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# The Concentration-Compactness Principle in the Calculus of Variations. The limit case, Part 1

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## Abstract

After the study made in the locally compact case for variational problems with some translation invariance, we investigate here the variational problems (with constraints) for example in $\mathbb{R}^N$ where the invariance of $\mathbb{R}^N$ by the group of dilatations creates some possible loss of compactness. This is for example the case for all the problems associated with the determination of extremal functions in functional inequalities (like for example the Sobolev inequalities). We show here how the concentration-compactness principle has to be modified in order to be able to treat this class of problems and we present applications to Functional Analysis, Mathematical Physics, Differential Geometry and Harmonic Analysis.

## Key-words

Concentration-compactness principle, minimization problems, unbounded domains, dilatations invariance, concentration function, nonlinear field equations, Dirac masses, Morse theory, Sobolev inequalities, convolution, Yamabe problem, scalar curvature, conformal invariance, trace inequalities.

## Subject AMS classification

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## Introduction

We have studied in the preceding parts (P. L. Lions [20], [21]) *variational problems* set in *unbounded domains*, where the unboundedness induces some possible *loss of compactness* (a classical example of such loss of compactness is the well-known fact that Rellich Teorem does not hold on unbounded domains like $\mathbb{R}^N$ for example). Roughly speaking we had to take care in [20], [21] of the difficulty caused by the invariance of $\mathbb{R}^N$ by the *non-compact group of translations*.

We want to study here, in a systematic way, variational problems where not only compactness may be lost because of translations but also because of the invariance of $\mathbb{R}^N$, say, by the *non-compact group of dilations*. This difficulty was absent from [20], [21] since we were interested there in the so-called locally compact case, while it is encountered there when studying the so-called limit-cases problems, or problems with limit exponents (see below for concrete examples).

Before giving examples and explaining the statements above, we would like to mention that most of the problems considered below have their origins in Geometry and in Mathematical Physics and have been studied by many authors. In particular we refer to the fundamental studies of T. Aubin [3] on the Yamabe problem; J. Sacks and K. Uhlenbeck [32], Y. T. Siu and S. T. Yau [34] on harmonic mappings; and of K. Uhlenbeck [41], [42], C. Taubes [36], [37], [38].

The dilations invariance of $\mathbb{R}^N$ is a typical difficulty in the study of the existence of *extremal functions* in *functional inequalities*; indeed if $A$ is a linear bounded operator from a Banach space $E$ into another Banach space $F$, one may consider the smallest positive constant $C_0$ such that the following inequality holds for all $u$ in $E$;

$$||Au||_F \leqslant C_0 ||u||_E; \tag{1}$$

and one may ask whether the *best constant* $C_0$ is obtained for some $u$. Now if $E, F$ are functional spaces, it is often the case that (1) is preserved if we perform a *scale change* that is if we replace $u(\cdot)$ by $u(\cdot/\sigma)$ for $\sigma > 0$. Of course the question concerning $C_0$ is equivalent to the solution of the following minimization problems:

$$\text{Min}\{ ||u||_E / u \in E, ||Au||_F = 1 \} \tag{2}$$

or

$$\text{Min}\{ -||Au||_F / u \in E, ||u||_E = 1 \}; \tag{2'}$$

and the invariance of (1) by scale changes is often reflected by the invariance of $||\cdot||_E$ or $||\cdot||_F$ by changes such as:

$$u(\cdot) \to \sigma^{-\alpha} u\left(\frac{\cdot}{\sigma}\right)$$

where $\alpha$ depends on $A, E, F$. And this invariance will imply compactness defects on minimizing sequences of problems (2)-(2').

Let us give a few examples of such situations:

EXAMPLE 1. *Sobolev inequalities.*

Let $1 \leqslant p < N/m$, $m \geqslant 1$ and let $E$ be the Banach space consisting of functions in $L^q(\mathbb{R}^N)$ with $q = Np/(N - mp)$ such that all their derivatives of order $m$ are in $L^p(\mathbb{R}^N)$; $E$ is equipped for example with the norm $||D^m u||_{L^p(\mathbb{R}^N)}$. The so-called Sobolev embedding theorem (or Sobolev inequality) yields that $E$ is continuously embedded in $F = L^q(\mathbb{R}^N)$. Therefore the question of extremal functions in the Sobolev inequealities

$$||u||_{L^q(\mathbb{R}^N)} \leqslant C_0 ||D^m u||_{L^p(\mathbb{R}^N)} \tag{3}$$

is an example of the above framework—$A$ being the injection of $E$ into $F$. The associated minimization problem is, for example,

$$\text{Min}\left[\int_{\mathbb{R}^N} |D^m u|^p \, dx / u \in E, \int_{\mathbb{R}^N} |u|^q \, dx = 1\right]. \tag{4}$$

One then checks easily that if we replace $u$ by $\sigma^{-N/q} u(\cdot/\sigma)$ for any $\sigma > 0$, the two functionals occuring in the above variational problem are preserved (this invariance being nothing else than the invariance of Sobolev inequalities (3) with respect to scale changes).

EXAMPLE 2. *Hardy-Littlewood-Sobolev inequalities.*

Let $0 < \mu < N$, $1 < p < (N/(N-\mu))$ and let $q$ satisfy: $(1/p) + (\mu/N) = 1 + (1/q)$. The Hardy-Littlewood-Sobolev inequality then states

$$||K * u||_{L^q(\mathbb{R}^N)} \leqslant C_0 ||u||_{L^p(\mathbb{R}^N)}, \qquad \forall u \in L^p(\mathbb{R}^N) \tag{5}$$

where $K = 1/|x|^\mu$. The determination of the best $C_0$ is then equivalent to

$$\operatorname{Min}\left[ - \int_{\mathbb{R}^N} |K * u|^q \, dx / u \in L^p(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^p \, dx = 1 \right] \tag{6}$$

(that is (2′), with $E = L^p$, $F = L^q$, $Au = K * u$). Again the two functionals are invariant by the transformation: $u \to \sigma^{-N/q} u(\cdot / \sigma)$ for all $\sigma > 0$.

EXAMPLE 3. *Trace inequalities.*

Let $1 \leqslant p < N$, $N \geqslant 2$, $m \geqslant 1$ and let $q$ be given by: $q = (N-1)p(N-mp)^{-1}$. It is well—known that there exists a bounded linear operator $A$—called the trace operator—from $E = \{u \in L^\alpha(\mathbb{R}^{N-1} \times \mathbb{R}_+), D^m u \in L^p(\mathbb{R}^{N-1} \times \mathbb{R}_+)\}$ with $\alpha = Np/(N-mp)$ equipped with the same norm as in Example 1 into $F = L^q(\mathbb{R}^{N-1})$ such that if $u \in \mathfrak{D}(\mathbb{R}^N)$, $Au$ is the usual trace of $u$ on $\mathbb{R}^{N-1} \times \{0\}$. For obvious reasons we still denote $Au$ by $u$. In this context, problem (2) becomes

$$\operatorname{Min}\left[ \int_{\mathbb{R}^{N-1} \times \mathbb{R}_+} |D^m u|^p \, dx / u \in E, \int_{\mathbb{R}^{N-1}} |u(x', 0)|^q \, dx' = 1 \right]. \tag{7}$$

And again both functionals are preserved if we replace $u$ by $\sigma^{-(N-1)/q} u(\cdot / \sigma)$.

There are of course may more examples of this type (some are discussed in the following section). Let us now explain on these examples what we mean by loss of compactness induced by the dilations group (or the scale change invariance). This can be easily seen on the fact that, even if we know there exists a minimum in (2), (2′), (4), (6) or (7), the set of minima is not relatively compact in $E$: indeed if $u$ is a minimum then $\sigma^{-\alpha} u(\cdot / \sigma) = u_\sigma$ would still be a minimum for all $\sigma > 0$ ($\alpha = N/q$ in Examples 1, 2, $\alpha = (N-1)/q$ in Example 3). Now if $\alpha \to 0$ or $\sigma \to \infty$, $u_\sigma$ converges weakly to 0 (which is not a minimum) and the probability $|u_\sigma|^q$ (or $|u_\sigma|^p$) either converges weakly as $\sigma \to 0$ to a Dirac mass or spreads out (vanishing in Lemma I.1 of [20]) as $\sigma \to \infty$. This loss of compactness may be also seen on the fact that $q$ in the various examples is a limit exponent and that if we consider only functions with support in a fixed bounded domain and if $q$ is replaced by a smaller exponent, then the various minimization problems are standard consequences of the Rellich theorem. Notice also that the set of minima in (2), (2′), (4), (6) or (7) is also translation invariant therefore we also have the loss of compactness induced by the translation invariance, as we had in the problems studied in [20], [21].

We present here a *general method to solve variational problems* (with constraints) where such difficulties are encountered, that is problems with *limit exponents* or with a *scale change invariance* or problems like (2), (2′) in functional spaces. In particular our methods enable us to prove that *any minimizing sequence of problems* (4), (6) *or* (7) *is relatively compact in E* up to a translation and a scale change(*). In particular there exists a minimum; this last assertion has been proved in Example 1 for the particular case of $m = 1$ by Rosen [31], G. Talenti [35], T. Aubin [4] and in Example 2 by E. H. Lieb [18] but all these works depend on the use of symmetrization and therefore cannot be extended to cover fully examples 1-3. Let us mention a few other applications of our methods.

EXAMPLE 4. *Yamabe problem in* $\mathbb{R}^N$.

An important problem in differential geometry is the so-called Yamabe conjecture or Yamabe problem (this problem will be explained in detail later on, see Yamabe [44], N. Trudinger [39], Eliasson [14] and T. Aubin [3]). We will come back below on the case when the problem is set on a compact manifold but here we restrict our attention to $\mathbb{R}^N$-prototype of a complete but non-compact manifold. We look for a positive function $u$ in $\mathbb{R}^N$ solution of

$$-\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + k(x)u = K(x)u^{(N+2)/(N-2)} \text{ in } \mathbb{R}^N, \quad u > 0 \text{ in } \mathbb{R}^N \qquad (8)$$

where $a_{ij}$, $k$, $K$ are smooth functions, $(a_{ij})$ is symmetric definite positive. First, if we look for a solution which vanishes at infinity, our general method enables us to study completely the variational problems associated with (8).

Next, if we consider solutions which remain positive at infinity, we also solve similar variational problems where we look for functions which converge to a given positive constant at infinity. However in that case, we need severe restrictions on $k, K$. In [28], Ni proposed a different approach of (8) by the method of sub and supersolutions —that we recall in an appendix— which gives a very general existence result. Roughly speaking, one can find an interval $]0, \bar{\mu}[$ such that if $0 < \mu < \bar{\mu}$, *there exists a minimum solution* $\underline{u}$ of (8) such that: $\underline{u}(x) \to \mu$ as $|x| \to \infty$. *We prove below* that under quite general assumptions, *there exists a second solution u of* (8) such that: $u(x) > \underline{u}(x)$ on $\mathbb{R}^N$, $u(x) \to \mu$ as $|x| \to \infty$. This is achieved in the appendix by looking at the problem satisfied by $(u - \underline{u})$ and by solving the associated variational problem by our concentration-compactness method.

(*) Of course if $p = 1$ in Examples 1,3; $L^1$ has to be replaced by the space of bounded measures.

EXAMPLE 5. *Nonlinear field equations.*

In various domains of Mathematical Physics one encounters the following nonlinear problem

$$-\Delta u = f(u) \quad \text{in} \quad \mathbb{R}^N, \qquad u(x) \to 0 \quad \text{as} \quad |x| \to \infty \tag{9}$$

(here to simplify the presentation, we take scalar functions $u$). Of particular interest is the so-called ground state solution which, if it exists, is the minimum of the following problem (see for instance Coleman, Glazer and Martin [13], H. Berestycki and P. L. Lions [6])

$$I = \text{Min}\Big[\int_{\mathbb{R}^N} |Du|^2\,dx \Big/ \int_{\mathbb{R}^N} F(u)\,dx = 1,\, u \in L^{2N/(N-2)}(\mathbb{R}^N), \\ Du \in L^2(\mathbb{R}^N), F(u) \in L^1(\mathbb{R}^N)\Big]. \tag{10}$$

where $F(t) = \int_0^t f(s)\,ds$, $N \geqslant 3$ (to simplify). In view of both known existence results on this problem (see the references above and their bibliographies), the behaviour of $F$ at 0 and at $\infty$ is known to be determinant: more precisely in all known existence results of a minimum in (10), $F$ is supposed to satisfy

$$\lim_{|t| \to 0_+} F(t)|t|^{-(2N/(N-2))} \leqslant 0, \qquad \lim_{|t| \to \infty} F(t)|t|^{-(2N/(N-2))} \leqslant 0.$$

Our method enables us to give a much more general condition for the existence of a minimum in (10) which will cover both the situation above and the case of the best Sobolev constant i.e. $F(t) = |t|^{2N/(N-2)}$. We assume that $F \in C(\mathbb{R})$, $F(0) = 0$ and

$$\exists \zeta \in \mathbb{R}, \qquad F(\zeta) > 0 \qquad \text{and} \tag{11}$$

$$\lim_{|t| \to 0_+} F^+(t)|t|^{-(2N/(N-2))} = \alpha \geqslant 0, \qquad \lim_{|t| \to \infty} F^+(t)|t|^{-(2N/(N-2))} = \beta \geqslant 0 \tag{12}$$

(of course if $\alpha, \beta > 0$, $F^+$ may be replaced by $F$); and we denote by

$$I^\infty = \text{Min}\Big[\int_{\mathbb{R}^N} |Du|^2\,dx \Big/ \int_{\mathbb{R}^N} |u|^{2N/(N-2)}\,dx = 1\Big];$$

(cf. Example 1 above). Then *we prove that any minimizing sequence* $(u_n)$ *is relatively compact in* $L^{2N/(N-2)}(\mathbb{R}^N)$ (and $F(u_n)$ is relatively compact in $L^1(\mathbb{R}^N)$) *up to a translation if and only if*

$$I < \{\max(\alpha, \beta)\}^{-(N-2)/N} I^\infty \tag{13}$$

(if $\alpha = \beta = 0$, (13) holds automatically). We also prove that, if we allow the equality, (13) always holds, any minimizing sequence is always relatively compact in $L^{2N/(N-2)}(\mathbb{R}^N)$ up to a translation and a scale change (and we can analyse what happens exactly when the minimizing sequence is not compact up to a translation).

On those two examples, we see that the problem is not invariant under the action of the group of dilations; nevertheless the underlying invariance of $\mathbb{R}^N$ by dilations plays a crucial role in our solution of such problems. But even the full invariance of $\mathbb{R}^N$ by dilations is not needed, only the local part of it plays a role and this explains why our method also applies to problems set in regions different from $\mathbb{R}^N$. These regions may be compact —in which case the group of translations does not induce any more some form of loss of compactness— as is the case in the following typical example of such problems.

EXAMPLE 6. *Yamabe problem on compact manifolds.*

Let $(M, g)$ be some $N$ dimensional compact Riemannian manifold, a general open question is the determination of the class $\mathcal{C}$ of functions on $M$ which can be achieved as scalar curvatures of metrics $\tilde{g}$ (pointwise) conformal to $g$. To solve this problem, one introduces for some positive function $u$ on $M$ a new metric given by: $\tilde{g} = u^{4/(N-2)}g$; we assume $N \geqslant 3$. Then if we denote by $\Delta$ the Laplace-Beltrami operator on $(M, g)$ and by $k$ the scalar curvature, one checks (see [3]) that the scalar curvature of $\tilde{g}$ is given by

$$\tilde{k} = \left\{ -\frac{4(N-1)}{N-2}\Delta u + ku \right\} u^{-(N+2)/(N-2)}.$$

Therefore $K$ —a given function on $M$— belongs to $\mathcal{C}$ if there exists $u$ solution of

$$-\frac{4(N-1)}{N-2}\Delta u + ku = Ku^{(N+2)/(N-2)} \quad \text{in} \quad M, u > 0 \quad \text{on} \quad M. \tag{14}$$

And up to some multiplicative constants such a $u$ exists if we find a minimum of

$$I = \operatorname{Inf}\left[\int_M |\nabla u|^2 + ku^2 dV / u \in H^1(M), \int_M K|u|^{2N/(N-2)}\, dV = 1\right], \tag{15}$$

where $k, K$ are given functions in $C(M)$.

Under natural assumptions on $(-\Delta + k)$ and $K$, we prove below that for any minimizing sequence $(u_n)$ weakly convergent to some $u$ then: either $u$ is a minimum of (15) and $(u_n)$ converges in $H^1$ to $u$, or $u \equiv 0$ and there exists $x_0 \in M$ such that

$$K(x_0) = \max_M K, \qquad |u_n|^{2N/(N-2)} \rightarrow \alpha\delta_{x_0}, \qquad |\nabla u_n|^2 \rightarrow \beta\delta_{x_0}$$

for some $\beta > 0$, and where $\alpha = 1/\max K$ (the above convergence is for the weak topology of bounded measures on $M$). This immediately yields the following result due to $T$. Aubin [3]: if we have

$$I < \left(\max_M K\right)^{-(N-2)/N} I^\infty \tag{16}$$

(where $I^\infty$ is given as in Example 5), then there exists a minimum in (15) (actually we prove that any minimizing sequence is relatively compact in $H^1(M)$ if and only if (16) holds). And we refer to [3] for a sharp discussion of (16).

In fact, we present below more examples: in particular we will present the recent results of $H$. Brézis and J. M. Coron on the Rellich conjecture [8] and on harmonic maps [9] in the light of our systemtic tratment of such problems; and we will explain how it is possible to recover the results of Jacobs [15] on holomorphic functions by our general approach. . . .

At that stage, we would like to explain the main lines of our method: roughly speaking in all the problems listed above, the main difficulty — created by the possible loss of compactness— is due to the fact that *some functional is not weakly continuous* and that *strong compactness is not a priori available*. Then, in the same spirit as in Parts 1 and 2 [20], [21] where we explained what were the two possible forms of "non compactness" due to unbounded domains, we investigate here what happens when passing to the limit on those functionals along weakly convergent sequences. We use basically some general compactness lemma which, rougfily speaking, tells that weakly convergent sequences are converging strongly except possibly at "isolated" points where Dirac masses appear in the densities of the functionals. And this is of course a local property. A typical example is the following.

**Lemma.** *Let $(u_n)_n$ be a bounded sequence in $W^{m,p}(\Omega)^{(*)}$ for some $m > 0$, $p \in [1, N/m[$, and a bounded smooth domain $\Omega$ of $\mathbb{R}^N$. We may assume that $u_n$ converges weakly in $W^{m,p}$ to some $u$ and that $|u_n|^q$ converges weakly in the sense of measures to some $\nu$, where $q = Np(N-mp)^{-1}$. Then there exist $(x_i)_{i\geq 1}$ in $\bar{\Omega}$, $(\nu_i)_{i\geq 1}$ in $[0, \infty[$ such that*

$$\nu = |u|^q + \sum_{i=1}^{\infty} \nu_i \delta_{x_i}, \qquad \sum_{i=1}^{\infty} \nu_i^{p/q} < \infty.$$

Actually we obtain more information on $\nu_i, x_i$ and we show that any such measure $\nu$ can be obtained as the weak limit of $|u_n|^q$ for some bounded sequence $(u_n)$ in $W^{m,p}(\Omega)$ weakly convergent to $u$. In addition such a result is not at all restricted to Sobolev spaces but is based upon the underlying invariance by dilations.

Let us also emphasize that such a phenomenon of "energy" concentrations at points was first observed by J. Sacks and K. Uhlenbeck [32] in the study of harmonic mappings: see also Y. T. Siu and S. T. Yau [34]; S. Sedlacek [33]

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If $p = 1$, we replace $L^1(\Omega)$ by the bounded measures on $\Omega$.

for similar observations in the context of Yang-Mills equations and K. Uhlenbeck [43] for a general presentation. Let us only mention that this lemma is very simple and holds for *arbitrary* sequences $(u_n)$.

With the help of such results, we are able to decide what happens to the functionals if the minimizing sequence $(u_n)$ is not compact. Roughly speaking, $u_n$ breaks in two parts $u$ and $(u_n - u) = \tilde{u}_n$ which "concentrates around the isolated points $x_i$". Then this enables us to conclude that all minimizing sequences are relatively compact if and only if some strict subadditivity inequalities hold, exactly like in [20], [21]. Those inequalities with equalities allowed always hold and they involve, like in [20], [21], a notion of problem at infinity which is essentially obtained by using the dilation invariance of $\mathbb{R}^N$ (or the local invariance for other domains) and concentrating a test function around any fixed point of the domain.

In examples 1, 2, 3, those inequalities hold because of the homogeneity of the problem and the conclusion is reached, while in examples 4, 5, 6, only one of these collections of inequalities does not always hold and this explains the role of the strict inequalities that we mentioned in Examples 5, 6.

Despite the generality of the argument and of the approach, we postpone its general presentation until section III, while in section I we treat examples 1, 4, 5, in section II we treat examples 2, 3. Finally section IV is devoted to various problems in compact regions like example 6.

The results presented here were announced in [25], [26] and combined with those of P.L. Lions [20], [21] are the subject of lectures given at Collège de France for the Cours Peccot.

Finally, it is a pleasure to thank H. Brézis and J.M. Coron for several discussions and their interest in this work and to acknowledge that some of the questions treated here are motivated by E.H. Lieb's work [18].

Let us warn the reader that this work is divided in two parts: Part 1 consists of Section I, while the remainder is contained in Part 2. Notations are identical for both parts.

