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The Concentration-Compactness Principle in the Calculus of Variations. The Limit Case, Part. 2

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ABSTRACT. This paper is the second part of a work devoted to the study of variational problems (with constraints) in functional spaces defined on domains presenting some (local) form of invariance by a non-compact group of transformations like the dilations in $\mathbb{R}^N$. This contains for example the class of problems associated with the determination of extremal functions in inequalities like Sovolev inequalities, convolution or trace inequalities... We show how the concentration-compactness principle and method introduced in the so-called locally compact case are to be modified in order to solve these problems and we present applications to Functional Analysis, Mathematical Physics, Differential Geometry and Harmonic Analysis.

Key-words. Concentration-compactness principle, minimization problems, unbounded domains, dilation invariance, concentration function, nonlinear field equations, Dirac masses, Morse theory, Sobolev inequalities, convolution, weak L^p spaces, trace theorems, Yamabe problem, scalar curvature, conformal invariance.

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Summary

1. Introduction

This paper is the second part of a work devoted to variational problems with compactness defects. We use the notations of Part 1 [65] and we refer the reader to Part 1 [65] for a general introduction to the problems studied here.

Finally, formula $(1 - n)$ will mean formula (n) of Part 1 while formula (n) is the n^{th} formula of this paper.

2. Extremal functions in unbounded domains

2.1 *Hardy-Littlewood-Sobolev inequality*

In this section we are mainly concerned with the minimization problem associated to the «best constant» C_0 in the following classical inequality —called Hardy-Littlewood-Sobolev inequality—; see [39], [40], [32], [74]:

$$\|u * |x|^{-\lambda}\|_{L^q} \leqslant C\|y\|_{L^p}, \qquad \forall u \in L^p(\mathbb{R}^N) \tag{2.1}$$

for some C depending only on N, p, q, λ and where:

$$0 < \lambda < N, \quad 1 < p < \frac{N}{N-\lambda}, \qquad \frac{1}{p} + \frac{\lambda}{N} = 1 + \frac{1}{q} \cdot \tag{2.2}$$

Following E. H. Lieb [53], we consider the minimization problem:

$$= \mathrm{Inf}\left\{ -\int_{\mathbb{R}^N} |K * u|^q \, dx \,/\, u \in L^p(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} |u|^p \, dx = 1 \right\}; \tag{2.3}$$

we will also denote by I_μ the corresponding infimum where l is replaced by $\mu > 0$ so that: $I_\mu = \mu^{q/p_I} < 0$. Of course we consider in this section $K(x) = |x|^{-\lambda}$ but our goal is to show how our general method applies in this example and gives the existence of a minimum (and compactness of minimizing sequences) without using the very particular properties of K namely the fact that K is spherically symmetric and decreasing. And this will enable us to treat general classes of potentials $K(x)$.

We prove here the following:

Theorem 2.1. *Under assumption* (2.3), *let* $(u_n)_n$ *be a minimizing sequence of problem* (2.3). *There exist* $(y_n)_n$ *in* $\mathbb{R}^N$, $(\sigma_n)_n$ *in* $]0, \infty[$ *such that the new minimizing sequence* $\tilde{u}_n$ *defined by*:

$$\tilde{u}_n(\,\cdot\,) = \sigma_n^{-N/p} u_n((\,\cdot\, - y_n)/\sigma_n)$$

is relatively compact in $L^p(\mathbb{R}^N)$. *In particular there exists a minimum of problem* (2.3).

Remark 2.1. The existence of a minimum is proved in E. H. Lieb [53], hence the above result is a minor extension of [53], but we emphasize the fact that the proof in [53] relies on symmetrization arguments which prevents any generality on the class of potentials K while our methods does not depend on the particular form of K.

Remark 2.2. Using —a posteriori— the symmetrization, one easily sees that any minimum of (2.3) is spherically symmetric, decreasing (up to a translation)— see [53] for more details. In addition if $p = q'$ or $p = 2$ or $q = 2$, the explicit values of I and of the minima are given in [53].

PROOF OF THEOREM 2.1. Let $(u_n)_n$ be a minimizing sequence of (2.3); since both functionals $\int_{\mathbb{R}^N}|K * u|^q\,dx$, $\int_{\mathbb{R}^N}|u|^p\,dx$ are invariant by the change: $\sigma^{-N/p}u(\cdot/\sigma)$, we have to get rid of the possibility of «vanishing» exactly as we did in Part 1 for Sobolev inequalities (Section 1). We thus consider a new minimizing sequence —that we still denote by u_n— obtained by dilating u_n such that:

$$Q_n(1) = \tfrac{1}{2} \tag{2.4}$$

where

$$Q_n(t) = \operatorname*{Sup}_{y \in \mathbb{R}^N} \int_{y + B_t} \rho_n(x)\,dx \quad (\forall t \geqslant 0), \qquad \rho_n = |u_n|^p.$$

Exactly as in [55], [56], [65], we exclude vanishing by (2.4) and dichotomy as in [55], [65] since $I_\lambda = \lambda^{q/p}I(<0)$ is strictly subadditive ((S.2) holds!) —here since $(u_n)_n$ is only in L^p, we do not have to use smooth cut-off functions to perform the dichotomy and the argument is exactly the one described in [55], [58]. In conclusion, there exists $(y_n)_n$ in $\mathbb{R}^N$ such that $\tilde{u}_n(\cdot) = u_n(\cdot + y_n)$, satisfies: $|\tilde{u}_n|^p$ is tight. In the remainder of the proof, we still denote by u_n the new minimizing sequence $\tilde{u}_n$. We may of course assume that: $u_n \to u$ weakly in $L^p(\mathbb{R}^N)$. Let us also observe that $|K * u_n|^q$ is tight and that $K * u_n \to K * u$ a.e. on $\mathbb{R}^N$: indeed for all $R < M < \infty$:

$$\begin{aligned}\int_{|x| \geqslant M} |K * u_n|^q\,dx &\leqslant C_0 \|u_n\|^q_{L^p(|x| > R)} + \\ &+ \int_{|x| \geqslant M} \left| \int_{|y| \leqslant R} \frac{1}{|x - y|^\lambda} u_n(y)\,dy \right|^q dx \leqslant \\ &\leqslant \epsilon(R) + \int_{|x| \geq M} (|x| - R)^{-\lambda q}\,\mathrm{dx}\, \|u_n\|^q_{L^1(B_R)} \leqslant \\ &\leqslant \epsilon(R) + \delta_R(M)\end{aligned}$$

where $\epsilon(R) \to 0$ if $R \to +\infty$ and $\delta_R(M) \to 0$ if $M \to +\infty$ for any fixed $R < \infty$. This shows the tightness of $|K * u_n|^q$.

Concerning the a.e. convergence of $K * u_n$, we just observe that we have the following series of inequalities:

$$\|K * (u_n 1_{B_R}) - K * u_n\|_{L^q} \leqslant \epsilon(R),$$

$$\|K * (u 1_{B_R}) - K * u\|_{L^q} \leqslant \epsilon(R)$$

where $\epsilon(R) \to 0$ if $R \to +\infty$;

$$K_d * (u_n 1_{B_R}) \to K_\delta * (u 1_{B_R}) \quad (\forall x \in \mathbb{R}^N, \quad \forall \delta > 0, \ \forall R < \infty)$$

where $K_\delta = 1_{|x| \geq \delta} K$;

$$\| K_\delta * (u_n 1_{B_R}) - K * (u_n 1_{B_R}) \|_m \leqslant \mu_R(\delta) \qquad (\text{ind}^t \text{ of } n)$$

where $\mu_R(\delta) \to 0$ as $\delta \to 0_+$, $m \in]N/\lambda, q[$.

And this yields the convergence in measure.

There just remains to prove that: $\int_{\mathbb{R}^N} |u|^p \, dx = 1$ (we will denote by $C_0 = -I$). To this end we adapt to our setting the method of sections I.2-I.3 of Part 1 [65]: a basic ingredient being the following lemma corresponding to lemma I.1 in [65]:

Lemma 2.1. *Let u_n converge weakly in $L^p(\mathbf{R}^N)$ to u and assume $|u_n|^p$ is tight. We may assume without loss of generality that $|K * u_n|^q$, $|u_n|^p$ converge weakly (or tightly) in the sense of measures to some bounded nonnegative measures ν, μ on $\mathbb{R}^N$. Then we have:*

iii) *There exist some at most countable set (possibly empty) and two families $(x_j)_{j\in J}$ of distinct points in $\mathbb{R}^N$, $(\nu_j)_{j \in J}$ in $]0, \infty[$ such that:*

$$\nu = |K * u|^q + \sum_{j \in J} \nu_j \delta_{x_j}. \tag{2.5}$$

ii) *In addition we have:*

$$\mu \geqslant |u|^p + \sum_{j \in J} \nu_j^{p/q} C_0^{-p/q} \delta_{x_j} \tag{2.6}$$

iii) *If $u \equiv 0$, $C_0 \mu(\mathbb{R}^N)^{q/p} \leqslant \nu(\mathbb{R}^N)$; then J is a singleton and $\nu = c_0 \delta_{x_0}$, $\mu = (c_0/C_0)^{+p/q} \delta_{x_0}$ for some $c_0 > 0$, $x_0 \in \mathbb{R}^N$.*

Remark 2.3. Exactly as in Remark I.3 ([65]), if ν is given by (2.5) with $u \in L^p(\mathbb{R}^N)$, $\sum_{j \in J} \nu_j^{p/q} < \infty$ then ν is the tight limit of $(|u_n|^p)_n$ where u_n converges weakly in $L^p(\mathbb{R}^N)$ to u.

Remark 2.4. Both lemma 2.1 and 2.1 have the same consequence: for example in the context of lemma 2.1, if $u_n \to u$ weakly in $L^p(\mathbb{R}^N)$ and if $|u_n|^p$ converges tightly to a measure μ without atoms then $K * u_n$ converges strongly in $L^p(\mathbb{R}^N)$ to $K * u$.

Using lemma 2.1, we may now conclude the proof of Theorem 2.1 following the scheme of the proof of section 1.2: first, if $u \equiv 0$ and if $|K * u_n|^q$, $|u_n|^p$ converge tightly to some bounded nonnegative measures ν, μ, we have

$$\nu(\mathbb{R}^N) = C_0, \qquad \mu(\mathbb{R}^N) = 1.$$

Hence we may apply part (iii) of lemma 2.1: $\mu = \delta_{x_0}$ for some $x_0 \in \mathbb{R}^N$. And we obtain a contradiction with (2.4).

Next, if $\alpha = \int_{\mathbb{R}^N} |u|^p \, dx \in]0, 1[$, we observe:

$$I_1 = I = -\int_{\mathbb{R}^N} |K * u|^q \, dx - \sum_{j \in J} \nu_j.$$

In view of the homogeneity of $(\lambda \to I_\lambda)$ we deduce:

$$I_1 \geqslant I_\alpha + \sum_{j \in J} I_{\mu_j}, \quad \text{with} \quad \mu_j = (\nu_j / C_0)^{p/q}$$

Since $\sum_{j \in J} \mu_j \leqslant 1 - \alpha$ by (2.6), we finally obtain:

$$I_1 \geqslant I_\alpha + I_{1-\alpha}$$

and this contradicts the fact that (S.2) holds for all $\mu > 0$:

$$I_\mu < I_\alpha + I_{\mu - \alpha}, \qquad \forall \alpha \in]0, \mu[. \tag{S.2}$$

Therefore $\alpha = 1$ and we conclude.

PROOF OF LEMMA 2.1. Many of the arguments below are identical to those introduced in the proof of Lemma 1.1 [65]; only technical details differ!

We first observe that since $(K * u_n)$ converges a.e. to $K * u$ and $(|K * u_n|^q)_n$ is tight, applying the Brézis-Lieb lemma [21] we just need to prove (2.5) in the case when $u \equiv 0$. Using Lemma 1.2 [21], we only have to prove that:

$$\int |\varphi|^q \, d\nu \leqslant C_0 \Big(\int |\varphi|^p \, d\mu \Big)^{q/p}, \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}^N). \tag{2.7}$$

This inequality will then prove i) and iii).

