Polynomial Invariants of 2-component Links

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1. Introduction

Let $L = X \cup Y$ be an oriented 2-component link in S^3. In this paper, we will define two different types of polynomials which are ambient isotopic invariants of L. One is associated with a cyclic cover branched along one of their components, and the other is associated with a metabelian cover of L. These invariants are defined for any link unless the linking number, $lk(X, Y)$, is ± 1.

The invariants a_n^*, h_n^* defined in [5] can be considered one of the special cases of our polynomial invariants. In fact, we can prove that a_n^* depends only on $lk(X, Y)$; therefore, for all n, a_n^* coincides for two links with the same linking number. (See Theorem 5.7.)

It should be noted that our metabelian representation of the link group differs completely from those studied in [2], [3], [9] or [10], where in most of the cases there exist only finitely many metabelian representations. We will prove in this paper that every 2-component link L with $lk(X, Y) \neq \pm 1$ has infinitely many metabelian coverings. In particular, if $lk(X, Y)$ is even, then the link group $G(L)$ has a representation on the dihedral group of order 2^{k+1} for each $k \geqslant 1$. (See Proposition 3.3 and Theorem 4.1.) Our polynomials are, in fact, the covering linkage invariants associated with these (infinite) sequences of coverings.

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In this paper, some of the basic formulas involving Fox free differential calculus [1] will be used without proofs, since they have already been proved in [9] or are easy consequences of the results in [9].

2. Group Actions

Let F be a free group (of rank 2) freely generated by x and y.

Let $P = Z[[t]]$ be the ring of formal power series over the ring of integers Z. Denote by $Sym(S)$ the group of permutations on a set S.

Associated with an ordered sequence $s = \{j_1, j_2, \ldots, j_k, \ldots\}$ of 1 or 2 is an action ϕ of F on P; that is, a homomorphism $\phi\colon F \to Sym(P)$ defined as follows:

$$\phi(x)\left(\sum_{i=0}^{\infty} a_i t^i\right) = a_0 + \sum_{i=1}^{\infty} (a_i + \delta(i)a_{i-1})t^i$$

$$\phi(y)\left(\sum_{i=0}^{\infty} a_i t^i\right) = a_0 + \sum_{i=1}^{\infty} (a_i + (1 - \delta(i))a_{i-1})t^i \tag{2.1}$$

where $\delta(i) = 1$ or 0 according to $j_i = 1$ or 2.

Throughout this paper, we do not distinguish between an action $\phi\colon F \times P \to P$ and the homomorphism $\phi\colon F \to Sym(P)$, associated with ϕ, and therefore the same symbol will be used.

EXAMPLE 2.1.   Let $s = \{1, 2, 1, 2, \ldots\}$, where $j_k = 1$ if and only if k is odd. Then for $f(t) = \sum\limits_{i=0}^{\infty} a_i t^i \in P$,

$$\phi(x)(f(t)) = f(t) + \sum_{i=0}^{\infty} a_{2i} t^{2i+1}$$

$$\phi(y)(f(t)) = f(t) + \sum_{i=0}^{\infty} a_{2i+1} t^{2i+2}.$$

Using Fox free derivative [1], we can now express the action ϕ more precisely. The following proposition is essentially Proposition 3.1 in [9].

Proposition 2.1.   *Let $s = \{j_1, j_2, \ldots, j_k, \ldots\}$ be an ordered sequence of 1 or 2. For $u \in F$, write $\phi(u)\left(\sum\limits_{i=0}^{\infty} a_i t^i\right) = \sum\limits_{i=0}^{\infty} b_i t^i$. Then*

(1) $b_0 = a_0$,

(2) For $q \geqslant 1$,

$$b_q = a_q + a_{q-1}\left(\frac{\partial u}{\partial z_q}\right)^o + a_{q-2}\left(\frac{\partial^2 u}{\partial z_{q-1}\partial z_q}\right)^o + \ldots + a_0\left(\frac{\partial^q u}{\partial z_1 \ldots \partial z_q}\right)^o, \quad (2.2)$$

where z_i is x or y according to $j_i = 1$ or 2, and o denotes the trivializer.

For the proof, see Proposition 3.1 in [9].

Let $\hat{F}$ be the free group freely generated by $\{x_{f(t)}, y_{f(t)} \mid f(t) \in P\}$. Let $\{\mathfrak{D}_f(t)\}$ be the Reidemeister-Schreier rewriting function of $\hat{F}$ associated with the action ϕ ([4] or [9]). $\mathfrak{D}_{f(t)}: F \to \hat{F}$ is characterized by the following two properties: For any $f(t) \in P$ and $u, v \in F$,

(1) $\mathfrak{D}_{f(t)}(x) = x_{f(t)}$ and $\mathfrak{D}_{f(t)}(y) = y_{f(t)}$,

(2) $\mathfrak{D}_{f(t)}(uv) = \mathfrak{D}_{f(t)}u \cdot \mathfrak{D}_{\phi(u)f(t)}v.$ $\hspace{3cm}$ (2.3)

The following properties will easily be proved from (2.3).

(1) $\mathfrak{D}_{f(t)}(u^{-1}) = (\mathfrak{D}_{\phi(u^{-1})f(t)}(u))^{-1}$

(2) If $\phi(uv^{-1}) = 1$, then $\mathfrak{D}_{f(t)}(uv^{-1}) = (\mathfrak{D}_{f(t)}u) \cdot (\mathfrak{D}_{f(t)}v)^{-1}.$ $\hspace{1.5cm}$ (2.4)

Now let $\psi: \hat{F} \to P$ be a homomorphism defined by $\psi(x_{f(t)}) = 0$ and $\psi(y_{f(t)}) = f(t)$.

Proposition 2.2. *Let* $s = \{j_1, j_2, \ldots, j_k, \ldots\}$ *be an ordered sequence of* 1 *or* 2. *For* $u \in F$ *and* $f(t) = \sum_{i=0}^{\infty} a_i t^i \in P$, *write* $\psi\mathfrak{D}_{f(t)}u = \sum_{i=0}^{\infty} b_i t^i$. *Then*

(1) $b_0 = a_0$,

(2) For $q \geqslant 1$,

$$b_q = a_q\left(\frac{\partial u}{\partial y}\right)^o + a_{q-1}\left(\frac{\partial^2 u}{\partial z_q \partial y}\right)^o + \ldots$$

$$+ a_1\left(\frac{\partial^q u}{\partial z_2 \ldots \partial z_q \partial y}\right)^o + a_0\left(\frac{\partial^{q+1} u}{\partial z_1 \ldots \partial z_q \partial y}\right)^o, \quad (2.5)$$

where z_i is x or y according to $j_i = 1$ or 2.

For a proof, see Proposition 6.1 in [9].

In this paper we are particularly interested in the group action associated with a sequence $\{1, 1, \ldots, 1, \ldots\}$ or $\{1, 2, 2, \ldots, 2, \ldots\}$. Our approach, however, is different from what we did in [9] and hence, we obtain different representations of the link groups. To be more precise, we define two actions σ and τ of F on P.

Definition 2.1.   *For $f(t) \in P$, define*

$$\sigma(x)f(t) = (1 + t)f(t)$$
$$\sigma(y)f(t) = f(t) \tag{2.6}$$

$$\tau(x)f(t) = f(t) + f(0)t$$
$$\tau(y)f(t) = (1 + t)f(t) - f(0)t. \tag{2.7}$$

Let P^* be the set of power series $f(t)$ for which $f(0) = 1$. Then σ and τ induce actions of F on P^*, since for any $u \in F$, $[\sigma(u)f(t)]_{t=0} = [\tau(u)f(t)]_{t=0} = 1$. Furthermore, let $q(t)$ be an element of P and $\langle q(t) \rangle$ the ideal of P generated by $q(t)$. Denote by R the quotient ring $P/\langle q(t) \rangle$.

Proposition 2.3.

(1) σ *induces an action $\bar{\sigma}$ of F on R.*

(2) *If $q(0) = 0$, then τ induces an action $\bar{\tau}$ of F on R.*

Proof.   It suffices (and is easy) to show that $\langle q(t) \rangle$ is closed under the actions σ and τ.

Remark 2.1.   Let $q(t) = \sum_{i=0}^{\infty} s_i t^i$ and let s_m be the first non zero coefficient of $q(t)$. Then every element $f(t)$ in R has a unique representative $\bar{f}(t)$ of the form: $\bar{f}(t) = \bar{a}_0 + \bar{a}_1 t + \ldots + \bar{a}_k t^k + \ldots$, where $\bar{a}_0, \ldots, \bar{a}_{m-1}$ are integers, and if s_m is positive, then $\bar{a}_k(k \geqslant m)$ is a non-negative integer less than s_m, but if s_m is negative, $\bar{a}_k(k \geqslant m)$ is a non-positive integer greater than s_m. We call this unique representative $\bar{f}(t)$ the *normal* form of $\bar{f}(t)$.

Example 2.2.   Let $q(t) = 2 + 3t$. Then the normal form of $\bar{f}(t) = 3 + 6t - t^2 - 3t^3 + 3t^4$ is $1 + t + t^3$.

Since Propositions 2.1 and 2.2 for $\phi = \sigma$ or τ will be used quite extensively in this paper, it will be convenient to state them as separate propositions.

