfrak{n})}$— open subvariety of V_{ω}^{*}.

The conditions for $\psi \in \mathfrak{A}(\mathbb{K}, \mathbf{Q}_{ab})$ to belong to the $\mathbf{Q}$-structure are equivalent to the following: If $\omega \in \Omega$, let

$$\psi_{\omega}(z) = \Sigma\, a(\mu, \omega) e^{2\pi i tr(\mu z)}, \qquad \mu \in (\mathfrak{n}\mathfrak{b}\mathfrak{d}_k)^{-1},$$

be the Fourier expansion of the modular form with respect to $\Gamma_{\omega} = \Gamma({}^{\omega}\mathbb{K})$ induced by ψ on X_{ω}. Then for every $\tilde{\sigma} \in \mathrm{Gal}(\mathbf{Q}_{ab}/\mathbf{Q}) = \mathfrak{G}$, one has

$$a(\mu, \omega)^{\tilde{\sigma}} = a(\mu, \mu(\tilde{\sigma})\omega).$$

6. Splitting of the Class Polynomial

6.1. The multiplier polynomial. We consider the class polynomial

$$P_{\mathfrak{v},\mathfrak{R},\phi}(X) = \prod_{\tilde{\Sigma}} P_{\mathfrak{v},\mathfrak{R},\tilde{\Sigma},\phi}(X)$$

defined for any order $\mathfrak{R}$ of K containing the maximal order of k and an arithmetic modular function ϕ for $\Gamma_{\mathfrak{v}}^{+}$ holomorphic on all points of the set $\Xi_{\mathfrak{v}}(\mathfrak{R})$ for given $\mathfrak{v}$, as defined in §3.3 of [3], and where

$$P_{\mathfrak{v},\mathfrak{R},\tilde{\Sigma},\phi}(X) = \prod_{j=1}^{h'} (X - \phi(o_j')),$$

$o_1', \ldots, o_{h'}'$ being a set of representatives of the orbits of $\Gamma_{\mathfrak{v}}^{+}$ in $\Xi_{\mathfrak{v}}(\mathfrak{R}, \tilde{\Sigma})$. In case $\mathfrak{R} = \mathfrak{O}$, the maximal order of K, it has been shown in [3: §3.3, Theorem 4] that $P_{\mathfrak{v},\mathfrak{O},\phi}(X) \in \mathbf{Q}[X]$. We shall show here, in a later section, that

$$P_{\mathfrak{v},\mathfrak{O},\tilde{\Sigma},\phi}(X) \in K^*(\tilde{\Sigma})_0[X], \tag{30}$$

where $K^*(\tilde{\Sigma})_0$ is the totally real subfield of the reflex field $K^*(\tilde{\Sigma})$ associated to the lifting $\tilde{\Sigma}$ of Σ (cf. §§2, 6.2).

If $S \in G_+(\mathbf{A}) \simeq G'_+(\mathbf{A}(k))$ and S written as a two by two matrix over $\mathbf{A}(k)$ is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, let $\nu = ad - bc \in I_+(k)$, $\mathfrak{a}$ be the $g.c.d.$ of a_f, b_f, c_f, and d_f, and $\mathfrak{U} = \mathfrak{U}(S)$ be the integral ideal $\mathfrak{a}^{-2}\nu$. If $\phi \in \mathfrak{A}(\mathbb{K}, w)$, define $\phi \mid S \in \mathfrak{A}({}^S\mathbb{K}, w)$ by $(\phi \mid S)(g) = \phi(gS)$ for $g \in G_+(\mathbf{A})$. Then, following the definition of §2.4(48) of [3], put (where $N = N_{k/\mathbf{Q}}$)

$$\phi \,||\, S = N(\mathfrak{U})^w \cdot \phi \mid S.$$

By Lemma 1 (*loc. cit.*), if $\phi \in \mathfrak{A}(\mathbb{K}, w, R')$, where R' is a finitely generated subring of a number field, then $\phi \,||\, S$ belongs to $\mathfrak{A}({}^S\mathbb{K}, w, R'')$, where R'' is integral over R' in some finite extension of that field. (N. B. The second sentence of the proof of Lemma 1 of *loc. cit.* should be corrected to read: "We may write for that $\omega \in \Omega$ such that $S \in G_\omega(\mathbf{A})$ and for each $\omega' \in \Omega$: $\omega^{-1}\omega' S = S'_{\omega' f}\omega' \mathbf{k}$ for some $S'_{\omega'} \in G_+(Q)$, $\mathbf{k} \in \mathbb{K}$." and then S' replaced by $S'_{\omega'}$ in the rest of the proof.)

Fix $S_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G'(\mathbf{A}(k)_f)$ and suppose $S_0 \in \mathfrak{C}(\omega_1)$. We write

$$C_{\mathbb{K}}(S_0) = \mathbb{K} S_0 \mathbb{K} = \bigcup_{j=1}^{N} S_j \mathbb{K}.$$

Then $\det(C_{\mathbb{K}}(S_0)) \subset \det(S_0)\det(\mathbb{K})$, so that by §1, $C_{\mathbb{K}}(S_0) \subset \mathfrak{C}_{\mathbb{K}}(\omega_1)$ and for each $\omega \in \Omega$, $\omega C_{\mathbb{K}}(S_0) \subset \mathfrak{C}_{\mathbb{K}}(\omega\omega_1)$; therefore, $\omega S_j \in S'_{j\omega f}\omega\omega_1 \mathbb{K}$ for some $S'_{j\omega} \in$ $\in G_+(Q)$, $j = 1, \dots, N$. We introduce the so-called multiplier polynomial for any non-zero-divisor $\phi \in \mathfrak{A}(\mathbb{K}, w)$

$$M_{\phi, S_0}(g)(X) = \prod_{j=1}^{N} (X - (\phi \,||\, S_j/\phi)(g)) = X^N + \sum_{l<N} \mu_{l,\phi,S_0}(g) X^l, \tag{31}$$

where clearly μ_{l,ϕ,S_0} is a modular function for $\mathbb{K}$, and on each component X_ω, it induces a modular function $m_{l,\phi,S_0,\omega}$ for Γ_ω defined by

$$m_{l,\phi,S_0,\omega}(z) = j(g, \mathrm{i.e})^{-w(N-l)} \mu_{l,\phi,S_0}(g\omega), \qquad z \in \mathfrak{H}^n,$$

where $g \in G_+(\mathbb{R})$ is such that $z = g(\mathrm{i.e})$. Let σ_t denote de t-th elementary symmetric function in N variables and observe that for each $\omega \in \Omega$, $j = 1, \dots, N$, and $g \in G_+(\mathbb{R})$ we have