To show (2.7), we first remark that for all $\varphi \in \mathcal{D}(\mathbb{R}^N)$:

$$\int |K * (\varphi u_n)|^q \, dx \leqslant C_0 \Big(\int |\varphi|^p |u_n|^p \, dx \Big)^{q/p}.$$

Then (2.7) is deduced from the following claim:

$$\Big| \int |K * (\varphi u_n)|^q \, dx - \int |\varphi|^q |K * u_n|^q \, dx \Big| \to 0.$$

Using the argument we already made on the tightness of $|K * u_n|^q$ we just have to show that for all $M < \infty$: $K * (\varphi u_n) - \varphi(K * u_n)$ converges to 0 in $L^q(B_M)$. But for almost all x we have:

$$K * (\varphi u_n)(x) - \varphi(x)(K * u_n)(x) = \\ = \int_{|y| \leqslant R} \frac{1}{|x-y|^\lambda} (\varphi(y) - \varphi(x)) u_n(y) \, dy + \varphi K * (u_n 1_{C^R})$$

where $C^R = \{ y \in \mathbb{R}^N, |y| > R \}$.

Since $|u_n 1_{C^R}|_{L^p} \leqslant \epsilon(R)$ with $\epsilon(R) \to 0$ if $R \to +\infty$; we just have to bound for any $R < \infty$:

$$\left| \int_{|y| \leqslant R} \frac{1}{|x-y|^\lambda} (\varphi(y) - \varphi(x)) u_n(y)\, dy \right| = v_n(x).$$

Denoting by $R(x,y) = (\varphi(y) - \varphi(x))|x-y|^{-\lambda}$ and observing that $R(x,y) 1_{|y| \leqslant R} \in L^r(\mathbb{R}^N)$ for each x where $r < \frac{N}{\lambda - 1}$ if $\lambda > 1$, $r \leqslant +\infty$ if $\lambda \leqslant 1$, we see that: $v_n \to 0$ a.e. on $\mathbb{R}^N$. Finally for some $s > q$

$$\| v_n \|_{L^s(B_M)} \leqslant C(M,R) \| u_n \|_{L^p} = C(M,R)$$

and thus $v_n \to 0$ in $L^q(B_M)$; and (2.7) is proved.

We next show part ii) of Lemma 2.1: since $\mu \geqslant |u|^p$, we just have to show that for each fixed $j \in J$:

$$\mu(\{x_j\}) \geqslant (\nu_j / C_0)^{p/q}.$$

Let $\sigma_\epsilon = \left(\frac{\cdot - x_j}{\epsilon}\right)$, where $\varphi \in \mathcal{D}_+(\mathbb{R}^N)$, $\varphi(0) = 1$, $\varphi \leqslant 1$ and $\operatorname{Supp} \varphi \subset B_1$. We have:

$$\int |K * (\varphi_\epsilon u_n)|^q \, dx \leqslant C_0 \left(\int \varphi_\epsilon^p |u_n|^p \, dx \right)^{q/p}. \tag{2.8}$$

We fix ϵ and we let n go to $+\infty$: we estimate the left-hand side of (2.8) as follows:

$$K * (\varphi_\epsilon u_n) - (K * u_n)\varphi_\epsilon = K * (\varphi_\epsilon u_n 1_{C^R}) - [K * (u_n 1_{C^R})]\varphi_\epsilon + \psi$$

where $\psi = \psi(\epsilon, n, R)$ satisfies: $\| \psi \|_{L^q} \leqslant \delta(R) \to 0$ as $R \to +\infty$. In addition exactly as we did before:

$$\| K * (\varphi_\epsilon u_n) - (K * u_n)\varphi_\epsilon \|_{L^q(|x| > M)} \leqslant \mu(M) \to 0 \quad \text{as} \quad M \to \infty.$$

Finally by easy arguments identical to those given to prove (2.7), we show:

$$K * (\varphi_\epsilon u_n) - (K * u_n)\varphi_\epsilon \to K * (\varphi_\epsilon u) - (K * u)\varphi_\epsilon \quad \text{in} \quad L^q(B_M).$$

This together with (2.8) yields:

$$\begin{aligned} \left(\int |\varphi_\epsilon|^q \, d\nu \right)^{1/q} &\leqslant C_0^{1/q} \left(\int |\varphi_\epsilon|^p \, d\mu \right)^{1/p} + \\ &+ \delta(R) + \mu(M) + \| A_\epsilon^R u \|_{L^q(B_M)} + \| \{K * (u 1_{C^R})\} \varphi_\epsilon \|_{L^q(B_M)} \end{aligned} \tag{2.9}$$

where $A_\epsilon^R v = K * (v 1_{C^R} \varphi_\epsilon)$. Since A_ϵ^R is a family of uniformly bounded operators from L^p to $L^q(B_M)$, in order to show that $A_\epsilon^R u$ converges in $L^q(B_M)$ to 0 as ϵ goes to 0, we just need to check it for $v \in \mathcal{D}(\mathbb{R}^N)$ and this is then obvious since $\varphi_\epsilon v_\epsilon \to 0$ in $L^p(\mathbb{R}^N)$. Therefore using the fact that $\varphi_\epsilon(x_j) = 1$, Supp $\varphi_\epsilon \subset B(x_j, \epsilon)$ and (2.9), we obtain letting ϵ go to 0, then R, M go to $+\infty$:

$$\nu(\{x_j\})^{1/q} = \nu_j^{1/q} \leqslant C_0^{1/q}\mu(\{x_j\})^{1/p}$$

and this yields (2.6).

Remark 2.5. Of course, we also have the analogue of part iii) of Lemma 1.1: namely under the assumptions of Lemma 2.1, and if $v \in L^p(\mathbb{R}^N)$, $|v + u_n|^p$ converges weakly to some measure $\tilde{\mu}$ then $\tilde{\mu} - \mu \in L^1(\mathbb{R}^N)$ and

$$\tilde{\mu} \geqslant |u + v|^p + \sum_{j\in J}(\nu_j/C_0)^{p/q}\delta_{x_j}.$$

Remark 2.6. Another proof of (2.6) consists in using Brézis-Lieb lemma [21] to deduce

$$C_0\left(\int |\varphi_\epsilon|^p \, d\mu\right)^{q/p} \geqslant \int |K * (\varphi_\epsilon u)|^q \, dx + \lim_n \int |K * (\varphi_\epsilon v_n)|^q \, dx$$

where $v_n(u_n - u) \to 0$. Thus in view of the proof of part i) we deduce:

$$\begin{cases} \lim \int |K * (\varphi_\epsilon v_n)|^q \, dx = \int |\varphi_\epsilon|^q \, d\tilde{\nu} \\ \tilde{\nu} = \sum_{j\in J} \epsilon_j \delta_{x_j}. \end{cases}$$

Therefore we have:

$$C_0\,\mu(B(x_j, \epsilon))^{q/p} \geqslant \nu_j.$$

2.2 *Other potentials*

In this section we consider various questions related to problem (2.3) where K is now a general potential. To simplify the presentation (see the remarks below) we consider only the following situation:

$$K(x) = \varphi(x)\bar{K}(x) + \psi(x) \tag{2.10}$$

where

$$\varphi \in C_b(\mathbb{R}^N), \quad \varphi(x) \to \beta \quad \text{as} \quad |x| \to \infty, \quad \psi(x) \in L^{N/\lambda}(\mathbb{R}^N) \tag{2.11}$$

$$t^{-\lambda}\bar{K}(x) = \bar{K}(tx), \quad \forall t > 0, \quad \forall x \in \mathbb{R}^N - \{0\},$$

$$\bar{K} \in C(\mathbb{R}^N - \{0\}), \quad \bar{K} > 0 \quad \text{on} \quad \mathbb{R}^N - \{0\}. \tag{2.12}$$

We will denote by $\alpha = \varphi(0)$.

Clearly enough, except in the case when φ is constant ($\not\equiv 0$) and $\psi \equiv 0$, (2.3) is no more invariant by dilations. But, still, the invariance of $\mathbb{R}^N$ by dilations induces possible loss of compactness; to understand this possibility we compute for any $u \in L^p(\mathbb{R}^N)$, $\sigma > 0$:

$$\int_{\mathbb{R}^N} \left| K * \left[\sigma^{-N/p} u\left(\frac{\cdot}{\sigma}\right)\right]\right|^q dx = \int_{\mathbb{R}^N} |K_\sigma * u|^q \, dx$$

with $K_\sigma(x) = \sigma^\lambda K(\sigma x) = \varphi(\sigma x)\bar{K}(x) + \sigma^\lambda \psi(\sigma x)$.

Therefore the value I of the infimum is not changed if we replace K by K_σ for all $\sigma > 0$ and letting $\sigma \to +\infty$, or $\sigma \to 0$ we deduce:

$$I \leqslant \operatorname{Inf}\left\{ -\int_{\mathbb{R}^N} |\beta \bar{K} * u|^q \, dx / u \in L^p(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^p \, dx = 1 \right\}$$

or

$$I \leqslant \operatorname{Inf}\left\{ -\int_{\mathbb{R}^N} |\alpha \bar{K} * u|^q \, dx / u \in L^p(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^p \, dx = 1 \right\}$$

and we denote by I^∞ the minimum of these two upper bounds i.e. if $\gamma = \max(|\alpha|, |\beta|)$:

$$I^\infty \leqslant \operatorname{Inf}\left\{ -\int_{\mathbb{R}^N} |\gamma \bar{K} * u|^q \, dx / u \in L^p(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^p \, dx = 1 \right\}. \tag{9.13}$$

Denoting by I_μ, $I_\mu{}^\infty$ the values of the infima in (2.3), (2.13) where 1 replaced by $\mu > 0$ and observing that $I_\mu = \mu^{q/p} I$, $I_\mu^\infty = \mu^{q/p} I^\infty$ with I, $I^\infty < 0$ we conclude these considerations by observing that we have proved:

$$I_\mu \leqslant I_\mu^\infty, \qquad \forall \mu > 0; \quad I \leqslant I^\infty \tag{2.14}$$

$$I_\mu < I_\alpha + I_{\mu-\alpha} \leqslant I_\alpha + I_{\mu-\alpha}^\infty, \qquad \forall \alpha \in [0, \mu[. \tag{2.15}$$

Therefore, we expect the:

Theorem 2.2. *We assume* (2.2), (2.11), (2.12).

i) *If* $\varphi \equiv \beta \neq 0$, $\psi \equiv 0$, *then every minimizing sequence* $(u_n)_n$ *of* (2.3) *is relatively compact up to a dilation* $(\sigma_n)_n$, *and a translation* $(y_n)_n$ *in* $L^p(\mathbb{R}^N)$ *i.e.* $\sigma_n^{-N/p} u_n((\cdot - y_n)/\sigma_n)$ *is relatively compact in* $L^p(\mathbb{R}^N)$ *for some* y_n *in* $\mathbb{R}^N$, σ_n *in* $]0, \infty[$. *And* (2.3) *has a minimum.*

ii) *Any minimizing sequence of* (2.3) is relat*ively compact in* $L^p(\mathbb{R}^N)$ *up to a translation if and only if*:

$$I < I^\infty. \tag{2.16}$$

Remark 2.7. We first observe that if $\alpha = \beta = 0$ i.e. $K \in L^{N/\lambda}(\mathbb{R}^N)$ then (2.16) automatically holds since $I^\infty = 0$ and all minimizing sequences are compact up to translations. But as it will be observed in Step 1 of the proof, this is due to an easy compactness argument which shows that this case is actually treated by the concentration-compactness method in the locally compact case [55], [56]. In fact this compactness property still holds for Lorentz spaces i.e. when $K \in L^{N/\lambda,\alpha}(\mathbb{R}^N)$ for any $1 < \alpha < \infty$ (same proof as below).