Proposition 2.4.   *For $u \in F$, write*

$$\sigma(u)\left(\sum_{i=0}^{\infty} a_i t^i \right) = \sum_{i=0}^{\infty} b_i t^i \quad \text{and} \quad \tau(u)\left(\sum_{i=0}^{\infty} a_i t^i \right) = \sum_{i=0}^{\infty} c_i t^i.$$

Then for any $q \geqslant 0$,

$$(1) \quad b_q = a_q + a_{q-1}\left(\frac{\partial u}{\partial x} \right)^o + a_{q-2}\left(\frac{\partial^2 u}{\partial x^2} \right)^o + \ldots + a_1\left(\frac{\partial^{q-1} u}{\partial x^{q-1}} \right)^o + a_0\left(\frac{\partial^q u}{\partial x^q} \right)^o$$

$$(2) \quad c_q = a_q + a_{q-1}\left(\frac{\partial u}{\partial y} \right)^o + a_{q-2}\left(\frac{\partial^2 u}{\partial y^2} \right)^o + \ldots$$

$$+ a_1\left(\frac{\partial^{q-1} u}{\partial y^{q-1}} \right)^o + a_0\left(\frac{\partial^q u}{\partial x \partial y^{q-1}} \right)^o. \tag{2.8}$$

In particular, $a_0 = b_0 = c_0$.

Proposition 2.5. *Let $\mathfrak{D}^\sigma_{f(t)}$ and $\mathfrak{D}^\tau_{f(t)}$ be the Reidemeister-Schreier rewriting functions associated with the actions σ and τ, respectively. For*

$$f(t) = \sum_{i=0}^{\infty} a_i t^i \in P,$$

write

$$\psi\mathfrak{D}^\sigma_{f(t)}(u) = \sum_{i=0}^{\infty} b_i t^i \quad and \quad \psi\mathfrak{D}^\tau_{f(t)}(u) = \sum_{i=0}^{\infty} c_i t^i.$$

Then for $q \geqslant 0$,

(1) $\quad b_q = a_q \left(\dfrac{\partial u}{\partial y}\right)^o + a_{q-1} \left(\dfrac{\partial^2 u}{\partial x \partial y}\right)^o + \ldots + a_1 \left(\dfrac{\partial^q u}{\partial x^{q-1} \partial y}\right)^o + a_0 \left(\dfrac{\partial^{q+1} u}{\partial x^q \partial y}\right)^o,$

(2) $\quad c_q = a_q \left(\dfrac{\partial u}{\partial y}\right)^o + a_{q-1} \left(\dfrac{\partial^2 u}{\partial y^2}\right) + \ldots + a_1 \left(\dfrac{\partial^q u}{\partial y^q}\right)^o + a_0 \left(\dfrac{\partial^{q+1} u}{\partial x \partial y^q}\right)^o.$ $\qquad$ (2.9)

3. Representations of a Free Group

For an integer n (positive, negative or 0), let

$$q_n(t) = \sum_{i=1}^{\infty} \binom{n}{i} t^i = (1 + t)^n - 1.$$

As usual,

$$\binom{n}{i} \quad \text{denotes} \quad \frac{n(n-1)\ldots(n-i+1)}{i!}.$$

Lemma 3.1. *If $m \equiv 0 \pmod{n}$, then $q_m(t) \equiv 0 \pmod{q_n(t)}$.*

A proof is easy.

Now let $R(n)$ be the quotient ring $P/\langle q_n(t)\rangle$, and let $R^*(n)$ be the set of elements $f(t)$ in $R(n)$ such that $f(0) = 1$. Since $q_n(0) = 0$, it follows from Propositions 2.3 and 2.4 that σ and τ, respectively, induce actions σ_n and τ_n of F on $R^*(n)$. Let $\Omega_\sigma(n)$ and $\Omega_\tau(n)$ denote the orbits of 1 in $R^*(n)$ under σ_n and τ_n respectively. Namely,

$$\Omega_\sigma(n) = \{\sigma_n(u)(1) \mid u \in F\} \quad \text{and} \quad \Omega_\tau(n) = \{\tau_n(u)(1) \mid u \in F\}.$$

σ_n and τ_n define homomorphisms

$$\sigma_n \colon F \to Sym(\Omega_\sigma(n)) \quad \text{and} \quad \tau_n \colon F \to Sym(\Omega_\tau(n)).$$

Proposition 3.2.

(1) $\sigma_n(F)$ *is a cyclic group of order* $|n|$.

(2) $\tau_n(F)$ *is a metabelian group.*

Proof. (1) Since $(1 + t)^n \equiv 1 \,(\mathrm{mod}\, q_n(t))$, $\Omega_\sigma(n)$ consists of exactly $|n|$ elements $\{1, 1 + t, (1 + t)^2, \ldots, (1 + t)^{|n|-1}\}$. Since $\sigma_n(1 + t)^k = (1 + t)^{k+1}$, it follows that $\sigma(x^n) = 1$, but $\sigma(x^k) \neq 1$ for $1 \leqslant k < |n|$, and hence $\sigma_n(F)$ is a cyclic group of order $|n|$.

(2) Let $G = \tau_n(F)$. As a special case of Proposition 9.1 in [9], we see that $G'' = [G', G'] = 1$. Therefore, G is metabelian.

Generally, $\Omega_\tau(n)$ is not a finite set. Therefore, to obtain a finite representation of F, we need to "truncate" higher terms of $f(t)$. Let I_{k+1} be the ideal of P generated by t^{k+1} and $q_n(t)$. Let $R_k(n) = P/I_{k+1}$ and let $R_k^*(n)$ be the set of elements $f(t)$ in $R_k(n)$ such that $f(0) = 1$. An element of $R_k^*(n)$ is a polynomial of degree at most k, and it has the (unique) normal form of degree $\leqslant k$. (See Remark 2.1.) Obviouly, τ_n induces an action $\tau_{k,n}$ of F on $R_k^*(n)$.

Let $\Omega_k(n)$ be the orbit of 1 under $\tau_{k,n}$; i.e., $\Omega_k(n) = \{\tau_{k,n}(u)(1) \mid u \in F\}$. $\tau_{k,n}$ defines a (transitive) homomorphism $\tau_{k,n} \colon F \to Sym(\Omega_k(n))$.

Proposition 3.3.

(1) $\tau_{k,n}(F)$ *is nilpotent of class at mos* k.

(2) *If* n *is a prime* p, *then* $\tau_{k,n}(F)$ *is a finite* p-group.

(3) *In particular, if* $n = 2$, *then* $\tau_{k,n}(F)$ *is the dihedral group of order* 2^{k+1}.

Proof.

(1) follows from Proposition 3.2 in [9];

(2) since a proof will be done by an easy induction on k, the details will be omitted. Note that $(\tau_{k,p}(x))^{p^k} = 1$ and $(\tau_{k,p}(y))^p = 1$;

(3) denote $X = \tau_{k,n}(x)$ and $Y = \tau_{k,n}(y)$. Then a straight-forward calculation shows that for any $f(t) \in P^*$

(1) $X^{2^k}(f(t)) \equiv f(t) \,\mathrm{mod}\, I_{k+1}$

(2) $Y^2(f(t)) \equiv f(t) \,\mathrm{mod}\, I_{k+1}$

(3) $(XY)^2(f(t)) \equiv f(t) \,\mathrm{mod}\, I_{k+1}$.   (3.1)

Therefore, $\tau_{k,2}(F)$ is a quotient group of the dihedral group

$$D_{2^k} = \langle A, B \mid A^{2^k} = B^2 = (AB)^2 = 1 \rangle.$$

But it is easy to see that they are, in fact, isomorphic. The details will be omitted.

Remark 3.1.   We can prove, further, that for a prime p,

$$\begin{cases} \tau_{k,p}[x,y,x] = 1 \\ \tau_{k,p}[x,y,y,\ldots,y] = 1, \end{cases} \tag{3.2}$$
$$k \text{ times}$$

where $[u_1, u_2] = u_1 u_2 u_1^{-1} u_2^{-1}$ and $[u_1, u_2, \ldots, u_m] = [[u_1, u_2, \ldots, u_{m-1}], u_m]$. In particular, $\tau_{2,p}$ is isomorphic to the group

$$M(p) = \langle x, y \mid x^p = y^p = [x,y]^p = 1, \ [x,y,x] = [x,y,y] = 1 \rangle.$$

Remark 3.2. p-group representations of F obtained in Proposition 3.3 (2) and (3) are quite different from those discussed in [9, §10] or [10, §§2-3].

4. Representations of Link Groups

Let $L = X \cup Y$ be an oriented 2-component link in S^3. In this section, we will define a homomorphism from the link group $G(L)$ onto the group $\sigma_n(F)$ or $\tau_{k,n}(F)$ for various n and k.

For the first group $\sigma_n(F)$, such a homomorphism $\Sigma_n: G(L) \to \sigma_n(F)$ always exists, since $\sigma_n(F)$ is cyclic of order $|n|$. In fact, let m_x and m_y be meridian elements of X and Y, respectively. Then for any integer n, a mapping $\Sigma_n: G(L) \to \sigma_n(F)$ defined by

$$\begin{cases} \Sigma_n(m_x) = \sigma_n(x) \\ \Sigma_n(m_y) = \text{id} \end{cases} \tag{4.1}$$

gives a required homomorphism. However, it will be seen later that Σ_n is only interesting in our purpose when n divides $lk(X, Y)$, the linking number between X and Y.