$$(\phi \mid S_j)(g\omega) = \phi(g\omega S_j) = \phi(g S'^{-1}_{j\omega f}\omega\omega_1) = \phi(S'_{j\omega\infty} g\omega\omega_1),$$

hence, $(\phi \mid S_j)_\omega(g) = \phi_{\omega\omega_1}(S'_{j\omega\infty}g)$, and therefore for $g \in G_+(\mathbb{R})$

$$\mu_{l,\phi,S_0,\omega}(g) = \sigma_{N-l}((\phi \,||\, S_1)_\omega(g), \ldots, (\phi \,||\, S_N)_\omega(g))/\phi_\omega(g)^{N-l}$$
$$= \sigma_{N-l}(\phi_{\omega\omega_1}(S'_{1\omega\infty}g), \ldots, \phi_{\omega\omega_1}(S'_{N\omega\infty}g))\phi_\omega(g)^{l-N}N\mathfrak{A}^{\omega(N-l)}.$$

From this and from the definitions we have for $z = g(\text{i.e}) \in \mathfrak{H}^n$

$$m_{l,\phi,S_0,\omega}(z) = \sigma_{N-l}((f_{\omega\omega_1} \,||\, S'_{1\omega}/f_\omega)(z), \ldots, (f_{\omega\omega_1} \,||\, S'_{N\omega}/f_\omega)(z)). \tag{31'}$$

Now for the present, we assume $S_0 \in G_+(\mathbf{Q})_f$ and $\omega_1 = 1$, and fix the order $\mathfrak{R} \supset \mathfrak{o}$ in K and assume ϕ is non-zero at all the cusps and at all the finite number of double cosets in $V_{\mathbb{K}}$ representing points of $\Xi_{\mathbf{A}}(\mathfrak{R})$. Then on each component $X_\omega = X_\mathfrak{v}$, $m_{l,\phi,S_0,\omega}$ is holomorphic at all the cusps and at all the points of $\Xi_\mathfrak{v}(\mathfrak{R})$. Let f_ω be the modular form for Γ_ω induced on X_ω by ϕ. Then as a polynomial with meromorphic coefficients on $\mathfrak{H}^n$, the multiplier polynomial takes the form

$$M_{f_\omega,S_0,\omega}(z)(X) = \prod_{j=1}^{N} (X - N(\mathfrak{U})^w(f_\omega \mid S'_{j\omega}/f_\omega)(z)).$$

We now wish to consider the roots of this polynomial for certain $z \in \Xi_\mathfrak{v}(\mathfrak{R})$ and for a conveniently chosen double coset $C_{\mathbb{K}}(S_0)$, for the particular case $\mathbb{K} = G(\hat{\mathbb{Z}})$.

6.2. We refer to the notation of §3.3 of [3]. Let $\tilde{\Sigma} = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n)$ be a lifting of Σ, where each $\tilde{\sigma}_j$ is an injection of K into $\mathbf{C}$. Let $\theta \in \Theta$ be one of the double coset representatives appearing in (6), let $\omega = \theta$, $\mathbb{K} = G(\hat{\mathbb{Z}})$, and $v = \text{id.}(\det(\omega)^{-1})$. Then, as in §3.3 of [3], we let $h' = h_{\mathfrak{v},\tilde{\Sigma}}$ be the number of orbits of $\Gamma_\mathfrak{v}^+$ in $\Xi_\mathfrak{v}(\mathfrak{O}, \tilde{\Sigma})$, and form the polynomial $P_{\mathfrak{v},\mathfrak{O},\tilde{\Sigma},\phi}$ where ϕ is a $\mathbf{Q}$-arithmetic modular function holomorphic on $o'_1, \ldots, o'_{h'}$. Our purpose is to show that for all such ϕ this is a polynomial with coefficients in the totally real subfield $K^*(\tilde{\Sigma})_0$ of the reflex field $K^*(\tilde{\Sigma}) = \mathbf{Q}(\{\sum_{\sigma\in\Sigma}\eta^{\tilde{\sigma}} \mid \eta \in K\})$. Showing this is evidently equivalent to showing that the image $A_\mathfrak{v}(\mathfrak{O}, \tilde{\Sigma})$ of $\Xi_\mathfrak{v}(\mathfrak{O}, \tilde{\Sigma})$ in $V_\mathfrak{v}^*$ is a zero cycle rational over $K^*(\tilde{\Sigma})_0$. Recall that $V_\mathfrak{v}^*$ is itself defined over $\mathbf{Q}$.

To show this, we need to consider certain of the roots of

$$M_{\eta,S_0,\theta}(\xi)(X)$$

for $\xi \in \Xi_\mathfrak{v}(\mathfrak{O})$, suitable $C(S_0) = C_{G(\hat{\mathbb{Z}})}(S_0)$, and a suitably chosen modular form η with respect to $\Gamma_\mathfrak{v}^+$.

Let $\mathfrak{p}$ be a prime ideal of first degree in $\mathfrak{o}$ and unramified over $\mathbf{Q}$ such that $\mathfrak{p}$ splits, $\mathfrak{p} = \mathfrak{P} \cdot \bar{\mathfrak{P}}$ in the maximal order $\mathfrak{O}$ of K. Let L be the smallest Galois extension of $\mathbf{Q}$ containing K and assume that the rational prime p contained in $\mathfrak{P}$ does not ramify in L. Denote by $\mathfrak{P}$ also the extension of $\mathfrak{P}$ to an ideal in the maximal order of L, and let H be the Galois group of L over K and G be the (absolute) Galois group of L over $\mathbf{Q}$. Let $\mathfrak{P} = \mathfrak{Q}_1 \cdot \cdots \cdot \mathfrak{Q}_m$ be the factorization of $\mathfrak{P}$ into prime ideals in the maximal order of L, let f be the degree of each of these over K, and h be the order of H. Then $h = mf$, f is the order of the decomposition group of each $\mathfrak{Q}_j$ over K, and is also the order of the decomposition group of each $\mathfrak{Q}_j$ over $\mathbf{Q}$, since the absolute degree of $\mathfrak{P}$ is 1, and H permutes $\mathfrak{Q}_1, \ldots, \mathfrak{Q}_m$ transitively; therefore, H consists of precisely those $\sigma \in G$ for which σ carries any $\mathfrak{Q}_i$ into the same or some other $\mathfrak{Q}_j$, $1 \leqslant i, j \leqslant m$, so that if $\sigma \in G - H$, then the ideals $\mathfrak{P}$ and $\sigma\mathfrak{P}$ are prime to each other. Let a be a positive integer such that $\mathfrak{P}^a$ is principal, say $\mathfrak{P}^a = (\Pi)$. Then $\sigma \in G - H$ also implies that Π and Π^σ are relatively prime to each other. Thus, if $\tilde{\Sigma} = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n)$ is any lifting of Σ and if s denotes complex conjugation, the elements