# I. Sobolev inequalities and extremal functions

## I.1 The main result

Let $m$ be an integer (to simplify) $\geqslant 1$, let $p \in [1, \infty[$. If $N \geqslant 2$, $\varphi \in \mathcal{D}(\mathbb{R}^N)$ we denote by $|D^m\varphi(x)|$ any product norm of all derivatives of order $m$ at the point $x$. The classical Sobolev inequality states that if $p < (N/m)$, $q = Np(N - mp)^{-1}$ then there exists a positive constant $C_0$ such that for all $\varphi \in \mathcal{D}(\mathbb{R}^N)$

$$\left(\int_{\mathbb{R}^N} |\varphi|^q \, dx\right)^{1/q} \leqslant C_0\left(\int_{\mathbb{R}^N} |D^m\varphi|^p \, dx\right)^{1/p}. \tag{3}$$

We then denote by $\mathfrak{D}^{m,p}$ the completion of $\mathfrak{D}(\mathbb{R}^N)$ for the norm

$$||u|| = \left(\int_{\mathbb{R}^N} |D^m\varphi|^p\,dx\right)^{1/p};$$

actually for the special case $p = 1$, we consider directly $\mathfrak{D}^{m,1}$ as the space of $u$ in $L^q(\mathbb{R}^N)$ such that $D^m u \in M_b(\mathbb{R}^N)$. The Sobolev inequality then holds for any $\varphi \in \mathfrak{D}^{m,p}$. In order to decide whether the best constant $C_0$ is achieved, we have to determine whether the following minimization problem has a minimum

$$I = \mathrm{Inf}\left(\int_{\mathbb{R}^N} |D^m u|^p\,dx / u \in \mathfrak{D}^{m,p}, \int_{\mathbb{R}^N} |u|^q\,dx = 1\right); \tag{4}$$

(we will also write $I = I_1$ and $I_\lambda$ will be the value of the infimum of the same problem but with 1 replaced by $\lambda$).

**Theorem I.1.** *Every minimizing sequence $(u_n)_n$ of* (4) *is relatively compact in $\mathfrak{D}^{m,p}$ up to a translation and a dilation i.e. there exist $(y_n)_n$ in $\mathbb{R}^N$, $(\sigma_n)_n$ in $]0,\infty[$ such that the new minimizing sequence $\tilde{u}_n = \sigma_n^{-N/q} u_n(\cdot - y_n/\sigma_n)$ is relatively compact in $\mathfrak{D}^{m,p}$ for $p > 1$ and in $L^q$ for $p = 1$ (in this case $|D^m u_n|^p$ is tight).*

*In particular there exists a minimum of* (4).

In the case when $m = 1$, this result implies easily the

**Corollary I.1.** *If $m = 1$, $p > 1$; any minimum $u$ of* (4) *is given by*

$$u(x) = \sigma^{-N/q} u_1\left(\frac{\cdot - y}{\sigma}\right) \quad \text{where} \quad y \in \mathbb{R}^N, \quad \sigma > 0$$

*and $u_1(x) = \{1 + bx^{p/(p-1)}\}^{(p-N)/p}$, where $b > 0$ depends explicitly on $p, N$ and $I$ below. Moreover we have*

$$I = \pi^{p/2} N\{(p-1)(N-p)^{-1}\}^{-(p-1)/p} \left\{\frac{\Gamma(1+N/2)\Gamma(N)}{\Gamma(N/p)\Gamma(1+N-N/p)}\right\}^{p/N}.$$

*Remark* I.1. The value I and the fact that $u, u_1$ are minima were found by G. Rosen [31], G. Talenti [35], T. Aubin [4] and this was based upon a symmetrization argument and some optimal one-dimensional bounds discovered by G. A. Bliss [7].

*Remark* I.2. Of course Corollary I.1 holds with the norm $|Du|$ chosen to be the usual norm on $\mathbb{R}^N$.

We begin with the proof of Corollary I.1 using Theorem I.1: by Theorem I.1 we know there exists a minimum $u$ of (4). Now by a simple use of Schwarz symmetrization we see that $v = u^*$ is also a minimum and thus $v_\sigma = \sigma^{-N/q} \cdot$ $\cdot\, v(x/\sigma)$ is a minimum for all $\sigma > 0$. In addition, $v_\sigma$ being spherically symmetric, $v$ solves the O.D.E. form of the Euler-Lagrange equation associated with (4) namely

$$\begin{cases} -(p-1)|v_\sigma'|^{p-2}v_\sigma'' - \dfrac{N-1}{r}|v_\sigma'|^{p-1} = Iv_\sigma^{q-1} & \text{for} \quad r > 0 \\ v_\sigma'(0) = 0, \qquad v_\sigma \geqslant 0, \qquad v_\sigma' \leqslant 0, \qquad v_\sigma(0) = \sigma^{-N/q}. \end{cases}$$

And remarking that for any constant $I$, there exists a constant $b$ (which can be computed explicitly) such that the unique solution of this O.D.E. is given by

$$v_\sigma(r) = \sigma^\mu \{\sigma^{p/(p-1)} + br^{p/(p-1)}\}^{(p-N)/p}$$

where $\mu = (N-p)/(p(p-1))$. Computing the $L^q$ norm of $v$ (or $v_\sigma$) one then gets the values of $b, I$. Finally from the fact that both $u$ and $v = u^*$ solve the same Euler equation, we conclude as in A. Alvino, P. L. Lions and G. Trombetti [1] there exists $y \in \mathbb{R}^N$, $u(y + \cdot) = u^*(\cdot)$.

Theorem I.1 is proved in the next section but we would like to explain the general scheme of proof here. First of all we saw in P.L. Lions [20], [21] that, if $(u_n)$ is a minimizing sequence of (4), a crucial quantity is the concentration function of $|u_n|^q$. For technical reasons we have to consider the concentration function of

$$\rho_n = \sum_{j=0}^{m} |D^j u_n|^{q_j}$$

where $q_j = Np(N - (m-j)p)$. We denote by $L_n = \int_{\mathbb{R}^N} \rho_n \, dx$, of course: $L_n \geqslant$ $\geqslant \int_{\mathbb{R}^N} |u_n|^q + |D^m u_n|^p \, dx \geqslant 1 + I$, and $L_n$ being bounded we may assume without loss of generality that $L_n \underset{n}{\to} L \geqslant 1 + I$.

In "locally compact" problems the occurrence of vanishing (see [20], [21] for more details) was easily avoided. On the other hand, here vanishing may occur since the concentration function $Q_n{}^\sigma$ of $u^\sigma(\cdot) = \sigma^{-N/q} u_n(\cdot/\sigma)$ is given by

$$Q_n^\sigma(t) = Q_n(t/\sigma) \quad \text{for} \quad t \geqslant 0,$$

(and playing with $\sigma = \sigma_n \underset{n}{\to} \infty$, one may build minimizing sequences for which vanishing occurs). We will avoid vanishing by choosing $(\sigma_n)_n$ in $[0, -\infty[$ such that

$$Q_n^{\sigma_n}(1) = 1/2 \tag{17}$$

(of course we could replace 1 by any $R \in ]0, \infty[$, 1/2 by any $\theta \in ]0, 1[$) indeed $Q_n^\sigma(1) = Q_n(1/\sigma)$ and $Q_n$ is a non-decreasing continuous function such that

$$Q_n(0) = 0, \qquad 1 \leqslant \lim_{t \uparrow \infty} Q_n(t).$$

In what follows, we will still denote by $u_n$ the new minimizing sequence $u_n^{\sigma_n}$ and by $Q_n$ the associated concentration function, hence we have by (17): $Q_n(1) = 1/2$.

The proof given in the next section is organized as follows: *Step* 1: Using (17) and the concentration-compactness argument of [20] [21], we will show that $\rho_n$ is up to a translation a tight sequence of bounded measures on $\mathbb{R}^N$; *Step* 2: Using again (17), we will check that $u_n$ does not converge weakly to 0; *Step* 3: we conclude by proving that $u_n$ converges weakly to $u$ satisfying: $\int_{\mathbb{R}^N} |u|^q \, dx = 1$. Both Steps 2 and 3 will rely on a Lemma stated in section I.2 and proved in section I.3.

I.2. PROOF. In what follows we will denote by $(u_n)$ all subsequences extracted from the original sequence $(u_n)$.

*Step* 1. In view of (17), if $Q_n(t) \underset{n}{\to} Q(t)$ for some non-decreasing, non-negative function $Q$ on $\mathbb{R}_+$, we have

$$0 < 1/2 = Q(1) \leqslant Q(t) \leqslant C, \qquad \forall 1 \leqslant t < +\infty.$$

Applying the method of [20], [21], in order to prove that *there exists* $(y_n)$ *in* $\mathbb{R}^N$ *such that* $\rho_n(\cdot - y_N)$ *is tight on* $\mathbb{R}^N$, *we* just have to show that *dichotomy cannot occur.* In order to prove this claim, we assume that dichotomy occurs and we will reach a contradiction since: $I_\lambda = \lambda^{p/q} I$, thus

$$I = I_1 < I_\alpha + I_{1-\alpha}, \qquad \forall \alpha \in ]0, 1[$$

(i.e. (S. 2) holds!). Therefore we assume that there exists $\bar{\alpha} \in ]0, L[$ such that for all $\epsilon > 0$

$$\begin{cases} \exists y_n \in \mathbb{R}^N, \quad \exists R_0, R_n > 0, \quad R_n \geq R_0 \quad \text{and} \quad R_n \underset{n}{\to} \infty. \\ |\bar{\alpha} - \int_{y_n + B_{R_0}} \rho_n dx|, \quad \int_{R_0 \leq |x - y_n| \leq R_n} \rho_n \, dx \leqslant \epsilon. \end{cases} \tag{18}$$

Let $\xi, \eta \in C_b^\infty(\mathbb{R}^N)$ satisfying: $0 \leqslant \xi \leqslant 1$, $0 \leqslant \eta \leqslant 1$, $\xi = 1$ if $|x| \leqslant 1$, $\xi = 0$ if $|x| \geqslant 2$, $\eta = 1$ if $|x| \geqslant 1$, $\eta = 0$ if $|x| \leqslant 1/2$. We denote by $\xi_n = \xi((x - y_n)/R_1)$, $\eta_n = \eta((x - y_n)/R_n)$ where $R_1 \geqslant R_0$ is determined below. We then have

$$\left| \int_{\mathbb{R}^N} |D^m u_n|^p \, dx - \int_{\mathbb{R}^N} |D^m(\xi_n u_n)|^p \, dx - \int_{\mathbb{R}^N} |D^m(\eta_n u_n)|^p dx \right| \leqslant C(X_n^p + X_n) + \epsilon$$

provided $n$ is large enough so that: $4R_1 \leqslant R_n$; and where

$$X_n = \left( \int_{\mathbb{R}^N} \sum_{j=0}^{m-1} \{ |D^{m-j} \xi_n|^p + |D^{m-j} \eta_n|^p \} |D^j u_n|^p \, dx \right)^{1/p}.$$

Using Hölder inequalities, we obtain

$$X_n^p \leqslant C \sum_{j=0}^{m-1} \left( \int_{\mathbb{R}^N} |D^{m-j}\xi_n|^{p_j} + |D^{m-j}\eta_n|^{p_j}\, dx \right)^{p/p_j} \cdot$$
$$\cdot \left( \int_{R_0 \le |x-y_n| \le R_n} |D^j u_n|^{q_j}\, dx \right)^{p/q_j}$$

where $p_j/p = (q_j/p)'$. We deduce in view of (18)

$$X_n^p \leqslant C\epsilon \sum_{j=0}^{m-1} \left( \int_{\mathbb{R}^N} |D^{m-j}\xi_n|^{p_j} + |D^{m-j}\eta_n|^{p_j}\, dx \right)^{p/p_j}$$

and

$$\int_{\mathbb{R}^N} |D^{m-j}\xi_n|^{p_j} + |D^{m-j}\eta_n|^{p_j}\, dx = \int_{\mathbb{R}^N} |D^{m-j}\xi|^{p_j} + |D^{m-j}\eta|^{p_j}\, dx$$

since $(m-j)p_j = N$. We obtain finally

$$\left| \int_{\mathbb{R}^N} |D^m u_n|^p\, dx - \int_{\mathbb{R}^N} |D^m u_n^1|^p\, dx - \int_{\mathbb{R}^N} |D^m u_n^2|^p\, dx \right| \leqslant C(\epsilon^{1/p} + \epsilon). \quad (19)$$

where $u_n^1 = \xi_n u_n$, $u_n^2 = \eta_n u_n$.

Without loss of generality we may assume that

$$\int_{\mathbb{R}^N} |u_n^1|^q\, dx \underset{n}{\to} \alpha, \qquad \int_{\mathbb{R}^N} |u_n^2|^q\, dx \underset{n}{\to} \beta$$

and $0 \leqslant \alpha, \beta \leqslant 1$, $|\beta - (1-\alpha)| \leqslant \epsilon$.

We claim that for all $\epsilon$ small enough $|D^m u_n^i|_{L^p}$ remains for $i = 1, 2$ bounded away from 0: indeed the above proof shows that

$$\left| \int \sum_{j=0}^{m} |D^j u_n^1|^{q_j}\, dx - \bar{\alpha} \right| \leqslant C(\epsilon^{1/p} + \epsilon)$$

$$\left| \int \sum_{j=0}^{m} |D^j u_n^2|^{q_j}\, dx - (L - \bar{\alpha}) \right| \leqslant C(\epsilon^{1/p} + \epsilon)$$

and $\bar{\alpha} \in ]0, L[$. Therefore let us denote by $\gamma > 0$ some constant such that for all $\epsilon$ small and for all $n$: $\gamma \leqslant |D^m u_n^i|_{L^p}^p$. Next, if for some sequence $\epsilon_k \underset{k}{\to} 0$, the constant $\alpha_k = \alpha(\epsilon_k)$ either goes to 0 or to 1, we deduce from (19)

$$I \geqslant I + \gamma - \delta(\epsilon_k)$$

where $\delta(t) \to 0$ as $t \to 0_+$; and this is not possible. On the other hand if $\alpha_k \underset{k}{\to} \alpha \in ]0, 1[$, $\beta_k \underset{k}{\to} 1 - \alpha$ and we obtain from (19): $I \geqslant I_\alpha + I_{1-\alpha}$ and again this is not possible.

*In conclusion* we have proved that there exists $(y_n)$ in $\mathbb{R}^N$ such that: $\forall \epsilon > 0$, $\exists R \in ]0, \infty[$

$$\int_{|x-y_n| \ge R} \sum_{j=0}^{m} |D^j u_n|^{q_j}\, dx \leqslant \epsilon. \quad (20)$$

We still denote by $(u_n)_n$ the new minimizing sequence $(\tilde{u}_n)_n$ obtained by

$$\tilde{u}_n(x) = u_n(x + y_n), \qquad \forall x \in \mathbb{R}^N.$$

Without loss of generality we may assume that $u_n$ converges weakly in $\mathfrak{D}^{m,p}$ and a.e. to some $u \in \mathfrak{D}^{m,p}$; and that $D^j u_n$ converges weakly and a.e. to $D^j u$ in $L^{q_j}(\mathbb{R}^N)$.

The next result —that we will call below the *second concentration compactness lemma*— is the crucial tool for the next two steps of the proof of Theorem I.1. Before stating this result let us observe that if $(u_n)_n \subset W^{m,p}(\Omega)$ for some smooth bounded region $\Omega$ of $\mathbb{R}^N$, by standard extension theorems we may assume without loss of generality that $(u_n)_n \subset W^{m,p}(\mathbb{R}^N)$ and $|u_n|^q$ is tight (even with some uniform compact support!).

**Lemma I.1.** *Let $(u_n)_n$ be a bounded sequence in $\mathfrak{D}^{m,p}$ converging weakly to some $u$ and such that $|D^m u_n|^p$ converges weakly to $\mu$ and $|u_n|^q$ converges tightly to $\nu$ where $\mu$, $\nu$ are bounded nonnegative measures on $\mathbb{R}^N$. Then we have*:

(i) *There exist some at most countable set $J$ and two families $(x_j)_{j\in J}$ of distinct points in $\mathbb{R}^N$, $(\nu_j)_{j\in J}$ in $]0, \infty[$ such that*

$$\nu = |u|^q + \sum_{j\in J} \nu_j \delta_{x_j}. \tag{21}$$

(ii) *In addition we have*

$$\mu \geqslant |D^m u|^p + \sum_{j\in J} \mu_j \delta_{x_j} \tag{22}$$

*for some $\mu_j > 0$ satisfying*

$$\nu_j^{p/q} \leqslant \mu_j / I, \quad \textit{for all } j \tag{23}$$

*hence*

$$\sum_{j\in J} \nu_j^{p/q} < \infty.$$

(iii) *If $v \in \mathfrak{D}^{m,p}(\mathbb{R}^N)$ and $|D^m(u_n + v)|^p$ converges weakly to some measure $\tilde{\mu}$, then $\tilde{\mu} - \mu \in L^1(\mathbb{R}^N)$; and therefore*

$$\tilde{\mu} \geqslant |D^m(u + v)|^p + \sum_{j\in J} \mu_j \delta_{x_j}.$$

(iv) *If $u \equiv 0$ and: $\left(\int d\mu\right) \leqslant I\left(\int d\nu\right)^{p/q}$; then $J$ is a singleton and: $\nu = \gamma\delta_{x_0} = \mu(I\gamma^{p/q})^{-1}$ for some $\gamma > 0$, $x_0 \in \mathbb{R}^N$.*

The proof of this lemma is given in the next section.

*Remark* I.3. We claim that if $u \in \mathfrak{D}^{m,p}(\mathbb{R}^N)$, $J$ is an at most countable set, $(x_j)_{j\in J}$ are distinct points in $\mathbb{R}^N$ and $(\nu_j)_{j\in J}$ are positive numbers such that $\sum_{j\in J} \nu_j^{p/q} < \infty$, then the measure $\nu = |u|^q + \sum_{j\in J} \nu_j \delta_{x_j}$ is the tight limit of a sequence $|u_n|^q$ where $u_n$ converges in $\mathfrak{D}^{m,p}$ to $u$. Hence, the above result completely characterizes the limits of $|u_n|^q$ for weakly convergent sequences of $\mathfrak{D}^{m,p}$. Of course $u_n$ converges in $L^q$ to $u$ if and only if $\nu = |u|^q$; therefore the loss of compactness (for the Sobolev limit exponent) occurs at a countable number of points $x_j$ (with weights $\nu_j$ such $\sum \nu_j^{p/q} < \infty$ and $p/q < 1$).

To prove the above claim, we consider $\varphi \in \mathfrak{D}(\mathbb{R}^N)$ with $\int |\varphi|^q \, dx = 1$ (say) — observe that we can take $\int |D^m\varphi|^p \, dx$ as close to $I$ as we wish — . Then for any $x_0 \in \mathbb{R}^N$, $\varphi_n = \varphi_n = n^{N/q}\varphi(\cdot - x_0/n)$ satisfies

$$\begin{cases} \int |D^m\varphi_n|^p \, dx = \int |D^m\varphi|^p \, dx, & \int |\varphi_n|^q \, dx = 1 \\ |\varphi_n|^q \to \delta_{x_0}, & \varphi_n \underset{n}{\rightharpoonup} 0 \quad \text{in} \quad \mathfrak{D}^{m,p} \quad \text{weakly}. \end{cases}$$

Next for any finite subfamily $J'$ of $J$, we consider for $n \geqslant n_0(J')$: $\psi_n = \sum_{j\in J'} \nu_j^{1/q}\varphi_n^{x_j}$, Supp $\varphi_n^{x_j}$ are disjoint for $j \in J'$. Clearly we have

$$\begin{cases} \int |D^m\psi_n|^p \, dx = \Big(\sum_{j\in J'} \nu_j^{p/q}\Big)\int |D^m\varphi|^p \, dx \leqslant \Big(\sum_{j\in J} \nu_j^{p/q}\Big)\int |D^m\varphi|^p \, dx \\ \int |\psi_n|^q \, dx = \sum_{j\in J'} \nu_j, \qquad |\psi_n|^q \to \sum_{j\in J'} \nu_j\delta_{x_j}, \qquad \psi_n \underset{n}{\rightharpoonup} 0 \text{ in } \mathfrak{D}^{m,p} \text{ weakly}. \end{cases}$$

Increasing $J'$ to $J$, we obtain by a diagonal procedure a sequence $\tilde{\psi}_n$ such that

$$\begin{cases} \int |D^m\tilde{\psi}_n|^p \, dx \leqslant \Big(\sum_{j\in J} \nu_j^{p/q}\Big)\int |D^m\varphi|^p \, dx \\ \int |\tilde{\psi}_n|^q \, dx \underset{n}{\to} \sum_{j\in J} \nu_j, \qquad \tilde{\psi}_n \underset{n}{\rightharpoonup} 0 \quad \text{in} \quad \mathfrak{D}^{m,p} \quad \text{weakly} \\ |\tilde{\psi}_n|^q \to \sum_{j\in J} \nu_j\delta_{x_j} \quad \text{tightly}. \end{cases}$$

We finally set: $u_n = u + \tilde{\psi}_n$, and one easily checks that $u_n$ has the required properties. Actually one even checks that

$$|D^m u_n|^p \to |D^m u|^p + \Big(\int |D^m\varphi|^p \, dx\Big) \sum_{j\in J} \nu_j^{p/q}\delta_{x_j}.$$

We go back now to the proof of Theorem I.1:

*Step* 2. $u$, the weak limit of the minimizing sequence $u_n$, is not identically 0.

Indeed, in view of (20), we may apply Lemma I.1 (extracting if necessary some subsequences) and we know by (20)

$$\int_{\mathbb{R}^N} d\mu = I, \qquad \int_{\mathbb{R}^N} d\nu = 1. \tag{24}$$

Now if $u \equiv 0$, we may apply part iv) of lemma I.1 and we deduce: $\nu = \frac{1}{I}\mu = \delta_{x_0}$, for some $x_0 \in \mathbb{R}^N$.

On the other hand

$$\tfrac{1}{2} = Q_n(1) \geqslant \int_{B(x_0,1)} |u_n|^q \, dx \underset{n}{\to} 1;$$

this contradiction shows that $u \not\equiv 0$.

*Step* 3. $u_n$ converges strongly to $u$.

Let us denote by $\alpha = \int_{\mathbb{R}^N} |u|^q \, dx$: by step 2 we know that $\alpha \in ]0, 1]$ and we have to prove that $\alpha = 1$. Suppose that $\alpha \neq 1$, then applying Lemma I.1, we see

$$\begin{cases} \alpha = \int_{\mathbb{R}^N} |u|^q \, dx, & \sum_{j \in J} \nu_j = 1 - \alpha \\ \mu_j \geqslant I \nu_j^{p/q}, & \int_{\mathbb{R}^N} |D^m u|^p \, dx \leqslant I - \sum_{j \in J} \mu_j. \end{cases}$$

Hence, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} |D^m u|^p \, dx &\leqslant I - \sum_{j \in J} \mu_j \\ &\leqslant I\Big(1 - \sum_{j \in J} \nu_j^{p/q}\Big) \\ &< I\Big(1 - \Big(\sum_{j \in J} \nu_j\Big)\Big)^{p/q} = I\alpha^{p/q} \end{aligned}$$

while $\int_{\mathbb{R}^N} |D^m u|^p \, dx \geqslant I_\alpha = I_\alpha = I\alpha^{p/q}$. The contradiction shows that $\alpha = 1$ and we conclude easily.