Remark 2.8. Condition (2.16) clearly holds if, for example,

$$\varphi(x) \geqslant \alpha = \beta > 0, \qquad \varphi \not\equiv \alpha, \qquad \psi \geqslant 0$$

Indeed let u_0 be a minimum of I^∞ (which exists by case i) of the above result, we may assume that $u_0 \geqslant 0$ (replace u_0 by $|u_0|$) then:

$$I \leqslant -\int_{\mathbb{R}^N} |K * u_0|^q \, dx \leqslant -\int_{\mathbb{R}^N} |\varphi \bar{K}) * u_0|^q \, dx$$
$$< -\int_{\mathbb{R}^N} |\bar{K} * u_0|^q \, dx \leqslant I^\infty$$

On the other hand the same type of argument shows that if

$$0 \leqslant \varphi(x) \leqslant \gamma = \alpha = \beta, \qquad \varphi \not\equiv \gamma, \qquad \gamma > 0; \qquad \psi \equiv 0$$

not only (2.16) does not hold i.e. $I = I^\infty$ but (2.3) does not have a minimum. This class of K contains the example mentioned in [2.53].

Remark 2.9. In fact, the method of proof enables us to treat much more general potentials K. First of all in (2.12), the condition that $\bar{K} > 0$ may be replaced by $\bar{K} \not\equiv 0$; next we could treat

$$K = \sum_{i=1}^{\infty} \varphi_i(x)\bar{K}_i(x - x_i) + \psi(x)$$

with $\psi \in L^{N/\lambda}(\mathbb{R}^N)$, $\varphi_i \in C_b(\mathbb{R}^N)$ and $\sum_{i=1}^{\infty} |\varphi_i(x)| \in C_b(\mathbb{R}^N)$; $\bar{K}_i \in C(\mathbb{R}^N - \{0\})$, $\bar{K}_i(tx) = t^{-\lambda}\bar{K}_i(x)$ $\forall t > 0$, $\forall x \neq 0$, $|\bar{K}_i(x)| \leqslant (C/|x|^\lambda)$; $(x_i)_i \geqslant 1$ is a family of distinct points in $\mathbb{R}^N$. Denoting by $\alpha_i = \varphi_i(0)$, $\beta_i = \lim_{|x| \to \infty} \varphi_i(x)$ (which we assume exists). Theorem 2.2 still holds provided we define I^∞ by

$$I^\infty = \mathrm{Min}\left(\inf_i I_i^\infty, I_\infty^\infty\right)$$

with I_i^∞, I_∞^∞ corresponding to the potentials $\alpha_i\bar{K}_i$, $\sum_i \beta_i\bar{K}_i$.

Other technical extensions of (2.10) are possible and we will skip them.

Remark 2.10. Another possible extension is to replace $K * u$ by some $\int_{\mathbb{R}^N} K(x,y)u(y)\,dy$. For instance if we consider $K(x,y) = R(x,y)\bar{K}(x-y)$ where $\bar{K}$ satisfies (2.12), $R(x,y) \in C_b(\mathbb{R}^N \times \mathbb{R}^N)$ and $R(x,y) \to \beta$ if $|x-y| \to \infty$. Then part (ii) of Theorem 2.2 still holds if we replace I^∞ by

$$I^\infty = \mathrm{Min}\left\{ \inf_{y \in \mathbb{R}_N} I_y^\infty, I_\infty^\infty \right\}$$

where I_y^∞, I_∞^∞ correspond to (2.3) with the potentials $R(y, y)\bar{K}$, $\beta\bar{K}$.

Remark 2.11. Even if we may extend the classes of K for which we may analyse completely problem (2.3) (see also Corollary 2.1 bellow) we are unable to treat (2.3) for an arbitrary $K \in M^{N/\lambda}(\mathbb{R}^N) = L^{N/\lambda,\infty}(\mathbb{R})^N$. This is due to the fact that Lemma 2.1 which still holds for potentials like (2.10) is not true in general for arbitrary $K \in M^{N/\lambda}(\mathbb{R}^N)$. Indeed consider:

$$K(x) = |x_1|^{-\alpha}\varphi(x_2), \qquad x_1 \in \mathbb{R}^n, \quad x_2 \in \mathbb{R}^m$$

and (for example) $0 < \alpha < n$, $\varphi \in D_+(\mathbb{R}^m)$ $(\varphi \not\equiv 0)$, $x = (x_1, x_2)$. Then $K \in M^{N/\lambda}$ if $N/\lambda = n/\alpha$. In this example, one remarks that if $(u_n)_n$ converges weakly in $L^p(\mathbb{R}^N)$ to u, if $|u_n|^p$ is tight and if we choose $u_n(x_1, u_2) = v_n(x_1)w_n(x_2)$ where v_n converges weakly to v in $L^p(\mathbb{R}^n)$, w_n converges weakly to w in $L^p(\mathbb{R}^m)$ then denoting by μ, ν the tight limits of the measures $|u_n|^p$, $|k * u_n|^q$ (or subsequences) we have:

$$\nu = |K * u|^q + \sum_{i \in J} \nu_j \delta_{x_1^j} \otimes (\varphi * w)^q$$

$$\mu \geqslant |u|^p + \sum_{j \in J} (\nu_j/C)^{p/q} \delta_{x_1^j} \otimes \tilde{\mu}$$

for some at most countable family J, distinct points x_1^j in $\mathbb{R}^n$ bounded nonegative measure $\tilde{\mu}$ on $\mathbb{R}^m$, $C > 0$.

We now turn to the *proof of Theorem* 2.2:

Step 1: *Preliminary reductions*

We first explain why ψ may be assumed to be 0: indeed we just need to observe that if

$$u_n \to u \quad \text{weakly in} \quad L^p(\mathbb{R}^N), \qquad (|u_n|^p)_n \quad \text{is tight} \tag{2.17}$$

then $\psi * u_n \to_n \psi * u$ strongly in $L^p(\mathbb{R}^N)$.

By the density of $D(\mathbb{R}^N)$ in $L^{N/\lambda}(\mathbb{R}^N)$, we may without loss of generality assume that $\psi \in D(\mathbb{R}^N)$ since

$$\|\psi * u_n - \tilde{\psi} * u_n\|_{L^q} \leqslant C\|\psi - \tilde{\psi}\|_{L^{N/\lambda}};$$

But if $\psi \in D(\mathbb{R}^N)$, $\psi * u_n$ converges a.e. to $\psi * u$ and $\psi * u_n$ is bounded in $L^p \cap L^\infty$. Finally since $|u_n|^p$ is tight, $|\psi * u_n|^q$ is tight and we conclude easily.

This easy observation indicates that ψ creates no difficulty in the argument below, hence we assume from now on: $\psi \equiv 0$. Next, if we still denote by Q_n the concentration function of $|u_n|^p$, where u_n is a minimizing sequence of (3) and if we denote by $u_n^\sigma = \sigma^{-N/p} u_n(\cdot/\sigma)$ we observe that the concentration function of $|u_n^\sigma|^p$ satisfies:

$$Q_n^\sigma(t) = Q_n(t/\sigma), \qquad \forall t \geqslant 0.$$

Therefore there exists $(\sigma_n)_n$ in $]0, \infty[$ such that (4) holds. We denote by $\tilde{u}_n = u_n^{\sigma_n}$. Observe that:

$$\int |K * u_n|^q \, dx = \int |K_n * \tilde{u}_n|^q \, dx$$

where $K_n = \varphi(x/\sigma_n)\bar{K}(x)$ and recall that we already saw that for each $n \geqslant 1$ the value of the infimum (2.3) is not changed if we replace K by K_n.

We now apply the standard concentration compactness method ([58], [59], [55]): vanishing is ruled out by (2.4). If dichotomy occurs we find $\alpha \in]0, 1[$ such that for all $\epsilon > 0$, there exist R_0, R_n, y_n satisfying:

$$\left| \int |\tilde{u}_n^1|^p \, dx - \alpha \right| \leqslant \epsilon, \qquad \int |\tilde{u}_n^2|^p \, dx \to 1 - \alpha, \qquad R_n \to \infty,$$
$$\tilde{u}_n^1 = \tilde{u}_n \cdot \chi_{\{|x - y_n| \leq R_0\}}, \qquad \tilde{u}_n^2 = \tilde{u}_n \cdot \chi_{\{|x - y_n| \leq R_n\}}.$$

Let $v_n = \tilde{u}_n - (\tilde{u}_n^1 + \tilde{u}_n^2)$, we have clearly:

$$\begin{cases} \left| \int |K_n * \tilde{u}_n|^q \, dx - \int |K_n * (\tilde{u}_n^1 + \tilde{u}_n^2)|^q \, dx \right| \leqslant C\epsilon \\ \int |K^n * \tilde{u}_n^1|^q \, dx \geqslant I_{\alpha - \epsilon}, \qquad \underline{\lim}_n \int |K_n * \tilde{u}_n^2|^q \, dx \geqslant I_{1-\alpha}. \end{cases}$$

Since (S.2) holds (cf. (2.15)), we reach a contradiction since

$$\left| \int |K_n * (\tilde{u}_n^1 + \tilde{u}_n^2)|^q \, dx - \int |K_n * \tilde{u}_n^1|^q \, dx - \int |K_n * \tilde{u}_n^2|^q \, dx \right| \leqslant$$
$$\leqslant C \int \| \bar{K} * |\tilde{u}_n^1| \| \, |\bar{K} * |\tilde{u}_n^2| \|^{q-1} + |\bar{K} * |\tilde{u}_n^2| \| \, |\bar{K} * |\tilde{u}_n^1| \|^{q-1} \, dx.$$

To conclude we prove that this integral goes to 0; and since both terms are basically equivalent, we will only treat the first one: first

$$\int_{|x| > M} |\bar{K} * \tilde{u}_n^1| \, |\bar{K} * \tilde{u}_n^2|^{q-1} \, dx \leqslant C \int_{|x| > M} |\bar{K} * |\tilde{u}_n^1| |^q \, dx.$$

Translating if necesary $\tilde{u}_n$, we may assume $y_n = 0$. Then $\tilde{u}_n^1$ has its support in a fixed ball B_{R_0} and we deduce as in section 2.1 that the above integral is bounded by $\delta(M) \to 0$ as $M \to \infty$ (ind. of n). Next we consider:

$$\int_{|x|<M} |\bar{K} * |\tilde{u}_n^1|| \, |\bar{K} * |\tilde{u}_n^2||^{q-1} \, dx =$$
$$= \int_{|x|<M} \Big| \int_{|x-y| \le R_0 + M} \bar{K}(x-y) |\tilde{u}_n^1(y)| \, dy \cdot$$
$$\cdot \Big| \int_{|y| \ge R_n} \bar{K}(x-y) |\tilde{u}_n^2(y) \, dy|^{q-1} \, dx \Big| \geqslant$$
$$\leqslant R_n^{-\epsilon} \int_{|x|<M} z_n(x) \Big| \int \bar{K}(x-y) |x-y|^{\epsilon} |\tilde{u}_n^2(y)| dy \Big|^{q-1} dx$$

where $0 < \epsilon < \lambda$, $z_n = (\bar{K} \cdot \chi_{\{|x| \leqslant R_0 - M\}}) * |\tilde{u}_n^1|$: z_n is bounded in $L^1 \cap L^q$. We conclude observing that

$$|(\bar{K}|x|^{\epsilon}) * |\tilde{u}_n^2||^{q-1} \quad \text{is bounded in} \quad L^{r_\epsilon}, \quad \text{with} \quad r_\epsilon = q_\epsilon/(q-1)$$

and q_ϵ is given by: $\frac{1}{p} + \frac{\lambda - \epsilon}{N} = 1 + \frac{1}{q_\epsilon}$ (choose ϵ small enough), hence $q_\epsilon > q$, $r_\epsilon > q'$ and $r'_\epsilon \in [1, q]$.