$$\Pi^{\tilde{\sigma}_1}, \ldots, \Pi^{\tilde{\sigma}_n}, \qquad s \cdot \Pi^{\tilde{\sigma}_1}, \ldots, s \cdot \Pi^{\tilde{\sigma}_n} \tag{33}$$

are the images of Π under representatives of the distinct $2n = [K\colon Q]$ cosets of H in G, and are therefore pairwise relatively prime. Let

$$N_{\tilde{\Sigma}}(\Pi) = \prod_{\tilde{\sigma} \in \tilde{\Sigma}} \Pi^{\tilde{\sigma}},$$

and suppose $\tilde{\Sigma}'$ is some lifting of Σ to K distinct from $\tilde{\Sigma}$ and from $s \cdot \tilde{\Sigma}$. If $\sigma \in \Sigma$, let $\tilde{\Sigma}(\sigma) = \tilde{\sigma}$ be the element of $\tilde{\Sigma}$ which extends σ to K. Let Σ' be the set of $\sigma \in \Sigma$ such that $\tilde{\Sigma}(\sigma) = \tilde{\Sigma}'(\sigma)$ and Σ'' be the set of $\sigma \in \Sigma$ such that $\tilde{\Sigma}(\sigma) = s \cdot \tilde{\Sigma}'(\sigma)$; then $\Sigma = \Sigma' \cup \Sigma''$ and Σ' and Σ'' are both non-empty. Let

$$\mathrm{M} = \mathrm{M}(\tilde{\Sigma}) = \prod_{\sigma \in \Sigma'} \Pi^{\tilde{\sigma}}, \qquad \mathrm{M}' = \mathrm{M}'(\tilde{\Sigma}) = \prod_{\sigma \in \Sigma''} \Pi^{\tilde{\sigma}}.$$

Lemma 3. $N_{\tilde{\Sigma}}(\Pi) = \mathrm{M} \cdot \mathrm{M}'$ *and, under the assumption* $\tilde{\Sigma}' \neq \tilde{\Sigma}, s \cdot \tilde{\Sigma}$, *we have for any positive integer* k

$$N_{\tilde{\Sigma}}(\Pi)^k + N_{s \cdot \tilde{\Sigma}}(\Pi)^k \neq N_{\tilde{\Sigma}'}(\Pi)^k + N_{s \cdot \tilde{\Sigma}'}(\Pi)^k.$$

PROOF. The first assertion follows from the definitions of $N_{\tilde{\Sigma}}(\Pi)$, M, and M′. Moreover we then see that $N_{s \cdot \tilde{\Sigma}}(\Pi) = s\mathrm{M} \cdot s\mathrm{M}'$, $N_{\tilde{\Sigma}'}(\Pi) = \mathrm{M} \cdot s\mathrm{M}'$, $N_{s \cdot \tilde{\Sigma}'}(\Pi) = s\mathrm{M} \cdot \mathrm{M}'$ and therefore.

$$N_{\tilde{\Sigma}}(\Pi)^k + N_{s \cdot \tilde{\Sigma}}(\Pi)^k - N_{\tilde{\Sigma}'}(\Pi)^k - N_{s \cdot \tilde{\Sigma}'}(\Pi) = (\mathrm{M}^k - s\mathrm{M}^k)(\mathrm{M}'^k - s \cdot \mathrm{M}'^k).$$

For the lemma to be false, one would have to have either $\mathrm{M}^k = (s \cdot \mathrm{M})^k$ or $\mathrm{M}'^k = (s \cdot \mathrm{M}')^k$. However, Σ' and Σ'' are non-empty. Therefore, by the discussion preceding the lemma, neither equality can occur since the two sides in each case have distinct prime ideal factorizations in L because s is in the center of G (since L is a CM-field) and takes any set of n of the $2n$ quantities in (33) onto its complement. (This argument occurs in Hecke's thesis [13], §12, for the case when k is a real quadratic field, but the idea is the same.)

6.3. Let $\xi \in \Xi_{\mathfrak{v}}(\mathfrak{O}, \tilde{\Sigma})$ so that $\mathfrak{v}\xi + \mathfrak{o}$ is a fractional ideal of the maximal order $\mathfrak{O}$ in K. (Cf. [3], §3.1) Since $\Pi \in \mathfrak{O}$, where Π is as in §6.2, we have

$$\begin{aligned} \Pi\xi &= \alpha\xi + \beta, & s\Pi \cdot \xi &= \alpha'\xi + \beta', \\ \Pi \cdot 1 &= \gamma\xi + \delta & s\Pi \cdot 1 &= \gamma'\xi + \delta', \end{aligned} \tag{34}$$

where α, δ, α', $\delta' \in \mathfrak{o}$, β, $\beta' \in \mathfrak{v}^{-1}$, γ, $\gamma' \in \mathfrak{v}$ are such that $\alpha\delta - \beta\gamma = \alpha'\delta' - \beta'\gamma' = N_{K/k}(\Pi) = \pi \gg 0$, while $g.c.d.(\alpha, \mathfrak{v}\beta, \mathfrak{v}^{-1}\gamma, \delta) = g.c.d.(\alpha', \mathfrak{v}\beta', \mathfrak{v}^{-1}\gamma', \delta') = (1)$. Let S_0, $S_0' \in R_{\mathfrak{v}}$ be defined by

$$S_0 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad S_0' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}.$$

Then S_0, $S_0' \in T_{\mathfrak{v}}(\pi)$ (as defined in §3.1 of [3]) and (under linear fractional operation) we have

$$S_0 \cdot (\xi) = S_0' \cdot (\xi) = (\xi), \tag{35}$$

so that the image of (ξ) in $V_\omega = V_{\mathfrak{v}}$ is a fixed point of the Hilbert modular correspondence associated to $T_{\mathfrak{v}}(\pi)$. Moreover, $\mathfrak{U}(S) = (\pi)$ for all $S \in T_{\mathfrak{v}}(\pi)$.