*Remark* I.4. We may rewrite the above argument in a way which clearly shows the role of sub-additivity inequalities like (S.2). Indeed

$$I = I_1 \geqslant \int_{\mathbb{R}^N} |D^m u|^p \, dx + \sum_{j \in J} \mu_j \geqslant I_\alpha + I \sum_{j \in J} \nu_j^{p/q} \geqslant I_\alpha + \sum_{j \in J} I_{\nu_j} > I_1 \qquad (!),$$

since we know that $I_\lambda$ is strictly sub-additive and $\alpha + \sum_{j \in J} \nu_j = 1$.

### I.3 The second concentration-compactness lemma

We now prove Lemma I.1: we *first treat the case when* $u \equiv 0$. The goal is to obtain some reversed Hölder inequality between $\nu$ and $\mu$ which will give the various informations contained in Lemma I.1 via Lemma I.2 below.

Let $\varphi \in \mathcal{D}(\mathbb{R}^N)$, by Sobolev inequalities we have

$$\Big(\int_{\mathbb{R}^N} |\varphi|^q |u_n|^q \, dx\Big)^{1/q} I^{1/p} \leqslant \Big(\int_{\mathbb{R}^N} |D^m(\varphi u_n)|^p \, dx\Big)^{1/p}. \qquad (25)$$

The left-hand side member of (25) goes to $\big(\int_{\mathbb{R}^N} |\varphi|^q \, d\nu\big)^{1/q} I^{1/p}$ as $n$ goes to

$\infty$. Now the right-hand side member is estimated as follows

$$\left| \left( \int_{\mathbb{R}^N} |D^m(\varphi u_n)|^p \, dx \right)^{1/p} - \left( \int_{\mathbb{R}^N} |\varphi|^p |D^m u_n|^p \, dx \right)^{1/p} \right| \leqslant$$
$$\leqslant C \sum_{j=0}^{m-1} \left( \int_{\mathbb{R}^N} |D^{m-j}\varphi|^p |D^j u_n|^p \, dx \right)^{1/p}.$$

And using the fact that $\varphi$ has compact support and the Rellich theorem we see that this bound goes to 0 as $n$ goes to $\infty$. Therefore, passing to the limit in (25), we obtain for all $\varphi \in \mathcal{D}(\mathbb{R}^N)$

$$\left( \int_{\mathbb{R}^N} |\varphi|^q \, d\nu \right)^{1/q} \leqslant I^{-1/p} \left( \int_{\mathbb{R}^N} |\varphi|^p \, d\mu \right)^{1/q}. \tag{26}$$

And lemma I.1 is proved in the case $u \equiv 0$, by the application of

**Lemma 1.2.** *Let $\mu$, $\nu$ be two bounded nonnegative measures on $\mathbb{R}^N$ satisfying for some constant $C_0 \geqslant 0$*

$$\left( \int_{\mathbb{R}^N} |\varphi|^q \, d\nu \right)^{1/q} \leqslant C_0 \left( \int_{\mathbb{R}^N} |\varphi|^p \, d\mu \right)^{1/p}, \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}^N) \tag{26'}$$

*where $1 \leqslant p < q \leqslant +\infty$. Then, there exist an at most countable set $J$, families $(x_j)_{j \in J}$ of distinct points in $\mathbb{R}^N$, $(\nu_j)_{j \in J}$ in $]0, \infty[$ such that*

$$\nu = \sum_{j \in J} \nu_j \delta_{x_j}, \qquad \mu \geqslant C_0^{-p} \sum_{j \in J} \nu_j^{p/q} \delta_{x_j}.$$

*Thus, in particular*

$$\sum_{j \in J} \nu_j^{p/q} < \infty.$$

*If in addition: $\nu(\mathbb{R}^N)^{1/q} \geqslant C_0 \mu(\mathbb{R}^N)^{1/p}$, $J$ reduces to a single point and $\nu = \gamma \delta_{x_0} = \gamma^{-p/q} C_0^p \mu$, for some $x_0 \in \mathbb{R}^N$ and for some $\gamma \geqslant 0$.*

Lemma I.2 is proved below; we first conclude the proof of Lemma I.1. We now consider the general case of a weak limit $u$ not necessarily 0. Of course (25) still holds and, if we denote by $v_n = u_n - u$, Brézis-Lieb lemma [10] yields for all $\varphi \in \mathcal{D}(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} |\varphi|^q |u_n|^q \, dx - \int_{\mathbb{R}^N} |\varphi|^q |v_n|^q \, dx \to \int_{\mathbb{R}^N} |\varphi|^q |u|^q \, dx$$

But, clearly, $v_n$ is bounded in $\mathcal{D}^{m,p}$ and $|v_n|^q$ is tight; therefore applying what we proved above we obtain the representation (21) of $\nu$. Next, passing to the limit in (25) and using as before Rellich theorem we find for all $\varphi \in \mathcal{D}(\mathbb{R}^N)$

$$\left( \int_{\mathbb{R}^N} |\varphi|^q \, d\nu \right)^{1/q} I^{1/p} \leqslant \left( \int_{\mathbb{R}^N} |\varphi|^p \, d\mu \right)^{1/p} + C \sum_{i=0}^{m-1} \left( \int_{\mathbb{R}^N} |D^{m-i}\varphi|^p |D^i u|^p \, dx \right)^{1/p}.$$

If $\varphi$ satisfies: $0 \leqslant \varphi \leqslant 1$, $\varphi(0) = 1$, $\operatorname{Supp}\varphi = B(0,1)$, $\varphi \in \mathcal{D}(\mathbb{R}^N)$; we apply the above inequality with $\varphi((x - x_j)/\epsilon)$ for $\epsilon > 0$ and where $j$ is fixed in $J$. We obtain

$$\nu_j^{1/q} I^{1/p} \leqslant \mu(B(x_j, \epsilon))^{1/p} + \\ + C \sum_{i=1}^{m-1} \left( \int_{B(x_j,\epsilon)} \epsilon^{-p(m-i)} \left| D^{m-i} \varphi\left(\frac{x - x_j}{\epsilon}\right) \right|^p |D^i u|^p dx \right)^{1/p}.$$

Now we may estimate each term of the sum by Hölder inequalities recalling that $D^i u \in L^{q_i}(\mathbb{R}^N)$ (by Sobolev inequalities)

$$\epsilon^{-p(m-i)} \int_{B(x_j,\epsilon)} \left| D^{m-i} \varphi\left(\frac{x - x_j}{\epsilon}\right) \right|^p |D^i u|^p \, dx \leqslant$$

$$\leqslant \left( \int_{B(x_j,\epsilon)} |D^i u|^{q_i} |D^i u|^{q_i} \, dx \right)^{p/q_i} \epsilon^{-p(m-i)} \left( \int_{\mathbb{R}^N} \left| D^{m-i} \varphi\left(\frac{x}{\epsilon}\right) \right|^{p_i} dx \right)^{(q_i - p)/q_i}$$

where $p_i = q_i p (q_i - p)^{-1}$, $(q_i - p)/q_i = (m - i)p/N$. Hence, we have

$$\nu_j^{1/q} I^{1/p} \leqslant \mu(B(x_j, \epsilon))^{1/p} + C \sum_{i=1}^{m-1} \left( \int_{B(x_j,\epsilon)} |D^i u|^{q_i} \, dx \right)^{p/q_i}.$$

This implies that $\mu(\{x_j\}) > 0$ and

$$\mu \geqslant \nu_j^{p/q} I \delta_{x_j}, \qquad \forall j \in J$$

and thus

$$\mu \geqslant \sum_{j \in J} I \nu_j^{p/q} \delta_{x_j} =: \mu_1.$$

Since by weak convergence we also have: $\mu \geqslant |D^m u|^p$ and since $|D^m u|^p$ and $\mu_1$ are orthogonal, (22)-(23) are proved.

Finally to prove part iii) of Lemma I.1 we just observe that for all $\varphi \in C_b(\mathbb{R}^N)$, $\varphi \geqslant 0$

$$\left( \int_{\mathbb{R}^N} \varphi |D^m(u_n + v)|^p \, dx \right)^{1/p} - \left( \int_{\mathbb{R}^N} \varphi |D^m u_n|^p \, dx \right)^{1/p} \Big| \leqslant \\ \left( \int_{\mathbb{R}^N} \varphi |D^m v|^p \, dx \right)^{1/p}.$$

Passing to the limit in $n$, we find

$$\left| \left( \int_{\mathbb{R}^N} \varphi \, d\tilde{\mu} \right)^{1/p} - \left( \int_{\mathbb{R}^N} \varphi \, d\mu \right)^{1/p} \right| \leqslant \left( \int_{\mathbb{R}^N} \varphi h \, dx \right)^{1/p}$$

where $h \in L_+^1(\mathbb{R}^N)$. And this shows that the singular parts of $\tilde{\mu}$ and $\mu$ are the same; and we conclude.

*We next turn to the proof of Lemma I.2.* We first remark that (26′) holds by density for all $\varphi$ bounded measurable. Therefore we see that in particular

$\nu$ is absolutely continuous with respect to $\mu$ i.e.: $\nu = f\mu$ where $f \in L^1_+(\mu)$. Since

$$\nu(A) \leqslant C_0 \mu(A)^{q/p}, \qquad \forall A \quad \text{Borel} \quad \subset \mathbb{R}^N$$

we have in fact $f \in L^\infty_+(\mu)$. Next, if $\mu = g\nu + \sigma$ where $g \in L^1_+(\nu)$, $\sigma$ is a bounded nonnegative measure such that if $K = \operatorname{Supp} \varphi$, $\nu(K) = 0$; considering $\tilde{\mu} = 1_K \mu$ and taking $\varphi$ in (26′) of the form $1_K \psi$ where $\psi$ is bounded measurable, we see that without loss of generality we may assume that $\sigma \equiv 0$. We next denote by $\nu_k = g^\alpha 1_{(g \le k)} \nu$, where $\alpha = q/(q-p)$. We are going to prove that $\nu_k$ is given by a finite number of Dirac masses; this will prove that $\nu 1_{(g \le k)}$ is a finite number of Dirac masses for all $k < \infty$ and letting $k \to \infty$, the claim on $\nu$ will be proved (since $\nu(\{g = +\infty\}) = 0$).

To prove our claim on $\nu_k$, we take in (26′) $\varphi$ of the form

$$\varphi = g^{1/(q-p)} 1_{(g \le k)} \psi$$

where $\psi$ is an arbitrary bounded measurable function. We thus obtain for all $\psi$

$$\left( \int_{\mathbb{R}^N} |\psi|^q \, d\nu_k \right)^{1/q} \leqslant C_0 \left( \int_{\mathbb{R}^N} |\psi|^p \, d\nu_k \right)^{1/p}$$

(indeed: $g^{p/(q-p)} 1_{(g \le k)} \mu = g^{q/(a-p)} 1_{(g \le k)} \nu$).

This reversed Hölder inequality now yields our claim on $\nu_k$: a short proof of this standard statement is the following. For any Borel set $A$ the above inequality gives

$$\nu_k(A)^{1/q} \leqslant C_0 \nu_k(A)^{1/ap}.$$

Therefore either $\nu_k(A) = 0$, or $\nu_k(A) \geqslant \delta = C_0^{-\, p/(q-p)} > 0$. Since for each $x \in \mathbb{R}^N$, $\nu_k(\{x\}) = \lim_{\epsilon \downarrow 0} \downarrow \nu_k(B(x, \epsilon))$, we have for all $x \in \mathbb{R}^N$

$$\text{either} \quad \nu_k(\{x\}) \geqslant \delta, \quad \text{or} \quad \exists \epsilon > 0, \qquad \nu_k(B(x, \epsilon)) = 0.$$

Thus there exists a finite number of distinct points $x_j$ in $\mathbb{R}^N$ such that

$$\begin{cases} \nu_k(\{x_j\}) \geqslant \delta & \forall i \leqslant j \leqslant m \\ \nu_k(B(x, \epsilon)) = 0 & \text{for some} \quad \epsilon = \epsilon(x) > 0, \qquad \forall x \notin \{x_j / 1 \leqslant j \leqslant m\}. \end{cases}$$

Let $K$ be any compact set in $O = \{x / x \neq x_j \text{ for all } 1 \leqslant j \leqslant m\}$, we have by a finite covering of $K$ by balls $B(x, \epsilon(x))$: $\nu_k(K) = 0$, therefore $\nu_k(O) = 0$; and our claim is proved.

At this point, we have proved the representation of $\nu$ and by (26′) we have

$$\mu(\{x_j\}) \geqslant C_0^{-p} \nu(\{x_j\})^{p/q}$$

Finally if $\nu(\mathbb{R}^N)^{1/q} \geqslant C_0 \mu(\mathbb{R}^N)^{1/p}$, taking $\varphi \equiv 1$ in (26′) we see that $\nu(\mathbb{R}^N)^{1/q} =$ $= C_0 \mu(\mathbb{R}^N)^{1/p}$; and using Hölder inequality we find for all $\varphi \in \mathcal{D}(\mathbb{R}^N)$

$$\left( \int_{\mathbb{R}^N} |\varphi|^q \, d\nu \right)^{1/q} \leqslant C_0 \mu(\mathbb{R}^N)^\theta \left( \int_{\mathbb{R}^N} |\varphi|^q d\mu \right)^{1/p}$$

where $\theta = (q-p)/(pq)$. Observing that

$$\nu(\mathbb{R}^N) = C_0^q \mu(\mathbb{R}^N)^{q/p} = \{C_0\mu(\mathbb{R}^N)^\theta\}^q \mu(\mathbb{R}^N)$$

we deduce from the above inequality: $\nu = \{C_0\mu(\mathbb{R}^N)^\theta\}^q\mu$. Therefore we have for all $\varphi \in \mathfrak{D}(\mathbb{R}^N)$

$$\left(\int_{\mathbb{R}^N} |\varphi|^q \, d\nu\right)^{1/q} \leqslant \nu(\mathbb{R}^N)^{-\theta}\left(\int_{\mathbb{R}^N} |\varphi|^p \, d\nu\right)^{1/q}.$$

And the above proof already shows that: $\nu = \sum_{i=1}^m \nu_i \delta_{x_i}$, where $m \geqslant 1$, $(x_i)_i$ are $m$ distinct points in $\mathbb{R}^N$ and $\nu_i > 0$.

We choose $\varphi \in \mathfrak{D}(\mathbb{R}^N)$ such that $\varphi(x_i) = \alpha_i > 0$; thus we find for all $\alpha_i > 0$

$$\left(\sum_{i=1}^m \alpha_i^q \nu_i\right)^{1/q}\left(\sum_{i=1}^m \nu_i\right)^{(q-p)/pq} \leqslant \left(\sum_{i=1}^m \alpha_i^p \nu_i\right)^{1/p}.$$

And this is possible if and only if $m = 1$.

*Remark* I.5. Lemma I.2 is of course valid in an arbitrary measure space and the various conclusions hold provided one replaces points in $\mathbb{R}^N$ by atoms...

## I.4 Variants

We briefly mention here a few related problems and inequalities which can be treated in a similar way. In particular in all the cases mentioned below *all minimizing sequences are relatively compact up to a translation and a scale change*; and *the analogue of Lemma* I.1 *holds in each case*. The proofs being very similar to the previous ones, we skip them.

i) *Other norms in* $\mathfrak{D}^{m,p}(\mathbb{R}^N)$.
Of course we may replace the norm on $\mathfrak{D}^{m,p}$ by the following one

$$\begin{aligned} \mathcal{E}(u) &= |(-\Delta)^{m/2} u|^p_{L^p(\mathbb{R}^N)} && \text{if } m \text{ is even} \\ &= |\nabla(-\Delta)^{(m-1)/2} u|^p_{L^p(\mathbb{R}^N)} && \text{if } m \text{ is odd.} \end{aligned}$$

We could in fact take any norm in $\mathfrak{D}^{m,p}$ but the particular one chosen above is of interest since some additional information on extremal functions is available (see Corollary I.2 below) and since we have by easy integrations by parts

$$\mathcal{E}(u) = \sum_{|\alpha| = m} |D^\alpha u|^2_{L^2}. \tag{27}$$

if $p = 2$, $u \in \mathfrak{D}^{m,2}(\mathbb{R}^N)$ —and we recover the previous norm! The existence of extremal functions is determined by the following minimization problem

$$\operatorname{Inf}\{\mathcal{E}(u)/u \in \mathfrak{D}^{m,p}(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^q \, dx = 1\}. \tag{28}$$

**Corollary I.2.** *Let $(u_n)_n$ be a minimizing sequence of* (28). *There exist $(y_n)_n$ in $\mathbb{R}^N$, $(\sigma_n)_n$ in $]0, \infty[$ such that the new minimizing sequence $\tilde{u}_n = \sigma_n^{-N/q} u_n((\cdot - y_n)/\sigma_n)$ is relatively compact in $\mathcal{D}^{m,p}$ (for $p > 1$, and in $L^q$ for $p = 1$). In particular the minimum is achieved. And if $p > 1$, for any minimum $u$ of* (28), *there exists $y \in \mathbb{R}^N$ such that $\tilde{u} = u(\cdot - y)$ satisfies*

$$\begin{cases} (-\Delta)^\alpha \tilde{u} \text{ is spherically symmetric, nonnegative and decreasing} \\ \text{in } |x| \text{ for all } \alpha \in \mathbb{N} \text{ such that } \alpha \leqslant m/2. \end{cases}$$

The statement about the geometry of the minima is obtained as follows: let $u$ be a minimum of (28). We set

$$f = (-\Delta)^{m/2} u \quad \text{if} \quad m \quad \text{is even,} = (-\Delta)^{(m-1)/2} u \quad \text{if} \quad m \quad \text{is odd.}$$

If $m$ is even, $f \in L^p(\mathbb{R}^N)$ and if $m$ is odd $f \in L^{\bar{p}}(\mathbb{R}^N)$ with $\bar{p} = Np/(N-p)$ (and $\nabla f \in L^p(\mathbb{R}^N)$). If we denote by $\varphi^*$ the Schwarz symmetrization of $\varphi$, we introduce $v$ solution of

$$(-\Delta)^\alpha v = f^* \quad \text{in} \quad \mathbb{R}^N, \qquad v \in L^q(\mathbb{R}^N),$$

with $\alpha = m/2$ if $m$ is even, $\alpha = (m-1)/2$ if $m$ is odd.

Smoothing and truncating $f$, we see that we may apply ($\alpha$ times) Talenti comparizon theorem on linear elliptic problems to deduce

$$u^* \leqslant v \qquad \text{a.e. in } \mathbb{R}^N.$$

In particular we have

$$\int_{\mathbb{R}^N} |u|^q dx = \int_{\mathbb{R}^N} |u^*|^q \, dx \leqslant \int_{\mathbb{R}^N} |v|^q \, dx$$

and

$$\mathcal{E}(v) = \int_{\mathbb{R}^N} |f^*|^p \, dx = \int_{\mathbb{R}^N} |f|^p \, dx = \mathcal{E}(u) \quad \text{if} \quad m \quad \text{is even}$$

while

$$\mathcal{E}(v) = \int_{\mathbb{R}^N} |\nabla f^*|^p \, dx \leqslant \int_{\mathbb{R}^N} |\nabla f|^p \, dx = \mathcal{E}(v) \quad \text{if} \quad m \quad \text{is odd.}$$

Hence $v$ is also a minimum of (28) and all inequalities are equalities. We then conclude using the results and methods of A. Alvino, P. L. Lions and G. Trombetti [1] (in particular the method with Green functions).

ii) *Systems.*

Let $k \geqslant 1$, we want to consider here systems analogues of Sobolev inequalities (our motivation comes from the problem of nonlinear field equations —see section I.6 below). Let $u \in (\mathcal{D}^{m,p}(\mathbb{R}^N))^k$; $u = (u^1, \ldots u^k)$ with $u^i \in \mathcal{D}^{m,p}(\mathbb{R}^N)$.