Therefore dichotomy does not occur and we conclude: there exists $(y_n)_n$ in $\mathbb{R}^N$ such that $|\tilde{u}_n(\cdot + y_n)|^p$ is tight. We still denote by $\tilde{u}_n$ this translated sequence. We may assume that $\tilde{u}_n$ converges weakly to $\tilde{u} \in L^p(\mathbb{R}^N)$, that $\sigma_n \to \sigma \in [0, \infty]$. We denote by $C = -I$, $\bar{C} = -\bar{I}$ where $\bar{I}$ corresponds to (2.3) with $K = \alpha\bar{K}$, $K^\infty = \beta\bar{K}$ if $\sigma = 0$, $= \alpha\bar{K}$ if $\sigma = +\infty$, $= \varphi(\sigma x)\bar{K}$ if $\sigma \in]0, \infty[$.

Step 2. *A variation of Lemma* 2.1.

Lemma 2.2. *Let $\tilde{u}_n$ converge weakly in $L^p(\mathbb{R}^N)$ to $\tilde{u}$ and assume $|\tilde{u}_n|^p$ is tight. We may assume that $|K_n * \tilde{u}|^q$, $|\tilde{u}|^p$ converge weakly to some measures ν, μ. Then part* i) *of Lemma* 2.1 *holds with K replaced by K^∞ in* (2.5); *and we have:*

$$\begin{cases} \mu \geqslant |\tilde{u}|^p + \sum_{j \in J} (\nu_j/\bar{C})^{p/q} \delta_{x_j} & \text{if} \quad \sigma \in]0, \infty[\\ \mu \geqslant |\tilde{u}|^p + \sum_{j \in J} (\nu_j/C)^{p/q} \delta_{x_j} & \text{if} \quad \sigma = 0. \end{cases} \tag{2.18}$$

And if $\tilde{u} = 0$ and $C_1 \mu(\mathbb{R}^N)^{q/p} \leqslant \nu(\mathbb{R}^N)$ with $C_1 = \bar{C}$ if $\sigma \in]0, \infty]$, $C_1 = C$ if $\sigma = 0$; then J is a singleton and $\nu = c_0 \delta_{x_0}$, $\mu = (c_0/C_1)^{p/q} \delta_{x_0}$ for some $x_0 \in \mathbb{R}^N$, $c_0 > 0$.

PROOF. The proof is very similar to the one of Lemma 2.1, and we will only sketch it. Since $|K_n| \leqslant C|x|^{-\lambda}$; it is clear that $(|K_n * \tilde{u}_n|^q)_n$ is tight. In addition $(K_n * \tilde{u}_n) \to K^\infty * \tilde{u}$ a.e. in $\mathbb{R}^N$, and $K_n * \tilde{u} - K^\infty * \tilde{u} \to 0$ in $L^q(\mathbb{R}^N)$. Furthermore if $\sigma > 0$, $(K_n - K^\infty) * \tilde{u} \to 0$ in $L^q(\mathbb{R}^N)$ and this proves the above result if $\sigma > 0$. In the case when $\sigma = 0$, we just go through the proof of Lemma 2.1 and we find if $u \equiv 0$:

$$\Big(\int_{\mathbb{R}^N} |\xi|^q \, d\nu\Big) \leqslant C\Big(\int_{\mathbb{R}^N} |\xi|^p \, d\mu\Big)^{q/p}, \qquad \forall \xi \in \mathcal{D}(\mathbb{R}^N).$$

And this reverse Hölder inequality allows us to conclude.

Step 3. *$\tilde{u}_n$ is compact in $L^p(\mathbb{R}^N)$.*

If $\tilde{u} \equiv 0$, by Lemma 2.2 since $\nu(\mathbb{R}^N) = -I = C$, $\mu(\mathbb{R}^N) = 1$, $\nu = C\delta_{x_0}$, $\mu = C\delta_{x_0}$ for some $x_0 \in \mathbb{R}^N$. But this contradicts (2.4). Hence $\tilde{u} \not\equiv 0$; now let $\theta = \int_{\mathbb{R}^N} |\tilde{u}|^p \, dx$. If $\theta \in]0, 1[$ we argue as follows: first of all if $\sigma = -\infty$, then by (2.18) and (2.5):

$$\begin{cases} 1 \geqslant \theta + \sum_{j \in J} \mu_j \quad \text{with} \quad \mu_j = (\nu_j/\bar{C})^{p/q} \\ I = I_1 \geqslant \bar{I}_\theta + \sum_{j \in J} \bar{I}_{\mu_j} \geqslant I_\theta + I_{1-\theta}^\infty \end{cases}$$

and we reach a contradiction in view of (2.15).

On the other hand if $\sigma \in]0, \infty[$, still by (2.18) and (2.5):

$$\begin{cases} 1 \geqslant \theta + \sum_{j \in J} \mu_j \\ I = I_1 \geqslant I_\theta + \sum_{j \in J} \bar{I}_{\mu_j} \geqslant I_\theta + I_{1-\theta}^\infty \end{cases}$$

and again we conclude.

Finally if $\sigma = 0$, we again use (2.18) and (2.5):

$$\begin{cases} 1 \geqslant \theta + \sum_{j \in J} \mu_j \\ I = I_1 \geqslant I_\theta^\infty + I_{1-\theta}; \end{cases}$$

and we conclude: $\theta = 1$ i.e. $\tilde{u}_n$ converges to $\tilde{u}$ in $L^p(\mathbb{R}^N)$.

Step 4. *Conclusión.*

If we had $\sigma = +\infty$, then $(K_n - \alpha\bar{K}) * \tilde{u} \to 0$ in $L^q(\mathbb{R}^N)$; indeed $|\tilde{u}_n|^p$ and $||x|^{-\lambda} * \tilde{u}_n|^q$ are tight hence we may restrict the integrals on $|x - y| \leqslant K$. But $\varphi\left(\frac{x-y}{\sigma_n}\right)$ converges uniformly to α if $\sigma_n \to +\infty$ on such a set and we conclude. Now this would imply:

$$I = \lim_n \int |K_n * \tilde{u}_n|^q \, dx \geqslant \bar{I} \geqslant I^\infty;$$

and if (2.16) holds this is not possible.

On the other hand if we had $\sigma = 0$, then we claim that

$$(K_n - \beta\bar{K}) * \tilde{u}_n \to 0 \quad \text{in} \quad L^q(\mathbb{R}^N)$$

And again (2.16) would rule out this possibility. To prove the claim we just have to prove for any $R < \infty$ that

$$\int_{|x| \leqslant R} \left| \int_{|y| \leqslant R} \left| \varphi\left(\frac{x-y}{\sigma_n}\right) - \beta \right| \bar{K}(x-y) |\tilde{u}_n(y)| \, dy \right|^q dx \to 0.$$

Taking subsequences if necessary, we may assume that $|\tilde{u}_n| \leqslant \bar{u}$ which belongs to $L^p(\mathbb{R}^N)$, and thus the above integral is estimated by:

$$C \int_{|x| \leqslant R} \left| \int_{|y| \leqslant R} \chi_{\{|x-y| \leqslant \delta\}} \cdot |x-y|^{-\lambda} \bar{u}(y)\, dy \right|^q dx + \epsilon_n^\delta$$

where $\epsilon_n^\delta \to 0$, for any fixed $\delta > 0$. And we conclude since the first integral converges to 0 as $\delta \to 0_+$.

Therefore $\sigma \in]0, +\infty[$, but this is equivalent to the compactness of u_n.

We have actually proved the:

Corollary 2.1. *We assume* (2.2), (2.11), (2.12) *and we denote by* I_1^∞, I_2^∞ *the infima given by* (2.3) *where K is repalced by* $\alpha\bar{K}$, $\beta\bar{K}$. *Let* $(u_n)_n$ *be a minimizing sequence of* (2.3), *then there exist* $(\sigma_n)_n$ *in* $]0, \infty[$, $(y_n)_n$ *in* $\mathbb{R}^N$ *such that* $\tilde{u}_n(\cdot) = \sigma_n^{-N/p} u_n((\cdot - y_n)/\sigma_n)$ *is relatively compact in* $L^p(\mathbb{R}^N)$. *In addition, if* $I = I_1^\infty < I_2^\infty$, *all limit points of* (σ_n) *lie in* $]0, -\infty]$, *and there exists* $(u_n)_n$ *such that* $\sigma_n \to +\infty$; *while if* $I = I_2^\infty < I_1^\infty$, *all limit points of* (σ_n) *lie in* $[0, \infty[$ *and there exists* $(u_n)_n$ *such that* $\sigma_n \to 0$. *Finally if* $I = I_1^\infty = I_2^\infty$ *both cases occur.*

2.3 *Trace inequalities*

We first recall the well-known trace theorems (see for example Amdas [1]): let $u \in \mathfrak{D}^{m,p}(\mathbb{R}^N)$ with $p \in [1, N/m[$, m integer $\geqslant 1$ (for example!), $N \geqslant 2$, then there exists a bounded linear operator γu mapping $\mathfrak{D}^{m,p}(\mathbb{R}^N)$ into $L^q(\mathbb{R}^{N-1})$ —where q is given by: $q = (N-1)p(N-mp)^{-1}$— such that if u es smooth, then γu is the restriction of u on $\mathbb{R}^{N-1} \times \{0\}$. For obvious reasons we will still denote by u the trace operator γu.

The minimization problem associated with the question of the attainability of the norm of γ is of course:

$$I = \operatorname{Inf}\left\{ \int_{\mathbb{R}^N} |D^m u|^p\, dx / u \in \mathfrak{D}^{m,p}(\mathbb{R}^N), \qquad \int_{\mathbb{R}^{N-1}} |u(x',0)|^q\, dx' = 1 \right\} \quad (2.19)$$

here $(\int |D^m u|^p)^{1/p}$ is just any norm on $\mathfrak{D}^{m,p}(\mathbb{R}^N)$ which is «scale invariant» like for example:

$$\left(\sum_{|\alpha| = \alpha = m} \|D^\alpha u\|_{L^p}^r \right)^{1/r} \text{(for any } r \in [1, \infty],$$

$\|\Delta^{m/2} u\|_{L^p}$ if m is even, $\|\nabla(\Delta^k u)\|_{L^p}$ if m is odd...

It is clear that both functionals are invariant under the change $\{ u \to \sigma^{-(N-1)/q} u(\frac{\cdot}{\sigma}) \}$ for any $\sigma > 0$; and that if I_λ denotes the infimum given by (2.19) where 1 is replaced by λ: $I_\lambda = \lambda^{p/q} I_1 = \lambda^{p/q} I$; and thus (S.2) holds.

Theorem 2.3. *Let* $(u_n)_n$ *be a minimizing sequence of* (2.19), *then there exist* $(\sigma_n)_n$ *in* $]0, \infty[$, $(y'_n)_n$ *in* $\mathbb{R}^{N-1}$ *such that the new minimizing sequence* $\tilde{u}_n$ *given by*:

$$\tilde{u}_n(x', x_N) = \sigma_n^{-(N-1)/q} u_n((x' - y'_n)/\sigma_n, x_N/\sigma_n), \qquad \forall x' \in \mathbb{R}^{N-1}, \quad \forall x_N \in \mathbb{R}$$

is relatively compact in $\mathfrak{D}^{m,p}(\mathbb{R}^N)$ (*). *In particular* (2.19) *has a minimum.*

Remark 2.12. Just as in section 1.4 the above result admits many variants like: m non integer, Korn-trace inequalities, convolution-trace inequalities, «time-dependent» spaces, limit cases $(mp = N)$... Let us also mention the following extension of Theorem 2.3, we may instead consider the trace of u on $\mathbb{R}^k$ for $1 \leqslant k \leqslant N-1$ (i.e. on $\mathbb{R}^k \times \{0\}$) then $q = kp(N - mp)^{-1}$ and the above result still holds with $y'_n \in \mathbb{R}^k$ (provided $q > q$ i.e. $p > (N-k)/m$.