So we let ϕ be a $\mathbf{Q}$-arithmetic modular form of weight $w \equiv 0 \bmod 2$ and f_ω be the holomorphic modular form with respect to Γ_ω it induces on each component X_ω, and P, p, Π, π, S_0, and S_0' be as just now described. As S runs over a set of representatives $S_1, \ldots, S_N$ of the right cosets of $\Gamma_{\mathfrak{v}}^+$ contained in

$$C_{\mathfrak{v}}(S_0) = C_{\mathfrak{v}}(S_0') = \Gamma_{\mathfrak{v}}^+ S_0 \Gamma_{\mathfrak{v}}^+, \tag{36}$$

then

$$\mu_S(z) = N(\mathfrak{U})^w((f_\omega \mid S)(z)/f_\omega(z)) \tag{37}$$

runs over the roots of $M_{f_\omega, S_0, \omega}(z)(X)$ as functions of $z \in \mathfrak{H}^n$, where we continue to assume f_ω is non-vanishing on all $\xi \in \Xi_{\mathfrak{v}}(\mathfrak{O})$. Let $\Xi = \Xi_{\mathfrak{v}}(\mathfrak{O})$ and $\Xi(\tilde{\Sigma}) = \Xi_{\mathfrak{v}}(\mathfrak{O}, \tilde{\Sigma})$. By definition, $f_\omega \mid S$ takes the form

$$(f_\omega \mid S)(z) = f_\omega(S \cdot z) j(S, z)^w,$$

hence, if $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $\mu_S(z)$ is equal to

$$\begin{aligned} &N(\mathfrak{U} \det(S))^w f_\omega(S \cdot z) f_\omega(z)^{-1} N(cz + d)^{-2w} = \\ &\qquad = N_{k/\mathbf{Q}}(\pi)^{2w} f_\omega(S \cdot z) f_\omega(z)^{-1} N(cz + d)^{-2w}. \end{aligned} \tag{38}$$

Let $f_1, \ldots, f_m \in \mathfrak{M}(\Gamma_{\mathfrak{v}}^+, \mathcal{O}, \{\varkappa\})$ be **Q**-arithmetic modular functions generating the field of **Q**-arithmetic modular functions with respect to $\Gamma_{\mathfrak{v}}^+$ such that each is holomorphic on

$$\Xi(S_0) = \Xi \cup T_{\mathfrak{v}}(S_0)\Xi,$$

($T_{\mathfrak{v}}(S_0)$ being the correspondence on V_{ω}^* associated to $C_{\mathfrak{v}}(S_0) = C_{\mathfrak{v}}(S_0')$), and such that if $\xi \in \Xi$ and $z \in \mathfrak{H}^n - \Xi$, then there is an index $1 \leqslant j \leqslant m$ such that f_j is holomorphic at z also and $f_j(z) \neq f_j(\xi)$. We may assume $f_1, \ldots, f_m$ are the affine coordinates on an affine **Q**-open subset U of V_{ω}^* containing $\Xi(S_0)$ and all the cusps. Since V_{ω}^* is a **Q**-normal projective variety, U is a **Q**-normal affine variety and we may assume that if g is any **Q**-arithmetic modular function regular at all $\xi \in \Xi(S_0)$ and at all cusps $\varkappa$, then g may be written as a quotient of polynomials in $f_1, \ldots, f_m$ with coefficients in **Q** such that the denominator does not vanish at any $\xi \in \Xi(S_0)$ or at any cusp $\varkappa$. Let $u_1, \ldots, u_M$ be indeterminates (at first), define

$$F_u(z) = \sum_{j=1}^{m} u_j f_j(z), \qquad z \in \mathfrak{H}^n,$$

and define the polynomial

$$G_{\pi,u}(X; \{f_j\}) = \prod_{i=1}^{N} (X - F_u \mid S_i),$$

where $S_1, \ldots, S_N$ are as above and, of course, $(F_u \mid S)(z) = F_u(Sz)$. $G_{\pi,u}$ is a polynomial in $X, u_1, \ldots, u_m$ whose coefficients are **Q**-arithmetic modular functions holomorphic on Ξ, hence, expressible as rational functions of $f_1, \ldots, f_m$ with non-vanishing denominators on Ξ. By appropiate choice of ϕ, we may assume it does not vanish on $\Xi(S_0)$, hence μ_{S_i} is holomorphic at every point of Ξ. We then form another polynomial

$$\Phi_{\pi,u}(X, z) = \Phi_{\pi,u}(X) = \prod_{i=1}^{N} (X - (\mu_{S_i}(z) + F_u(S_i z)))$$

whose coefficients as a polynomial in $X, u_1, \ldots, u_m$ are **Q**-arithmetic modular functions holomorphic on Ξ, hence expressible as rational functions in $f_1, \ldots, f_m$ with denominators that do not vanish for $z = (\xi) \in \Xi$.

For any $(\xi) \in \Xi$, say $(\xi) = (\xi^{\tilde{\sigma}_1}, \ldots, \xi^{\tilde{\sigma}_n})$, $\xi \in K - k$, we consider the system of equations

$$\Phi_{\pi,u}(X + F_u(S^*(\xi)), f_1(\xi), \ldots, f_m(\xi)) = 0 \tag{39}$$

depending on $u \in \mathbf{Q}^n$, with $S^* = S_0$ or S_0'. Now

$$S_0((\xi)) = S_0'((\xi)) = (\xi) \tag{40}$$

because of (35) (apply $\tilde{\sigma}_j \in \tilde{\Sigma}$ to (35) for $j = 1, \ldots, n$); therefore, $F_u((S_0(\xi))) = F_u((S_0'(\xi))) = F_u((\xi))$. So consider the roots of the polynomial $\psi(X, u; (\xi))$ defined to be

$$\Phi_{\pi,u}(X + F_u(\xi), f_1(\xi), \ldots, f_m(\xi)).$$

For all complex $u = (u_1, \ldots, u_m)$, $\mu = \mu_{S_0}((\xi))$ and $\mu' = \mu_{S_0'}((\xi))$ are roots of it. Written out, we have