We denote by

$$\mathcal{E}(u) = \sum_{i=1}^{k} \int_{\mathbb{R}^N} |D^m u^i|^p \, dx.$$

Let $F \in C(\mathbb{R}^k)$ satisfy: $F(\zeta) > x > 0$ if $\zeta \neq 0$, $F$ is homogeneous of degree $q$ on $\mathbb{R}^k$. We deduce from Sobolev inequalities

$$\left(\int_{\mathbb{R}^N} F(u) \, dx\right)^{1/q} \leqslant C_0 \mathcal{E}(u)^{1/q} \tag{29}$$

And the existence of extremal functions is determined by the following minimization problem

$$\operatorname{Inf}\{\mathcal{E}(u)/u \in (\mathcal{D}^{m,p}(\mathbb{R}^N))^k, \int_{\mathbb{R}^N} F(u) \, dx = 1\}. \tag{30}$$

Exactly as before, any minimizing sequence is relatively compact up to a translation and a scale change, and there exists a minimum of (30). In addition the analogue of Lemma I.1 holds with $|u_n|^q$ replaced by $F(u_n)$. The only technical point we have to explain is why Brézis-Lieb lemma [10] still applies; and this in an application of the following remark:

**Lemma I.3.** *The nonlinearity $F$ satisfies for all $a, b \in \mathbb{R}^k$*

$$|F(a + b) - F(a)| \leqslant \epsilon |a|^q + C_\epsilon(|b|^q + 1) \tag{31}$$

*for all $\epsilon > 0$.*

Proof. Recall that $F$ satisfies: $|F(t)| \leqslant C(1 + |t|^q)$ on $\mathbb{R}^k$. Hence to prove (31), we may assume without loss of generality that $|a| \geqslant 1$, $|a + b| \geqslant 1$ since if, for example, $|a + b| \leqslant 1$, we have

$$\begin{aligned} |F(a + b) - F(a)| &\leqslant C + C(1 + |a|^q) \\ &\leqslant C + C(1 + |b|^q) \end{aligned}$$

and (31) holds. Furthermore we may also assume that, for any $\delta > 0$ fixed: $|b| \leqslant \delta |a|$. Indeed if this is not the case we have

$$\begin{aligned} |F(a + b) - F(a)| &\leqslant C(1 + |a + b|^q) + C(1 + |a|^q) \\ &\leqslant C + C|a|^q + C|b|^q \\ &\leqslant C + (C\delta^{-q} + C)|b|^q. \end{aligned}$$

But if $|b| \leqslant \delta |a|$, we deduce

$$||a + b| - |a|| \leqslant \delta |a|, \frac{|a| - |a + b|}{|a + b|} \leqslant \frac{\delta}{1 - \delta}, \frac{|b|}{|a + b|} \leqslant \frac{\delta}{1 - \delta}.$$

And we obtain

$$|F(a+b)-F(a)| = \left| |a+b|^q F\left(\frac{a+b}{|a+b|}\right) - |a|^q F\left(\frac{a}{|a|}\right) \right|$$

$$\left| \frac{a+b}{|a+b|} - \frac{a}{|a|} \right| \leqslant \frac{|b|}{|a+b|} + \frac{||a|-|a+b||}{|a+b|} \leqslant \frac{2\delta}{1-\delta}.$$

Hence choosing $\delta$ small enough, we find for any fixed $\epsilon > 0$

$$\left| F\left(\frac{a+b}{|a+b|}\right) - F\left(\frac{a}{|a|}\right) \right| \leqslant \frac{\epsilon}{2},$$

$$\begin{aligned} |F(a+b)-F(a)| &\leqslant \left| F\left(\frac{a}{|a|}\right) \right| \left| |a+b|^q - |a|^q \right| + \frac{\epsilon}{2}|a|^q \\ &\leqslant C\left| |a+b|^q - |a|^q \right| + \frac{\epsilon}{2}|a|^q \\ &\leqslant \epsilon|a|^q + C_\epsilon |b|^q. \end{aligned}$$

We next turn to some other extension to systems (again motivated by nonlinear field equations): let $k > 1$, let $q_i \in ]0, q[$ for $1 \leqslant i \leqslant k$ be such that: $\sum_{i=1}^{k} q_i = q$. We denote by $\theta_i = q_i/q$. Clearly Hölder and Sobolev inequalities yield that for any $u = (u^1, \dots, u^k) \in (\mathcal{D}^{m,p}(\mathbb{R}^N))^k$, we have

$$\left(\int_{\mathbb{R}^N} |u^1|^{q_1} \dots |u^k|^{q_k}\, dx\right)^{1/q} \leqslant C_0 \mathcal{E}(u)^{1/p} \tag{32}$$

where

$$\mathcal{E}(u) = \prod_{i=1}^{k} \left(\int_{\mathbb{R}^N} |D^m u^i|^p\, dx\right)^{\theta_i}.$$

In view of the homogeneity of the problem in each $u_i$, the existence of extremal functions is equivalent to the existence of a minimum of

$$I = \operatorname{Inf}\Big\{ -\int_{\mathbb{R}^N} |u^1|^{q_1} \dots |u^k|^{q_k}\, dx / u \in (\mathcal{D}^{m,p})^k, \quad \forall 1 \leqslant i \leqslant k, \int_{\mathbb{R}^N} |D^m u^i|^p\, dx = 1 \Big\}. \tag{33}$$

We denote by $I(\lambda_1, \dots, \lambda_k)$ (for $\lambda_i > 0$) the value of the infimum where the constraints of norm 1 are replaced by

$$\int_{\mathbb{R}^N} |D^m u_i|^p\, dx = \lambda_i.$$

Clearly

$$I(\lambda_1, \dots, \lambda_k) = \left(\prod_{i=1}^{k} \lambda_i^{\theta_i}\right)^{q/p} I < 0.$$

Therefore we have

$$I(1, \ldots, 1) < I(\lambda_1, \ldots, \lambda_k) + I(1 - \lambda_1, \ldots, 1 - \lambda_k)$$

for all $\lambda_i \in (0, 1)$ such that

$$0 < \sum_{i=1}^{k} \lambda_i < k.$$

This strict sub-additivity inequality shows (cf P. L. Lions [21] and the arguments above) that any minimizing sequence $(u_n)_n$ is such that: $|D^m u_n^i|^p$ are tight, $\prod_{i=1}^{k} |u_n^i|^{q_i}$ is tight. And this enables us to argue as before, therefore any minimizing sequence is relatively compact up to a translation (the same for all $u_n^i$) and a scale change and a minimum of (33) exists.

*Remark* I.6. Of course in (28), (33) we may take any norm on $\mathcal{D}^{m,p}$ and the choice may depend on $i \in \{1, \ldots, k\}$.

iii) *Fractional derivatives.*

We first recall that a norm on $W^{m,p}(\mathbb{R}^N)$ for $0 < m$, $1 \leqslant p < \infty$ is given by

$$||u||_{m,p}^p = \sum_{|\alpha| \leq \alpha_0} ||D^\alpha u||_{L^p}^p + \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|D^{\alpha_0} u(x) - D^\alpha u(y)|^p}{|x - y|^{N+sp}} dx\, dy$$

where $\alpha_0$ is the integer part of $m$ and we assume: $\alpha_0 < m < \alpha_0 + 1$; and $s = (m - \alpha_0)$. The Sobolev inequality still holds

$$||u||_{L^q(\mathbb{R}^N)} \leqslant C_0 ||u||_{m,p}, \qquad \forall u \in W^{m,p}(\mathbb{R}^N);$$

where $q = Np/(N - mp)$. But if we replace $u$ by $\sigma^{-N/q} u(\cdot/\sigma)$ in this inequality we find

$$||u||_{L^q(\mathbb{R}^N)} \leqslant C_0 \Big\{ \sum_{|\alpha| \leq \alpha_0} \sigma^{(m-\alpha)p} ||D^\alpha u||_{L^p}^p + \\ + \iint |D^{\alpha_0} u(x) - D^{\alpha_0} u(y)|^p |x - y|^{-(N+sp)} dx\, dy \Big\}^{1/p}.$$

Therefore sending $\sigma$ to 0, we find for all $u \in W^{m,p}(\mathbb{R}^N)$

$$||u||_{L^q(\mathbb{R}^N)} \leqslant C_0 \mathcal{E}(u)^{1/p} \tag{34}$$

where

$$\mathcal{E}(u) = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|D^{\alpha_0} u(x) - D^{\alpha_0} u(y)|^p}{|x - y|^{N+sp}} dx\, dy.$$

We then denote by $\mathcal{D}^{m,p}(\mathbb{R}^N)$ the space of functions $u$ satisfying $u \in L^q$,

$\mathcal{E}(u) < \infty$; it is a reflexive Banach space equipped with the norm $\mathcal{E}(u)^{1/p}$. Exactly as before the best constant in (34) is achieved (and all minimizing sequences are compact up to translations and dilations). In lemma I.1 which still holds we have to replace $|D^m u_n|^p(x)$ by

$$\int |D^{\alpha_0}u_n(x) - D^{\alpha_0}u_n(y)|^p |x-y|^{-(N+sp)}\, dy.$$

*Remark* I.7. Of course we may replace $\mathcal{E}(u)^{1/p}$ by any norm on $\mathcal{D}^{m,p}(\mathbb{R}^N)$. For example if $p = 2$ and if $\hat{u}$ is the Fourier transform of $u$, we may take

$$\left\{\int_{\mathbb{R}^N} |\hat{u}(\xi)|^2 |\xi|^{2m}\, d\xi\right\}^{1/2}.$$

In lemma I.2, $|D^m u_n|^2$ is to be replaced by

$$|F^{-1}(\xi^m \hat{u})|^2.$$

iv) *Convolution and Sobolev inequalities.*

The general Choquard-Pekar equations (cf. E. H. Lieb [19], P.L. Lions [22]) use the following limit embeddings

$$\left(\int\int_{\mathbb{R}^N \times \mathbb{R}^N} |u|^q(x)|u|^q(y)|x-y|^{-\alpha}\, dx\, dy\right)^{1/(2q)} \leqslant C_0 ||D^m u||_{L^p}; \tag{35}$$

where $m \geqslant 1$, $1 \leqslant p < \infty$, $0 < \alpha < N$ and $q$ is given by

$$2q = (2N - \alpha)p(N - mp)^{-1}.$$

For example if $p = q = 2$, $m = 1$ then $\alpha = 4$ (and $N \geqslant 5$). All the results proved above adapt to this situation and in particular Lemma I.1 holds with $|u_n|^q$ replaced by

$$|u_n|^q(x) \cdot \int_{\mathbb{R}^N} |u_n|^q(y)|x-y|^{-\alpha}\, dy.$$

v) *Korn-Sobolev inequalities.*

To simplify we will consider only $N = 3$, $p = 2$, $m = 1$. Let $u \in H^1(\mathbb{R}^3)^3$, we denote by $\epsilon_{ij}(u)$ the linear deformations tensor; $\epsilon_{ij}(u) = \frac{1}{2}\{(\partial u_i/\partial x_j) + (\partial u_j/\partial x_i)\}$. A fundamental inequality in elasticity theory is the Körn inequality which yields the Korn-Sobolev inequalities: for all $u \in H^1(\mathbb{R}^3)^3$ we have

$$||u||_{L^6(\mathbb{R}^3)} \leqslant C\{||u||_{L^2(\mathbb{R}^3)} + ||\epsilon(u)||_{L^2(\mathbb{R}^3)}\}$$

where $\epsilon(u) = \{\sum_{i,j} |\epsilon_{ij}(u)|^2\}^{1/2}$. The same dimensional analysis ($u(\cdot) \to \sigma^{-1/2} u(\cdot/\sigma)$) shows that in fact

$$||u||_{L^6(\mathbb{R}^3)} \leqslant C_0 ||\epsilon(u)||_{L^2(\mathbb{R}^3)} \tag{36}$$

and thus (36) holds for all $u \in (\mathfrak{D}^{1,2}(\mathbb{R}^3))^3$ and $||\epsilon(u)||_{L^2}$ is an equivalent norm on $(\mathfrak{D}^{1,2}(\mathbb{R}^3))^3$. Therefore all the results given above still hold in that case.

vi) *Time-dependent problems.*

Let $Q = \mathbb{R}^N \times \mathbb{R}$. Then Sobolev inequalities give in that case

$$||u||_{L^q(Q)} \leqslant C_0||u||, \qquad \forall u \in \mathfrak{D}(Q)$$

where $||u||^p = ||u_t||^p_{L^p(Q)} + ||D_x^m u||^p_{L^p(Q)}$ (or any other equivalent norm, for example if $m = 2$, $p = 2$, we may choose: $||u|| = ||u_t - \Delta u||_{L^2(Q)}$). Here and below we have: $m \geqslant 1$, $1 < p < (N + m)/m$ and $q = (N + m)p(N + m - mp)^{-1}$. We then denote by $\mathfrak{D}^{m,1,p}(Q)$ the completion of $\mathfrak{D}(Q)$ for the norm $||\cdot||$. The existence of an extremal function is equivalent to the existence of a minimum of

$$\operatorname{Inf}\{\, ||u||^p / u \in \mathfrak{D}^{m,1,p}(Q), \int_Q |u|^q \, dx\, dt = 1\,\} \tag{37}$$

**Corollary I.3.** *For any minimizing sequence* $(u_n)_n$ *of* (37), *there exist* $((y_n, t_n))_n$ *in* $Q$, $(\sigma_n)_n$ *in* $]0, \infty[$ *such that the new minimizing sequence*

$$\tilde{u}_n = \sigma_n^{-(N+m)/q} u_n\left(\frac{\cdot - y_n}{\sigma_n}, \frac{\cdot - t_n}{\sigma_n^m}\right)$$

*is relatively compact in* $\mathfrak{D}^{m,1,p}(Q)$. *In particular there exists a minimum.*

Of course there are many extensions that we skip such as:
$D_t^k u \in L^q, D_x^m u \in L^p \ldots$

vii) *Nonlinear embeddings.*

We just give one example of many situations which can be treated by the methods described above. Let $u \in (\mathfrak{D}^{s,2}(\mathbb{R}^N))^N$ with $s = (N + 2)/4$; then we have

$$|(u \cdot \nabla)u|_{L^2} = \left\{\int_{\mathbb{R}^N} \sum_j \left(\sum_i u^i \frac{\partial u^j}{\partial x_i}\right)^2 dx\right\}^{1/2} \leqslant C_0||u||_{\mathfrak{D}^{s,2}} \tag{38}$$

—such norms have been defined above even if $s$ is not an integer. Such equalities are interesting in the context of Navier-Stokes equations. Our methods yield the compactness up to translations and dilations of all minimizing sequences, the existence of extremal functions in (38) and informations on the weak convergence such as Lemma I.1 (one replaces $|u_n|^q$ by $|(u_n \cdot \nabla)u_n|^2 \ldots$).

Another application of these methods to the existence of extremal funtions for Sobolev-type inequalities is given in D. Jerison and J. M. Lee [16].

### I.5 Yamabe problem in $\mathbb{R}^N$

We already explained in the Introduction the motivation for the study of the following equation

$$-\sum_{i,j}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)+k(x)u=K(x)|u|^{4/(N-2)}u \quad \text{in} \quad \mathbb{R}^N \tag{39}$$

with $N \geqslant 3$. One is particularly interested in positive solutions of (39). We will assume (to simplify) all throughout the section

$$\begin{cases} a_{ij}=a_{ji}\in C_b(\mathbb{R}^N), & a_{ij}\to a_{ij}^\infty \quad \text{as} \quad |x|\to\infty \\ \exists\nu>0, \quad \forall x\in\mathbb{R}^N, & (a_{ij}(x))\geqslant \nu I_N \\ k,K\in C_b(\mathbb{R}^N), & k\to k^\infty, \quad K\to K^\infty \quad \text{as} \quad |x|\to\infty. \end{cases} \tag{40}$$

And our first approach of (39) will require either

$$\begin{cases} \exists\alpha>0, \quad \forall u\in\mathcal{D}(\mathbb{R}^N), \\ \int_{\mathbb{R}^N}\sum_{i,j}a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}+k(x)u^2\,dx\geqslant\alpha|Du|_{L^2}^2 \\ \sup_{\mathbb{R}^N}K>0, \quad k^\infty>0 \quad \text{or} \quad k\in L^{N/2}(\mathbb{R}^N) \end{cases} \tag{41}$$

or

$$\begin{cases} \exists\alpha>0, \quad K(x)\geqslant\alpha \quad \text{on} \quad \mathbb{R}^N \\ k^\infty>0 \quad \text{or} \quad k\in L^{N/2}(\mathbb{R}^N) \end{cases} \tag{42}$$

If one is interested in *solutions of* (39) *which vanish at infinity*, then a minimum of the following minimization problem will provide such a solution

$$I=\operatorname{Inf}\left\{\int_{\mathbb{R}^N}\sum_{i,j}a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}+k(x)u^2\,dx/u\in\mathcal{D}^{1,2}(\mathbb{R}^N),\right.$$
$$\left. k(x)u^2\in L^1(\mathbb{R}^N),\ \int_{\mathbb{R}^N}K(x)u^2\,dx=1\right\}. \tag{43}$$

Then if (41) or (42) holds, the class of minimizing functions is not empty and minimizing sequences are bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$; observe also that if (41) holds then $k^\infty\geqslant 0$ and that if $k^\infty>0$, the minimizing class is included in $H^1(\mathbb{R}^N)$ and minimizing sequences are bounded in $H^1$.

We have seen in the previous sections that if $a_{ij}, K$ are independent of $x$ and if $k\equiv 0$, then, if $u$ is a minimum, $\tilde{u}=\sigma^{-N/q}u(\cdot/\sigma)$ is still a minimum and nothing may prevent losses of compactness for minimizing sequences due to dilations (i.e. scale changes as above). Here of course, in general, the problem is not invariant by those scale changes anymore but we have to decide when and how the non-compactness in $L^q$ of a minimizing sequence (for

*$q = 2N/(N-2)$) —i.e. when Dirac masses (as in Lemma I.1) do appear in the limits of* $|u_n|^q$— is avoided. Of course we have also to avoid the non-compactness due to translations and we know (cf. [20], [21]) that this is done using the problem at infinity

$$\bar{I} = \operatorname{Inf}\left\{ \int_{\mathbb{R}^N} \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + k^\infty u^2 \, dx / u \in \mathcal{D}^{1,2}(\mathbb{R}^N), K^\infty \int_{\mathbb{R}^N} |u|^q \, dx = 1 \right\}$$

and $\bar{I} = +\infty$ if $K^\infty \leqslant 0$.

Here to avoid the non-compactness due to dilations we have to introduce a different notion of problem at infinity: to this end we denote by

$$\begin{cases} \mathcal{E}(u) = \displaystyle\int_{\mathbb{R}^N} \sum_{i,j} a_{ij}(x) \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + k(x) u^2 \, dx \\ J(u) = \int_{\mathbb{R}^N} K(x) |u|^q \, dx. \end{cases}$$

Then for any fixed point y $\in \mathbb{R}^N$, we consider for $u \in \mathcal{D}^{1,2}$ or $H^1$

$$\mathcal{E}_y^\infty(u) = \lim_{\sigma \to 0} \mathcal{E}(\sigma^{-N/q} u((\cdot - y)/\sigma)) = \int_{\mathbb{R}^N} \sum_{i,j} a_{ij}(y) \frac{\partial u}{\partial x_i}(x) \frac{\partial u}{\partial x_j}(x) \, dx, \tag{44}$$

$$J_y^\infty(u) = \lim_{\sigma \to 0} J(\sigma^{-N/q} u((\cdot - y)/\sigma)) = \int_{\mathbb{R}^N} K(y) |u|^q \, dx, \tag{45}$$

$$I_y^\infty = \operatorname{Inf}\{ \mathcal{E}_y^\infty(u) / u \in \mathcal{D}^{1,2}(\mathbb{R}^N), J_y^\infty(u) = 1 \} \tag{46}$$

and $I_y^\infty = +\infty$ if $K(y) \leqslant 0$. We could say that $I_y^\infty$ is the value of the infimum of the problem "at infinity at $y$". We finally introduce

$$I^\infty = \operatorname{Inf}\{ I_y^\infty / y \in \mathbb{R}^N \} \tag{47}$$

Observe that $I_y^\infty \to \bar{I}$ as $|y| \to \infty$, and thus: $I^\infty \leqslant \bar{I}$.

In the particular situation at hand $I_y^\infty$ and $I^\infty$ may be computed using dilations, homogeneity and symmetry arguments

$$I_y^\infty = K^+(y)^{-2/q} \det(a_{ij}(y))^{1/N} I^0$$

where $I^0$ corresponds to the best Sobolev exponent

$$I^0 = \operatorname{Min}\{ \int_{\mathbb{R}^N} |\nabla u|^2 \, dx / u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} |u|^q \, dx = 1 \}.$$

Therefore

$$I^\infty = \operatorname*{Inf}_{y \in \mathbb{R}^N} \{ K^+(y)^{-2/q} \det(a_{ij}(y))^{1/N} \} I^0. \tag{48}$$

The above construction of $I^\infty$ easily yields

$$I \leqslant I^\infty \tag{49}$$

and if we denote by $I_\lambda$, $I_\lambda^\infty$ the values of the infima of the same minimization problems but with 1 replaced by $\lambda > 0$, observing that

$$I_\lambda = \lambda^{2/q} I, \qquad I_\lambda^\infty = \lambda^{2/q} I^\infty;$$

we deduce from (49)

$$I = I_1 < I_\alpha + I_{1-\alpha}^\infty, \qquad \forall \alpha \in ]0,1[. \tag{50}$$

Therefore condition (S. 1) (of [20], [21]) holds if and only if $I < I^\infty$. By analogy with [20], [21], we expect the

**Theorem I.2.** *We assume* (40) *and* (41) *or* (42). *Let* $(u_n)_n$ *be a minimizing sequence of* (43).

i) *If* $I < I^\infty$, $(u_n)_n$ *is relatively compact in* $\mathfrak{D}^{1,2}(\mathbb{R}^N)$ (*and in* $H^1(\mathbb{R}^N)$, *if* $k^\infty > 0$). *In particular there exists a minimum and any minimum is, when* $I > 0$, *a positive solution of* (39) *up to a multiplicative constant.*

*If* $I = I^\infty$, *there exist minimizing sequences which are not relatively compact in* $\mathfrak{D}^{1,2}(\mathbb{R}^N)$.

(ii) *If* $I = \bar{I} < I_y^\infty$ *for all* $y \in \mathbb{R}^N$, *and if* $(u_n)_n$ *is not relatively compact, there exist* $(y_n)_n$ *in* $\mathbb{R}^N$, $(\sigma_n)_n$ *in* $]0,\infty[$ *such that:* $|y_n| \underset{n}{\to} \infty$, $\sigma_n^{-N/q} u_n((\cdot - y_n)/\sigma_n)$ *is relatively compact in* $\mathfrak{D}^{1,2}(\mathbb{R}^N)$. *In addition if* $k^\infty > 0$, $\sigma_n \underset{n}{\to} \infty$. *And there exist such sequences* $(u_n)_n$.

iii) *If* $I = I^\infty < \bar{I}$, $u_n$ *converges weakly to* 0, $|u_n|^q$, $|Du_n|^2$ *are tight. And if we denote by* $C = \{y \in \mathbb{R}^N, I_y^\infty = I^\infty\}$ *and if* $|u_{n_k}|^q$ *converges weakly to some measure* $\nu$, *we have*

$$\begin{cases} \nu = K(y)\delta_y \quad \textit{for some} \quad y \in C; \\ \exists \sigma_{n_k} \underset{k}{\to} \infty, \qquad \exists y_k, y_k/\sigma_{n_k} \underset{k}{\to} y \quad \text{and} \\ \sigma_{n_k}^{-N/q} u_{n_k}((\cdot\ + y_k)/\sigma_{n_k}) \text{ is relatively compact in } \mathfrak{D}^{1,2}(\mathbb{R}^N). \end{cases}$$

*And such sequences* $(u_n)_n$ *exist for any* $y \in C$.

iv) *If* $I = \bar{I} = I^\infty = I_y^\infty$ *for some* $y \in \mathbb{R}^N$ *then the conclusions of either* ii), *or* iii) *hold for subsequences. And both cases occur.*

We see that, even when compactness is not available, parts ii), iii), iv) describe exactly the phenomena involved. We will not explain here how to check the condition: $I < I^\infty$. Let us just mention that this is by no means easy and one may use the techniques of T. Aubin [3] (see also H. Brézis and L. Nirenberg [12]): this method is illustrated in the example following the proof of Theorem I.2. Let us also observe that if (42) holds and $k = \mu k_0$ for some $k_0 \leqslant 0$, $k_0 \not\equiv 0$, $k_0 \in L^{N/2}$, then $I < I^\infty$ for $\mu$ large. Of course if $I \leqslant 0$, then $I < I^\infty$!