Remark 2.13. If $m = 1$ and $|Du|$ is the usual norm on $\mathbb{R}^N$, then if u is minimum of (2.19), the Steiner symmetrization of u —that we denote by u^*— is still a minimum of (2.19): u^* is spherically symmetric in $x' \in \mathbb{R}^{N-1}$, non increasing with respect to $|x'|$, and even in x_N, non increasing for $x_N \geqslant 0$.

Remark 2.14. We could of course replace $W^{m,p}(\mathbb{R}^N)$ by $W^{m,p}(Q)$ where $Q = \mathbb{R}^{N-1}x\,]0, \infty[$, then Theorem 2.3 still holds. If $m = 1$ the corresponding value of the infimum $\tilde{I}$ is given by:

$$\tilde{I} = \tfrac{1}{2} I_2 = 2^{p/q - 1} I.$$

PROOF. We are going to apply the concentration compactness method to the bounded measures $(P_n)_n$:

$$P_n = \sum_{j=0}^{m} |D^j u_n|^{q_j} + |u_n|^q(x', 0) \otimes \delta_0(x_N)$$

where $q_j = Np/(N - (m-j)p)$, and where $(u_n)_n$ is a miminizing sequence. Hence, we consider:

$$Q_n(t) = \sup_{y \in \mathbb{R}^N} P_n(y + B_t), \qquad \forall t \geqslant 0.$$

Remarking that if we replace u_n by $\sigma^{-(N-1)/q} u_n\left(\frac{\cdot}{\sigma}\right)$, $Q_n(t)$ is replaced by $Q_n\left(\frac{t}{\sigma}\right)$, we may always assume choosing $\sigma = \sigma_n$ conveniently:

$$Q_n(1) = \tfrac{1}{2}.$$

Such a choice prevents vanishing from occuring while, as usual, dichotomy does not occur (cf. sections 1, 2.1-2). Therefore there exists $y_n = (y'_n, y''_n) \in$

$\in \mathbb{R}^{N-1} \times \mathbb{R}$ such that $P_n(\cdot + y_n)$ is tight i.e.;

$$\forall \epsilon > 0, \quad \exists R < \infty, \quad \forall n \geqslant 1, \qquad P_n(\mathbb{R}^N - (y_n + B_R)) \leqslant \epsilon \tag{20}$$

We next claim that we may choose $y_n'' = 0$; indeed if $\epsilon < 1$ then $|y_n''| \leqslant R$ since if $|y_n''| > R$

$$y_n + \bar{B}_R \subset \mathbb{R}^{N-1} \times \mathbb{R}^*, \quad \text{thus} \quad \int_{x_N=0} |u_n|^q \, dx \leqslant \epsilon$$

and this contradicts the constraint. Therefore taking $\tilde{y}_n = (y_n', 0)$, (2.20) still holds if we replace R by $2R$; and we may thus assume $y_n'' = 0$.

The remainder of the proof is then an easy adaptation of arguments given in the sections above in view of the

Lemma 2.3. *Let $(u_n)_n$ be a bounded sequence in $\mathfrak{D}^{m,p}(\mathbb{R}^N)$ such that $(|D^m u_n|^p)$ is tight. We may assume u_n converges a.e. to $u \in \mathfrak{D}^{m,p}$, $|D^m u_n|^p$, $|u_n|^q(x', 0) \otimes \delta_0(x_N)$ converges weakly to some bounded, nonnegative measures on $\mathbb{R}^N$ μ, ν —and* Supp $\nu \subset \{x_N = 0\}$.

i) *Then we have for some at most countable family J, for some families $(x_j)_{j \in J}$ fo distinct points in $\mathbb{R}^{N-1} \times \{0\}$, $(\nu_j)_{j \in J}$ in* $]0, \infty[$

$$\nu = |u|^q(x', 0) \otimes \delta_0(x_N) + \sum_{j \in J} \nu_j \delta_{x_j} \tag{2.21}$$

$$\mu \geqslant |D^m u|^p + \sum_{j \in J} I \nu_j^{p/q} \delta_{x_j} \tag{2.22}$$

ii) *If $u \equiv 0$ and $\mu(\mathbb{R}^N) \leqslant I \nu(\mathbb{R}^N)^{p/q}$ then $J = \{x_0\}$ for some $x_0 \in \mathbb{R}^{N-1} \times \{0\}$ and $\nu = c_0 \delta_{x_0}$, $\mu = I c_0^{p/q} \delta_{x_0}$ for some $c_0 > 0$.*

We skip the proof of this lemma which is totally similar to the one of Lemma 1.1 (or Lemma 2.1).

2.4. *Singular inequalities*

Let us first recall the following inequality

$$\int_{\mathbb{R}^N} |u|^p |x|^{-p} \, dx \leqslant C \int_{\mathbb{R}^N} |\nabla u|^p \, dx \tag{2.23}$$

for all $u \in \mathfrak{D}^{1,p}(\mathbb{R}^N)$, with $1 \leqslant p < N$ —this inequality is easily proved by the use of Schwarz symmetrization and standard one dimensional inequalities

$$\int_0^\infty |u|^p t^\beta \, dt \leqslant C(p, \beta) \int_0^\infty |u'|^p t^{\beta + p} \, dt \tag{2.24}$$

for $1 \leqslant p < \infty$, $\beta \in \mathbb{R}$, $u \in \mathfrak{D}(]0, \infty[)$.

We next want to observe that there exists a general class of inequalities like (2.23) namely

$$\int_{\mathbb{R}^N} |u|^p |x|^{-mp}\, dx \leqslant C \int_{\mathbb{R}^N} |D^m u|^p\, dx, \qquad \forall u \in \mathfrak{D}^{m,p}(\mathbb{R}^N) \tag{2.25}$$

where $m \geqslant 1$, $1 \leqslant p < (N/m)$, and $\left(\int_{\mathbb{R}^N} |D^m u|^p\, dx\right)^{1/p}$ is any norm on $\mathfrak{D}^{m,p}$ which is «scale-invariant». In particular to prove (2.25), we will choose the norm

$$\|(-\Delta)^{m/2} u\|_{L^p(\mathbb{R}^N)} \text{ if } m \text{ is even}, \quad \|\nabla(-\Delta)^{(m-1)/2} u\|_{L^p(\mathbb{R}^N)} \text{ if } m \text{ is odd}.$$

By density we may consider only $u \in \mathfrak{D}(\mathbb{R}^N - \{0\})$. We then observe that if $f = (-\Delta)^{m/2} u$ or $(-\Delta)^{(m-1)/2} u$ depending on the parity of m, and if we denote by v $(\in \mathfrak{D}^{m,p}(\mathbb{R}^N))$ the solution of

$$(-\Delta)^k v = f^* \quad \text{in} \quad \mathbb{R}^N \quad \left(k = \tfrac{m}{2} \text{ if } m \text{ is even, } k = \tfrac{m-1}{2} \text{ if } m \text{ is odd}\right)$$

where φ^* denotes the Schwarz symmetrization of φ, then by Talenti comparizon theorem [77]: $u^* \leqslant v$ a.e. on $\mathbb{R}^N$, and thus

$$\|(-\Delta)^{m/2} u\|_{L^p} = \|(-\Delta)^{m/2} v\|_{L^p} \quad \text{if } m \text{ is even}$$

$$\|\nabla(-\Delta)^{(m-1)/2} u\|_{L^p} = \|\nabla f\|_{L^p} \geqslant \|\nabla f^*\|_{L^p} = \|\nabla(-\Delta)^{(m-1)/2} v\|_{L^p} \quad \text{if } m \text{ is odd}$$

$$\int_{\mathbb{R}^N} |u|^p |x|^{-mp}\, dx \leqslant \int_{\mathbb{R}^N} |u^*|^p |x|^{-mp}\, dx \leqslant \int_{\mathbb{R}^N} |v|^p |x|^{-mp}\, dx$$

and thus it is enough to prove (2.25) for spherically symmetric functions.

Now for spherically symmetric functions v we may assume by density that $v \in \mathfrak{D}(\mathbb{R}^N - \{0\})$ and we remark using (2.24)

$$\int_{\mathbb{R}^N} |v|^p |x|^{\beta}\, dx \leqslant C(p, \beta) \int_{\mathbb{R}^N} |\nabla v|^p |x|^{\beta + p}\, dx$$

for v spherically symmetric, $\in \mathfrak{D}(\mathbb{R}^N - \{0\})$, $\beta \in \mathbb{R}$, $p \in [1, \infty[$. Then we obtain

$$\int_{\mathbb{R}^N} |v|^p |x|^{-pm}\, dx \leqslant C_1 \int_{\mathbb{R}^N} |Dv|^p |x|^{-p(m-1)}\, dx \leqslant$$
$$\leqslant C_2 \int_{\mathbb{R}^N} |D^2 v|^p |x|^{-p(m-2)}\, dx \leqslant C \int_{\mathbb{R}^N} |D^m v|^p\, dx$$

and (2.25) holds. Another proof (communicated to us by H. Brézis) uses Lorentz spaces: if $u \in \mathfrak{D}^{m,p}(\mathbb{R}^N)$ then $u \in L^{q,p}(\mathbb{R}^N)$ and thus $|u|^p \in L^{q/p,1}(\mathbb{R}^N)$ while $|x|^{-mp} \in L^{N/(mp),\infty}(\mathbb{R}^N)$. This proves the claim since $(q/p)' = N/(mp)$.

In addition if we combine (2.25) with Hölder and Sobolev inequalities we find

$$\left(\int_{\mathbb{R}^N} \frac{|u|^q}{|x|^r}\, dx\right)^{1/q} \leqslant C_0 \left(\int_{\mathbb{R}^N} |D^m u|^p\, dx\right)^{1/p}, \qquad \forall u \in \mathfrak{D}^{m,p}(\mathbb{R}^N) \tag{2.26}$$

where $p < q < Np/(N - mp)$, $1 \leqslant p < N/m$, $m \geqslant 1$; and r is given by

$$\frac{N-r}{q} = \frac{N-mp}{p} \quad \text{or} \quad r = N - q(N-mp)/p. \tag{2.27}$$

The associated minimization problem is then

$$I = \operatorname{Inf}\left\{\int_{\mathbb{R}^N} |D^m u|^p\, dx / u \in \mathfrak{D}^{m,p}(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} |u|^q |x|^{-r}\, dx = 1\right\} \tag{2.28}$$

Observe that this minimization problem is *not* invariant by translations and is invariant by dilations or more precisely by the change

$$u \to \sigma^{-(N-r)/q} u\left(\frac{\cdot}{\sigma}\right), \qquad \forall \sigma > 0$$

Let us also remark that if I_λ denotes the infimum corresponding to the constraint where 1 is replaced by $\lambda > 0$

$$I_\lambda = \lambda^{p/q} I_1 = \lambda^{p/q} I, \quad \forall \lambda > 0$$

and thus (S.2) holds.

Theorem 2.4. *Any minimizing sequence $(u_n)_n$ of* (2.28) *is relatively compact in $\mathfrak{D}^{m,p}(\mathbb{R}^N)$ up to a dilation i.e. there exists $(\sigma_n)_n$ in $]0, \infty[$ such that the new minimizing sequence $\tilde{u}_n(\cdot) = \sigma_n^{-(N-r)/q} u_n\left(\frac{\cdot}{\sigma_n}\right)$ is relatively compact in $\mathfrak{D}^{m,p}(\mathbb{R}^N)$. In particular there exists a minimum in* (2.28).

Remark 2.15. Exactly as in section 1.4, Remark 2.12, there are many variants and extensions of the above inequalities and results in particular we may replace $|x|^{-r}$ by various potentials K satisfying for example

$$\lim_{x \to 0} K(x)|x|^{-r} = \alpha \geqslant 0, \qquad \lim_{|x| \to \infty} K(x)|x|^r = \beta > 0.$$

Remark 2.16. If $m = 1$, by a symmetrization argument and an O.D.E. analysis one may compute explicitely the expression of I and of any minimum. The existence of a minimum and these explicit expressions are given in Glaser, Martin, Grosse and Thirring [38], E.H. Lieb [53].