On the other hand, the second group $\tau_{k,n}(F)$ is not an obvious group. In fact, $\tau_{k,n}(F)$ is metabelian, but not abelian. Nevertheless, for any k, we can find a homomorphism from $G(L)$ onto $\tau_{k,n}(F)$ when $lk(X, Y)$ is divisible by n.

In this section, we will prove the following theorem.

Theorem 4.1.   *Let n be an integer. Suppose $lk(X, Y) \equiv 0 \pmod{n}$. Then for each $k \geqslant 1$, there is a homomorphism*

$$T_{k,n}: G(L) \to \tau_{k,n}(F) \subset Sym(\Omega_k(n))$$

such that $T_{k,n}(m_x) = \tau_{k,n}(x)$ *and* $T_{k,n}(m_y) = \tau_{k,n}(y)$. n *can be* 0 *only when* $lk(X, Y) = 0$.

PROOF. Since there is no essential difference in proving the theorem we may assume that n is a positive integer. Also we may assume *w.l.o.g.* that $lk(X, Y) \geqslant 0$.

Now, if $k = 1$, then $\tau_{k,n}(F)$ is a cyclic group of order n, and the theorem is trivially true. Therefore, we assume that $k \geqslant 2$.

Let $G(L) = \langle x_i, y_j \mid r_i, s_j \rangle$, $1 \leqslant i \leqslant \lambda$, $1 \leqslant j \leqslant \mu$, be a "modified" Wirtinger presentation of $G(L)$ in the following sense.

x_1 and y_1 correspond to prescribed meridian elements m_x and m_y, respectively, and relators are of the form

$$\begin{cases} r_i = u_i x_1 u_i^{-1} x_{i+1}^{-1}, & 1 \leqslant i \leqslant \lambda - 1 \\ r_\lambda = \eta x_1 \eta^{-1} x_1^{-1}, \end{cases}$$

$$\begin{cases} s_j = v_j y_1 v_j^{-1} y_{j+1}^{-1}, & 1 \leqslant j \leqslant \mu - 1 \\ s_\mu = \xi y_1 \xi^{-1} y_1^{-1}, \end{cases}$$

where u_i, v_j are words in $\{x_i, y_j\}$, and η and ξ represent longitudes of X and Y, respectively, so that $\{x_1, \eta\}$ and $\{y_1, \xi\}$ form peripheral subgroups of $G(L)$.

Let F^* be the free group freely generated by $\{x_i, y_j, 1 \leqslant i \leqslant \lambda, 1 \leqslant j \leqslant \mu$. As before, F denotes the free group $\langle x, y \mid \rangle$. Let $\rho: F^* \to F^*$ and $\nu: F^* \to F$ be homomorphisms defined by

$$\begin{cases} \rho(x_1) = x \quad \text{and} \quad \rho(y_1) = y \\ \rho(x_{i+1}) = u_i x_1 u_i^{-1}, & 1 \leqslant i \leqslant \lambda - 1 \\ \rho(y_{j+1}) = v_j v_1 v_j^{-1}, & 1 \leqslant j \leqslant \mu - 1. \end{cases}$$

$$\begin{cases} \nu(x_i) = x, & 1 \leqslant i \leqslant \lambda \\ \nu(y_j) = y, & 1 \leqslant j \leqslant \mu. \end{cases}$$

Using ρ and ν, we define the third homomorphism

$$\theta_{k+1} = \nu \rho^k : F^* \to F \quad \text{for} \quad k \geqslant 0.$$

(θ_{k+1} will be called the Chen-Milnor homomorphism.)

Let $T = \tau_{k,n} \theta_{k+1}$ be a homomorphism from F^* to $\tau_{k,n}(F)$. Then T will induce the homomorphism $T_{k,n}: G(L) \to \tau_{k,n}(F)$ if

$$T(r_i) = 1, \quad 1 \leqslant i \leqslant \lambda, \quad \text{and} \quad T(s_j) = 1, \quad 1 \leqslant j \leqslant \mu. \tag{4.2}$$

Now, Proposition 5.1 in [9] proves (4.2) except the last two relations $T(r_\lambda) = 1$ and $T(s_\mu) = 1$. Therefore, it only remains to show that

$$T[\eta, x_1] = 1 \quad \text{and} \quad T[\xi, y_1] = 1. \tag{4.3}$$

Since one of the relations in (4.2) is redundant, it is enough to show that $T[\eta, x_1] = 1$.

For simplicity, write $\theta_{k+1}(\eta) = h$. Since $\theta_{k+1}(x_1) = x$, it suffices to prove that

$$\tau_{k,n}(hx) = \tau_{k,n}(xh). \tag{4.4}$$

Denote $u = hx$ and $w = xh$, and write

$$\tau_{k,n}(u)\left(\sum_{i=0}^{\infty} a_i t^i\right) = \sum_{i=0}^{\infty} b_i t^i \quad \text{and} \quad \tau_{k,n}(w)\left(\sum_{i=0}^{\infty} a_i t^i\right) = \sum_{i=0}^{\infty} c_i t^i.$$

Then, since $a_0 = 1$, it follows from (2.8) (2) that for $q \geqslant 1$,

$$(1) \quad b_q = a_q + a_{q-1}\left(\frac{\partial u}{\partial y}\right)^o + \ldots + a_1\left(\frac{\partial^{q-1} u}{\partial y^{q-1}}\right)^o + \left(\frac{\partial^q u}{\partial x \partial y^{q-1}}\right)^o \tag{4.5}$$

and

$$(2) \quad c_q = a_q + a_{q-1}\left(\frac{\partial w}{\partial y}\right)^o + \ldots + a_1\left(\frac{\partial^{q-1} w}{\partial y^{q-1}}\right)^o + \left(\frac{\partial^q w}{\partial x \partial y^{q-1}}\right)^o.$$

Now

$$\left(\frac{\partial^r u}{\partial y^r}\right)^o = \left(\frac{\partial^r w}{\partial y^r}\right)^o = \binom{m}{r}, \quad \text{where} \quad m = \left(\frac{\partial h}{\partial y}\right)^o = lk(X, Y),$$

[1], and $(\partial h/\partial x)^o = 0$. Further,

$$\left(\frac{\partial^q u}{\partial x \partial y^{q-1}}\right)^o = \left[\frac{\partial}{\partial x}\left(\frac{\partial^{q-1} u}{\partial y^{q-1}}\right)\right]^o = \left(\frac{\partial^q h}{\partial x \partial y^{q-1}}\right)^o$$

and

$$\left(\frac{\partial^q w}{\partial x \partial y^{q-1}}\right)^o = \left[\frac{\partial}{\partial x}\left(x \frac{\partial^{q-1} h}{\partial y^{q-1}}\right)\right]^o = \left(\frac{\partial^{q-1} h}{\partial y^{q-1}}\right)^o + \left(\frac{\partial^q h}{\partial x \partial y^{q-1}}\right)^o.$$

Therefore, it follows from (4.5) that

$$b_0 = c_0 = 1, \quad b_1 = \left(\frac{\partial u}{\partial x}\right)^o = 1 \quad \text{and} \quad c_1 = \left(\frac{\partial w}{\partial x}\right)^o = 1,$$

and hence,

$$\sum_{q=0}^{\infty} c_q t^q = \sum_{q=0}^{\infty} b_q t^q + \sum_{q=2}^{\infty} \left(\frac{\partial^{q-1}h}{\partial y^{q-1}}\right)^o t^q = \sum_{q=0}^{\infty} b_q t^q + \sum_{q=2}^{\infty} \binom{m}{q-1} t^q.$$

Since n devides m by the assumption, it follows from Lemma 3.1 that

$$\sum_{q=2}^{\infty} \binom{m}{q-1} t^q \equiv 0 \quad (\text{mod } q_n(t)),$$

and hence

$$\sum_{q=0}^{\infty} c_q t^q \equiv \sum_{q=0}^{\infty} b_q t^q \quad (\text{mod } I_{k+1}).$$

This proves Theorem 4.1.