$$\Phi_{\pi,u}(X) = X^N + \sum_{\nu < N} P_\nu(u_1, \ldots, u_m; z)x^\nu,$$

where the coefficients of $P_\nu(u_1, \ldots, u_m; z)$ as a polynomial in $u_1, \ldots, u_m$ are $\mathbf{Q}$-arithmetic modular functions of $z \in \mathfrak{H}^n$ having no singularities on the set Ξ. The roots of $\Phi_{\pi,u}(X; (\xi))$ are $\mu_L((\xi)) + F_u((L\xi))$, $L \in C_\mathfrak{v}(S_0)$, which equals $\mu_L(\xi) + F_u(\xi)$ for all u if and only if $L(\xi)$ is in the $\Gamma_\mathfrak{v}^+$-orbit of (ξ). But suppose $L' \in C_\mathfrak{v}(S_0)$, so that $\det(L') = \eta'\pi \gg 0$ for some totally positive unit η' and so that L' is everywhere locally a *primitive* element of the $\mathfrak{o}$-lattice $R_\mathfrak{v}$. Suppose also that $L'(\xi) \in \Gamma_\mathfrak{v}^+(\xi)$, $L'(\xi) = \sigma^{-1}(\xi)$, $\sigma \in \Gamma_\mathfrak{v}^+$ or $L(\xi) = (\xi)$, where $L = \sigma L' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, say, and $\det L = \eta\pi \gg 0$, $\eta \in \mathfrak{o}_+^{\times}$. As before this implies there exists $M \in \mathbf{C}$ such that

$$\begin{aligned} M \cdot \xi &= \alpha\xi + \beta \\ M &= \gamma\xi + \delta. \end{aligned} \tag{41}$$

By assumption, $\xi \in \Xi = \Xi_\mathfrak{v}(\mathfrak{O})$, so that $M \in \mathfrak{O}$, and $M \cdot sM = \alpha\delta - \beta\xi = \eta\pi$ (where, as usual, s is complex conjugation). Then the prime factorization of M is of the form $\mathfrak{P}^b \cdot s\mathfrak{P}^c$, $b, c \geqslant 0$, $b + c = a$. But if $bc \neq 0$, M would be devisible by $\mathfrak{p}$, hence at the prime $\mathfrak{p}$, $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ would not be a primitive element of the lattice $R_{\mathfrak{v}\mathfrak{p}}$, contrary to definition of $C_\mathfrak{v}(S_0)$. Hence $(M) = \mathfrak{P}^a$ or $s\mathfrak{P}^a$ and so we may assume $M = \Pi$ or $s \cdot \Pi$, hence $L = S_0$ or S_0'; in other words, $L' \in \Gamma_\mathfrak{v}^+ \cdot S_0$ or $L' \in \Gamma_\mathfrak{v}^+ \cdot S_0'$. Then the roots of $\Psi(X; u; (\xi))$ as a polynomial in X are

$$\rho_L = \mu_L(\xi) + F_u(L(\xi)) - F_u(\xi), \qquad L \in C_\mathfrak{v}(S_0).$$

If $F_u(L\xi) - F_u(\xi)$ is not identically zero as a function of u, i.e., if $f_i(\xi) \neq f_i(L\xi)$ for some i, $1 \leqslant i \leqslant m$, then there is $u \in \mathbf{C}^m$ for which the above root ρ_L will not be equal to any root of

$$\Psi(X; 0; (\xi)) = M_{\pi, f_\omega}(X; (\xi)).$$

Hence the only common roots of $\Psi(X; u; (\xi))$ for all u are

$$\mu_1 = \rho_{S_0}, \qquad \mu_2 = \rho_{S_0'}. \tag{42}$$

The values of u for which other roots in common with $\Psi(X; 0; (\xi))$ exist then form a proper Zariski-closed subset $Z(\xi)$ of u-space depending on ξ. Therefore there exists $(u_0) = (u_1, \ldots, u_m)_0 \in \mathbf{Q}^m$ such that $\Psi(X; 0\colon (\xi))$ and $\Psi(X : u_0; (\xi))$ have monic *g.c.d.* of degree two having only the two roots μ_1, μ_2 in common. Since the number of $\Gamma_{\mathfrak{v}}^+$-orbits among such (ξ) is finite, we may assume u_0 is always the same. By the Weber-Perron theorem [32, 23], the monic *g.c.d.* of these two polynomials has coefficients which are rational functions of $f_1(\xi), \ldots, f_m(\xi)$ defined over $\mathbf{Q}$; thus,

$$\mu_1 + \mu_2 = P(f_1(\xi), \ldots, f_m(\xi))/Q(f_1(\xi), \ldots, f_m(\xi)), \tag{43}$$

where the polynomials P and Q belong to $\mathbf{Q}[X_1, \ldots, X_M]$ and are independent of $\xi \in \Xi$ and $Q(f_1(\xi), \ldots, f_m(\xi)) \neq 0$.

We now obtain expressions for the roots μ_1, μ_2. Take $\xi \in K - k$ as above such that $(\xi) \in \Xi$, $A = \mathfrak{v}\xi + \mathfrak{o}$ being a fractional $\mathfrak{O}$-ideal of K, and let $\Pi \in \mathfrak{O}$ be a fixed generator of $\mathfrak{P}^a$ as before. We use the relations (34) and the fact, just observed, that

$$g.c.d.(\alpha, \mathfrak{v}\beta, \mathfrak{v}^{-1}\gamma, \delta) = g.c.d.(\alpha', \mathfrak{v}\beta', \mathfrak{v}^{-1}\gamma', \delta') = (1).$$

Then $\Pi = \gamma\xi + \delta, s \cdot \Pi = \gamma' \cdot \xi + \delta'$, and according to (38) we have (since $(\xi) = (\xi^{\tilde{\sigma}_1}, \ldots, \xi^{\tilde{\sigma}_n})$ if $\tilde{\Sigma} = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n)$ and $\xi \in \Xi(\tilde{\Sigma})$)

$$\begin{aligned}\mu_S(\xi) = N(\pi)^{2w} f_\omega(S \cdot \xi) f(\xi)^{-1} \prod_{\tilde{\sigma} \in \tilde{\Sigma}} (c^\sigma \xi^{\tilde{\sigma}} + d^\sigma)^{-2w} &= \\ = N_{k/\mathbf{Q}}(\pi)^{2w} N_{\tilde{\Sigma}}(\Pi)^{-2w} &= N_{s\tilde{\Sigma}}(\Pi)^{2w}\end{aligned}$$

because $\pi = \Pi \cdot s\Pi$. Similarly, $\mu_{S'}(\xi) = N_{\tilde{\Sigma}(\Pi)^{2w}}$; therefore,

$$\mu_1 + \mu_2 = N_{\tilde{\Sigma}}(\Pi)^{2w} + N_{s\tilde{\Sigma}}(\Pi)^{2w},$$

which depends only on $\tilde{\Sigma}$ and Π, and not on ξ.