Remark 2.17. Clearly if $p = q$, $I_\lambda = \lambda I$ and (S.2) fails; and neither does our method continue to apply, but also —at least if $m = 1$— there does not exist a minimum of (2.28).

PROOF OF THEOREM 2.4. Again the proof follows the general scheme of our method: if $(u_n)_n$ is a minimizing sequence and if

$$\rho_n = \sum_{j=0}^{m} |D^j u_n|^{p_j}$$

where $p_j = Np/(N - (m - j)p)$, we may choose $\sigma_n > 0$ such that, if we still denote by $(u_n)_n$ the new minimizing sequence $[\sigma_n^{-(N-r)/q} u_n(\cdot/\sigma_n)]$, we have

$$Q_n(1) = \frac{I}{2}, \quad \text{with} \quad Q_n(t) = \operatorname*{Sup}_{y \in \mathbb{R}^N} \int_{y + B_t} \rho_n \, dx, \qquad \forall t \geqslant 0.$$

Since (S.2) holds we prove easily that u_n is tight up to a translation i.e. there exists $(y_n)_n$ in $\mathbb{R}^N$ such that

$$\forall \epsilon > 0, \qquad \exists R < \infty, \qquad \int_{|x - y_n| \geq R} \rho_n(\lambda) \, dx \leqslant \epsilon.$$

We claim that $(y_n)_n$ remains bounded and we argue by contradiction: $|y_n|$ (or a subsequence) goes to $+\infty$ as $n \to \infty$. Then let $\xi \in \mathcal{D}_+(\mathbb{R}^N)$, $\xi \equiv 1$ on B_1, $0 \leqslant \xi \leqslant 1$, Supp $\xi \subset B_2$ and let us denote by $\xi_n = \xi((\cdot - y_n)/R)$. The above inequality easily yields

$$\int_{\mathbb{R}^N} |D^m(u_n - v_n)|^p \, dx \leqslant \delta(\epsilon) \to 0 \quad \text{as} \quad \epsilon \to 0$$

where $v_n = \xi_n u_n$. Therefore for ϵ small enough

$$\int_{\mathbb{R}^N} |v_n|^q |x|^{-r} \, dx \geqslant \tfrac{1}{2}.$$

On the other hand

$$\int_{\mathbb{R}^N} |v_n|^q |x|^r \, dx \leqslant \int_{|x - y_n| \leqslant 2R} |u_n|^q |x|^{-r} \, dx \leqslant C(|y_n| - 2R)^{-r}, \quad \text{for } n \text{ large}$$

and we reach a contradiction which proves our claim. Hence $(y_n)_n$ is bounded and we may as well take $y_n = 0$.

The remainder of the proof is then a repetition of arguments made above and in Part 1 [65] in view of the following lemma —which is also proved by similar methods as before.

Lemma 2.4. *Let $(u_n)_n$ be a bounded sequence in $\mathcal{D}^{m,p}(\mathbb{R}^N)$ such that $|D^m u_n|^p$ is tight. We may assume that u_n converges a.e. to $u \in \mathcal{D}^{m,p}(\mathbb{R}^N)$ and that $|D^m u_n|^p$, $|u_n|^q |x|^{-r}$ converge weakly to some measures μ, ν. Then we have:*

i) $$\nu = |u|^q |x|^{-r} + \nu_0 \delta_0 \quad \textit{with} \quad \nu_0 \geqslant 0;$$

ii) $$\mu \geqslant |D^m u^p| + I \nu_0^{p/q} \delta_0$$

Remark 2.15. If $|u_n|^{q^*}$, $|u_n|^p |x|^{-mp}$ converge weakly to some measures ν^*, ν^0 where $q^* = Np(N - mp)$, we have

$$\begin{cases} \nu^0 = |u|^p|x|^{-mp} + \nu_1\delta_0 \\ \nu^* = |u|^{q^*} + \sum_{j\in J} \nu_j\delta_{x_j} \quad \text{and if} \quad \nu_0 > 0, \quad 0 \in \{x_j/j \in J\} \\ \nu^*(0)^{(1-\theta)/q^*}\nu_1^{\theta/mp} \geqslant \nu_0 \end{cases}$$

Remark 2.16. The fact that only δ_0 occurs is clear: since u_n is bounded in $L^{q^*}(\mathbb{R}^N)$, $|u_n|^q|x|^{-r}$ is bounded in $L^\alpha_{\text{loc}}(\mathbb{R}^N - \{0\})$ for some $\alpha > 1$ (and part i) above is obvious!).

2.5. *Nonlinear problems in unbounded domains*

We want to give in this section a few examples of nonlinear problems in unbounded domains which possess a variational structure and that we treat by our concentration-compactness method.

We begin with a model problem namely the Yamabe equation in infinite streps: let $N \geqslant 1$, $\Omega = 0 \times \mathbb{R}^p$ where 0 es a bounded domain in $\mathbb{R}^m$ and $m + p = N$. We consider positive, nontrivial solutions (vanishing at infinity) of

$$-\Delta u - \lambda u = u^{\frac{N+2}{N-2}} \quad \text{in} \quad \Omega,\ u > 0 \quad \text{in} \quad \Omega,\ u \in H_0^1(\Omega) \tag{2.29}$$

where $\lambda > 0$. This problem —somewhat related to the Yamabe problem— was investigated by H. Brézis and L. Nirenberg [23] in the case when Ω is bounded— see also sections 4.1-2 below.

In view of the homogeneity of the nonlinearity, we obtain a solution of (2.29) if we solve the following minimization problem

$$I = \operatorname{Inf}\left\{\int_\Omega |\nabla u|^2 - \lambda u^2\, dx \Big/ \int_\Omega |u|^{\frac{2N}{N-2}} dx = 1\right\}, \quad u \in H_0^1(\Omega) \tag{2.30}$$

and we denote by

$$I^\infty = \operatorname{Inf}\left\{\int_{\mathbb{R}^N} |\nabla u|^2\, dx \Big/ \int_{\mathbb{R}^N} |u|^{2N/(N-2)}\, dx = 1\right\}\cdot$$

We denote by λ_1 the first eigenvalue of $-\Delta$ in $H_0^1(0)$ (λ_1 is also the infimum of the spectrum of $-\Delta$ in $H_0^1(\Omega)$). The methods of Part 1 and the sections above immediately yield:

Theorem 2.5. *For any minimizing sequence* $(u_n)_n$ *of* (2.30), *there exists* $(y_n)_n \subset \{0\} \times \mathbb{R}^p$ *such that* $(u_n(\cdot + y_n))_n$ *is relatively compact in* $H_0^1(\Omega)$ *if and only if* (2.16) *holds*

$$I < I^\infty \tag{2.16}$$

In particular if (2.16) *holds, there exists a minimum of* (2.30) *and a solution of* (2.29). *In addition* (2.16) *holds if* $N \geqslant 4$ *or if* $N = 3$ *and* $\lambda \in]\bar{\lambda}_1, \lambda_1[$ *where* $\bar{\lambda}_1 \in [0, \lambda_1[$.

Remark 2.17. The result —as long as the existence of a minimum and (2.16) are concerned— is very much the same as in *H.* Brézis and L. Nirenberg [23]. And the quick discussion of (2.16) we mention above is deduced from [23]: indeed if B_R is a ball in $\mathbb{R}^p$ of radius R we have

$$I \leqslant I_R = \operatorname{Inf}\left\{\int_{OxB_R} |\nabla u|^2 - \lambda u^2\, dx \Big/ \int_{OxB_R} |u|^{\frac{2N}{N-2}} dx = 1, \quad u \in H_0^1(OxB_R)\right\}$$

and in [23] it is proved that: $I_R < I^\infty$ if $N \geqslant 4$, $I_R < I^\infty$ if $N = 3$ and $\lambda \in]\bar{\lambda}_1^R, \lambda_1[$ for some $\bar{\lambda}_1^R$. Clearly $\bar{\lambda}_1^R \downarrow \bar{\lambda}_1$ as $R \uparrow +\infty$ and we do not know if $\bar{\lambda}_1 > 0$ or $\bar{\lambda}_1 = 0$.

Remark 2.18. The above problem and result is only an example of our method: we could as well treat general minimization problems (combining the methods of P.L. Lions [55], [56] and of Part 1 [65]) such as

$$I = \operatorname{Inf}\left\{\int_\Omega \sum_{i,j} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + c(x) u^2\, dx \Big/ \int_\Omega F(x, u)\, dx = 1\right\}$$

where (a_{ij}) is uniformly elliptic and a_{ij}, c, $F(x, t)$ satisfy various assumptions and where Ω is an arbitrary unbounded domain (strip, halfspace, exterior domain...). In particular this could allow us to study the Yamabe equation

$$\begin{cases} -\sum_{i,j} \dfrac{\partial}{\partial x_i}\left(a_{ij}(x) \dfrac{\partial u}{\partial x_j}\right) + c(x) u = K(x) u^{\frac{N+2}{N-2}} \quad \text{in} \quad \Omega \\ u \in \mathfrak{D}^{1,2}(\Omega), \quad u > 0 \quad \text{in} \quad \Omega,\ u = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Remark 2.19. Concerning semilinear equations in infinite strips

$$-\Delta u = f(u) \quad \text{in} \quad \Omega, \qquad u > 0 \quad \text{in} \quad \Omega, \qquad u \in H_0^1(\Omega)$$

where $\Omega = \mathfrak{D} \times \mathbb{R}^p$, O bounded domain in $\mathbb{R}^m$, $N \geqslant 3$ and $f \in C^1(\mathbb{R})$, $f(0) = 0$, $f'(0) > -\lambda_1$.

Such problems have been studied in M. J. Esteban [34]; C. J. Amick and J. F. G. Toland [3]; J. Bona, D. K. Bose and R. E. L. Turner [15]; P. L. Lions [56] in the «locally compact» case. If we assume, for instance, that f is odd and

$$0 \leqslant \frac{f(t)}{t} - f'(0) \leqslant \theta\{f'(t) - f'(0)\}, \quad \forall t \in \mathbb{R}, \tag{2.31}$$

for some $\theta \in]0, 1[$;

$$\lim_{|t| \to \infty} f(t)|t|^{-\frac{N+2}{N-2}} = \alpha \geqslant 0; \tag{2.32}$$

then the above problem will be solved if we find a minimum of

$$I = \operatorname{Inf}\{\mathcal{E}(u), u \in H_0^1(\Omega), \quad u \not\equiv 0, \quad J(u) = 0\} \tag{2.33}$$

where

$$\mathcal{E}(u) = \int_\Omega \tfrac{1}{2}|\nabla u|^2 - F(u)\,dx, \qquad J(u) = \int_\Omega |\nabla u|^2 - f(u)u\,dx,$$

and

$$F(t) = \int_0^t f(s)\,ds.$$

To this end we introduce

$$I^\infty = \operatorname{Inf}\{\mathcal{E}^\infty(u), \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \quad u \not\equiv 0, \quad J^\infty(u) = 0\}$$

with

$$\mathcal{E}^\infty(u) = \int_{\mathbb{R}^N} \frac{1}{2}|\nabla u|^2 - \alpha\frac{N-2}{2N}\int_{\mathbb{R}^N}|u|^{\frac{2N}{N-2}}dx,$$

$$J^\infty = \int_{\mathbb{R}^N}|\nabla u|^2 - \alpha|u|^{\frac{2N}{N-2}}dx \qquad (\text{if } \alpha = 0, \quad I^\infty = +\infty).$$

If $\alpha > 0$, by an easy scaling argument I^∞ is also given by

$$I^\infty = I_0^{N/2}\alpha^{-(N-2)/2}\frac{N+2}{2N},$$

$$I_0 = \operatorname{Inf}\left\{\int_{\mathbb{R}^N}|\nabla u|^2\,dx \Big/ \int_{\mathbb{R}^N}|u|^{\frac{2N}{N-2}}dx = 1\right\}.$$

Then any minimizing sequence of (2.33) converges up to a translation (of the form $(0, \bar{y}_n)$) if and only if $I < I^\infty$.