Now, let $g = \theta_{k+1}(\xi)$ and $v = gy$ and $z = yg$. Note that $(\partial g/\partial y)^o = 0$. Since $T_{k,n}[\xi, y_1] = 1$, we have $\tau_{k,n}(v) = \tau_{k,n}(z)$. Write

$$\tau_{k,n}(v)\left(\sum_{i=0}^{\infty} a_i t^i\right) = \sum_{i=0}^{\infty} b_i t^i, \qquad a_0 = 1$$

and

$$\tau_{k,n}(z)\left(\sum_{i=0}^{\infty} a_i t^i\right) = \sum_{i=0}^{\infty} c_i t^i, \qquad a_0 = 1.$$

Then by (2.8) (2) we obtain

(1) $a_0 = b_0 = c_0 = 1$ (4.6)

For $q \geqslant 1$,

(2) $$b_q = a_q + a_{q-1}\left(\frac{\partial v}{\partial y}\right)^o + \ldots + a_1\left(\frac{\partial^{q-1}v}{\partial y^{q-1}}\right)^o + \left(\frac{\partial^q v}{\partial x \partial y^{q-1}}\right)^o$$

and

(3) $$c_q = a_q + a_{q-1}\left(\frac{\partial z}{\partial y}\right)^o + \ldots + a_1\left(\frac{\partial^{q-1}z}{\partial y^{q-1}}\right)^o + \left(\frac{\partial^q z}{\partial x \partial y^{q-1}}\right)^o.$$

Since $(\partial v/\partial y)^o = (\partial z/\partial y)^o = 1$, it follows that for $q \geqslant 1$,

$$\left(\frac{\partial^q v}{\partial y^q}\right)^o = \left(\frac{\partial^q z}{\partial y^q}\right)^o = \binom{1}{q}$$

and hence

$$\left(\frac{\partial^q v}{\partial x \partial y^{q-1}}\right)^o = \left(\frac{\partial^q g}{\partial x \partial y^{q-1}}\right)^o + \left(\frac{\partial^{q-1} g}{\partial x \partial y^{q-2}}\right)^o$$

and

$$\left(\frac{\partial^q z}{\partial x \partial y^{q-1}}\right)^o = \left(\frac{\partial^q g}{\partial x \partial y^{q-1}}\right)^o.$$

Therefore, $\tau_{k,n}(v) = \tau_{k,n}(z)$ yields the following

Proposition 4.2. *Let* $g = \theta_{k+1}(\xi)$. *Then for* $k \geqslant 1$,

$$\left(\frac{\partial g}{\partial x}\right)^o t^2 + \left(\frac{\partial^2 g}{\partial x \partial y}\right)^o t^3 + \ldots + \left(\frac{\partial^q g}{\partial x \partial y^{q-1}}\right)^o t^{q+1} + \ldots \equiv 0 \qquad (\text{mod } I_{k+1}). \tag{4.7}$$

Remark 4.1. A homomorphism $\Sigma_n: G(L) \to \sigma_n(F)$ is formally given as follows. First, define a homomorphism $\Sigma^*: F^* \to \sigma_n(F)$ by

$$\begin{cases} \Sigma^*(x_i) = \sigma_n(x) & \text{for} \quad i = 1, 2, \ldots, \lambda, \\ \Sigma^*(y_j) = id & \text{for} \quad j = 1, 2, \ldots, \mu. \end{cases} \tag{4.8}$$

Then $\Sigma^*(r_i) = \Sigma^*(s_j) = 1$ for any i, j. Therefore, Σ^* induces the homomorphism $\Sigma_n: G(L) \to \sigma_n(F)$. This rather obvious observation will be used in the next section.

5. Covering Space (I) Cyclic Covering

In the previous section we found representations Σ_n and $T_{k,n}$ of $G(L)$ on $\sigma_n(F)$ and $\tau_{k,n}(F)$.

To each finite representation ϕ we can associate a (unbranched) covering space M. Let $U(X)$ and $U(Y)$ denote tubular neighborhoods of X and Y in S^3, respectively. Then the covering space M associated with ϕ is a compact 3-manifold with boundary consisting of tori.

Suppose we have a homomorphism

$$\phi: \pi_1(M) \to A$$

from $\pi_1(M)$ to an abelian group A. Then ϕ induces the homomorphism $\hat{\phi}$ from $H_1(M)$ to A. The most characteristic element of $H_1(M)$ is a "longitude" $\tilde{\zeta}$ of each boundary torus of M. In many cases, such a "longitude" can be

realized as a "lift" of a longitude $\mathfrak{f}$ of $\partial U(X)$ or $\partial U(Y)$, and then $\phi(\hat{\mathfrak{f}})$ will be an invariant of the original link type L. By taking A as the polynomial ring $R_k(n)$, we will obtain our polynomial invariants.

In this section, we define such polynomial invariants for a finite representation $\Sigma_n\colon G(L) \to \sigma_n(F) \subset Sym(\Omega_\sigma(n))$.

Let M_σ be the (unbranched) covering space of $S^3 - L$ associated with Σ_n. M_σ is in fact the n-fold cyclic covering space of $S^3 - X$.

Let $\mathfrak{D}^{*\sigma}_{f(t)}$ be the Reidemeister-Schreier rewriting function associated with the action $\Sigma^*\colon F^* \to \sigma_n(F) \subset Sym(\Omega_\sigma(n))$, where Σ^* is defined in Remark 4.1. (4.8).

Now the set $S_\sigma = \{\mathfrak{D}^{*\sigma}_{f(t)}(x_i), \mathfrak{D}^{*\sigma}_{f(t)}(y_j) \mid 1 \leqslant i \leqslant \lambda, 1 \leqslant j \leqslant \mu, f(t) \in \Omega_\sigma(n)\}$ generates a free group F^*_σ and $\pi_1(M_\sigma)$ has a presentation $\langle S_\sigma\colon R_\sigma, U_\sigma \rangle$ where $R_\sigma = \{\mathfrak{D}^{*\sigma}_{f(t)}(r_i), \mathfrak{D}^{*\sigma}_{f(t)}(s_j) \mid 1 \leqslant i \leqslant \lambda, 1 \leqslant j \leqslant \mu, f(t) \in \Omega(n)\}$ and $U_\sigma = \{\mathfrak{D}^{*\sigma}_{f(t)}(x_1), f(t) \in \Omega_\sigma(n)\}$.

Theorem 5.1.   *Let n be an integer. Suppose $lk(X, Y) \equiv 0 \pmod n$. Then for $k \geqslant 1$, there exists a homomorphism $\Phi_\sigma\colon \pi_1(M_\sigma) \to R_k(n)$ such that for any $f(t) \in \Omega_\sigma(n)$,*

$$\Phi_\sigma(\mathfrak{D}^{*\sigma}_{f(t)}(x_1)) = 0 \quad and \quad \Phi_\sigma(\mathfrak{D}^{*\sigma}_{f(t)}(y_1)) = f(t).$$

Remark 5.1. n can be 0 only if $lk(X, Y) = 0$, and then M_σ is an infinite cyclic covering space of $S^3 - X$.

Proof of Theorem 5.1.   To prove the theorem it suffices to define a homomorphism $\Phi^*_\sigma\colon F^*_\sigma \to R_k(n)$ such that $\Phi^*_\sigma(w) = 0$ for $w \in R_\sigma$ or U_σ.

Now let F_σ be the free group freely generated by a set $\{x_{f(t)}, y_{f(t)} \mid f(t) \in \Omega_\sigma(n)\}$, and let ψ_σ be a homomorphism from F_σ to $R_k(n)$ given by

$$\psi_\sigma(x_{f(t)}) = 0 \quad and \quad \psi_\sigma(y_{f(t)}) = f(t). \tag{5.2}$$

Using ψ_σ, we define, for $f(t) \in \Omega_\sigma(n)$ and for any i, j,

$$\begin{cases} \Phi^*_\sigma(\mathfrak{D}^{*\sigma}_{f(t)}(x_i)) = \psi_\sigma \mathfrak{D}^\sigma_{f(t)} \theta_{k+1}(x_i) \\ \Phi^*_\sigma(\mathfrak{D}^{*\sigma}_{f(t)}(y_j)) = \psi_\sigma \mathfrak{D}^\sigma_{f(t)} \theta_{k+1}(y_j). \end{cases} \tag{5.3}$$

Note that

$$\Phi^*_\sigma(\mathfrak{D}^{*\sigma}_{f(t)}(x_1)) = \psi_\sigma \mathfrak{D}^\sigma_{f(t)} \theta_{k+1}(x_1) = \psi_\sigma \mathfrak{D}^\sigma_{f(t)}(x) = \psi_\sigma(x_{f(t)}) = 0,$$

and

$$\Phi^*_\sigma(\mathfrak{D}^{*\sigma}_{f(t)}(y_1)) = \psi_\sigma \mathfrak{D}^\sigma_{f(t)} \theta_{k+1}(y_1) = \psi_\sigma \mathfrak{D}^\sigma_{f(t)}(y) = \psi_\sigma(y_{f(t)}) = f(t),$$

and therefore, Φ_σ^* satisfies (5.1). We will prove further, for any $u \in F^*$,

$$\Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(u) = \psi_\sigma \mathfrak{D}_{f(t)}^\sigma \theta_{k+1}(u). \tag{5.4}$$

A proof will be done by induction on the length $l(u)$ of u.

If $l(u) = 1$, then $u = x_i^{\pm 1}$ of $y_j^{\pm 1}$. If $u = x_i$ or y_j, (5.4) is trivially true. Suppose $u = x_i^{-1}$. Since $\sigma_n(x_i) = \sigma_n(\theta_{k+1}(x_i))$, it follows from (2.4) (1) that

$$\begin{aligned}
\Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(x_i^{-1}) &= \Phi_\sigma^* [\mathfrak{D}_{\sigma_n(x_i^{-1})f(t)}^{*\sigma}(x_i)]^{-1} \\
&= \psi_\sigma [\mathfrak{D}_{\sigma_n(x_i^{-1})f(t)}^\sigma \theta_{k+1}(x_i)]^{-1} \\
&= \psi_\sigma [\mathfrak{D}_{\sigma_n(\theta_{k+1}(x_i)^{-1})f(t)}^\sigma \theta_{k+1}(x_i)]^{-1} \\
&= \psi_\sigma \mathfrak{D}_{f(t)}^\sigma (\theta_{k+1}(x_i))^{-1} \\
&= \psi_\sigma \mathfrak{D}_{f(t)}^\sigma \theta_{k+1}(x_i^{-1}).
\end{aligned}$$

Similarly, (5.4) holds for $u = y_j^{-1}$.