PROOF OF THEOREM I.2. In all cases $(u_n)_n$ is bounded in $\mathcal{D}^{1,2}$ and if $k^\infty > 0$, $(u_n)_n$ is bounded in $H^1$. Depending whether $k^\infty = 0$ or $k^\infty > 0$, we consider $\rho_n \in L^1_+(\mathbb{R}^N)$ given by

$$\rho_n = |\nabla u_n|^2 + |u_n|^q \quad \text{or} \quad \rho_n = |\nabla u_n|^2 + |u_n|^q + u_n^2.$$

Applying the arguments of P. L. Lions [20], [21], we conclude that $\rho_n$ is tight up to a translation if vanishing does not occur: indeed observe that we have

$$I = I_1 < I_\alpha + \bar{I}_{1-\alpha} \qquad \forall \alpha \in ]0, 1[.$$

Now if vanishing occurs i.e.

$$\forall R < \infty, \qquad \operatorname*{Sup}_{y \in \mathbb{R}^N} \int_{y+B_R} \rho_n \, dx \underset{n}{\rightarrow} 0; \tag{51}$$

we have clearly

$$\left| \int_{\mathbb{R}^N} K(x)|u_n|^q \, dx - K^\infty \int_{\mathbb{R}^N} |u_n|^q \, dx \right| \leqslant C \sup_{|x| \geq R} |K(x) - K^\infty| + C \int_{B_R} |u_n|^q \, dx;$$

$$\left| \int_{\mathbb{R}^N} \sum_{i,j} a_{ij}(x) \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial x_j} dx - \int_{\mathbb{R}^N} \sum_{i,j} a_{ij}^\infty \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial x_j} dx \right| \leqslant$$

$$\leqslant C \sum_{i,j} \sup_{|x| \geq R} |a_{ij}(x) - a_{ij}^\infty| + C \int_{B_R} |\nabla u_n|^2 \, dx;$$

$$\left| \int_{\mathbb{R}^N} k(x) u_n^2 \, dx - \int_{\mathbb{R}^N} k^\infty u_n^2 \, dx \right| \leqslant C \sup_{|x| \geq R} |k(x) - k^\infty| + C \int_{B_R} u_n^2 \, dx \quad \text{if} \quad k^\infty > 0;$$

$$\leqslant C \|k\|_{L^{N/2}(\mathbb{R}^N - B_R)} + C_R \left( \int_{B_R} |u_n|^q \, dx \right)^{2/q}.$$

Therefore choosing $R$ large and then $n$ large, we see that vanishing implies: $I \geqslant \bar{I} = I^\infty$.

In a similar way if $\rho_n$ (or a subsequence) is tight up to a translation $y_n$ such that $|y_n| \underset{n}{\rightarrow} \infty$, then $I = \bar{I} = I^\infty$. Therefore if $I < I^\infty$, we have

$$\forall \epsilon > 0, \exists R < \infty, \forall n, \qquad \int_{|x| \geq R} \rho_n \, dx \leqslant \epsilon.$$

*We now complete the proof of Part i) of Theorem* I.2. We may now assume that $(u_n)_n$ converges weakly to some $u$ (and a.e.): *We first show that* $u \not\equiv 0$: if it were the case we would have by Lemma I.1

$$|u_n|^q \rightarrow \sum_{k \in J} \nu_k \delta_{x_k}$$

for some at most countable set $J$ of distinct points $x_k$ in $\mathbb{R}^N$ and of positive real numbers $\nu_k$. In addition we have the:

**Lemma I.4.** *Let $a_{ij} = a_{ji} \in C_b(\mathbb{R}^N)$ and assume $(a_{ij}) \geqslant 0$ on $\mathbb{R}^N$. If $u_n \underset{n}{\rightharpoonup} u$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and $|u_n|^q$ is tight, we know by Lemma* I.1 *that: $|u_n|^q \to |u|^q + \sum_{k\in J} \nu_k \delta_{x_k}$. Extracting if necessary a subsequence, we may assume that*

$$\sum_{i,j} a_{ij}(x) \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial x_j} \underset{n}{\rightharpoonup} \mu$$

*for some positive bounded measure $\mu$, then*

$$\mu \geqslant \sum_{i,j} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \sum_{k\in J} \nu_k^{(N-2)/N} I^0 (\det a_{ij}(x_k))^{1/N} \delta_{x_k}.$$

This lemma is proved after the proof of Theorem I.2. Of course it is valid with various adaptations in any $\mathcal{D}^{m,p}$.

Now if we go back to the proof of Theorem I.2, we see that

$$\begin{cases} I \geqslant \sum_{k\in J} \nu_k^{(N-2)/N} I^0 (\det a_{ij}(x_k))^{1/N} + \lim_n \int_{\mathbb{R}^N} k(x) u_n^2 \, dx \\ 1 \leqslant \sum_{k\in J} K(x_k)\nu_k. \end{cases}$$

Next we claim that:

$\int_{\mathbb{R}^N} |k| u_n^2 \, dx \underset{n}{\to} 0.$

Indeed if $k^\infty > 0$, since $u_n \underset{n}{\to} 0$ in $L^2(B_R)$ strongly for all $R < \infty$ by Rellich theorem and $\rho_n$ is tight, we see that $u_n \underset{n}{\to} 0$ in $L^2(\mathbb{R}^N)$ and our claim is proved.

On the other hand if $k^\infty = 0$ and thus $k \in L^{N/2}$, we remark

$$\int_{\mathbb{R}^N} |k| u_n^2 \, dx \leqslant \int_{B_R} |k| u_n^2 \, dx + C \| k \|_{L^{N/2}(\mathbb{R}^N - B_R)} \leqslant M \int_{B_R} u_n^2 \, dx + \\ + C \| (|k| - M)^+ \|_{L^{N/2}(B_R)} + C \| k \|_{L^{N/2}(\mathbb{R}^N - B_R)}$$

and we conclude choosing $R$ large, then $M$ large and finally $n$ large.

Therefore (52) yields

$$\begin{cases} I \geqslant \sum_{k\in J} \nu_k^{(N-2)/N} I^0 (\det a_{ij}(x_k))^{1/N} \\ 1 \leqslant \sum_{k\in J} K(x_k)\nu_k. \end{cases}$$

On the other hand since $I \leqslant I^\infty$ and $I^\infty$ is given by the formula (48), this implies that $J$ reduces to a single point $x_0$ which is a minimum point of $K^{-2/q} \det(a_{ij})^{1/N}$ i.e. a minimum point of $I_y^\infty$ and $I = I_{x_0}^\infty = I^\infty$. Of course if $I < I^\infty$, this is not possible and $u \not\equiv 0$.

*We next conclude the proof of part i) of Theorem* I.2 by showing that

$\int_{\mathbb{R}^N} K|u|^q\,dx = 1$. Let us denote by $\alpha = \int_{\mathbb{R}^N} K|u^q|\,dx$. By Lemma I.4 we know

$$I \geqslant \int_{\mathbb{R}^N} \sum_{i,j} a_{ij}(x) \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\,dx + \sum_{k\in J} \nu_k^{(N-2)/N} I^0 (\det a_{ij}(x_k))^{1/N} + \varliminf_n \int_{\mathbb{R}^N} k(x) u_n^2\,dx$$

$$1 = \alpha + \sum_{k\in J} \nu_k K(x_k);$$

and exactly as before we prove that

$$\int_{\mathbb{R}^N} k(x) u_n^2\,dc \xrightarrow{n} \int_{\mathbb{R}^N} k(x) u^2\,dx.$$

Hence, we have

$$\begin{cases} I \geqslant \mathcal{E}(u) + \sum_{k\in J} \nu_k^{(N-2)/N} I^0 (\det a_{ij}(x_k))^{1/N} \\ 1 - \alpha = \sum_{k\in J} \nu_k K(x_k). \end{cases}$$

Using (48), we deduce

$$\begin{aligned} I &\geqslant \mathcal{E}(u) + I^\infty \sum_{k\in J} \nu_k^{(N-2)/N} K^+(x_k)^{(N-2)/N} \\ &\geqslant \mathcal{E}(u) + I^\infty \Big\{ \sum_{k\in J} \nu_k K^+(x_k) \Big\}^{(N-2)/N} = \mathcal{E}(u) + I_\beta^\infty \end{aligned}$$

where

$$\beta = \sum_{k\in J} \nu_k K^+(x_k) \geqslant 1 - \alpha.$$

If $\alpha \leqslant 0$, $\beta \geqslant 1$ and we obtain: $I \geqslant \mathcal{E}(u) + I^\infty > I^\infty$, a contradiction with the large inequality $I \leqslant I^\infty$ which always holds.

If $\alpha < 1$, we find: $I \geqslant \mathcal{E}(u) \geqslant I_\alpha > I$, another contradiction.

Finally if $\alpha \in ]0, 1[$, we find

$$I = I_1 \geqslant \mathcal{E}(u) + I_{1-\alpha}^\infty \geqslant I_\alpha + I_{1-\alpha}^\infty$$

and this contradicts (50). And part i) is proved.

*To prove part ii)*, we observe that from the second part of the proof that, in the situation described in ii), either $\rho_n$ "vanishes" or $\rho_n$ is tight up to a translation $y_n$ such that $|y_n| \xrightarrow[n]{} \infty$. In the first case $\rho_n(\cdot - y_n)$, for some arbitrary $y_n$ satisfying $|y_n| \xrightarrow[n]{} \infty$, "vanishes" and by the arguments given above $\tilde{u}_n = u_n(\cdot - y_n)$ in both cases is a minimizing sequence of $\bar{I}$. If $k^\infty = 0$, we apply Theorem I.1 and we conclude. If $k^\infty > 0$, remarking that $\bar{I} = (K^\infty)^{-(N-2)/N} I_0$

$$k^\infty \int_{\mathbb{R}^N} \tilde{u}_n^2\,dx \xrightarrow[n]{} 0,$$

and thus $(K^\infty)^{-(N-2)/N} \tilde{u}_n$ is also a minimizing sequence of $I_0$. And applying

Theorem I.1 we find $(z_n)_n$ in $\mathbb{R}^N$, $(\sigma_n)_n$ in $]0, \infty[$ such that

$$\hat{u}_n = \sigma_n^{-N/q} \tilde{u}_n((\cdot - z_n)/\sigma_n)$$

is relatively compact in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. And this implies

$$0 < \delta \leqslant \int_{\mathbb{R}^N} (\hat{u}_n)^2 \, dx = \sigma_n^{-2N/q} \sigma_n^N \int_{\mathbb{R}^N} \tilde{u}_n^2 \, dx$$

therefore $\sigma_n \underset{n}{\to} +\infty$. And part ii) is proved.

Part iii) is easily deduced from the above arguments: we just need to observe that if $u_n \underset{n}{\rightharpoonup} 0$, $|u_n|^q \to K(y)\delta_y$, then $(u_n)_n$ is a minimizing sequence of $I_y^\infty$ and by Theorem I.1 we conclude easily. Finally part iv) is a consequence of the proof already made.

*Remark* I.8. Of course if we know that there does not exist a minimum of $I$, then the conclusions of Parts ii), iii), iv) hold. In particular this is the case when

$$(a_{ij}(x)) \geqslant (a_{ij}^\infty), \qquad K(x) \leqslant K^\infty, \qquad k(x) \geqslant k^\infty.$$

PROOF OF LEMMA I.4. We take the notations of the proof of Lemma I.1 and we have for all fixed $k \in J$ and for all $\epsilon > 0$

$$\left| \int_{\mathbb{R}^N} \sum_{i,j} a_{ij}(x) \frac{\partial}{\partial x_i} \left\{ \varphi\left(\frac{x - x_k}{\epsilon}\right) u_n \right\} \frac{\partial}{\partial x_j} \left\{ \varphi\left(\frac{x - x_k}{\epsilon}\right) u_n \right\} dx + \right.$$
$$\left. - \int_{\mathbb{R}^N} \varphi^2\left(\frac{x - x_k}{\epsilon}\right) \left\{ \sum_{i,j} a_{ij} \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial x_j} \right\} dx \right| \leqslant \delta(\epsilon)$$

where $\delta(\epsilon)$ denotes various quantities (ind. of $n$) which go to 0 as $\epsilon$ goes to 0. But $\operatorname{Supp} \varphi \subset B(0, 1)$ and $a_{ij}$ is continuous, hence

$$\int_{\mathbb{R}^N} \varphi^2\left(\frac{x - x_k}{\epsilon}\right) \left\{ \sum_{i,j} a_{ij}(x) \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial x_j} \right\} dx \geqslant$$
$$\geqslant -\delta(\epsilon) + \int_{\mathbb{R}^N} \sum_{i,j} a_{ij}(x_k) \frac{\partial}{\partial x_i} \left( \varphi\left(\frac{x - x_k}{\epsilon}\right) u_n \right) \frac{\partial}{\partial x_j} \left( \varphi\left(\frac{x - x_k}{\epsilon}\right) u_n \right) dx$$
$$\geqslant -\delta(\epsilon) + I_0 (\det a_{ij}(x_k))^{1/N} \left( \int_{\mathbb{R}^N} \left| \varphi\left(\frac{x - x_k}{\epsilon}\right) \right|^q |u_n|^q \, dx \right).$$

And sending $n$ to $\infty$, we deduce

$$\mu(B(x_k, \epsilon)) \geqslant -\delta(\epsilon) + I^0 (\det a_{ij}(x_k))^{1/N} \nu_k^{2/q}.$$

We conclude letting $\epsilon \to 0$.

EXAMPLE. We want to mention a simple situation where $I < I^\infty$. We take $k \equiv 0$. Observe that if $a_{ij}$ does not depend on $x$ for all $i, j$, the minimum is achieved if and only if $K(x) \equiv K^\infty > 0$. Indeed if $K \equiv \sup K$, any minimum $u$ of $I$ will satisfy

$$\mathcal{E}(u) = I \leqslant I^\infty, \qquad (\sup K) \int_{\mathbb{R}^N} |u|^{2N/(N-2)}\, dx > 1$$

and this contradicts the choice of $I^\infty$.

Now we take for example: $a_{ij}(x) = a(x)\delta_{ij}$, $0 < \alpha < a(x)$, $K(x) \leqslant 1$, $K(x) \to K^\infty$, $a(x) \to a^\infty$ as $|x| \to \infty$. We will assume $N \geqslant 5$ and we may always normalize $K, a$ by assuming $K(0) = a(0) = 1$ (the choice of the origin is arbitrary).

In order to try to prove $I < I^\infty$, it is natural to use the extremal functions of $I^\infty = I^0$ i.e. $u_\epsilon(x) = (\epsilon^2 + |x|^2)^{-(N-2)/2}$. This method was first used by T. Aubin [3] (see also H. Brézis and L. Nirenberg [12]). We compute

$$\mathcal{E}(u_\epsilon) = (N-2)^2\left(\int_{\mathbb{R}^N} a(\epsilon y)\frac{|y|^2}{(1+|y|^2)^N}\,dy\right)\epsilon^{(N-2)}$$

$$J(u_\epsilon) = \epsilon^{-N}\int_{\mathbb{R}^N} K(\epsilon y)(1+|y|^2)^N\,dy$$

And if $a$ is twice differentiable at 0, we deduce easily (using symmetry arguments for the first expansion terms)

$$\mathcal{E}(u_\epsilon) = \epsilon^{-(N-2)} I^0 ||u_1||_{L^q} + \\ + \frac{(N-2)^2}{2}\epsilon^{4-N}\int_{\mathbb{R}^N}\Big(\sum_{i,j} a_{ij}y_iy_j\Big)|y|^2(1+|y|^2)^{-N}\,dy + o(\epsilon^{4-N})$$

$$J(u_\epsilon) = \epsilon^{-N}||u_1||^q_{L^q} + \epsilon^{2-N}\int_{\mathbb{R}^N}\Big(\sum_{i,j} K_{ij}y_iy_j\Big)(1+|y|^2)^{-N}\,dy + o(\epsilon^{2-N})$$

where

$$a_{ij} = \frac{\partial^2 a}{\partial x_i \partial x_j}, \qquad K_{ij} = \frac{\partial^2 K}{\partial x_i \partial x_j}(O).$$

Since $I \leqslant \mathcal{E}(u_\epsilon)J(u_\epsilon)^{-(N-2)/N}$, *we conclude that* $I < I^\infty$ by choosing $\epsilon$ small enough provided

$$\int_{\mathbb{R}^N}\Big(\sum_{i,j} a_{ij}y_iy_j\Big)|y|^2(1+|y|^2)^{-N}\,dy < C_0\int_{\mathbb{R}^N}\Big(\sum_{i,j} K_{ij}y_iy_j\Big)(1+|y|^2)^{-N}\,dy$$

(recall that the origin is arbitrary !), where

$$C_0 = 2(N(N-2)||u_1||^q_{L^q})^{-1}.$$

It is possible to treat more general potentiels $k$: one possible extension relies on the following result. If $N \geqslant 3$, we have

$$\int_{\mathbb{R}^N} |u|^2 |x|^{-2}\, dx \leqslant \frac{4}{(N-2)^2} \int_{\mathbb{R}^N} |\nabla u|^2\, dx, \qquad \forall u \in \mathfrak{D}^{1,2}(\mathbb{R}^N) \tag{53}$$

(this classical inequality is sometimes called the "uncertainty principle"!). Hence if we consider for $\alpha > -((N-2)^2/4$ the coercive quadratic form

$$\mathcal{E}(u) = \int_{\mathbb{R}^N} |\nabla u|^2 + \alpha \frac{u^2}{|x|^2}\, dx,$$

we may study the question of the existence of an extremal function for the best constant of the Sobolev embedding when $\mathfrak{D}^{1,2}(\mathbb{R}^N)$ is endowed with the norm $\mathcal{E}(u)^{1/2}$ i.e.

$$I_\alpha = \operatorname{Inf}\{\mathcal{E}(u)/u \in \mathfrak{D}^{1,2}(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |u|^{2N/(N-2)}\, dx = 1\}. \tag{54}$$

**Theorem I.3.** *For any minimizing sequence* $(u_n)_n$ of (54), *there exists* $(\sigma_n)_n$ *in* $]0, \infty[$ *such that the new minimizing sequence* $\tilde{u}_n = \sigma_n^{-(N-2)/N} u_n(\cdot/\sigma_n)$ *satisfies*:

i) *If* $\alpha < 0$, $\tilde{u}_n$ *is relatively compact in* $\mathfrak{D}^{1,2}(\mathbb{R}^N)$ *and thus a minimum of* (54) *exists.*

ii) *If* $\alpha \geqslant 0$, *there exists* $(y_n)_n$ *in* $\mathbb{R}^N$ *such that* $\tilde{u}_n(\cdot - y_n)$ *is relatively compact in* $\mathfrak{D}^{1,2}(\mathbb{R}^N)$ *and if* $\alpha > 0$, $|y_n| \underset{n}{\to} \infty$. *In addition if* $\alpha > 0$, $I^\alpha = I^0$ *and no minimum exists.*

The proof of Theorem I.3 is very similar to the above proofs and we will skip it. Let us just mention that if $v_n(\cdot y_n)$ is relatively compact in $\mathfrak{D}^{1,2}(\mathbb{R}^N)$, and if $|y_n| \underset{n}{\to} \infty$ then

$$\int_{\mathbb{R}^N} |v_n|^2 |x|^{-2}\, dx \underset{n}{\to} 0,$$

Next, we explain without even stating a theorem, what is one possible extension of Theorem I.2. Take, to simplify, $a_{ij} \equiv \delta_{ij}$, $K \equiv 1$, and $k \in C_0(\mathbb{R}^N)$ satisfies

$$\lim_{|x| \to \infty} k(x)|x|^2 = \alpha > -\frac{(N-2)^2}{4}$$

Then we set: $I^\infty = I^\alpha$ (thus $I^\infty = I^0$ if $\alpha \geqslant 0$). With these notations we can prove that if $I < I^\infty$, all minimizing sequences are relatively compact while if $I = I^\infty$, there is a least one minimizing sequence which is not relatively compact —and we may analyse as in Theorem I.2 what are the possible losses of compactness.

At this point let us observe that *everything we did concerned positive solutions of* (39) *which vanish at infinity* and if we go back to the original motivation of the Yamabe equation (39) —given in the Introduction— it *is not clear that the new metric* —basically given by $|u|^{4/(N-2)}a_{ij}$— *is complete* (and in general it is not complete). On the other had *if we consider positive solutions of* (39) *such that*: $u(x) \geqslant \alpha$ on $\mathbb{R}^N$ for some $\alpha > 0$; the *new metric will be automatically complete*. This is why in the remainder of this section we will consider *bounded solutions of* (39) *positive uniformly on* $\mathbb{R}^N$. At this stage, let us mention the work of Ni [28] (see also [29], Kenig and Ni [17]) where, in the particular case $a_{ij}(x) \equiv \delta_{ij}$, $k \equiv 0$, general results are obtained by the elementary method of sub and supersolutions. This approach is recalled in the appendix where we also explain how it is possible to obtain twice more solutions —and this is done by variational arguments involving our general method. Here, we present still another approach which in the special case afore mentioned does not cover the full generality of Ni's results since more severe restrictions are made on $K$ but on the other hand we obtain additional information on the solution and the approach also provides a general way to check the assumptions necessary in order to apply the method of sub and supersolutions of Ni.