Sketch of the proof of Theorem 2.5. We apply the general scheme of proof we used before: in particular we use the first concentration compactness lemma ([58], [55]) with the density

$$\rho_n = |\nabla u_n|^2 + u_n^2 + |u_n|^{\frac{2N}{N-2}}.$$

And we just have to explain how we avoid i) vanishing of ρ_n, ii) that the weak limit u of u_n is not trivial if μ_n is tight.

First, if ρ_n vanishes i.e. if

$$\sup_{y\in\mathbb{R}^N}\int_{y+B_R}\rho_n\,dx \to 0, \qquad \forall R < \infty;$$

where ρ_n is defined on $\mathbb{R}^N$ by extending u_n by 0—then we know (cf. P. L. Lions

[55], [56]) that $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < \frac{2N}{N-2}$. Thus for all $\delta > 0$, $|u_n| \quad \delta \to 0$ 0 in $L^p(\mathbb{R}^N)$ for $2 < p < \infty$. Let $v_n = (|u_n| - \delta)^+$, we have

$$\begin{cases} \text{meas}\{v_n > 0\} = \text{meas}\{|u_n| > \delta\} \leqslant \dfrac{1}{\delta^2}\displaystyle\int_{\mathbb{R}^N} |u_n|^2\,dx \leqslant \dfrac{C}{\delta^2} \\ \displaystyle\int v_n^2\,dx \leqslant C\left(\int v_n^p\,dx\right)^{2/p} \quad \text{for all} \quad p > 2 \end{cases}$$

and thus $v_n \to 0$ in $L^p(\mathbb{R}^N)$ for $2 \leqslant p < \frac{2N}{N-2}$. Therefore

$$\begin{aligned} I &= \underline{\lim}_n \int |\nabla u_n|^2 - \lambda u_n^2\,dx \geqslant \\ &\geqslant \underline{\lim}_n \int_{\mathbb{R}^N} |\nabla v_n|^2 + \underline{\lim}_n \int_{\mathbb{R}^N} |\nabla w_n|^2 - w_n^2\,dx \geqslant \\ &\geqslant \underline{\lim}_n \int_{\mathbb{R}^N} |\nabla v_n|^2\,dx, \qquad \text{where } w_n = |u_n| \wedge \delta \end{aligned}$$

and $\int_{\mathbb{R}^N} |v_n|^{\frac{2N}{N-2}}\,dx \to 1$. Hence $I \geqslant I^\infty$ and this contradicts (2.16).

Next, if ρ_n is tight and if u_n converges weakly and a.e. to some $u \in H_0^1(\Omega)$, we want to check that $u \not\equiv 0$. But if $u \equiv 0$, since u_n^2 is tight, $u_n \to 0$ in $L^2(\Omega)$ thus

$$I = \underline{\lim}_n \int |\nabla u_n|^2\,dx \geqslant I^\infty$$

and again this contradicts (16).

The remainder of the proof consists then of straightforward adaptations of previous arguments.

We now turn to a nonlinear problem involving nonlinear boundary conditions: this problem —in the locally compact case-was investigated in P. L. Lions [56] and we refer to [56] for various considerations on its solutions— may be formulated as follows

$$I_\lambda = \text{Inf}\left\{\int_\Omega |\nabla u|^2\,dx \Big/ \int_{\partial\Omega} F(u)\,ds = \lambda\right\} \tag{2.31}$$

where $\lambda > 0$, u belongs to $\mathfrak{D}^{1,2}(\Omega)$ (closure of $C^1_{\text{comp}}(\bar{\Omega})$ for the seminorm $|\nabla u|_{L^2(\Omega)}$), F is a given nonlinearity and Ω is and unbounded domain (smooth) of $\mathbb{R}^N$. To simplify the presentation only, we will consider two examples

$$\Omega = \{x_N > 0\} \tag{2.32}$$

$$\Omega = \mathbb{R}^N - \bar{O}, \text{ where } O \text{ is a smooth bounded open set in } \mathbb{R}^N. \tag{2.33}$$

We will assume that $N \geqslant 3$ and that F satisfies

$$F \in C(\mathbb{R}), \qquad F(0) = 0 \tag{2.34}$$

$$\lim_{|t| \to 0} F^+(t)|t|^{-q} = \alpha \geqslant 0 \tag{2.35}$$

$$\lim_{|t| \to \infty} F^+(t)|t|^{-q} = \beta \geqslant 0 \tag{2.36}$$

(if α or $\beta > 0$, we may replace F^+ by F) and where $q = \frac{2(N-1)}{N-2}$. We denote by

$$I_0 = \operatorname{Inf}\left\{ \int_{(x_N > 0)} |\nabla u|^2 \, dx / u \in \mathcal{D}^{1,2}(x_N > 0), \int_{x_N = 0} |u|^q \, dx' = 1 \right\}.$$

(recall that this problem was solved in section 2.3).

Theorem 2.6. *If Ω is given by* (2.32), *we assume* (2.34), (2.35), (2.36) *and we denote by $I_\lambda^\infty = (\max(\alpha, \beta)\lambda^{-1})^{-2/q} I_0$ ($= +\infty$ if $\alpha = \beta = 0$) while if Ω is given by* (2.33), *we assume* (2.34), (2.36) *and we denote by $I_\lambda^\infty = (\beta\lambda^{-1})^{-2/q} I_0$. If* (2.32) *holds, every minimizing sequence is relatively compact in $\mathcal{D}^{1,2}(\Omega)$ up to a translation of the form $(y_n', 0)$ if and only if*

$$I_\lambda < I_\lambda^\infty \tag{2.37}$$

If (2.33) *holds then every minimizing sequence is relatively compact in $\mathcal{D}^{1,2}(\Omega)$ if and only if*

$$I_\lambda < I_\alpha + I_{\lambda-\alpha}^\infty, \qquad \forall \alpha \in [0, \lambda[. \tag{S.1}$$

Remark 2.20. By an obvious argument, if Ω is given by (2.32), $I_\lambda = \lambda^{\frac{N-2}{N-1}} I_1$ and thus (2.37) is equivalent to (S.1). On the other hand if Ω is given by (2.33), since $\partial\Omega$ is bounded, the problem at infinity (for the translations group) disappears and thus there only remains the problem at infinity obtained by focussing u at a boundary point via dilations.

Sketch of the proof of Theorem 2.6. We first explain why the large inequalities always hold (i.e. $I_\lambda \leqslant I_\lambda^\infty$ in the first case, $I_\lambda \leqslant I_\alpha + I_{\lambda-\alpha}^\infty$ $\forall \alpha \in [0, \lambda[$ in the second case). If (2.32) holds, we introduce $u_n \in \mathcal{D}(\mathbb{R}^N)$ satisfying: Supp $u_n \subset B(0, 1/n)$, $\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx \to I_0$, $\int_{x_N=0} |u_n|^q \, dx' = 1$; we consider $v_n = \beta^{-1/q}\lambda^{1/q} u_n$ and if $\beta > 0$, we deduce

$$\int_{x_N=0} F(v_n) \, dx' \to 1, \int_{(x_N > 0)} |\nabla v_n|^2 \, dx \to \beta^{-2/q}\lambda^{2/q} I_0.$$

In a similar way if $\alpha > 0$, we may choose $u_n \in \mathcal{D}(\mathbb{R}^N)$ satisfying

$$\max_{\mathbb{R}^N} |u_n| = 1, \int_{x_N=0} \alpha |u_n|^q \, dx' = \lambda,$$

$$\int_{x_N > 0} |\nabla u_n|^2 \, dx \to \alpha^{-2/q}\lambda^{2/q} I_0$$

and we let $v_n = \left(\frac{1}{n}\right)^{\frac{N-2}{2}} u_n(n\,\cdot)$. Then we find

$$\int_{x_N=0} F(v_n)\,dx' \to \lambda, \qquad \int_{x_N>0} |\nabla v_n|^2\,dx \to \alpha^{-2/q}\lambda^{2/q} I_0.$$

Hence $I_\lambda \leqslant I_\lambda^\infty$ and if $I_\lambda = I_\lambda^\infty$, there exists a minimizing sequence which is not relatively compact even up to a translation.

In the second case —i.e. if (2.33) holds— let $\alpha \in [0, \lambda[$ and let $(u_n^1)_n$ be a minimizing sequence of I_α. On the other hand let u^2 be a minimum of I_0, we may always assume that $0 \in \partial\Omega$ and that $e_N = (0, \ldots, 0, 1)$ is the unit inward normal to $\partial\Omega$ at 0. We then set

$$u_n^2 = \sigma_n^{-\frac{N-2}{2}}(\lambda - \alpha)^{2/q}\beta^{-2/q}u^2(\cdot/\sigma_n)$$

where $\sigma_n \to 0$ is to be determined. We finally set: $u_n = u_n^1 + u_n^2$. Observing that we may take u_n^1 in $\mathfrak{D}(\mathbb{R}^N)$ if we wish, it is easy to check that we may choose $(\sigma_n)_n$ in such a way that

$$\left|\int_{\partial\Omega} F(u_n)\,ds - \int_{\partial\Omega} F(u_n^1)\,ds - \int_{\partial\Omega} \beta|u_n^2|^q\,ds\right| \to 0,$$

$$\int_{\partial\Omega} \beta|u_n^2|\,ds \to \lambda - \alpha,$$

$$\int_\Omega |\nabla u_n|^2\,dx - \int_\Omega |\nabla u_n^1|^2\,dx - \int_\Omega |\nabla u_n^2|^2\,dx \to 0,$$

$$\int_\Omega |\nabla u_n^2|^2\,dx \to I_{\lambda-\alpha}^\infty.$$

Therefore $(u_n)_n$ satisfies: $\int_\Omega |\nabla u_n|^2\,dx \to I_\alpha + I_{\lambda-\alpha}^\infty$, $\int_{\partial\Omega} F(u_n)\,ds \to \lambda$ and $u_n - u_n^1 \to 0$ weakly in $\mathfrak{D}^{1,2}(\Omega)$. And this proves the large inequalities and the fact that strict inequalities are necessary for the compactness of all minimizing sequences.

The proof of the sufficiency of the conditions (2.37) of (S.1) is then very similar to proofs made in Part 1 and before. We will only explain how we conclude in the case when (2.33) holds once we know that $|\nabla u_n|^2 + |u_n|^{\frac{2N}{N-2}}$ is tight. By arguments similar to those made in section 1.6, we obtain:

Lemma 2.3. *Assume Ω is given by* (2.33), *that u_n converges weakly in $\mathfrak{D}^{1,2}(\Omega)$ to u and that $\rho_n = |\nabla u_n|^2 + |u_n|^{\frac{2N}{N-2}}$ is tight. We may assume that $|\nabla u_n|^2$, ν_n converge weakly to some measures μ, ν where* ν_n is the measure on $\bar{\Omega}$ *supported by $\partial\Omega$ such that: $\forall \varphi \in C_b(\bar{\Omega})$, $\int \varphi\,d\nu_n = \int_{\partial\Omega} \varphi F(u_n)\,ds$. Then there exist J at a most countable set* (*possibly empty*) of $(x_j)_{j\in J}$ *distinct points of* $\partial\Omega$, $(\nu_j)_{j\in J} \in]0, \infty[$ *such that*

$$\nu = \nu_\infty + \beta \sum_{j\in J} \nu_j \delta_{x_j}, \qquad \mu \geqslant |\nabla u|^2 + I_0 \sum_{j\in J} \nu_j^{2/q}\delta_{x_j}$$

where ν_∞ is defined by

$$\int \varphi \, d\nu_\infty = \int_{\partial\Omega} \varphi F(u) \, ds, \qquad \forall \varphi \in C_b(\bar{\Omega}).$$

Remark 2.21. Similar results hold of course for sequences in $\mathfrak{D}^{m,p}(\Omega)$.