Now suppose (5.4) holds for any element u with $l(u) < d$. Let w be an element of F^* with $l(w) = d$. Then $w = ux_i^{\pm 1}$ or $uy_j^{\pm 1}$ for some u with $l(u) = d - 1$.

Consider the case $w = ux_i$. Then

$$\begin{aligned}
\Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(w) &= \Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(ux_i) \\
&= \Phi_\sigma^* [(\mathfrak{D}_{f(t)}^{*\sigma}u) \cdot (\mathfrak{D}_{\sigma_n(u)f(t)}^*(x_i))] \\
&= \Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(u) + \Phi_\sigma^* \mathfrak{D}_{\sigma_n(u)f(t)}^{*\sigma}(x_i) \\
&= \psi_\sigma \mathfrak{D}_{f(t)}^\sigma \theta_{k+1}(u) + \psi_\sigma \mathfrak{D}_{\sigma_n(u)f(t)}^\sigma \theta_{k+1}(x_i) \\
&= \psi_\sigma [\mathfrak{D}_{f(t)}^\sigma \theta_{k+1}(u) \cdot \mathfrak{D}_{\sigma_n(u)f(t)}^\sigma \theta_{k+1}(x_i)].
\end{aligned}$$

Since $\sigma_n(u) = \sigma_n(\theta_{k+1}(u))$, the last expression becomes

$$\psi_\sigma [\mathfrak{D}_{f(t)}^\sigma (\theta_{k+1}(u) \cdot \theta_{k+1}(x_i))] = \psi_\sigma \mathfrak{D}_{f(t)}^\sigma \theta_{k+1}(ux_i). \tag{5.5}$$

Since similar computations provide the proofs for other cases, the details will be omitted.

Now we must show

$$\begin{aligned}
\Phi_\sigma^* (\mathfrak{D}_{f(t)}^w r_i) &= 0, && i = 1, 2, \dots, \lambda \\
\Phi_\sigma^* (\mathfrak{D}_{f(t)}^{*\sigma} s_j) &= 0, && j = 1, 2, \dots, \mu.
\end{aligned} \tag{5.6}$$

First consider $r_i = u_i x_1 u_i^{-1} x_{i+1}^{-1}$, $1 \leqslant i < \lambda$. Since $\theta_{k+1}(x_{i+1}) \equiv \theta_{k+2}(x_{i+1})$ (mod F_{k+2}) by Proposition 5.1 in [9], Propositions 2.5 and (5.1) in [1] imply that

$$\psi_\sigma \mathfrak{D}_{f(t)}^\sigma \theta_{k+1}(x_{i+1}) = \psi_\sigma \mathfrak{D}_{f(t)}^\sigma \theta_{k+2}(x_{i+1}).$$

Since $\theta_{k+2}(x_{i+1}) = \theta_{k+1}(u_i x_1 u_i^{-1})$ by the definition, we obtain

$$\Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(x_{i+1}) = \Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(u_i x_1 u_i^{-1}),$$

and hence

$$\Phi^*_\sigma[(\mathfrak{D}^{*\sigma}_{f(t)}(x_{i+1}))(\mathfrak{D}^{*\sigma}_{f(t)}(u_i x_1 u_i^{-1}))^{-1}] = 0,$$

which is equal to

$$\Phi^*_\sigma \mathfrak{D}^{*\sigma}_{f(t)}(x_{i+1}(u_i x_1 u_i^{-1})^{-1}) = 0,$$

by (2.4) (2).

Similarly, we can prove $\Phi^*_\sigma \mathfrak{D}^{*\sigma}_{f(t)}(s_j) = 0$ for $j \neq \mu$.

Now it remains to show that

(1) $\Phi^*_\sigma \mathfrak{D}^{*\sigma}_{f(t)}[\eta, x_1] = 0,$ or

(2) $\Phi^*_\sigma \mathfrak{D}^{*\sigma}_{f(t)}[\xi, y_1] = 0.$ (5.7)

Since $\sigma_n[\xi, y_1] = id$, (5.7) (2) is equivalent to

$$\Phi^*_\sigma \mathfrak{D}^{*\sigma}_{f(t)}(\xi y_1) = \Phi^*_\sigma \mathfrak{D}^{*\sigma}_{f(t)}(y_1 \xi). \tag{5.8}$$

To prove (5.8), we compute both sides separately. Note that $\theta_{k+1}(y_1) = y$ and $\sigma_n(y_1) = id$. Then

$$
\begin{aligned}
\Phi^*_\sigma \mathfrak{D}^{*\sigma}_{f(t)}(\xi y_1) &= \Phi^*_\sigma[\mathfrak{D}^{*\sigma}_{f(t)}(\xi) \cdot \mathfrak{D}^{*\sigma}_{\sigma_n(\xi)f(t)}(y_1)] \\
&= \Phi^*_\sigma \mathfrak{D}^{*\sigma}_{f(t)}(\xi) + \Phi^*_\sigma \mathfrak{D}^{*\sigma}_{\sigma_n(\xi)f(t)}(y_1) \\
&= \Phi^*_\sigma \mathfrak{D}^{*\sigma}_{f(t)}(\xi) + \psi_\sigma \mathfrak{D}^\sigma_{\sigma_n(\xi)f(t)} \theta_{k+1}(y_1) \\
&= \Phi^*_\sigma \mathfrak{D}^{*\sigma}_{f(t)}(\xi) + \sigma_n(\xi)f(t).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\Phi^*_\sigma \mathfrak{D}^{*\sigma}_{f(t)}(y_1 \xi) &= \Phi^*_\sigma(\mathfrak{D}^{*\sigma}_{f(t)}y_1 \cdot \mathfrak{D}^{*\sigma}_{\sigma_n(y_1)f(t)}\xi) \\
&= \Phi^*_\sigma \mathfrak{D}^{*\sigma}_{f(t)}y_1 + \Phi^*_\sigma \mathfrak{D}^{*\sigma}_{f(t)}(\xi) \\
&= f(t) + \Phi^*_\sigma \mathfrak{D}^{*\sigma}_{f(t)}(\xi).
\end{aligned}
$$

Therefore, it suffices to prove

$$f(t) \equiv \sigma_n(\xi)f(t) \qquad (\mathrm{mod}\, I_{k+1}). \tag{5.9}$$

Since $m = lk(X, Y) \equiv 0 \pmod{n}$, it follows from Lemma 4.1 that $q_m(t) \equiv 0 \pmod{q_n(t)}$. Also, since $\sigma_n(\xi) = \sigma_n(x^m)$, $\sigma_n(x^m)f(t) = (1 + t)^m f(t) \equiv f(t) \pmod{q_n(t)}$.

This now completes the proof of Theorem 5.1.

A straightforward computation of the relation $\Phi^*_\sigma \mathfrak{D}^*_{f(t)}[\eta, x_1] = 0$ yields the following

Corollary 5.2. *Let* $h = \theta_{k+1}(\eta)$. *Then*

$$\sum_{q=0}^{\infty} \left(\frac{\partial^{q+1} h}{\partial x^q \partial y}\right)^{o} t^{q+1} \equiv 0 \qquad (\mathrm{mod}\, I_{k+1}). \tag{5.10}$$

Remark 5.2. (5.10) can be considered as the "dual" form to (4.7).

Now before our polynomial invariants are introduced, we need a few propositions.

Proposition 5.3. *For any* $f(t) \in \Omega_\sigma(n)$,

$$\Phi_\sigma^* \mathfrak{D}_1^{*\sigma}(\eta) = \Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(\eta).$$

PROOF. Let $u = \theta_{k+1}(\eta)$. Then $\Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(\eta) = \psi_\sigma \mathfrak{D}_{f(t)}^{q} \theta_{k+1}(\eta) = \psi_\sigma \mathfrak{D}_{f(t)}^{q}(u)$.

For $f(t) = \sum_{i=0}^{\infty} a_i t^i \in P^*$, write $\psi_\sigma \mathfrak{D}_1^{q}(u) = \sum_{i=0}^{\infty} b_i t^i$ and $\psi_\sigma \mathfrak{D}_{f(t)}^{q}(u) = \sum_{i=0}^{\infty} c_i t^i$.

Then Proposition 2.5 yields, since $a_0 = 1$,

$$(1) \quad \sum_{i=0}^{\infty} b_i t^i = \sum_{q=0}^{\infty} \left(\frac{\partial^{q+1} u}{\partial x^q \partial y}\right)^{o} t^q$$

$$(2) \quad \sum_{i=0}^{\infty} c_i t^i = \sum_{j=1}^{\infty} a_j \left[\sum_{q=0}^{\infty} \left(\frac{\partial^{q+1} u}{\partial x^q \partial y}\right)^{o} t^{q+j}\right] + \sum_{q=0}^{\infty} \left(\frac{\partial^{q+1} u}{\partial x^q \partial y}\right)^{o} t^q. \tag{5.11}$$

By (5.10), for $j > 0$,

$$\sum_{q=0}^{\infty} \left(\frac{\partial^{q+1} u}{\partial x^q \partial y}\right)^{o} t^{q+j} \equiv 0 \qquad (\mathrm{mod}\, q_n(t))$$

and hence,

$$\sum_{i=0}^{\infty} b_i t^i \equiv \sum_{i=0}^{\infty} c_i t^i \qquad (\mathrm{mod}\, q_n(t)).$$

This proves Proposition 5.3.