As before, L is the Galois closure of K over $\mathbf{Q}$. Then $K^*(\tilde{\Sigma}) \subset L$ and it is known [25, §5] that L is a CM-field, too, and that the automorphism s of complex conjugation is in the center of $G = \mathrm{Gal}(L/\mathbf{Q})$. For $\eta \in K$, define $\theta(\eta) = \sum_{\tilde{\sigma} \in \tilde{\Sigma}} \eta^{\tilde{\sigma}}$. Then θ is just the trace of the representation $\bigoplus_{\tilde{\sigma} \in \tilde{\Sigma}} \tilde{\sigma}$ of the $\mathbf{Q}$-algebra K. Therefore $K^*(\tilde{\Sigma})$ contains all the determinants

$$N_{\tilde{\Sigma}}(\eta), \qquad \eta \in K,$$

of that representation. This shows that μ_1 and μ_2 both belong to $K^*(\tilde{\Sigma})$ Moreover, $\mu_2 = s\mu_1$, hence $\mu_1 + \mu_2 \in K^*(\tilde{\Sigma})_0$. Now we have shown that

$$\mu_1 + \mu_2 = R(f_1(\xi), \ldots, f_m(\xi))$$

and the left side is an element $A(\Pi, \tilde{\Sigma})$ of $K^*(\tilde{\Sigma})_0$. We also write $\mu_i = \mu_i(\tilde{\Sigma})$, $i = 1, 2$, for fixed Π.

Clearly $\Xi_{\mathfrak{d}}(\mathcal{O}, \tilde{\Sigma}) = \Xi_{\mathfrak{d}}(\mathcal{O}, s \cdot \tilde{\Sigma})$. Suppose now that $\tilde{\Sigma}'$ is another lifting of Σ such that $\Xi_{\mathfrak{d}}(\mathcal{O}, \tilde{\Sigma}')$ is non-empty and $\tilde{\Sigma}' \neq \tilde{\Sigma}$, $s\tilde{\Sigma}$.

Proposition 4. *If $\tilde{\Sigma}' \neq \tilde{\Sigma}$, $s \cdot \tilde{\Sigma}$, then $A(\Pi, \tilde{\Sigma}') \neq A(\Pi, \tilde{\Sigma})$.*

PROOF. This is the same statement as that of Lemma 3 in different notation. Consider then the equation

$$A(\Pi, \tilde{\Sigma}) = P(f_1(\xi), \ldots, f_m(\xi))/Q(f_1(\xi), \ldots, f_m(\xi)),$$

where $P(X_1, \ldots, X_m)$ and $Q(X_1, \ldots, X_m) \in \mathbf{Q}[X_1, \ldots, X_m]$ and are independent of $\xi \in \Xi$, and $Q(f_1(\xi), \ldots, f_m(\xi)) \neq 0$ for all $\xi \in \Xi$. Let

$$U_{\tilde{\Sigma}}(X_1, \ldots, X_m) = P(X_1, \ldots, X_m) - A(\Pi, \tilde{\Sigma})Q(X_1, \ldots, X_m).$$

Then $U_{\tilde{\Sigma}}$ has coefficients in $K^*(\tilde{\Sigma})_0$, and the set of points where it vanishes cuts out a hypersurface section $V(\Pi, \tilde{\Sigma})$ on the $\mathbf{Q}$-open affine subset U of $V_{\mathfrak{d}}^*$ such that $V(\Pi, \tilde{\Sigma})$ is itself defined over $K^*(\tilde{\Sigma})_0$ and such that $V(\Pi, \tilde{\Sigma})$ intersects $A_{\mathfrak{d}}(\mathcal{O})$, the image of $\Xi_{\mathfrak{d}}(\mathcal{O})$ in $V_{\mathfrak{d}}^*$, in the image $A_{\mathfrak{d}}(\mathcal{O}, \tilde{\Sigma})$ of $\Xi_{\mathfrak{d}}(\mathcal{O}, \tilde{\Sigma})$ because according to Proposition 4, $V(\Pi, \tilde{\Sigma}')$ cannot meet $A_{\mathfrak{d}}(\Pi, \tilde{\Sigma})$ if $\tilde{\Sigma}' \neq \tilde{\Sigma}, s\tilde{\Sigma}$. According to Theorem 4 of [3], $A_{\mathfrak{d}}(\mathcal{O})$ is a $\mathbf{Q}$-set (in the statement of that theorem, the first Ξ was mistakenly put in place of A). Therefore we have the following theorem, due to Hecke in the case of real quadratic k,

Theorem 1. *The finite zero-cycle $A_{\mathfrak{d}}(\mathcal{O}, \tilde{\Sigma})$ on $V_{\mathfrak{d}}^*$ is defined over $K^*(\tilde{\Sigma})_0$. Therefore if f is any $K^*(\tilde{\Sigma})_0$-arithmetic modular function for $\Gamma_{\mathfrak{d}}^+$ which is holomorphic on $\Xi_{\mathfrak{d}}(\mathcal{O}, \tilde{\Sigma})$, then*

$$P_{\mathfrak{d}, \mathcal{O}, \tilde{\Sigma}, f}(X) \in K^*(\tilde{\Sigma})_0[X].$$

Now let $A(\mathcal{O}) = \bigcup_{\mathfrak{d}} A_{\mathfrak{d}}(\mathcal{O})$ and $A(\mathcal{O}, \tilde{\Sigma})$ be respectively the images of $\bigcup_{\mathfrak{d}} \Xi_{\mathfrak{d}}(\mathcal{O})$ and of $\bigcup_{\mathfrak{d}} \Xi_{\mathfrak{d}}(\mathcal{O}, \tilde{\Sigma})$ in $V_{\hat{G}(\mathbb{Z})}^* = \bigcup_{\mathfrak{d}} V_{\mathfrak{d}}^*$ and if $\mathfrak{n}$ is any integral ideal of k, let $A_{\mathfrak{n}}(\mathcal{O})$ and $A_{\mathfrak{n}}(\mathcal{O}, \tilde{\Sigma})$ denote respectively the corresponding images of the same sets in $V_{\mathbb{K}(\mathfrak{n})}^*$. Since $A_{\mathfrak{n}}(\mathcal{O})$ is the pre-image of $A(\mathcal{O})$ under the canonical morphism $\pi: V_{\mathbb{K}(\mathfrak{n})}^* \to V_{\hat{G}(\hat{\mathbb{Z}})}^*$ (defined via inclusion of double cosets), and π is defined over $\mathbf{Q}$ with respect to the $\mathbf{Q}$-structure defined earlier on $V_{\mathbb{K}(\mathfrak{n})}^*$, it follows that as an algebraic zero-cycle on $V_{\mathbb{K}(\mathfrak{n})}^*$, $A_{\mathfrak{n}}(\mathcal{O})$ is defined over $\mathbf{Q}$, while $A_{\mathfrak{n}}(\mathcal{O}, \tilde{\Sigma})$ is defined over $K^*(\tilde{\Sigma})_0$.