We consider the following minimization problem

$$I_\lambda = \operatorname{Inf}\{\mathcal{E}(u)/u - \alpha \in \mathcal{D}^{1,2}(\mathbb{R}^N),\ J(u) = \lambda\} \tag{55}$$

where $\alpha > 0$, $N \geqslant 3$ are fixed. $\mathcal{E}, J$ still denote the same functionals and we assume

$$a_{ij} = a_{ji} \in C_b(\mathbb{R}^N); \qquad \exists \nu > 0 \qquad (a_{ij}(x)) \geqslant \nu I_N \tag{40'}$$

$$k, K \in L^1(\mathbb{R}^N) \cap C_0(\mathbb{R}^N); \qquad K > 0 \quad \text{on} \quad \mathbb{R}^N \tag{56}$$

We define the energy at infinity exactly as before, but since $K^\infty = 0$, we have

$$I_\lambda^\infty = \operatorname*{Min}_{y \in \mathbb{R}^N} \{K^+(y)^{-2/q} \det\{a_{ij}(y)\}^{1/N}\} I^0. \tag{48'}$$

**Theorem I.4.** *We assume* (40′) *and* (56):

i) *Every minimizing sequence of* (55) *is relatively compact in* $X = \{\alpha + v, v \in \mathcal{D}^{1,2}(\mathbb{R}^N)\}$ *if and only if*

$$I_\lambda < I_\beta + I_{\lambda-\beta}^\infty, \qquad \forall \beta \in [0, \lambda[. \tag{S.1}$$

ii) *If we assume in addition*

$$\exists \gamma > 0, \forall \varphi \in \mathcal{D}(\mathbb{R}^N), \qquad \mathcal{E}(\varphi) \geqslant \gamma |D\varphi|^2_{L^2(\mathbb{R}^N)} \tag{57}$$

*then there exists* $\lambda_0 > 0$ *such that*: $I_\lambda$ *is decreasing on* $[0, \lambda_0]$ *from* $-\infty$ to $I_{\lambda_0}$

*and $I_\lambda$ is increasing on $[\lambda_0, +\infty[$ from $I_{\lambda_0}$ to $+\infty$. In addition there exists a unique $\varphi_1 \in H^1_{\text{loc}}(\mathbb{R}^N) \cap BUC(\mathbb{R}^N)$ satisfying*

$$\begin{cases} -\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial \varphi_1}{\partial x_j}\right) + k(x)\varphi_1 = 0 \quad in \quad \mathbb{R}^N \\ \varphi_1 > 0 \quad in \quad \mathbb{R}^N, \qquad \varphi_1 \to \alpha \quad as \quad |x| \to \infty \end{cases} \tag{58}$$

*and $\mathcal{E}(\varphi_1) = I_{\lambda_0}$, $\quad J(\varphi_1) = \lambda_0$, $\quad \varphi_1 - \alpha \in \mathcal{D}^{1,2}(\mathbb{R}^N)$.*

iii) *Furthermore if $\lambda \in ]0, \lambda_0[$,* (S.1) *holds and there exists a unique minimum of* (55), *$u_\lambda \in X$ which satisfies*

$$\begin{cases} -\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u_\lambda}{\partial x_j}\right) + k(x)u_\lambda + \theta_\lambda K u_\lambda^{(N+2)/(N-2)} = 0 \quad in \quad \mathbb{R}^N \\ u_\lambda \in BUC(\mathbb{R}^N), \qquad u_\lambda > 0 \quad on \quad \mathbb{R}^N, \qquad u_\lambda \to \alpha \quad as \quad |x| \to \infty; \end{cases} \tag{59}$$

*where $\theta_\lambda$ is a positive Lagrange multiplier. In addition $u_\lambda$ is the unique solution of* (59) *(in $H^1_{\text{loc}}$ say) and $\theta_\lambda$ decreases continuously on $]0, \lambda_0[$ from $+\infty$ to 0, while $u_\lambda$ increases continuously from 0 to $\varphi_1$ on $\mathbb{R}^N$.*

iv) *Finally there exists $\delta > 0$, such that for $\lambda \in ]\lambda_0, \lambda_0 + \delta[$,* (S. 1) *holds. In particular there exists a minimum $u_\lambda$ of* (55) *which solves*

$$-\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u_\lambda}{\partial x_j}\right) + k(x)u_\lambda = \theta_\lambda K u_\lambda^{(N+2)/(N-2)} \quad in \quad \mathbb{R}^N \tag{60}$$

*and $u_\lambda \in X \cap BUC(\mathbb{R}^N)$, $u_\lambda \to \alpha$ as $|x| \to \infty$; where $\theta_\lambda$ is a positive Lagrange multiplier.*

*Remark* I.9. Assumptions (56), (57) may be relaxed but the main assumption $k, K \in L^1(\mathbb{R}^N)$ subsists. In part i), we have: $I_0 = +\infty$ and $I_\alpha \to +\infty$ as $\alpha \to 0_+$, hence the strict inequality in (S.1) holds for $\alpha$ small.

Setting $v_\lambda = \theta_\lambda^{(N-2)/4} u_\lambda$ and using the variant of Ni's method given in the appendix, we find the

**Corollary I.4.** *We assume* (40′), (56), (57). *Then for any $\mu > 0$, there exists a unique solution $u$ of*

$$\begin{cases} -\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + k(x)u + K(x)u^{(N+2)(N-2)} = 0 \quad in \quad \mathbb{R}^N \\ u \in H^1_{\text{loc}}(\mathbb{R}^N) \cap C_b(\mathbb{R}^N), \quad u > 0 \quad on \quad \mathbb{R}^N, \quad u \to \mu \quad as \quad |x| \to \infty \end{cases} \tag{39′}$$

*and $u$ increases continuously in $\mu$, $u < (\mu/\alpha)\varphi_1$ in $\mathbb{R}^N$, $u - \mu \in \mathcal{D}^{1,2}(\mathbb{R}^N)$.*

*There exists $\mu_0 > 0$, such that:*

i) *for $\mu > \mu_0$, there does not exist a solution $u$ of* (39) *such that:*

$$u \in H^1_{\text{loc}}(\mathbb{R}^N) \cap C_b(\mathbb{R}^N), \qquad u \geqslant 0 \quad in \quad \mathbb{R}^N, \qquad u \to \mu \quad as \quad |x| \to \infty.$$

ii) *for* $0 < \mu \leqslant \mu_0$, *there exists a solution u of* (39) *in* $H^1_{\mathrm{loc}}(\mathbb{R}^N) \cap C_b(\mathbb{R}^N)$ *satisfying*: $u \to \mu$ *as* $|x| \to \infty$, $u > (\mu/\alpha)\varphi_1 > 0$ *on* $\mathbb{R}^N$, $u - \mu \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ *and u is the minimum positive solution of* (39) *in* $H^1_{\mathrm{loc}} \cap C_b$ *converging to* $\mu$ *at infinity. In addition u increases continuously with* $\mu$ *on* $\mathbb{R}^N$.

PROOF OF THEOREM I.4. We will prove part i) since it is a straightforward repetition of arguments given before provided we show that if $(u_n)_n$ is a minimizing sequence of (55) then $u_n - \alpha$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Indeed if $v_n = (u_n - \alpha)$, since $J(u_n) = \lambda$ and $K > 0$ on $\mathbb{R}^N$, we find:

$$||u_n||_{L^{2N/(N-2)}(B_R)} + ||v_n||_{L^{2N/(N-2)}(B_R)} \leqslant C_R, \qquad \forall R < \infty.$$

Next, in view of (40′):

$$\begin{aligned} \nu|Dv_n|^2_{L^2} &\leqslant C_\lambda + 2\int_{\mathbb{R}^N} |k|\alpha|v_n|\,dx + \int_{\mathbb{R}^N} |k| v_n^2\,dx \\ &\leqslant C + C||v_n||_{L^{2N/(N-2)}} + \int_{\mathbb{R}^N/B_R} |k| v_n^2\,dx \\ &\leqslant C + C||v_n||_{L^{2N/(N-2)}} + ||k||_{L^{N/2}(\mathbb{R}^N - B_R)} ||v_n||^2_{L^{2N/(N-2)}} \end{aligned}$$

and we conclude using Sobolev inequalities and choosing $R$ large.

*We next prove part* ii): *we first show that* $I_\lambda \to +\infty$ as $\lambda \to 0_+$ or $\lambda \to +\infty$. Indeed if $I_\lambda$ remains bounded when $\lambda \to 0_+$, the above argument shows that $\{v \in \mathcal{D}^{1,2}(\mathbb{R}^N),\ \mathcal{E}(\alpha + v) \leqslant I_\lambda + 1, J(\alpha + v) \in ]0, 1[\}$ is bounded. Hence there exists $v_n$ bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ such that $J(\alpha + v_n) \underset{n}{\to} 0$. Since $K > 0$ in $\mathbb{R}^N$, this yields: $v_n \to -\alpha$ in measure locally, and this contradicts the boundedness of $v_n$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ since $-\alpha \notin \mathcal{D}^{1,2}(\mathbb{R}^N)$.

Next, if $\lambda \to +\infty$ and $I_\lambda \leqslant C$, there exists $v_n$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ such that

$$\mathcal{E}(\alpha + v_n) \leqslant C; \qquad J(\alpha + v_n) \underset{n}{\to} +\infty.$$

But

$$\mathcal{E}(\alpha + v_n) \geqslant \gamma|Dv_n|^2_{L^2} - C - C|Dv_n|_{L^2}$$

hence $v_n$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and this contradicts the assumption on $J(\alpha + v_n)$.

Next, denoting by $\lambda_0(>0)$ an absolute minimum of the continuous function $(\lambda \to I_\lambda)$, we show the *monotonicity properties of* $I_\lambda$: we first observe that if $\lambda \in ]0, \lambda_0[$ satisfies

$$I_\lambda = \min\{I_\mu / \mu \in ]0, \bar{\lambda}]\}, \quad \text{for some} \quad \bar{\lambda} > \lambda,$$

then necessarily $\lambda = \lambda_0$. Indeed, clearly for such a $\lambda$, (S. 1) holds. By part i), there exists a minimum $\varphi$ of $I_\lambda$ which is a local minimum of $\mathcal{E}$ on $X$, therefore $\varphi$ solves (58) and by standard regularity results $\varphi \in BUC(\mathbb{R}^N)$. We now prove the uniqueness of such a function $\varphi$ proving thus the equality between $\lambda$ and $\lambda_0$. Remark first that necessarily $\varphi > 0$ on $\mathbb{R}^N$.

Indeed for $R$ large enough, denoting by

$$A = -\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + k$$

$$A\varphi = 0 \quad \text{in} \quad B_R, \qquad \varphi > 0 \quad \text{on} \quad \partial B_R \tag{60}$$

and (57) implies that the first eigenvalue of operator $A$ on $H_0^1(B_R)$ is positive. We may thus apply the maximum principle to (60): $\varphi > 0$ on $B_R$.

Next if $\varphi, \psi \in H^1_{\text{loc}} \cap BUC$ solve: $A\varphi = A\psi = 0$ in $\mathbb{R}^N$, $\varphi, \psi \to \alpha$ as $|x| \to \infty$, we show that $\varphi \equiv \psi$. Indeed for $R$ large enough we have

$$\alpha - \epsilon \leqslant \varphi, \psi \leqslant \alpha + \epsilon \quad \text{on} \quad \partial B_R$$

where $\epsilon = \epsilon(R) \to 0$ as $R \to \infty$.

Since (60) holds for $\varphi, \psi$ and since we may apply the maximum principle, we obtain

$$\frac{\alpha+\epsilon}{\alpha-\epsilon}\varphi \geqslant \psi, \qquad \frac{\alpha+\epsilon}{\alpha-\epsilon}\psi \geqslant \varphi \quad \text{on} \quad \bar{B}_R;$$

and we conclude letting $R \to \infty$ ($\epsilon(R) \to 0$).

At this point we have proved that

$$I_{\lambda_0} < I_\lambda < I_\mu, \quad \text{if} \quad 0 < \mu < \lambda < \lambda_0.$$

Next if $\mu > \lambda \geqslant \lambda_0$, for each $\epsilon > 0$ fixed, there exists $u \in X$ such that

$$I_\lambda \leqslant \mathcal{E}(u) \leqslant I_\mu + \epsilon, \qquad J(u) = \mu$$

and considering $\tilde{u} = \theta u + (1-\theta)\varphi_1$ for $\theta \in [0,1]$, we find $\theta \in [0,1[$ such that: $J(\tilde{u}) = \lambda$. On the other hand since $u - \varphi_1 \in \mathcal{D}^{1,2}(\mathbb{R}^N)$

$$\mathcal{E}(\tilde{u}) = \mathcal{E}(\varphi_1 + \theta(u-\varphi_1)) = \mathcal{E}(\varphi_1) + \theta^2\mathcal{E}(u-\varphi_1)$$

and $\mathcal{E}(\tilde{u})$ is strictly convex with respect to $\theta$. Hence we find

$$I_\lambda \leqslant \mathcal{E}(\tilde{u}) < \theta\mathcal{E}(u) + (1-\theta)\mathcal{E}(\varphi_1) \leqslant \theta(I_\mu + \epsilon) + (1-\theta)I_{\lambda_0}$$

and this yields: $I_\lambda \leqslant I_\mu$. Next if $I_\lambda = I_\mu$, clearly $I_{\tilde{\lambda}} = I_\lambda = I_\mu$ for all $\tilde{\lambda} \in [\lambda, \mu]$ and thus (S. 1) holds for $\tilde{\lambda} \in ]\lambda, \mu[$. Therefore for $\tilde{\lambda} \in ]\lambda, \mu[$ fixed, there exists a minimum $\varphi$ of $I_{\tilde{\lambda}}$ which is clearly a local minimum of $\mathcal{E}$ on $X$, hence $\varphi = \varphi_1, \hat{\lambda} = \lambda_0$. The contradiction shows: $I_{\lambda_0} \leqslant I_\lambda < I_\mu$ if $\lambda_0 \leqslant \lambda < \mu$; and part ii) is proved.

*We now prove part iii)*: the properties of $I_\lambda$ we proved imply easily that (S.1) holds for $\lambda \in ]0, \lambda_0[$. Therefore there exists a minimum $u$ of $I_\lambda$ ($u \in X$) and observing that if for some $\varphi \in \mathcal{D}_+(\mathbb{R}^N)$: $\int_{\mathbb{R}^N} K|u|^{4/(N-2)} u\varphi\, dx > 0$, then for $t$ small enough

$$\mathcal{E}(u - t\varphi) \geqslant I_\mu > I_\lambda, \quad \text{for some} \quad \mu \in ]0, \lambda[.$$

This shows that the Lagrange multiplier $(-\theta_\lambda)$ is strictly negative and thus $u$ ($\in X \cap BUC(\mathbb{R}^N)$) solves (59). The remainder of part iii) is proved by showing that for each $\theta > 0$, there exists at most one $\bar{u} = \bar{u}_\theta$ solution of

$$A\bar{u} + \theta K|\bar{u}|^{4/(N-2)}\bar{u} = 0 \quad \text{in} \quad \mathbb{R}^N, \qquad \bar{u} \in X \cap BUC(\mathbb{R}^N). \tag{61}$$

and that $\bar{u}_\theta \geqslant \bar{u}_{\theta'}$, if $\theta \leqslant \theta'$. We first observe that $\bar{u} > 0$ on $\mathbb{R}^N$: indeed for $R$ large enough: $\bar{u} \geqslant (\alpha/2)$ if $|x| \geqslant R$. Next we have

$$A\bar{u} + \theta K|\bar{u}|^{4/(N-2)}\bar{u} = 0 \quad \text{in} \quad B_R, \qquad \bar{u} \geqslant (\alpha/2) \quad \text{on} \quad \partial B_R$$

and $\lambda_1(A + \theta K|\bar{u}|^{4/(N-2)}, H_0^1(B_R)) > \lambda_1(A, H_0^1(B_R)) > 0$; therefore applying the maximum principle, we find $\bar{u} > 0$ in $\bar{B}_R$.

Next, let $u, v$ be two solutions of (61) corresponding to $\theta > \theta' > 0$: we just need to prove that $u \leqslant v$ on $\mathbb{R}^N$. Indeed for $R$ large enough: $0 < \alpha - \epsilon \leqslant$ $\leqslant u, v \leqslant \alpha + \epsilon$ on $\partial B_R$, with $\epsilon = \epsilon(R) \to 0$ as $R \to \infty$. Let $w = ((\alpha + \epsilon)/(\alpha - \epsilon))v$, we have

$$Aw + \theta K w^{(N+2)/(N-2)} \geqslant \frac{\alpha + \epsilon}{\alpha - \epsilon}(Av + \theta K v^{(N+2)/(N-2)})$$

$$\geqslant \frac{\alpha + \epsilon}{\alpha - \epsilon}(Av + \theta' K v^{(N+2)/(N-2)}) = 0 \quad \text{on} \quad B_R$$

and $w \geqslant u$ on $\partial B_R$. Applying the maximum principle once more we conclude: $w \geqslant u$ on $\bar{B}_R$.

*We finally prove part* iv): we just have to prove that (S. 1) holds for $(\lambda - \lambda_0)$ small, positive. Let $\lambda > \lambda_0$, if (S. 1) does not hold there exists $\mu \in ]0, \lambda[$ such that

$$I_\lambda = I_\mu + I^\infty_{\lambda - \mu}.$$

We first observe that $\mu \in ]\lambda_0, \lambda[$: indeed $I_\lambda > I_{\lambda_0}$ and if $\mu \in ]0, \lambda_0[$, $I_\mu + I^\infty_{\lambda-\mu} >$ $> I_{\lambda_0} + I^\infty_{\lambda-\mu} > I_{\lambda_0} + I^\infty_{\lambda-\lambda_0} \geqslant I_\lambda$. Next we claim that (S. 1) holds for $I_\mu$: indeed for $\bar{\mu} \in ]0, \mu[$

$$(I_{\bar{\mu}} + I^\infty_{\mu-\bar{\mu}}) + I^\infty_{\lambda-\mu} > I_{\bar{\mu}} + I^\infty_{\lambda-\bar{\mu}} \geqslant I_\lambda = I_\mu + I^\infty_{\lambda-\mu}.$$

Therefore there exists a minimum $u_\mu$ of $I_\mu$ and one proves easily that $u_\mu > 0$ in $\mathbb{R}^N$ and $u_\mu - \alpha$ converges strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ to $\varphi_1 - \alpha$ as $\mu \to \lambda_0$. Observing that $\mu \to \lambda_0$ when $\lambda \to \lambda_0$, we conclude the proof of part iv) as follows: we denote by $h = \lambda - \mu$ and we consider $\varphi \in \mathcal{D}_+(\mathbb{R}^N)$ such that

$$\frac{N+2}{N-2}\int_{\mathbb{R}^N} K\varphi_1^{(N+2)/(N-2)}\varphi\, dx = 1,$$

obviously $((N+2)/(N-2)\int_{\mathbb{R}^N} K u_\mu^{(N+2)/(N-2)}\varphi\, dx = \theta_\mu \to 1$ as $\mu \to \lambda_0$. We introduce $v_\mu = u_\mu + h\theta_\mu^{-1}\varphi$, clearly $v_\mu \in X$ and

$$\begin{cases} J(v_\mu) \geqslant J(u_\mu) + \dfrac{N+2}{N-2}\displaystyle\int_{\mathbb{R}^N} K u_\mu^{(N+2)/(N-2)}(h\theta_\mu^{-1}\varphi)\, dx = \mu + h = \lambda \\ \mathcal{E}(v_\mu) \leqslant \mathcal{E}(u_\mu) + Ch \end{cases}$$

for some $C$ independent of $h \in ]0,1[$. The properties of $I_\lambda$ as a function of $\lambda$ then yield: $I_\lambda \leqslant \mathcal{E}(v_\mu) \leqslant \mathcal{E}(u_\mu) + Ch = I_\mu + Ch$. On the other hand: $I_\lambda = I_\mu + I_h^\infty$; and we reach a contradiction for $h$ small enough since $I_h^\infty = I_1^\infty h^{(N-2)/N}$.

## I.6 Nonlinear field equations and limit exponents

As we explained in the introduction, one is interested in the so-called ground state solution of

$$-\Delta u = f(u) \quad \text{in} \quad \mathbb{R}^N, \qquad u(x) \to 0 \quad \text{as} \quad |x| \to \infty; \tag{9}$$

where $u$ is (for example) a scalar function. The ground state is determined through the minimum, if it exists, of the following minimization problem

$$I = \operatorname{Inf}\left\{\int_{\mathbb{R}^N} |Du|^2\, dx \Big/ \int_{\mathbb{R}^N} F(u)\, dx = 1,\, u \in \mathcal{D}^{1,2}(\mathbb{R}^N),\, F(u) \in L^1(\mathbb{R}^N)\right\} \tag{10}$$

where $N \geqslant 3$, $F \in C(\mathbb{R})$, $F(0) = 0$. For more details concerning the relations between (9) and (10), we refer to H. Berestycki and P. L. Lions [6].

We assume

$$\exists \zeta \in \mathbb{R}, \qquad F(\zeta) > 0 \tag{11}$$

$$\lim_{|t| \to 0+} F^+(t)|t|^{-2N/(N-2)} = \alpha \geqslant 0, \tag{12}$$

$$\lim_{|t| \to \infty} F^+(t)|t|^{-2N/(N-2)} = \beta \geqslant 0;$$

and we denote

$$\mathcal{E}(u) = \int_{\mathbb{R}^N} |Du|^2\, dx, \qquad J(u) = \int_{\mathbb{R}^N} F(u)\, dx.$$

If $u_\sigma(\cdot) = \sigma^{-(N-2)/N} u(\cdot/\sigma)$, $\quad \mathcal{E}(u_\sigma) = \mathcal{E}(u)$ and

$$J(u_\sigma) = \sigma^N \int_{\mathbb{R}^N} F(\sigma^{-(N-2)/N} u(x))\, dx.$$

Therefore

$$J(u_\sigma) \to \beta \int_{\mathbb{R}^N} |u|^{2N/(N-2)}\, dx \quad \text{as} \quad \sigma \to 0_+,$$

while

$$J(u_\sigma) \to \alpha \int_{\mathbb{R}^N} |u|^{2N/(N-2)}\, dx \quad \text{as} \quad \sigma \to +\infty.$$

And this yields

$$I \leqslant I^\infty = \operatorname{Inf}\{\mathcal{E}(u)/\gamma \int_{\mathbb{R}^N} |u|^{2N/(N-2)}\, dx = 1, u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\}$$

with $\gamma = \max(\alpha, \beta)$; or

$$I \leqslant I^\infty = \gamma^{-(N-2)/N} I^0 \tag{62}$$

of course if $\gamma = 0$ i.e. $\alpha = \beta = 0$, $I^\infty = +\infty$ and the inequality is strict.