Proposition 5.4. *For any* $f(t) \in \Omega_\sigma(n)$,

$$\Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(\xi) = f(t)[\Phi_\sigma^* \mathfrak{D}_1^{*\sigma}(\xi)].$$

PROOF. Since $f(t) \in \Omega_\sigma(n)$, $f(t)$ is of the form $(1 + t)^r$ for some $0 \leqslant r < n$. (We assume that n is non-negative.)

Write

$$\Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(\xi) = \psi_\sigma \mathfrak{D}_{f(t)}^q \theta_{k+1}(\xi) = \sum_{i=0}^\infty c_i t^i \quad \text{and} \quad \Phi_\sigma^* \mathfrak{D}_1^{*\sigma}(\xi) = \psi_\sigma \mathfrak{D}_1^q \theta_{k+1}(\xi) = \sum_{i=0}^\infty b_i t^i.$$

Denote $w = \theta_{k+1}(\xi)$. Then Proposition 2.5 yields again

$$(1) \quad \sum_{i=0}^\infty b_i t^i = \sum_{q=0}^\infty \left(\frac{\partial^{q+1}w}{\partial x^q \partial y}\right)^o t^q, \quad \text{and}$$

$$(2) \quad \sum_{i=0}^\infty c_i t^i = \sum_{j=0}^\infty a_j \left[\sum_{q=0}^\infty \left(\frac{\partial^{q+1}w}{\partial x^q \partial y}\right)^o t^{q+j}\right]. \qquad (5.12)$$

Since

$$f(t) = (1 + t)^r = \sum_{i=0}^\infty a_i t^i \quad \text{and} \quad \left(\frac{\partial w}{\partial y}\right)^o = 0,$$

it follows from (5.12) (1) (2) that

$$f(t) \sum_{i=0}^\infty b_i t^i = \sum_{j=0}^\infty a_j t^j \left(\sum_{q=0}^\infty \left(\frac{\partial^{q+1}w}{\partial x^q \partial y}\right)^o t^q\right)$$

$$= \sum_{j=0}^\infty a_j \left[\sum_{q=0}^\infty \left(\frac{\partial^{q+1}w}{\partial x^q \partial y}\right)^o t^{q+j}\right]$$

$$= \sum_{i=0}^\infty c_i t^i.$$

Definition 5.1. *Let n be an integer that divides $lk(X, Y)$. Then for any integer $k \geq 1$, define the polynomials $\eta_k^{(n)}(t)$ and $\xi_k^{(n)}(t)$ in $R_k(n)$ by*

$$\eta_k^{(n)}(t) = \Phi_\sigma^* \mathfrak{D}_1^{*\sigma}(\eta), \quad \text{and}$$
$$\xi_k^{(n)}(t) = \Phi_\sigma^* \mathfrak{D}_1^{*\sigma}(\xi). \qquad (5.13)$$

Theorem 5.5. *For any $k \geq 1$, $\eta_k^{(n)}(t)$ and $\{f(t)\xi_k^{(n)}(t), f(t) \in \Omega_\sigma(n)\}$ are invariants of an oriented link type L.*

Remark 5.3. For any $u \in F$, $\theta_{k+1}(u) \equiv \theta_{k+2}(u) \mod F_{k+2}$, and hence, for any $l \geq k$,

$$\eta_l^{(n)}(t) \equiv \eta_k^{(n)}(t) \quad (\mathrm{mod}\, I_{k+1})$$
$$\xi_l^{(n)}(t) \equiv \xi_k^{(n)}(t) \quad (\mathrm{mod}\, I_{k+1}). \qquad (5.14)$$

Since k can be taken arbitrarily large, these invariants are formal power series.

Corollary 5.6. *Suppose $lk(X, Y) = n \neq 0$. Let $\sum_{i=0}^{\infty} \bar{a}_i t^i$ be the normal form of $\eta_k^{(n)}$. Then for any $i \geqslant 0$, $\bar{a}_i = a_i^*$, where a_i^* is the invariant defined in [5].*

Now, as we did in [4] or [9], these invariants can be interpreted as the "linking number" between one component of the lifts of X or Y and the characteristic link defined in [9]. Using this geometric interpretation, we can obtain more information on $\eta_k^{(n)}(t)$.

Let $\tilde{X}$ and $\tilde{Y} = \tilde{Y}_0 \cup \ldots \cup \tilde{Y}_{n-1}$ be the lifts of X and Y, respectively, in the covering space M_σ associated with the homomorphism $\Sigma_n \colon G(L) \to \to \mathrm{Sym}(\Omega_\sigma(n))$.

Theorem 5.7. *Suppose $lk(X, Y) = rn$, $n > 0$. Then, for any $k \geqslant 1$,*

$$\eta_k^{(n)}(t) = r \left\{ \binom{n}{1} + \binom{n}{2}t + \ldots + \binom{n}{n}t^{n-1} \right\}. \tag{5.15}$$

In particular, the invariant a_i^* defined in [5] is completely determined by the linking number $lk(X, Y)$.

PROOF. By [4] or [9], the characteristic link associated with the linking function or the homomorphism $\Phi_\sigma \colon \pi_1(M_\sigma) \to R_k(n)$ is a 1-cycle

$$\tilde{Y} = \sum_{i=0}^{n-1} (1 + t)^i \tilde{Y}_i \quad \text{in} \quad H_1(\tilde{Y}; R_k(n)),$$

and $\tilde{Y}$ bounds a 2-chain $\tilde{D}$ in $C_2(M_\sigma; R_k(n))$. Then by [4], $\eta_k^{(n)}(t) = \mathrm{Int}(\tilde{X}, \tilde{D})$, where Int denotes the intersection number. Since $\tilde{X}$ bounds a 2-chain $\tilde{C}$ in M_σ, $\mathrm{Int}(\tilde{X}, \tilde{D}) = lk(\tilde{X}, \tilde{Y}) = \mathrm{Int}(\tilde{C}, \tilde{Y})$. Obviously, $\mathrm{Int}(\tilde{C}, \tilde{Y}_i) = r$ and hence

$$\eta_k^{(n)}(t) = r \sum_{i=0}^{n-1} (1 + t)^i = r \sum_{j=1}^{n} \binom{n}{j} t^{j-1}.$$

This proves (5.15).

Corollary 5.8. *If $lk(X, Y) = 0$, then for any n and k, $\eta_k^{(n)}(t) = 0$.*

Corollary 5.9. *$\xi_k^{(n)}(0) = 0$ and $\eta_k^{(n)}(0) = lk(X, Y)$.*

PROOF. By Remark 5.3, $\xi_k^{(n)}(0) = \xi_0^{(n)}(0)$. Since $\theta_1(\xi) = x^m$, $m = lk(X, Y)$, we have

$$\xi_0^{(n)}(t) = \Phi_\sigma^* \mathfrak{D}_1^{*\sigma}(\xi) = \psi_\sigma \mathfrak{D}_1^q \theta_1(\xi) = \psi_\sigma \mathfrak{D}_1^q(x^m) = 0.$$

Therefore, $\xi_k^{(n)}(0) = 0$. The second part follows from (5.15).

Suppose $lk(X, Y) = 0$ and take $n = 0$. Then, since $q_n(t) = 0$, $\xi_k^{(0)}(t)$ is a rational function on t. Furthermore, if we let $s = 1 + t$ and express $\xi_k^{(0)}(t)$ as a Laurent polynomial on s, then it is essentially the η-function defined in [6]. In fact, we can prove the following theorem

Theorem 5.10.   *Suppose that Y is contractible $S^3 - X$. Let $\eta(L, X, Y; s)$ be the polynomial defined in [6]. Write $\xi_k^{(0)}(t)$ as a Laurent polynomial $\tilde{\xi}_k(s)$ on $s = 1 + t$. Then for a sufficiently large k,*

$$\tilde{\xi}_k(s) \doteq \eta(L, X, Y; s),$$

where $A \doteq B$ means that A and B are equal up to a unit in $Z[s, s^{-1}]$.
   A proof follows from the definition of $\xi_k^{(0)}(t)$ and Theorem 2 in [7].

6. Covering Space (II) Metabelian Covering

In this section, we consider the other representation $T_{k,n}: G(L) \to Sym(\Omega_k(n))$ and the covering space M_τ of $S^3 - L$ associated with $T_{k,n}$.