Let $L(\tilde{\Sigma})$ be the minimal field containing $K^*(\tilde{\Sigma})$ over which each of the points of $A(\mathcal{O}, \tilde{\Sigma})$ is rational and $L_{\mathfrak{n}}(\tilde{\Sigma})$ be the minimal field containing $K^*(\tilde{\Sigma})$ over which each of the points of $A_{\mathfrak{n}}(\mathcal{O}, \tilde{\Sigma})$ is rational. Each of these is a normal extension of $K^*(\tilde{\Sigma})$ because it is the splitting field of a polynomial over $K^*(\tilde{\Sigma})$. in fact, each is an abelian extension. Let $G(\tilde{\Sigma}) = \mathrm{Gal}(L(\tilde{\Sigma})/K^*(\tilde{\Sigma}))$ and $G_{\mathfrak{n}}(\tilde{\Sigma}) = \mathrm{Gal}(L_{\mathfrak{n}}(\tilde{\Sigma})/K^*(\tilde{\Sigma}))$. Then, following the same idea as utilized in

Hecke's thesis, one may see that $G(\tilde{\Sigma})$ is isomorphic to a certain subgroup of the group of those ideal classes C of K for which $N_{K/k}(C)$ lies in the narrow principal class of k. In [16] there is a generalization of the same idea for the classical (one-variable, $n = 1$) case (of elliptic modular functions) to the situation where one considers modular functions for principal congruence subgroups of the modular group and the extension of an imaginary quadratic field which their special values generate, which is closely related to Hasse's paper [12]; one of Karel's results [16] says that, without using the theory of elliptic functions as such, one may show such extensions are abelian (cf. §5 of [16]) with Galois group isomorphic to a subgroup of a certain ray class group. It is possible, using the results we have proved, and without using the theory of abelian varieties, to show that $G_{\mathfrak{n}}(\tilde{\Sigma})$ is also abelian and isomorphic to a subgroup of the ray class group mod $\mathfrak{n}$ of K. We intend to provide further details of this in a later paper.

One may also obtain the reciprocity law for the extension $L_{\mathfrak{n}}(\tilde{\Sigma})/K^*(\tilde{\Sigma})$, in form very similar to that of Shimura (with some possible modifications connected with the units of k). This is already indicated in Karel's paper. To deal with the reciprocity law we need a certain q-expansion principle to be proved in the next section. Other details will be supplied in a subsequent publication. We should like, however, to emphasize at this point the important influence on all these developments of Hecke's original ideas.

7. A q-expansion principle

We use the notation of [1]. In particular, Γ denotes an arithmetic group acting on a rational tube domain $\mathfrak{T}$, $V = \Gamma\backslash\mathfrak{T}$, and V^* is the Satake compactification of V. Let k be a number field and make the Assumptions 1′ and 2′, p. 649 of [1], namely that

$$(1') \qquad \mathfrak{A}^{(d_0)}(\Gamma) = \mathfrak{A}^{(d_0)}_{k,\infty}(\Gamma) \otimes_k \mathbf{C}$$

for some positive integer d_0, where $\mathfrak{A}^{(d_0)}_{k,\infty}(\Gamma)$ denotes the graded algebra of modular forms for Γ of weights $\equiv 0 \bmod d_0$ having the coefficients of their Fourier expansions at ∞ in k, and

(2′) If $f \in \mathfrak{A}_{k,\infty}(\Gamma)$, then only finitely many primes divide the denominators of the coefficients of the Fourier expansion of f(*at* ∞).

Denote by $\mathfrak{o}$ the ring of integers of k. Then according to Theorem *B*, *loc. cit.*, there exist a positive integer d_0 and a finite set $\mathfrak{S}$ of prime ideals of $\mathfrak{o}$ such that if $F = \mathfrak{o}[\mathfrak{S}^{-1}]$, then the graded algebra $\mathfrak{A}_{R,\infty}^{(d_0)}(\Gamma)$ of modular forms (with respect to Γ) of weights $\equiv 0 \bmod d_0$ having all Fourier coefficients in R is finitely generated as graded algebra over R, and in fact by a finite set $(b_0, b_1, \ldots, b_M)$ of elements of weight d_0. Since the modular forms with Fourier coefficients in R actually span the graded algebra of all modular forms as a vector space, we may trivially replace R by any larger finitely generated subring of k.

Let $\xi \in \mathfrak{T}$ be a point where all k-arithmetic modular functions holomorphic at ξ take values in a fixed algebraic number field K/k (of finite degree over k). In other words, the image x of ξ in V^* is a K-rational point of V^*, V^* itself being defined over $k \subset K$.

If f is a k-arithmetic modular function, or, equivalently, a rational function on V^* defined over k, and if $y \in V^*(K)$ for some finite extension K of k, we say f is defined and finite $\bmod \mathfrak{p}$ at y, for a prime ideal $\mathfrak{p}$ of k if:

a) f is defined and finite at y in the usual sense (and then $f(y) \in K$); and

b) y belongs to a k-open affine subset U of V^* such that for some system $\alpha_1, \ldots, \alpha_M$ of affine coordinates on U, $\alpha_1(y), \ldots, \alpha_M(y)$ are integral over the valuation ring of $\mathfrak{p}$ in k and f may be expressed in the form

$$f = P(\alpha_1, \ldots, \alpha_M)/Q(\alpha_1, \ldots, \alpha_M),$$

where $P(X_1, \ldots, X_M)$ and $Q(X_1, \ldots, X_M)$ belong to $\mathfrak{o}[X_1, \ldots, X_M]$ and are such that $Q(\alpha_1(y), \ldots, \alpha_M(y))$, which belongs to K and is integral over the localization $\mathfrak{o}_{(\mathfrak{p})}$ of $\mathfrak{o}$ at $\mathfrak{p}$, is not divisible (locally) by any prime ideal $\mathfrak{P}$ extending $\mathfrak{p}$ to K. (In particular, $f(y)$ is integral over $\mathfrak{o}_{(\mathfrak{p})}$.)