**Theorem I.5.** *Under assumptions* (11), (12), *any minimizing sequence* $(u_n)_n$ *of* (10) *is relatively compact in* $\mathcal{D}^{1,2}(\mathbb{R}^N)$ *up to a translation if and only if*: $I < I^\infty$. *In particular if this strict inequality holds, there exists a minimum of* (10).

*Remark* I.10. If $\gamma = 0$, $I < I^\infty = +\infty$ and we recover the most general existence result —for the ground state— due to H. Berestycki and P. L. Lions [6], H. Brézis and E. H. Lieb [11]: in [6], this result was proved by a symmetrization argument which does not show that all minimizing sequences are relatively compact up to translations. Of course if $\gamma = 0$, we are in the locally compact case and the result of P. L. Lions [21] also applies to that particular situation. Let us also mention that the fact that minima of (10) yield ground states of (9) is due to Coleman, Glazer and Martin [13] —see also [6], [21]—. Except for a particular case (covered by the result above) due to F. V. Atkinson and L. A. Peletier [2] obtained by an O.D.E. method, the above result is the first where $F$ is allowed to behave like $|t|^{2N/(N-2)}$ near 0 or at infinity.

*Remark* I.11. Combining the method below and those of P. L. Lions [21], we could treat as well $x$-dependent functionals or higher-order functionals. Let us also mention that the same result holds if $u$ takes its value in $\mathbb{R}^m$ ($m \geqslant 1$), then we just need to assume (11) and

$$\begin{cases} \lim\limits_{|t| \to 0_+} F^+(t) F_0(t)^{-1} = \alpha \geqslant 0 \\ \lim\limits_{|t| \to \infty} F^+(t) F_1(t)^{-1} = \beta \geqslant 0 \end{cases} \tag{63}$$

where $F_0$, $F_1$ are continuous, positive on $\mathbb{R}^m - \{0\}$, homogeneous of degree $2N/(N-2)$; $I^\infty$ is given in this situation by

$$I^\infty = \min_{i=0,1} [\mathrm{Inf}\{ \int_{\mathbb{R}^N} |\nabla u|^2 \, dx / u \in (\mathcal{D}^{1,2}(\mathbb{R}^N))^m, \int_{\mathbb{R}^N} F_i(u) \, dx = 1 \}].$$

Observe that both infima are achieved by the results of section I.4.

*Remark* I.12. If $\alpha = \beta = \gamma > 0$, then by Theorem I.1, there exists $u_0$ such that

$$u_0 \in \mathcal{D}^{1,2}(\mathbb{R}^N), \ \mathcal{E}(u_0) = I^\infty, \ \int_{\mathbb{R}^N} |u_0|^{2N/(N-2)} \, dx = \gamma^{-1}, \qquad u \geqslant 0.$$

i) If $F(t) \geqslant \gamma |t|^{2N/(N-2)}$ and $F(t) \not\equiv \gamma |t|^{2N/(N-2)}$, then $I < I^\infty$: indeed we observe that, choosing by dilation $u_0$ the maximum of $u_0$ large enough, we may assume

$$\int_{\mathbb{R}^N} F(u_0) \, dx > 1, \ \mathcal{E}(u_0) = I^\infty.$$

Let $v_0(\cdot) = u_0(\cdot / \lambda)$ with

$$\lambda^{-N} = \int_{\mathbb{R}^N} F(u_0) \, dx > 1;$$

then $J(u_0) = 1$ and $I \leqslant \mathcal{E}(u_0) = \lambda^{N-2} \mathcal{E}(u_0) < I^\infty$.

ii) If $F(t) \leqslant \gamma |t|^{2N/(N-2)}$ and $F \not\equiv \gamma |t|^{2N/(N-2)}$, a similar argument shows that there does not exist a minimum of (10).

*Remark* I.13. If $\alpha = \gamma \geqslant \beta$, $\alpha > 0$ and if for some $t_0 > 0$

$$F(t) \geqslant \alpha |t|^{2N/(N-2)} \quad \text{for} \quad t \in [0, t_0] \qquad (\text{or } t \in [-t_0, 0])$$

and $F \not\equiv \alpha |t|^{2N/(N-2)}$ on $[0, t_0]$, considering $\sigma^{-(N-2)/N} u_0(\cdot / \sigma)$ with $u_0$ as in Remark I.12 and $\sigma$ large enough, we deduce from the argument given above that $I < I^\infty$.

*Remark* I.14. By looking carefully at the proof below, we see that if $I = I^\infty$ and if $(u_n)_n$ is not relatively compact in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ then

i) if $\alpha < \beta$, there exist $(\sigma_n)_n$ in $]0, \infty[$, $(y_n)_n$ in $\mathbb{R}^N$ such that: $\sigma_n \underset{n}{\to} \infty$, $\tilde{u}_n = \sigma_n^{-(N-2)/N} u_n((\cdot - y_n)/\sigma_n)$ is relatively compact in $\mathcal{D}^{1,2}$ and the limits of its converging subsequences are minima of $I^\infty$;

ii) if $\alpha > \beta$, the above still holds but with $\sigma_n \underset{n}{\to} 0$;

iii) if $\alpha = \beta$, the above still holds but $(\sigma_n)_n$ is arbitrary.

*Remark* I.15. Of course, if we are only interested in finding a (non trivial) solution of (9), we may use the maximum principle and assume: $f = F'$,

$F \in C^1(\mathbb{R})$ (for example), (11), and if $\zeta_\pm = \inf(\zeta > 0, F(\pm\zeta) > 0)$ we assume in addition

$$\lim_{|t| \to 0_+} F^+(t)|t|^{-2N/(N-2)} = \alpha \geqslant 0$$

$$\text{either } \exists \zeta'_+ > \zeta_+, f(\zeta'_+) \leqslant 0, \quad \text{or} \quad \lim_{|t| \to +\infty} F^+(t)|t|^{-2N/(N-2)} = \beta \geqslant 0$$

and

$$\text{either } \exists \zeta'_- > \zeta_-, f(-\zeta'_-) \geqslant 0, \quad \text{or} \quad \lim_{|t| \to \infty} F^+(t)|t|^{-2N/(N-2)} = \beta \geqslant 0.$$

*We now turn to the proof of Theorem* I.5: first we observe that $(u_n)_n$ is bounded in $\mathfrak{D}^{1,2}(\mathbb{R}^N)$ and $F(u_n)$ is bounded in $L^1$, for any minimizing sequence $(u_n)_n$.

Indeed, since $\mathcal{E}(u_n)$ is bounded, $\nabla u_n$ is bounded in $L^2$ and $u_n$ is bounded in $L^{2N/(N-2)}$. But (12) implies the existence of a constant $C \geqslant 0$ such that: $F^+(t) \leqslant C|t|^{2N/(N-2)}$; hence $F^+(u_N)$ is bounded in $L^1$ and using the constraint, the bounds claimed are proved. The proof below will use the concentration-compactness method of P. L. Lions [20], [21] with the sequence:

$$\rho_n = |\nabla u_n|^2 + |u_n|^{2N/(N-2)} + |F(u_N)|$$

We may of course assume that: $\int_{\mathbb{R}^N} \rho_n \, dx \underset{n}{\to} M > 0$. In what follows we will still denote by $u_n$ all subsequences we extract. With these preliminaries, we prove below: Step 1, $\rho_n$ does not vanish; Step 2, dichotomy does not occur; Step 3, weak limits are non trivial; Step 4, we conclude.

*Step* 1: $\rho_n$ *does not vanish.*

Indeed if $\rho_n$ vanishes i.e. if (in particular) there exists $R \in ]0, \infty[$ such that

$$\sup_{y \in \mathbb{R}^N} \int_{y + B_R} \rho_n \, dx \underset{n}{\to} 0$$

then we denote by $G(t) = (F(t) - \gamma|t|^{2N/(N-2)})^+$ and we claim that

$$G(u_n) \underset{n}{\to} 0 \quad \text{in} \quad L^1(\mathbb{R}^N).$$

To this end we just need to observe that $G$ satisfies

$$\lim_{|t| \to 0_+} G(t)|t|^{-2N/(N-2)} = 0, \lim_{|t| \to \infty} G(t)|t|^{-2N/(N-2)} = 0.$$

Therefore, using Lemma II.2 of P. L. Lions [21], we deduce that $G(u_n) \underset{n}{\to} 0$ in $L^1(\mathbb{R}^N)$.

But this yields

$$\lim_n \gamma \int_{\mathbb{R}^N} |u_n|^{2N/(N-2)}\, dx \geqslant 1,\ \mathcal{E}(u_n) \underset{n}{\to} I$$

and this contradicts the strict inequality $I < I^\infty$. Hence if $I < I^\infty$, vanishing does not occur.

*Step 2: Dichotomy does not occur.*

In view of the concentration-compactness lemma (Lemma I.1 in P. L. Lions [20]), we check that dichotomy does not occur. To this end we first remark that if we replace 1 by $\lambda > 0$ in (10) and if we denote by $I_\lambda$ the corresponding infimum, then

$$I_\lambda = \operatorname{Inf}\left\{ \mathcal{E}u\left(\frac{\cdot}{\lambda^{1/N}}\right) \Big/ J(u) = 1, u \in \mathcal{D}^{1,2}(\mathbb{R}^N), F(u) \in L^1 \right\} = \lambda^{(N-2)/N} I$$

therefore the subadditivity inequality (S. 2) holds

$$I < I_\alpha + I_{1-\alpha}, \qquad \forall \alpha \in ]0,1[. \tag{S. 2}$$

We now prove —as in P.L. Lions [20], [21], see also P.L. Lions [23], [24]— that dichotomy does not occur by contradiction. We will use a variant of the explicit dichotomy procedure of [20], [21] in order to cover the full generality of functions $F$: the idea of this variant was given to us by H. Brézis (see also H. Brézis and E.H. Lieb [11]). If dichotomy occurs we find $\alpha \in ]0, M[$ such that for any fixed $\epsilon > 0$, there exist $(y_n)_n$ in $\mathbb{R}^N$, $0 < R_0 < \infty$, $R_n$ in $]R_0, +\infty[$ satisfying

$$\begin{cases} \left| \int_{y_n + B_{R_0}} \rho_n\, dx - \alpha \right| \leqslant \epsilon, & \left| \int_{|x-y_n| \geq R_n} \rho_n\, dx - (M-\alpha) \right| \leqslant \epsilon \\ \int_{R_0 \leq |x-y_n| \leq R_n} v_n\, dx \leqslant \epsilon, & R_n \underset{n}{\to} \infty. \end{cases} \tag{64}$$

If we still denote by $(u_n)_n$ the minimizing sequence translated by $y_n$, we are going to "cut" $u_n$ in two pieces such that both functionals $\mathcal{E}, J$ split in the sums of the corresponding functionals.

To this end, we introduce for $\lambda \geqslant 1$, $R \in ]0, \infty[$ the mapping: $Tx = x$ if $|x| \leqslant R$, $Tx = \lambda x - (\lambda - 1)Rx|x|^{-1}$, and we set $u_n^1(x) = u_n(Tx)$. We now compute $J(u_n^1)$, $\mathcal{E}(u_n^1)$

$$\begin{aligned} J(u_n^1) &= \int_{|x| \leq R} F(u_n)\, dx + \int_{|x| > R} F(u_n(Tx))\, dx \\ &= \int_{|x| \leq R} F(u_n)\, dx + \int_{|y| > R} F(u_n(y))\phi(y)\, dy \end{aligned}$$

with $\phi(y)^{-1} = \lambda(\lambda + R|T^{-1}y|^{-1}(1-\lambda))^{N-1} \geqslant \lambda$ if $|y| \geqslant R$.

Hence we deduce

$$\left| J(u_n^1) - \int_{|x| \le R} F(u_n)\, dx \right| \leqslant \frac{C}{\lambda} \tag{65}$$

Next, we compute $\mathcal{E}(u_n^1)$

$$\mathcal{E}(u_n^1) = \int_{|x| \le R} |\nabla u_n|^2\, dx + \int_{|x| \ge R} |\nabla T \cdot \nabla u_n(Tx)|^2\, dx$$

and

$$T_{i,j} = \lambda \delta_{ij} + (\lambda - 1) \frac{R}{|x|^3} (x_i x_j - \delta_{ij} |x|^2),$$

therefore

$$\begin{aligned} \int_{|x| \ge R} |\nabla T \cdot \nabla u_n(Tx)|^2\, dx &\leqslant C\lambda \int_{|y| \ge R} |\nabla u_n(y)|^2 \phi(y)\, dy \\ &\leqslant C \int_{R_n \ge |y| \ge R} |\nabla u_n|^2\, dy + C(\lambda - (\lambda - 1)R/\bar{R}_n)^{1-N} \end{aligned}$$

if $R \leqslant R_n$, with $\bar{R}_n = \frac{1}{\lambda}(R_n + (\lambda - 1)R)$. Thus we obtain

$$\begin{cases} \left| \mathcal{E}(u_n^1) - \int_{|x| \le R} |\nabla u_n|^2\, dx \right| \leqslant C\epsilon + C(\lambda - (\lambda - 1)R/\bar{R}_n)^{1-N} \\ \text{if} \quad R_0 \leqslant R \leqslant R_n, \qquad \bar{R}_n = \frac{1}{\lambda}(R_n + (\lambda - 1)R). \end{cases}$$

And thus choosing $R = R_0$, $\lambda$ large enough, we find for $n$ large

$$\begin{cases} \left| J(u_n^1) - \int_{|x| \le R_0} F(u_n)\, dx \right| \leqslant \epsilon, \\ \left| \mathcal{E}(u_n^1) - \int_{|x| \le R_0} |\nabla u_n|^2\, dx \right| \leqslant C\epsilon. \end{cases} \tag{67}$$

We build $u_n^2$ in a similar way: we consider the mapping

$$Sx = \mu x \quad \text{if} \quad |x| \leqslant R, \qquad = x + (\mu - 1)Rx|x|^{-1} \quad \text{if} \quad |x| \geqslant R,$$

where $\mu \geqslant 1$, $R \geqslant R_0$; and we denote by: $u_n^2(x) = u_n(Sx)$. We have

$$J(u_n^2) = \mu^{-N} \int_{|y| \le \mu R} F(u_n)\, dy + \int_{|y| \ge \mu R} F(u_n(y)) \psi(y)\, dy$$

with $\psi(y)^{-1} = (1 + (\mu - 1)R|S^{-1}y|^{-1})^{N-1}$. Therefore if $\mu R \leqslant R_n$

$$\begin{aligned} \left| J(u_n^2) - \int_{|y| \ge \mu R} F(u_n(y))\, dy \right| &\leqslant C\mu^{-N} + 2 \int_{\mu R \le |y| \le R_n} |F(u_n)|\, dy + \\ &\quad + C|(1 + (\mu - 1)R/\tilde{R}_n)^{-(N-1)} - 1|; \end{aligned}$$

where $\tilde{R}_n = R_n - (\mu - 1)R$.

On the other hand, we have

$$\mathcal{E}(u_n^2) = \mu^{-(N-2)} \int_{|y| \le \mu R} |\nabla u_n|^2\, dy + \int_{|x| \le R} |\nabla S \cdot \nabla u_n(Sx)|^2\, dx$$

and

$$S_{i,j} = \delta_{ij} + (\mu - 1)\frac{R}{|x|^3}(|x|^2\delta_{ij} - x_i x_j).$$

Therefore if $\mu R \leqslant R_n$

$$\left|\int_{|x|\geq R} |\nabla S \cdot \nabla u_n(Sx)|^2\,dx - \int_{|y|\geq \mu R} |\nabla u_n(y)^2|dy\right| \leqslant$$

$$\leqslant C\mu \int_{\mu R\leq |y|\leq R_n} |\nabla u_n(y)|^2\,dy +$$

$$+ \int_{|y|\geq R_n} ||\nabla S(S^{-1}y)\cdot \nabla u_n(y)|^2\psi(y) - |\nabla u_n|^2(y)|dy.$$

$$\leqslant C\mu \int_{\mu R\leq |y|\leq R_n} |\nabla u_n|^2\,dy + C(\mu - 1)\frac{R}{\tilde{R}_n} + \int_{|y|\geq R_n} |\nabla u_n|^2\{|\psi(y) - 1|\}\,dy.$$

Finally we obtain if $R_0 \leqslant \mu R \leqslant R_n$

$$\left|\mathcal{E}(u_n^2) - \int_{|y|\geq \mu R} |\nabla u_n|^2\,dy\right| \leqslant C\mu^{2-N} + C\epsilon\mu + C(\mu - 1)\frac{R}{\tilde{R}_n} +$$

$$+ C\{1 - (1 + (\mu - 1)R/\tilde{R}_n)^{-(N-1)}\}. \qquad (69)$$

Combining (68), (69) and choosing $\mu = 1/\sqrt{\epsilon}$, $R = R_0$ we deduce finally that for $n$ large enough

$$\begin{cases} \left|J(u_n^2) - \int_{|y|\geq R_n} F(u_n)\,dy\right| \leqslant C\epsilon, \\ \left|\mathcal{E}(u_n^2) - \int_{|y|\geq R_n} |\nabla u_n|^2\,dy\right| \leqslant C\sqrt{\epsilon}. \end{cases} \qquad (70)$$

We may now conclude: indeed if $J(u_n^1) \underset{n}{\to} \bar{\beta}$, we claim that $\bar{\beta}$ —which depends on $\epsilon$— belongs to $]0, 1[$ and remains bounded away from 0 or 1 as $\epsilon$ goes to 0. Indeed if $\bar{\beta} = \bar{\beta}_\epsilon \underset{\epsilon}{\to} \bar{\beta} = 0$, this means that:

$$\underline{\lim}_{\epsilon}\,\underline{\lim}_{n}\,\mathcal{E}(u_n^2) \geqslant I,$$

while (67) and (70) yield

$$\underline{\lim}_{\epsilon}\,\underline{\lim}_{n}\,\mathcal{E}(u_n^2) + \underline{\lim}_{\epsilon}\,\underline{\lim}_{n}\,\mathcal{E}(u_n^1) \leqslant I;$$

and we reach a contradiction since

$$\underline{\lim}_{\epsilon}\,\underline{\lim}_{n}\,\mathcal{E}(u_n) > 0.$$

If this were not the case we would have

$$\alpha = \underline{\lim_{\epsilon}}\,\underline{\lim_{n}} \int_{B_{R_0}} \rho_n\,dx = \underline{\lim_{\epsilon}}\,\underline{\lim_{n}} \int_{B_{R_0}} |F(u_n)|\,dx$$
$$= \underline{\lim_{\epsilon}}\,\underline{\lim_{n}} \int |F(u_n^2)|\,dx$$
$$= \underline{\lim_{\epsilon}}\,\underline{\lim_{n}} \int -F(u_n^2) + 2F^+(u_n^2)\,dx = 0.$$

If $\bar{\beta}_\epsilon \underset{\epsilon}{\to} \bar{\beta} < 0$, then for $\epsilon$ small and $n$ large, $J(u_n^2) < 1$ and we deduce from (67) and (70): $J(u_n{}^2) < 1$ and $\mathcal{E}(u_n^2) < I$, and this is not possible.

Finally if $\bar{\beta}_\epsilon \underset{\epsilon}{\to} \bar{\beta} \geqslant 1$, we argue as before replacing $u_n^2$ by $u_n^1$. Thus we may assume that $\bar{\beta}_\epsilon \underset{\epsilon}{\to} \bar{\beta} \in ]0, 1[$. We then deduce from (67) and (70) for $\epsilon$ small

$$I \geqslant \underline{\lim_{n}}\, \mathcal{E}(u_n^1) + \underline{\lim_{n}}\, \mathcal{E}(u_n^2) - C\sqrt{\epsilon} \geqslant I_{\bar{\beta}} = I_{1\bar{\beta}} - \delta(\epsilon)$$

where $\delta(\epsilon) \to 0$ as $\epsilon \to 0_+$, and this contradicts (S. 2).

Therefore, from Step 1 and the above contradiction we deduce, using the concentration-compactness lemma of P. L. Lions [20], [21] that, if $I < I^\infty$, there exists $(y_n)_n$ in $\mathbb{R}^N$ such that $\rho_n(\cdot - y_n)$ is tight

$$\forall \epsilon > 0, \exists R < \infty, \forall n \geqslant 1, \int_{|x - y_n| \geq R} \rho_n\,dx \leqslant \epsilon. \tag{71}$$

We still denote by $u_n$ the new minimizing sequence $u_n(\cdot - y_n)$. We may assume that $u_n$ converges weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and a.e. on $\mathbb{R}^N$ to some $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ (and $F(u) \in L^1(\mathbb{R}^N)$ by Fatou's lemma).

*Step* 3: $u \not\equiv 0$ *if* $I < I^\infty$.

If $u \equiv 0$, then we claim that $G(u_n) = (F(u_n) - \gamma|u_n|^{2N/(N-2)})^+$ converges to 0 in $L^1(\mathbb{R}^N)$. Indeed since $F(t) \leqslant C|t|^{2N/(N-2)}$, we may find, using (71), $R$ large enough such that

$$\int_{|x| \geq R} G(u_n)\,dx \leqslant \epsilon, \qquad \forall n \geqslant 1.$$

Now on $B_R$, we use the fact that $u_n \underset{n}{\to} 0$ in $L^1(B_R)$ and that

$$G(t) \leqslant \epsilon|t|^{2N/(N-2)} + C_\epsilon t, \qquad \forall t \in \mathbb{R}.$$

Therefore the claim is proved and we show exactly as in Step 1 that we reach a contradiction with $I < I^\infty$.

*Step* 4: *Conclusion.*

We first assume that $I < I^\infty$ and thus $u \not\equiv 0$. We just need to show that $J(u) = 1$. Of course $J(u) \leqslant 1$, since $\mathcal{E}(u) \leqslant I$.