Theorem 6.1.   *Let n be an integer and suppose $lk(X, Y) \equiv 0 \pmod{n}$. Then for each $k \geqslant 1$, there exists a homomorphism*

$$\Phi_\tau: \langle S_\tau : R_\tau \rangle \to R_k(n)$$

such that, for any $f(t) \in \Omega_k(n)$,

$$\Phi_\tau(\mathfrak{D}_{f(t)}^{*\tau} x_1) = 0 \quad and \quad \Phi_\tau(\mathfrak{D}_{f(t)}^{*\tau} y_1) = f(t), \tag{6.1}$$

where $\mathfrak{D}_{f(t)}^{*\tau}$ denotes the Reidemeister-Schreier rewriting function associated with $T_{k,n}$ and $S_\tau = \{\mathfrak{D}_{f(t)}^{*\tau}(x_i), \mathfrak{D}_{f(t)}^{*\tau}(y_j) \mid 1 \leqslant i \leqslant \lambda, 1 \leqslant j \leqslant \mu, f(t) \in \Omega_k(n)\}$ and $R_\tau = \{\mathfrak{D}_{f(t)}^{*\tau}(r_i), \mathfrak{D}_{f(t)}^{*\tau}(s_j) \mid 1 \leqslant i \leqslant \lambda, 1 \leqslant j \leqslant \mu, f(t) \in \Omega_k(n)\}$.

PROOF.   We can use the same argument employed in the proof of Theorem 5.1 using the Reidemeister-Schreier rewriting functions $\mathfrak{D}_{f(t)}^{*\tau}$, $\mathfrak{D}_{f(t)}^{\tau}$ and homomorphisms Φ_τ^*, ψ_τ instead of $\mathfrak{D}_{f(t)}^{*\sigma}$, $\mathfrak{D}_{f(t)}^{\sigma}$, Φ_σ^*, ψ_σ. What we need to prove here is the formula (6.2) below corresponding to (5.7) (1).

$$\Phi_\tau^* \mathfrak{D}_{f(t)}^{*\tau}(\eta x_1) = \Phi_\tau^* \mathfrak{D}_{f(t)}^{*\tau}(x_1 \eta). \tag{6.2}$$

First we compute both sides separately. The left hand side is

$$\begin{aligned}
\Phi_\tau^* \mathfrak{D}_{f(t)}^{*\tau}(\eta x_1) &= \Phi_\tau^* (\mathfrak{D}_{f(t)}^{*\tau} \eta \cdot \mathfrak{D}_{T_{k,n}(\eta)f(t)}^{*\tau} x_1) \\
&= \Phi_\tau^* \mathfrak{D}_{f(t)}^{*\tau}(\eta) + \Phi_\tau^* \mathfrak{D}_{T_{k,n}(\eta)f(t)}^{*\tau}(x_1) \\
&= \Phi_\tau^* \mathfrak{D}_{f(t)}^{*\tau}(\eta) \\
&= \psi_\tau \mathfrak{D}_{f(t)}^{\tau} \theta_{k+1}(\eta).
\end{aligned}$$

On the other hand, the right hand side is

$$\Phi_\tau^* \mathfrak{D}_{f(t)}^{*\tau}(x_1 \eta) = \Phi_\tau^* \mathfrak{D}_{f(t)}^{*\tau}(x_1) + \Phi_\tau^* \mathfrak{D}_{Tk, n(x_1)f(t)}^{*\tau}(\eta)$$

$$= \Phi_\tau^* \mathfrak{D}_{Tk, n(x_1)f(t)}^{*\tau}(\eta)$$

$$= \psi_\tau \mathfrak{D}_{Tk, n(x)f(t)}^{\tau} \theta_{k+1}(\eta).$$

To compare these terms, let $u = \theta_{k+1}(\eta)$ and $f(t) = \sum\limits_{i=0}^{\infty} a_i t^i$.

Write

$$\psi_\tau \mathfrak{D}_{f(t)}^{\tau} \theta_{k+1}(\eta) = \sum_{i=0}^{\infty} b_i t^i \quad \text{and} \quad \psi_\tau \mathfrak{D}_{Tk, n(x)f(t)}^{\tau} \theta_{k+1}(\eta) = \sum_{i=0}^{\infty} c_i t^i.$$

Then by (2.9) (2), we have, since $a_0 = 1$,

(1) $b_0 = \left(\dfrac{\partial u}{\partial y}\right)^o = m = lk(X, Y)$

(2) For $q \geqslant 1$,

$$b_q = a_q \left(\frac{\partial u}{\partial y}\right)^o + a_{q-1}\left(\frac{\partial^2 u}{\partial y^2}\right)^o + \ldots + a_1\left(\frac{\partial^q u}{\partial y^q}\right)^o + \left(\frac{\partial^{q+1} u}{\partial x \partial y^q}\right)^o. \tag{6.3}$$

Now $\tau_{k,n}(x) = f(t) + t = 1 + (1 + a_1)t + \sum_{i=2}^{\infty} a_i t^i$ and hence, (2.9) yields, again,

(1) $c_0 = \left(\dfrac{\partial u}{\partial y}\right)^o = m$

(2) $c_1 = (1 + a_1)\left(\dfrac{\partial u}{\partial y}\right)^o + \left(\dfrac{\partial^2 u}{\partial x \partial y}\right)^o$

(3) For $q \geqslant 2$,

$$c_q = a_q\left(\frac{\partial u}{\partial y}\right)^o + a_{q-1}\left(\frac{\partial^2 u}{\partial y^2}\right)^o + \ldots + a_2\left(\frac{\partial^{q-1} u}{\partial y^{q-1}}\right)^o +$$

$$+ (1 + a_1)\left(\frac{\partial^q u}{\partial y^q}\right)^o + \left(\frac{\partial^{q+1} u}{\partial x \partial y^q}\right)^o. \tag{6.4}$$

Therefore, for $q \geqslant 1$, $c_q - b_q = (\partial^q u/\partial y^q)^o$ and hence

$$\sum_{q=0}^{\infty} b_q t^q - \sum_{q=0}^{\infty} c_q t^q = \sum_{i=1}^{\infty} \left(\frac{\partial^i u}{\partial y^i}\right)^o t^i.$$

Since

$$\left(\frac{\partial^i u}{\partial y^i}\right)^o = \binom{(\partial u/\partial y)^o}{i} = \binom{m}{i}$$

for $i \geqslant 1$, it follows from Lemma 3.1 that

$$\sum_{i=1}^{\infty} \left(\frac{\partial^i u}{\partial y^i}\right)^o t^i = \sum_{i=1}^{\infty} \binom{m}{i} t^i = q_m(t) \equiv 0 \qquad (\text{mod } q_n(t)),$$

and therefore,

$$\sum_{q=0}^{\infty} b_q t^q - \sum_{q=0}^{\infty} c_q t^q = 0.$$

This proves (6.2).

Proposition 6.2.   *For any* $f(t) \in \Omega_k(n)$,

$$\Phi_\tau^* \mathfrak{D}_1^{*\tau}(\eta) = \Phi_\tau^* \mathfrak{D}_{f(t)}^{*\tau}(\eta). \tag{6.5}$$

PROOF.   Let $f(t) = \sum_{i=0}^{\infty} a_i t^i$ and write $\Phi_\tau^* \mathfrak{D}_1^{*\tau}(\eta) = \sum_{i=0}^{\infty} b_i t^i$ and $\Phi_\tau^* \mathfrak{D}_{f(t)}^{*\tau}(\eta) = \sum_{i=0}^{\infty} c_i t^i$. Then by (2.9) (2) we have, for $q \geqslant 0$ and $u = \theta_{k+1}(\eta)$,

$$(1) \quad b_q = \left(\frac{\partial^{q+1} u}{\partial x \partial y^q}\right)^o \quad \text{and}$$

$$(2) \quad c_q = a_q \left(\frac{\partial u}{\partial y}\right)^o + a_{q-1} \left(\frac{\partial^2 u}{\partial y^2}\right)^o + \ldots + a_1 \left(\frac{\partial^q u}{\partial y^q}\right)^o + \left(\frac{\partial^{q+1} u}{\partial x \partial y^q}\right)^o. \tag{6.6}$$

Therefore, $b_0 = c_0 = m = lk(X, Y)$ and

$$\sum_{q=0}^{\infty} c_q t^q - \sum_{q=0}^{\infty} b_q t^q = \sum_{q=1}^{\infty} \left\{ a_q \left(\frac{\partial u}{\partial y}\right)^o + a_{q-1} \left(\frac{\partial^2 u}{\partial y^2}\right)^o + \ldots + a_1 \left(\frac{\partial^q u}{\partial y^q}\right)^o \right\} t^q$$

$$= \sum_{j=1}^{\infty} a_j \left[\sum_{i=1}^{\infty} \binom{m}{i} t^{i+j-1} \right]$$

$$\equiv 0 \quad (\text{mod } q_n(t)),$$

since for $j \geqslant 1$,

$$\sum_{i=1}^{\infty} \binom{m}{i} t^{i+j-1} \equiv 0 \quad (\text{mod } q_n(t)).$$

Proposition 6.3. $T_{k,n}(\xi) = id.$

PROOF.   Let $w = \theta_{k+1}(\xi)$ and write $T_{k,n}(\xi)(\sum_{i=0}^{\infty} a_i t^i) = \sum_{i=0}^{\infty} b_i t^i$, where $a_0 = 1$. Then by (2.8) (2),

$$b_q = a_q + a_{q-1} \left(\frac{\partial w}{\partial y}\right)^o + \ldots + a_1 \left(\frac{\partial^{q-1} w}{\partial y^{q-1}}\right)^o + \left(\frac{\partial^q w}{\partial x \partial y^{q-1}}\right)^o.$$