According to [32, §§9-10; 27, 4(iii)], for every prime $\mathfrak{p}$ of k, V^* defines a $\mathfrak{p}$-variety. Let us fix some covering of V^* by a system of affine coordinate neighborhoods (in the Zariski topology). Then by Proposition 23 of [32], for almost all $\mathfrak{p}$ these provide a covering by affine coordinate systems of the $\mathfrak{p}$-variety associated to V^*. By the nature of the definition of $\mathfrak{p}$-variety, for f to be defined and finite $\bmod \mathfrak{p}$ at $y \in V^*(K)$, it does not matter which system of affine coordinates for the $\mathfrak{p}$-variety one uses for the criterion for f to be defined and finite $\bmod \mathfrak{p}$ at $y \in V^*(K)$. Then we have:

Theorem 2. *Let $x \in V^*(K)$ be the image of $\xi \in \mathfrak{T}$. Then there is a finite set $\mathfrak{S}'$ of primes $\mathfrak{p}$ of k with the following property: Let f be a k-arithmetic modular function for Γ. Suppose $\mathfrak{p}$ is a prime ideal of k, $\mathfrak{p} \notin \mathfrak{S}'$, and suppose f is defined and finite* $\bmod \mathfrak{p}$ *at x and defined and finite at the image $\bar{\infty}$ of the* *cusp at ∞ in V^*, and such that all the coefficients of the Fourier expansion of f at ∞ are in the maximal ideal of the valuation ring of $\mathfrak{p}$. Then*

$$f(x) \equiv 0 \bmod \mathfrak{p},$$

the congruence being taken in the intersection of the valuation rings of K containing $\mathfrak{p}$.

PROOF. With $b_0, \ldots, b_M \in \mathfrak{A}_{R,\infty}(\Gamma)_{d_0}$ being as before, we may assume, possibly after adding a finite set of primes to S, that $b_0 \equiv 1 \bmod \mathfrak{F}_L$ for some suitably large L and that $b_0(\xi) \neq 0$. (Cf. [1], Proposition 2.) The congruence means that, in a suitable ordering, all Fourier coefficients of b_0 corresponding to indices less than a certain bound (i.e., all the early terms in the Fourier series) are zero, except for the constant term which is 1. Then $\infty, \xi \in V^*(b_0)$, the affine open subset of V^* on which $b_0 \neq 0$.

Let $\alpha_j = b_j/b_0, j = 1, \ldots, M$, so that $\alpha_1, \ldots, \alpha_M$ is a system of affine coordinates on $V^*(b_0)$; then all the coefficients of the Fourier expansion of each α_j are in $R, j = 1, \ldots, M$. The system of affine coordinates $\alpha_1, \ldots, \alpha_M$ determines the structure of an affine R-scheme on $V^*(b_0)$, and for all but a finite number of primes $\mathfrak{p}$ their reductions mod $\mathfrak{p}$ are a system of affine coordinates on a neighborhood of any specialization ref $\mathfrak{p}$ of x on the reduction mod $\mathfrak{p}$ of $V^*(b_0)$. We may assume $\alpha_j(\xi)$, $\alpha_j(\infty)$ all belong to the integral closure of R in $\bar{Q}$. The statement that f is defined and finite mod $\mathfrak{p}$ at ξ means that

$$f = P(\alpha_1, \ldots, \alpha_M)/Q(\alpha_1, \ldots, \alpha_M) = \mathcal{P}/\mathcal{Q},$$

where P and Q are polynomials in M variables having $\mathfrak{p}$-integral coefficients in R such that $\mathcal{Q}(\xi) \not\equiv 0 \bmod \mathfrak{P}$ for every prime $\mathfrak{P}$ extending $\mathfrak{p}$ in the field generated over k by the coordinates $\{\alpha_j(\xi), \alpha_j(\infty) | j = 1, \ldots, M\}$. Then

$$Q(\alpha_1, \ldots, \alpha_M)f = P(\alpha_1, \ldots, \alpha_M) = b_0^{-N}P^*(b_0, b_1, \ldots, b_M),$$

where $P^*(X_0, X_1, \ldots, X_M)$ is a homogeneous polynomial of degree N in $M + 1$ variables. Thus

$$b_0^N Q(\alpha_1, \ldots, \alpha_M)f = P^*(b_0, \ldots, b_M),$$

which is a modular form of weight Nd_0. By hypothesis the Fourier coefficients at ∞ of $b_0^N Q(\alpha_1, \ldots, \alpha_M)$ all lie in R, and those of f in $\mathfrak{p}R$. By inverting a finite set of primes, we may assume R is a principal ideal domain ([20], Prop. 17, p. 22). Let π be a generator of $\mathfrak{p}R$: $\mathfrak{p}R = \pi \cdot R$. Then the Fourier coefficients of $\pi^{-1}P^*(b_0, \ldots, b_M)$ all lie in R. This means, by the choice of $b_0, \ldots, b_M$, that we may write

$$P^*(b_0, \ldots, b_M) = \pi \cdot P^{\#}(b_0, \ldots, b_M),$$

where $P^{\#}$ is a homogeneous polynomial of degree N with coefficients in R. Then

$$f = \pi \cdot b_0^{-N} P^{\#}(b_0, \ldots, b_M)/Q(\alpha_1, \ldots, \alpha_M) = \\ = \pi \cdot P_1(\alpha_1, \ldots, \alpha_M)/Q(\alpha_1, \ldots, \alpha_M),$$

and P_1 is a polynomial in M variables with coefficients in R. Therefore,

$$f(\xi) = \pi \cdot P_1(\alpha_1(\xi), \ldots, \alpha_M(\xi))/Q(\alpha_1(\xi), \ldots, \alpha_M(\xi)) = \\ \pi \cdot P_1(\alpha_1(\xi), \ldots, \alpha_M(\xi))/\mathfrak{Q}(\xi)$$

which is clearly an expression $\equiv 0 \bmod \mathfrak{p}$. Q.E.D.

8. As a Convenience to the Reader

We list here some minor corrections needed in [3] as a predecessor to this paper:

In the line immediately preceding equation (49) of 2.4 (of [3]), $M_2(\hat{\mathbb{Z}})$ should be $M_2(\hat{\mathfrak{o}})$.

On the next page after that in the fourth line of the proof of Lemma 2, following the last = sign there should be

$$\bigcup_{j=1}^{N} {}^{\omega\omega_1}\mathbb{K}S_{j\omega f}'^{-1} \cap G_+(\mathbf{Q})_f.$$

And on still the very next page after that, the second displayed equation, line eleven from the top, should read

$$a_{\nu\omega}(z) = \sigma_{N-\nu}(f_{\omega\omega_1}(S_{1\omega}'^{-1} \cdot z), \ldots, f_{\omega\omega_1}(S_{N\omega}'^{-1} \cdot z)).$$

These corrections are in addition to other needed corrections pointed out in the course of this paper.

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