By lemma I.1, we know there exist $(\nu_k)_{k\in K} \in ]0, \infty[$, $(x_k)_{k\in K}$ in $\mathbb{R}^N$ —where $K$ is at most countable and the points $x_k$ are distinct— such that

$$|u_n|^{2N/(N-2)} \underset{n}{\rightharpoonup} |u|^{2N/(N-2)} + \sum_k \nu_k \delta_{x_k} \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^N)$$

and

$$\int_{\mathbb{R}^N} |u_n|^{2N/(N-2)}\, dx \underset{n}{\rightarrow} \int_{\mathbb{R}^N} |u|^{2N/(N-2)}\, dx + \sum_k \nu_k .$$

We claim that

$$1 = \lim_n J(u_n) \leqslant J(u) + \beta \sum_k \nu_k . \tag{72}$$

We first choose $R$ —using (71)— such that for all $n \geqslant 1$

$$\begin{cases} \int_{|x| \geq R} |F(u_n)| + |u_n|^{2N/(N-2)}\, dx \leqslant \epsilon, \\ \int_{|x| \geq R} |F(u)| + |u|^{2N/(N-2)}\, dx \leqslant \epsilon \\ \sum_{k:x_k \notin B_R} \nu_k \leqslant \epsilon . \end{cases}$$

To simplify the notations we assume that $x_k \in B_R$, $\forall k \in K$. We next apply Brézis-Lieb [10] to obtain

$$\int_{B_R} |F(u_n) - F(u) - F(u_n - u)|\, dx \underset{n}{\rightarrow} 0;$$

this is possible in view of the following observation: $\forall \epsilon > 0$, $\exists C_\epsilon \geqslant 0$

$$|F(a + b) - F(a)| \leqslant \epsilon |a|^{2N/(N-2)} + C_\epsilon (1 + |b|^{2N/(N-2)})$$

for all $a, b \in \mathbb{R}$. Then (72) is proved provided we show

$$\varliminf_n \int_{B_R} F(u_n - u)\, dx \leqslant \beta \sum_k \nu_k .$$

But we already know that: $|u_n - u|^{2N/(N-2)} \rightarrow \sum_k \nu_k \delta_{x_k}$, and by the same proof as in Step 3, we conclude

$$\varliminf \int_{B_R} F(u_n - u)\, dx \leqslant \varliminf \beta \int_{B_R} |u_n - u|^{2N/(N-2)}\, dx = \beta \sum_k \nu_k .$$

Using (72) and Lemma I.1, we finally obtain

$$\begin{cases} 1 \leqslant J(u) + \beta \sum_k \nu_k \leqslant J(u) + \gamma \sum_k \nu_k \\ I \geqslant \mathcal{E}(u) + I^0 \sum_k \nu_k^{(N-2)/N} \geqslant \mathcal{E}(u) + I^0 \Big( \sum_k \nu_k \Big)^{(N-2)/N}. \end{cases}$$

If $J(u) \leqslant 0$, $\sum_k \nu_k \geqslant 1/\gamma$ and

$$I > I^0 \Big( \sum_k \nu_k \Big)^{(N-2)/N} \geqslant I^0 \gamma^{-(N-2)/N} = I^\infty.$$

Hence $\alpha = J(u) \in ]0, 1]$ and if $\alpha \in ]0, 1[$

$$I \geqslant I_\alpha + I^0 \gamma^{-(N-2)/N} (1-\alpha)^{(N-2)/N} = I_\alpha + I^\infty_{1-\alpha}$$

But this contradicts the condition (S. 1) which holds here if $I < I^\infty$. Therefore $\alpha = 1$ and the compactness is proved.

On the other hand if $I = I^\infty$, we build a sequence $(u_n)_n$ such that $F(u_n) \in L^1$, $\int_{\mathbb{R}^N} F(u_n)\, dx \underset{n}{\to} 1$, $\mathcal{E}(u_n) \underset{n}{\to} I^\infty$ and $u_n$ is not compact even up to a translation. Indeed if $\gamma = \alpha \geqslant \beta$, and $\gamma > 0$, we consider $u_0 \in C_0(\mathbb{R}^N)$ satisfying (cf. section I.1)

$$\alpha \int_{\mathbb{R}^N} |u_0|^{2N/(N-2)}\, dx = 1, \qquad \mathcal{E}(u_0) = I^\infty.$$

We set $u_n = n^{-(N-2)/N} u_0(\cdot/n)$ and we check easily the above properties. On the other hand if $\gamma = \beta > \alpha$ and $\beta > 0$, we take $(u_n)_n$ in $\mathcal{D}(\mathbb{R}^N)$, such that: Supp $u_n = \bar{B}_{1/n}$, $|\nabla u_n|^2 \to I^\infty \delta_0$, $\beta \int_{\mathbb{R}^N} |u_n|^{2N/(N-2)}\, dx = 1$. Again it is easy to check the above properties. And Theorem I.6 is proved.

*Remark* I.16. Using the particular quadratic structure of $\mathcal{E}$(and the $x$-independence of the functionals) one may give in Step 4 a slightly simpler argument: indeed if $u_n \underset{n}{\rightharpoonup} u$, then

$$\begin{cases} \mathcal{E}(u_n) - \mathcal{E}(u_n - u) = 2 \int_{\mathbb{R}^N} \nabla u_n \cdot \nabla u\, dx - \int_{\mathbb{R}^N} |\nabla u|^2\, dx \underset{n}{\to} \mathcal{E}(u) \\ J(u_n) - J(u_n - u) \underset{n}{\to} J(u) \end{cases}$$

and if $J(u) \in ]0, 1[$ we simply use (S. 2) to conclude.

However this argument is very dependent on the special structures of $\mathcal{E}, J$ and fails completely if $\mathcal{E}$ is $x$-dependent or if

$$\mathcal{E}(u) = \int_{\mathbb{R}^N} |\nabla u|^2\, dx \quad \text{is replaced by} \quad \int_{\mathbb{R}^N} |\nabla u|^p\, dx \quad \text{for} \quad p \neq 2!$$

### I.7 A remark on Moser's treatment of the limiting case of Sobolev inequalities

We want to discuss here a few properties of functions in $W_0^{1,N}(\Omega)$ where $\Omega$ is a bounded open set in $\mathbb{R}^N$ and $N \geqslant 2$. Clearly the Sobolev exponent becomes infinite but $W_0^{1,N}(\Omega)$ is not embedded in $L^\infty(\Omega)$. It is possible to check that if $|\nabla u|_{L^N(\Omega)} \leqslant 1$, there exists some $\alpha > 0$ (independent of $u$) such that

$$\int_\Omega \exp\{\alpha|u|^{N/(N-1)}\}\, dx \leqslant C.$$

This was proved by N. Trudinger [37] (see also S. I. Pohozaev [30], T. Aubin [5]...). This estimate was sharpened by J. Moser [27] who proved that if $\alpha_N = N\omega_{N-1}^{1/(N-1)}$ where $\omega_k$ is the volume of $S^k$ (for example $\alpha_2 = 4\pi$) then we have

$$\int_\Omega \exp\{\alpha_N|u|^{N/(N-1)}\}\, dx \leqslant C|\Omega| \tag{73}$$

and $\alpha_N$ is the best constant in the following sense: $\exp(\alpha|u|^{N/(N-1)}) \in L^1(\Omega)$ for any $\alpha > 0$ but $\alpha_N$ is the biggest constant such that $\exp(\alpha|u|^{N/(N-1)})$ is bounded in $L^1(\Omega)$ independently of $u$. In other words $W_0^{1,N}(\Omega)$ is embedded in the Orlicz space determined by $\phi(t) = \exp\{\alpha_N|t|^{N/(N-1)}\}$.

A natural question is then: is this embedding compact? The answer is no: indeed, if for example $\Omega$ is the unit ball, we consider $(u_n)_n$ defined by

$$u_n(x) = f_n(-N\operatorname{Log}|x|); \qquad f_n(t) = \left(\frac{n}{\alpha_N}\right)^{(N-1)/N}\frac{t}{n} \quad \text{if} \quad t \leqslant n,$$

$$= \left(\frac{n}{\alpha_N}\right)^{(N-1)/N} \quad \text{if} \quad t \geqslant n.$$

Clearly

$$|\nabla u_n|_{L^N}^N = N^{N-1}\omega_{N-1}\int_0^\infty |f_n'(t)|^N\, dt$$
$$= N^{N-1}\omega_{N-1}\int_0^n \left(\frac{n}{\alpha_N}\right)^{N-1}\left(\frac{1}{n}\right)^N dt = 1;$$

and

$$|\exp\{\alpha_N|u_n|^{N/(N-1)}\}|_{L^1} = \omega_{N-1}N^1\int_0^\infty \exp\{\alpha_N(f_n(t))^{N/(N-1)} - t\}\, dt \geqslant$$
$$\geqslant \omega_{N-1}N^{-1}\int_n^\infty \exp\{n - t\}\, dt = \omega_{N-1}N^{-1}.$$

Since $u_n \underset{n}{\rightarrow} 0$ a.e. and weakly in $W_0^{1,N}(\Omega)$, the embedding is not compact. Observe that $|\nabla u_n|^N \to \delta_0$ in $\mathcal{D}'(\Omega)$ and $\exp\{\alpha_N|u_n|^{N/(N-1)}\} \to c\delta_0$ for some $c > 0$.

The following result shows that this is the exceptional case

**Theorem I.6.** *Let $(u_n)_n \subset W_0^{1,N}(\Omega)$ satisfy: $|\nabla u_n|_{L^N(\Omega)} \leqslant 1$. Without loss of generality we may assume that $u_n \underset{n}{\rightharpoonup} u$, $|\nabla u_n|^2 \underset{n}{\rightharpoonup} \mu$ weakly. Then either $\mu = \delta_{x_0}$ for some $x_0 \in \bar{\Omega}$ and $u_n \underset{n}{\rightharpoonup} 0$, $\exp(\alpha_N |u_n|^{N/(N-1)}) \underset{n}{\rightharpoonup} c\delta_{x_0}$ for some $c \geqslant 0$, or there exists $\alpha > 0$ such that $\exp\{(\alpha_N + \alpha)|u_n|^{N/(N-1)}\}$ is bounded in $L^1(\Omega)$ and thus*

$$\exp\{\alpha_N |u_n|^{N/(N-1)}\} \underset{n}{\rightarrow} \exp\{\alpha_N |u|^{N/(N-1)}\} \quad in \quad L^1(\Omega).$$

*In particular this is the case if $u \not\equiv 0$.*

*Remark* I.17. We also deduce from this result that except for "small weak neighborhoods of 0" the embedding is compact and the best constant $\alpha_N$ may be improved.

PROOF OF THEOREM I.6. *We first treat the case when $u \equiv 0$.* Let $\xi \in C^1(\bar{\Omega})$, we have using Rellich theorem

$$|\nabla(\xi u_n)|_{L^N}^N = \int_\Omega |(\nabla\xi)u_n + \xi\nabla u_n|^N \, dx \underset{n}{\rightarrow} \int |\xi|^N \, d\mu$$

since $u_n \underset{L^N}{\rightarrow} 0$ (strongly). Without loss of generality we may assume that $|\nabla u_n|_{L^N} = 1$ (consider $v_n = u_n |\nabla u_n|_{L^N}^{-1}$), hence $\int d\mu = 1$ and $\operatorname{Supp} \mu \subset \bar{\Omega}$.

We first observe that if $\xi \in C^1(\bar{\Omega})$, $\xi \geqslant 0$ and $\int |\xi|^N d\mu = 1$, then $\exp\{\alpha_N |u_n|^{N(N-1)}\}$ is bounded in $L^p(\{\xi \geqslant 1 + \delta\})$ for $p = p_\delta > 1$, $\delta > 0$. In particular $\exp\{\alpha_N |u_n|^{N/(N-1)}\}$ converges to 1 in $L^1(\{\xi \geqslant 1 + \delta\})$ for any $\delta > 0$. Indeed $|\nabla(\xi u_n)|_{L^N} \underset{n}{\rightarrow} 1$ and

$$\int_\Omega \exp\{\alpha_N |\xi u_n|^{N/(N-1)} |\nabla(\xi u_n)|_{L^N}^{-N/(N-1)}\} \, dx \leqslant C.$$

thus for any $\gamma \in ]1, (1 + \delta)^{N/(N-1)}[$ we have for $n$ large enough

$$\int_{(\xi \geq 1+\delta)} \exp\{\alpha_N \gamma |u_n|^{N/(N-1)}\} \, dx < C. \tag{74}$$

Next if $\mu = \delta_{x_0}$ for some $x_0 \in \bar{\omega}$, taking $\xi$ with $\xi(x_0) = 1$, $\xi > 1$ on $\bar{\omega} - \{x_0\}$, and remarking that $\nabla u_n \to 0$ weakly in $L^N(\Omega)$ and thus $u_n \to 0$ weakly in $W_0^{1,N}(\Omega)$, we deduce

$$\exp\{\alpha_N |u_n|^{N/(N-1)}\} \to c\delta_{x_0}, \quad \text{for some} \quad c \geqslant 0.$$

On the other hand if $\mu$ is not a Dirac mass, we claim that we can find $F_1, F_2$ compact contained in $\bar{\Omega}$ such that

$$\mu(F_1), \mu(F_2) \in ]0, 1[ \quad \text{and} \quad F_1 \cup F_2 = \bar{\Omega}.$$

Indeed if $\mu$ is not a Dirac mass, there exists $F$ compact contained in $\bar{\Omega}$ such that: $\mu(F) = \theta \in ]0, 1[$. We denote by $O = \mathbb{R}^N - F$, $O_\epsilon = \{x \in \mathbb{R}^N, \operatorname{dist}(x, F) > \epsilon\}$. Clearly —considering $\mu$ as a measure on $\mathbb{R}^N$ supported in $\Omega$— we have:

$\mu(O) = 1 - \theta = \lim_{\epsilon \downarrow 0} \uparrow \mu(O_\epsilon)$. Hence, there exists $\epsilon$ small enough such that: $\mu(O_\epsilon) \in ]0, 1[$.

Then if $F_1 = \bar{\Omega} \cap O_\epsilon^c$, $F_2 = \bar{\Omega} \cap \bar{O}_\epsilon$, clearly $F_1 \cup F_2 = \bar{\Omega}$ and $\mu(F_1) = 1 - \mu(O_\epsilon) \in ]0, 1[$, $\mu(O_\epsilon) \leqslant \mu(F_2) \leqslant \mu(O)$.

But we may now consider $\xi_1, \xi_2 \in C^1(\bar{\Omega})$ satisfying

$$\begin{cases} \xi_1, \xi_2 \geqslant 0, & \xi_1 = \frac{1}{2}(1 + \mu(F_1)^{-1}) \text{ on } F_1, \\ & \xi_2 = \frac{1}{2}(1 + \mu(F_2)^{-1}) \text{ on } F_2, \\ \int \xi_1 \, d\mu = 1, & \int \xi_2 \, d\mu = 1. \end{cases}$$

And using (74) we deduce that for some $\gamma > 1$, we have for $n$ large enough and thus for all $n \geqslant 1$

$$\begin{cases} \int_{F_1} \exp\{\alpha_N \gamma |u_n|^{N/(N-1)}\} \, dx \leqslant C \\ \int_{F_2} \exp\{\alpha_N \gamma |u_n|^{N/(N-1)}\} \, dx \leqslant C \end{cases}$$

and we conclude.

*We next consider the case when* $u \not\equiv 0$ *and* $N = 2$: we claim that $v_n = \exp\{\alpha_2 |u_n|^2\}$ converges to $v = \exp\{\alpha_2 u^2\}$ in $L^p(\Omega)$ for

$$p < \bar{p} = (1 - |\nabla u|_{L^2}^2)^{-1} \qquad (\bar{p} = +\infty \quad \text{if} \quad |\nabla u|_{L^2} = 1).$$

Indeed we have

$$v_n = \exp\{\alpha_2[u^2 + 2u(u_n - u) + (u_n - u)^2]\} = v\bar{v}_n\tilde{v}_n$$

where

$$v = \exp\{\alpha_2 u^2\} \in L^q(\Omega) \qquad (\forall q < \infty), \qquad \bar{v}_n = \exp\{2\alpha_2 u(u_n - u)\}$$

converges to 1 in $L^q(\Omega)$ $(\forall q < \infty)$. Finally remarking that

$$C_n = \int_\Omega |\nabla(u_n - u)|^2 \, dx = 1 - 2\int_\Omega \nabla u_n \cdot \nabla u \, dx + \int_\Omega |\nabla u|^2 \, dx \xrightarrow[n]{} \frac{1}{\bar{p}},$$

we obtain

$$|\exp\{\alpha_2 C_n^{-1}(u_n - u)^2\}|_{L^1} = |\tilde{v}_n^{1/C_n}|_{L^1} \leqslant C$$

and we conclude easily.

*Finally if* $u \not\equiv 0$ *and* $N \geqslant 3$; we claim that $v_n = \exp\{\alpha_N |u_n|^{N/(N-1)}\}$ converges to $v = \exp\{\alpha_N |u|^{N/(N-1)}\}$ in $L^p(\Omega)$ for $p < \bar{p} = (1 - |\nabla u|_{L^N})^{-1/(N-1)}$. Since $v_n \xrightarrow[n]{} v$ a.e., we just need to prove that for all $p < \bar{p}$

$$\int_\Omega \exp\{\alpha_N p |u_n|^{N/(N-1)}\} \, dx \leqslant C \qquad (\text{inf. of } n). \tag{75}$$

By standard symmetrization argument we may assume that $\Omega$ is a ball, $u_n, u$ are spherically symmetric, non increasing with respect to $|x|$. Without loss of generality we may assume that $\Omega$ is the unit ball and we consider —following Moser [27]— $f_n$ defined on $]0, \infty[$ by

$$u_n(x) = f_n(-N \operatorname{Log}|x|), \qquad u(x) = f(-N \operatorname{Log}|x|);$$

$f_n, f$ are continuous, non decreasing and $f_n(0) = f(0) = 0$.

In addition we have for all $\alpha > 0$

$$\begin{cases} 1 = \int_\Omega |\nabla u_n|^N dx = N^{N-1}\omega_{N-1}\int_0^\infty |f_n'(t)|^N dt \\ \int_\Omega \exp\{\alpha|u_n|^{N/(N-1)}\}\, dx = N^{-1}\omega_{N-1}\int_0^\infty \exp\{\alpha|f_n(t)|^{N/(N-1)} - t\}\, dt. \end{cases}$$

We next consider $g_n(t) = (f_n')^*(t)$ the decreasing rearrangement of $f_n'$ on $]0, \infty[$ and we set: $\tilde f_n(t) = \int_0^t g_n(s)\, ds$, $\tilde u_n = \tilde f_n(-N \operatorname{Log}|x|)$. Then we have

$$\begin{cases} \int_\Omega |\nabla \tilde u_n|^N dx = N^{N-1}\omega_{N-1}\int_0^\infty |\tilde f_n'(t)|^N dt = N^{N-1}\omega_{N-1}\int_0^\infty |f_n'(t)|^N dt = 1 \\ \tilde u_n(x) = \int_0^{-N \operatorname{Log}|x|} g_n(s)\, ds \geqslant \int_0^{-N \operatorname{Log}|x|} f_n'(s)\, \mathrm{ds} = u_n(x), \qquad \forall x \in \Omega. \end{cases}$$

In addition $f_n' \to f'$ weakly in $L^N(0, \infty)$ and thus we may assume that $g_n \to g$ weakly in $L^N(0, \infty)$. And if $\tilde u_n \underset{n}{\to} \tilde u$ weakly in $W_0^{1,N}(\Omega)$, then

$$\int_\Omega |\nabla \tilde u|^N dx = N^{N-1}\omega_{N-1}\int_0^\infty |g(t)|^N dt \geqslant N^{N-1}\omega_{N-1}\int_0^\infty |f'(t)|^N dt = \int_\Omega |\nabla u|^N dx$$

Hence, we just need to prove our claim for the new sequence $\tilde u_n$ i.e. we may assume without loss of generality that not only $u_n, u$ are spherically symmetric, non increasing but $f_n'$ is non increasing i.e.

$$u_n'' + \frac{1}{|x|} u_n' \leqslant 0 \quad \text{on} \quad ]0, 1[.$$

But this yields that $\nabla u_n$ is relatively compact in $L^p$ $(\epsilon \leqslant |x| \leqslant R - \epsilon)$ for all $p < \infty$, $\epsilon > 0$ —where $R$ is the radius of $\Omega$. Hence we may assume that $\nabla u_n \to \nabla u$ a.e. in $\Omega$.

All these reductions enable us to adapt the proof made below in the case $N = 2$. Indeed using Brézis-Lieb lemma [10], we deduce that

$$|\nabla(u_n - u)|^N_{L^N(\Omega)} \underset{n}{\to} 1 - |\nabla u|^N_{L^N(\Omega)}$$

thus for $\delta > 0$ small enough and for $n$ large enough

$$\int_\Omega \exp\{\alpha_N p(1 + \delta)|u_n - u|^{N/(N-1)}\}\, dx \leqslant C$$

while $\exp\{\alpha_N |u|^{N/(N-1)}\} \in L^q(\Omega)$ for all $q < \infty$; and this proves (75) and the theorem is proved.

*Remark* I.18. In fact we have proved that:

i) if $u \not\equiv 0$, then $\exp\{\alpha |u_n|^{N/(N-1)}\}$ is bounded in $L^1(\Omega)$ for $0 < \alpha < \alpha_N(1 - |\nabla u|_{L^N}^N)^{-1/(N-1)}$.

ii) if $\mu$ is without atoms, then $\exp\{\alpha |u_n|^{N/(N-1)}\}$ is bounded in $L^1(\Omega)$ for all $\alpha > 0$.

In fact, we can prove by a close examination of the above proof that if we consider $\theta = \max_{x \in \Omega} \mu(\{x\})$ and if $\theta \in [0, 1[$ then $\exp\{\alpha |u_n|^{N/(N-1)}\}$ is bounded in $L^1(\Omega)$ for $0 < \alpha < \alpha_N(1 - \theta)^{-1/(N-1)}$.

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