However,

$$\left(\frac{\partial^i w}{\partial y^i}\right)^o = \left(\frac{(\partial w/\partial y)^o}{i}\right) = 0, \quad \text{since} \quad \left(\frac{\partial w}{\partial y}\right)^o = 0.$$

Therefore,

$$b_q = a_q + \left(\frac{\partial^q w}{\partial x \partial y^{q-1}}\right)^o.$$

By Proposition 4.2 (4.7),

$$\sum_{q \ge 1} \left(\frac{\partial^q w}{\partial x \partial y^{q-1}}\right)^o t^q \equiv 0 \quad (\text{mod } q_n(t))$$

and hence

$$\sum_{q \ge 0} b_q t^q - \sum_{q \ge 0} a_q t^q = \sum_{q \ge 1} \left(\frac{\partial^q w}{\partial x \partial y^{q-1}}\right)^o t^q \equiv 0 \quad (\text{mod } q_n(t)).$$

Proposition 6.4. *For any* $f(t) \in \Omega_k(n)$,

$$\Phi_\tau^* \mathfrak{D}_1^{*\tau}(\xi) = \Phi_\tau^* \mathfrak{D}_{f(t)}^{*\tau}(\xi).$$

PROOF. The details will be omitted, since a proof can be obtained, using similar computations shown in the proofs of Propositions 6.2 and 6.3.

Definition 6.1. *Let* n *be an integer that divides* $lk(X, Y)$. *Then for any integer* $k \ge 1$, *define the polynomials* $\bar{\eta}_k^{(n)}(t)$ *and* $\bar{\xi}_k^{(n)}(t)$ *in* $R_k(n)$ *as follows*:

$$\bar{\eta}_k^{(n)}(t) = \Phi_\tau^* \mathfrak{D}_1^{*\tau}(\eta),$$
$$\bar{\xi}_k^{(n)}(t) = \Phi_\tau^* \mathfrak{D}_1^{*\tau}(\xi). \tag{6.7}$$

Theorem 6.5. *For any* $k \ge 1$, $\bar{\eta}_k^{(n)}(t)$ *and* $\bar{\xi}_k^{(n)}(t)$ *are invariants for an oriented link type* L.

Remark 6.1. As is stated in Remark 5.3, for $l \ge k$, $\bar{\eta}_l^{(n)}(t) \equiv \bar{\eta}_k^{(n)}(t)$ $(\text{mod } I_{k+1})$ and $\bar{\xi}_l^{(n)}(t) \equiv \bar{\xi}_k^{(n)}(t)$ $(\text{mod } I_{k+1})$, and therefore, these invariants are formal power series.

Now, it follows from Proposition 7.1 in [4] that these invariants also can be interpreted as linking numbers between two cycles in the covering space M_τ associated with $T_{k,n}$. Since $T_{k,n}(\xi) = \text{id}$, every lift $\bar{\xi}$ of a longitude ξ of Y in M_τ is a simple closed curve in M_τ and hence, $\bar{\xi}_k^{(n)}(t)$ is interpreted as the intersection number between $\bar{\xi}$ and a 2-chain in $C_2(M_\tau; R_k(n))$ which bounds the characteristic link. On the other hand, $T_{k,n}(\eta)$ may not be an identity, and

therefore, a lift $\tilde{\eta}$ of a longitude η of X in M_τ may not be a closed curve. Let r be the smallest positive integer such that $T_{k,n}(\eta^r) = $ id. Then Proposition 6.2 shows that for any $f(t) \in \Omega_k(n)$, $\Phi_\tau^* \mathfrak{D}_{f(t)}^{*\tau}(\eta^r) = r\Phi_\tau^* \mathfrak{D}_1^{*\tau}(\eta)$ and therefore, $\Phi_\tau^* \mathfrak{D}_1^{*\tau}(\eta) = \bar{\eta}_t^{(n)}(t)$ can be considered the "linking number" between a "longitude" of a covering torus and the characteristic link in M_τ.

Corollary 6.6. *Suppose $lk(X, Y) = n \neq 0$. Let $\sum_{i=0}^{\infty} \bar{a}_i t^i$ be the normal form of $\bar{\eta}_k^{(n)}(t)$. Then $\bar{a}_0 = n$ and $\bar{a}_i = h_i^*$, where h_i^* is the invariant defined in* [5].

Corollary 6.7. *If the Alexander polynomial of L is 0, then all invariants $\eta(t)$, $\xi(t)$, $\bar{\eta}(t)$, $\bar{\xi}(t)$ vanish.*

Corollary 6.8. $\bar{\xi}_k^{(n)}(0) = 0$ *and* $\bar{\eta}_k^{(n)}(0) = lk(X, Y)$.

Proof.   The proof of the first part is similar to that of Corollary 5.9. On the other hand, $\bar{\eta}_k^{(n)}(0) = \bar{\eta}_0^{(n)}(0)$ and $\bar{\eta}_0^{(n)}(t) = \Phi_\tau^* \mathfrak{D}_1^{*\tau}(\eta) = \psi_\tau \mathfrak{D}_1^\tau \theta_1(\eta) = \psi_\tau \mathfrak{D}_1^\tau(y^m) = $ $= \psi_\tau(y_1^m) = m$, since $\tau_{0,n}(y) = $ id and $\theta_1(\eta) = y^m$, where $m = lk(X, Y)$.

Finally, we study the behavior of these invariants under simple transformations of the link. The following two propositions are easy to prove, and therefore, the details will be omitted.

Proposition 6.9. *Let L' be the mirror image of an oriented link L. Then for any n and k,*

$$\eta_k^{(n)}(t)_{L'} = -\eta_k^{(n)}(t)_L$$
$$\xi_k^{(n)}(t)_{L'} = -\xi_k^{(n)}(t)_L$$
$$\bar{\eta}_k^{(n)}(t)_{L'} = -\bar{\eta}_k^{(n)}(t)_L$$
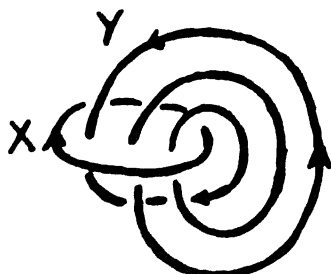$$\bar{\xi}_k^{(n)}(t)_{L'} = -\bar{\xi}_k^{(n)}(t)_L.$$

Proposition 6.10. *Let L^* be the link obtained from L by reversing the orientation of one component, X say. The n for any n and k,*

$$\begin{cases} \eta_k^{(n)}(t)_{L^*} = -\eta_k^{(n)}(t)_L \\ \xi_k^{(n)}(t)_{L^*} = \xi_k^{(n)}(t)_L. \end{cases}$$

7. Examples

In this section, we compute our invariants for two simple 2-component links.

Example 1. Torus link of type (6.2).

$G(L) = \langle x, y \colon [\eta, x] = 1, \ [\xi, y] = 1 \rangle$
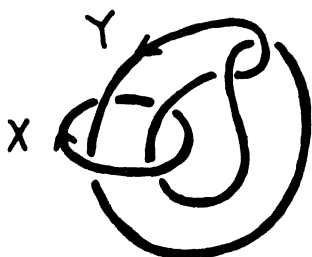where $\eta = x^{-1}yxyxyx^{-1}$ and
$\xi = xyxyxy^{-2}$.

$lk(X, Y) = 3$ and $\Delta(x, y) = 1 + xy + x^2 y^2$.

Let $n = 3$ and $q_n(t) = 3t + 3t^2 + t^3$. Then for $k \geqslant 4$.

$$\begin{cases} \eta_k(t) = 3 + t^2 + 2t^3 + t^4 \\ \xi_k(t) = t^2 + 2t^3 + t^4 \end{cases}$$

$$\begin{cases} \bar{\eta}_k(t) = 3 + t^2 + 2t^3 + t^4 \\ \bar{\xi}_k(t) = t^2 + 2t^3 + t^4. \end{cases}$$

Example 2.   Whitehead link.

$G(L) = \langle x, y \colon [\eta, x] = 1, \ [\xi, y] = 1 \rangle$
where $\eta = y^{-1}xyx^{-1}yxy^{-1}x^{-1}$ and
$\xi = x^{-1}yxy^{-1}xyx^{-1}y^{-1}$.

$lk(X, Y) = 0$ and $\Delta(x, y) = (1 - x)(1 - y)$.

Let $n = 3$ and $q_3(t) = 3t + 3t^3 + t^3$. Then for $k \geqslant 4$,

$$\begin{cases} \eta_k(t) = 0 \\ \xi_k(t) = t^2 + 2t^3 + t^4 \end{cases}$$

$$\begin{cases} \bar{\eta}_k(t) = t^2 + 2t^3 + t^4 \\ \bar{\xi}_k(t) = 0. \end{cases}$$

Let $n = 0$ and $q_0(t) = 0$. Then for any $k \geqslant 0$,

$$\begin{cases} \eta_k(t) = 0 \\ \xi_k(t) = (1 + t)^{-1} - 2 + (1 + t) \\ = t^2 - t^3 + t^4 - \dots. \end{cases}$$

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