We skip the proof of the lemma since it is very similar to arguments given in Part 1 and before: let us just observe that

$$\int_{\partial\Omega} |F(u_n - u) - \beta|u_n - u|^q| \, ds \to 0$$

and that the fact that the best constant I_0 (for half-spaces) occurs in the estimate for μ is due to a localization argument. Indeed if we follow the proof of Lemma I.1 ([55]) or Lemma II.1 we see that the lower bound on $\mu(\{x_j\})$ is obtained by multiplying u_n by some convenient cut-off function $\varphi\left(\frac{x-x_j}{\epsilon}\right)$. Thus all computations take place in the ball $B(x_j, \epsilon)$ and using local charts we may actually argue as if we were in a half-space.

We next conclude this section with another problem —motivated by geometric considerations, see Cherrier [25] and section 4.2 below—; we will consider it in an unbounded domain Ω, we look for positive solutions of

$$-\Delta u = f(u) \quad \text{in} \quad \Omega, \qquad \frac{\partial u}{\partial n} = g(u) \quad \text{on} \quad \partial\Omega, \quad u > 0 \quad \text{on} \quad \bar{\Omega} \tag{2.38}$$

where $f, g \in C(\mathbb{R})$, $f(0) = g(0) = 0$; n is the unit outward normal. Denoting by $F(t) = \int_0^t f(s) \, ds$, $G(t) = \int_0^t g(s) \, ds$ and assuming for example that f, g are odd, one way to solve problems "like" (2.38) is to consider the following minimization problem

$$I_\lambda = \operatorname{Inf}\left\{\int_\Omega |\nabla u|^2 \, dx \Big/ \int_\Omega F(u) \, dx + \int_{\partial\Omega} G(u) \, ds = \lambda\right\}. \tag{2.39}$$

However a solution of (2.39) leads only to a solution of (2.38) where f, g are multiplied by a Lagrange multiplier which can be taken care of only if F, G are homogeneous of the same degree —and this case is not really interesting— or if Ω is a half-space,

$$f(u) = |u|^{q_1 - 2}u, \qquad g(u) = |u|^{q_2 - 2}u \quad \text{with} \quad q_1 = \frac{2N}{N-2}, \qquad q_2 = \frac{2(N-1)}{N-2}$$

This is why we will not consider (2.39) —that we may analyse easily with our methods.

Instead, we will use the artificial constraint method (see for example C. V. Coffman [26], [27], P. L. Lions [56]) which will requiere the following structure conditions on f, g

$$f(t) = f_0(t) - mt, \qquad m \geqslant 0, \qquad 0 \leqslant f_0(t)t^{-1} \leqslant \theta f_0'(t) \qquad \forall t \in \mathbb{R} \tag{2.40}$$

$$g(t) = g_0(t) - \mu t, \qquad \mu \geqslant 0, \qquad 0 \leqslant g_0(t)t^{-1} \leqslant \theta g_0'(t) \qquad \forall t \in \mathbb{R} \tag{2.41}$$

$$\begin{cases} \lim_{|t| \to \infty} f_0(t)|t|^{2-q_1}t^{-1} = \beta_1 \geqslant 0 \\ \lim_{|t| \to \infty} g_0(t)|t|^{2-q_2}t^{-1} = \beta_2 \geqslant 0 \end{cases} \tag{2.42}$$

$$\text{if} \quad m = 0, \qquad \lim_{t \to 0_+} f_0(t)t^{-(q_1-1)} = \alpha_1 \geqslant 0 \tag{2.43}$$

$$\text{if} \quad \mu = 0, \qquad \lim_{t \to 0_+} g_0(t)t^{-(q_2-1)} = \alpha_2 \geqslant 0. \tag{2.44}$$

We consider now the following minimization problem

$$I = \operatorname{Inf}\{\mathcal{E}(u)/F(u) \in L^1(\Omega), \quad G(u) \in L^1(\partial\Omega), \quad J(u) = 0\} \tag{2.45}$$

where

$$\mathcal{E}(u) = \int_\Omega \tfrac{1}{2}|\nabla u|^2 - F(u)dx - \int_{\partial\Omega} G(u)ds,$$

$$J(u) = \int_\Omega |\nabla u|^2 - f(u)u\,dx - \int_{\partial\Omega} g(u)u\,ds.$$

Using (2.40)-(2.41), it is easy to check that a minimum of (2.45) is indeed a solution of (2.38).

To simplify the presentation, we will consider only the cases when Ω is given either by (2.32), or by (2.33). We need to introduce the following quantities

$$I^{\infty,i} = \operatorname{Inf}\{\mathcal{E}^{\infty,i}(u)/J^{\infty,i}(u) = 0, \quad u \not\equiv 0\}, \qquad i = 1, 2, 3$$

$$\mathcal{E}^{\infty,1}(u) = \int_{\mathbb{R}^N} \tfrac{1}{2}|\nabla u|^2 - F(u)dx, \quad J^{\infty,1}(u) = \int_{\mathbb{R}^N} |\nabla u|^2 - f(u)u\,dx$$

$$\mathcal{E}^{\infty,2}(u) = \int_{(x_N>0)} \tfrac{1}{2}|\nabla u|^2 - (\beta_1/q_1)|u|^{q_1}dx - \int_{(x_N=0)} (\beta_2/q_2)|u|^{q_2}dx'$$

$$J^{\infty,2}(u) = \int_{(x_N>0)} |\nabla u|^2 - \beta_1|u|^{q_1}dx - \int_{(x_N=0)} \beta_2|u|^{q_2}dx'$$

$$\mathcal{E}^{\infty,3}(u) = \int_{(x_N>0)} \tfrac{1}{2}|\nabla u|^2 - (\alpha_1/q_1)|u|^{q_1}dx - \int_{(x_N=0)} (\alpha_2/q_2)|u|^{q_2}dx'$$

$$J^{\infty,3}(u) = \int_{(x_N>0)} |\nabla u|^2 - \alpha_1|u|^{q_1}dx - \int_{(x_N=0)} \alpha_2|u|^{q_2}dx'$$

Of course if m (resp. μ) > 0 we set $\alpha_1 = 0$ (resp. $\alpha_2 = 0$) and if $\alpha_1 = \alpha_2 = 0$ (or $\beta_1 = \beta_2 = 0$) we set $I^{\infty,3} = +\infty$ (or $I^{\infty,2} = +\infty$).

To motivate the introduction of these various functionals, let us explain that $I^{\infty,1}$ corresponds to the "problem at $x_N^\infty = +\infty$" obtained by the action of the translation group if for example Ω is given by (2.32), while $I^{\infty,2}$ is obtained by "concentrating u" at a boundary point by the action of the dilation groups ("concentrating u" at an interior point is not necessary here since it is contained in $I^{\infty,1}$), and finally $I^{\infty,3}$ is obtained by "scaling out" u

($u \to \sigma^{-(N-2)/2}u(\cdot/\sigma)$ with $\sigma \to +\infty$) again by the action of the dilation group. The next two results correspond to the two domains Ω we consider:

Theorem 2.7. *We assume* (2.32), (2.40)-(2.44).

1) *If* $m = \mu = 0$, $f_0 = \gamma_1|t|^{q_1-1}t$, $g_0 = \gamma_2|t|^{q_2-1}t$ *with* $\gamma_1, \gamma_1 > 0$ *then any minimizing sequence of* (2.45) *is relatively compact in* $\mathfrak{D}^{1,2}(\Omega)$ *up to a scale change* ($\sigma \to \sigma^{-(N-2)/2}u(\cdot/\sigma)$ *and a translation of the form* $(y_n', 0)$. *In particular there exists a minimum.*

We denote $I^\infty = \mathrm{Min}(I^{\infty,1}, I^{\infty,2}, I^{\infty,3})$. *Then the condition*

$$I < I^\infty \tag{2.16}$$

is necessary and sufficient for the compactness of all minimizing sequences up to a translation of the form $(y_n', 0)$.

Theorem 2.8. *We assume* (2.33), (2.40)-(2.42) *and we denote by* $I^\infty = \mathrm{Min}(I^{\infty,1}, I^{\infty,2})$. *Then* (2.16) *is necessary and sufficient for the compactness of all minimizing sequences of* (2.45).

Remark 2.22. We could treat as well arbitrary unbounded domains such that: $\forall R < \infty$, $y \in \Omega$, $B(y, R) \subset \Omega$, or strip-like domains... Combining the methods of P. L. Lions [55], [56] and of Part 1 [65], we may treat exactly as below x-dependent problems and in particular

$$\begin{cases} -\dfrac{\partial}{\partial x_i}\left[a_{ij}(x)\dfrac{\partial u}{\partial x_j}\right] + k(x)u = K(x)u^{(N+2)/(N-2)} \quad \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial v^A} + k'u = K'u^{N/(N-2)} \quad \text{on} \quad \partial\Omega, \qquad u > 0 \quad \text{in} \quad \bar{\Omega} \end{cases}$$

where a_{ij}, k, k', K, K' are given functions having limits as $|x| \to \infty$, $x \in \Omega$, $(a_{ij}(x))$ is uniformly elliptic, K, K' are not everywhere nonpositive and are nonnegative at ∞ and the quadratic form associated with the linear part of the problem is positive on $\mathfrak{D}^{1,2}(\Omega)$. Of course v^A is the conormal associated with $(a_{ij}(x))$ i.e. $v_i^A = a_{ij}n_j \; \forall i$.

Remark 2.23. In fact our method not only shows Theorems 2.7-8 but also explains how compactness may be lost if $I = I^\infty$: for example if $I = I^{\infty,2} < I^{\infty,1} \wedge I^{\infty,3}$, a noncompact minimizing sequence $(u_n)_n$ will satisfy: $|\nabla u_n|^2 \to \delta_{x_0}$, $\beta_1|u_n|^{q_1} + \beta_2|u_n|^{q_2} \to \delta_{x_0}$ for some $x_0 \in \mathbb{R}^{N-1}x\{0\}$ and there exist $\sigma_n \to \infty$, $y_n = (y_n', 0)$, $-y_n/\sigma_n \to x_0$ such that $\sigma_n^{-(N-2)/2}u_n((\cdot - y_n)/\sigma_n)$ converges to a minimun of $I^{\infty,2}$ (up to subsequences...). And there exists such a sequence $(u_n)_n$.

Remark 2.24. We will not discuss here conditions (2.16): this strict inequality may be analyzed as in P. L. Lions [55], [56], [65], T. Aubin [6], H. Brézis and L. Nirenberg [23]... Let us only observe that by a symmetry argument similar to the one used below we have if Ω is given by (2.32): $I \leqslant \frac{1}{2} I^{\infty,1}$ and thus $I^{\infty} = \mathrm{Min}(I^{\infty,2}, I^{\infty,3})$.

Sketch of the proof of Theorems 2.7-8. First of all, in the case when (2.32) holds and $m = \mu = 0$, $f_0 = \gamma_1 |t|^{q_1 - 1} t$, $g_0 = \gamma_2 |t|^{q_2 - 1} t$, $\gamma_1, \gamma_2 > 0$, the minimization problem (2.45) is "scale invariant" and invariant by translations of the form $(y', 0)$. We claim that $I < I^{\infty,1}$. To check this strict inequality, we just observe that there exists $u \in \mathfrak{D}^{1,2}(\mathbb{R}^N)$ symmetric with respect to x_N (actually radial) such that

$$\begin{cases} I^{\infty,1} = \int_{\mathbb{R}^N} \frac{1}{2} |\nabla u|^2 - (\gamma_1/q_1) |u|^{q_1} \, dx \\ \int_{\mathbb{R}^N} |\nabla u|^2 - \gamma_1 |u|^{q_1} \, dx = 0. \end{cases}$$

Thus: $\int_{(x_N > 0)} |\nabla u|^2 - \gamma_1 |u|^{q_1} \, dx = 0$; and there exists $\theta \in]0, 1[$ such that if $v = \theta u$

$$\int_{(x_N > 0)} |\nabla v|^2 - \gamma_1 |v|^{q_1} \, dx - \int_{(x_N = 0)} \gamma_2 |v|^{q_2} \, dx' = 0.$$

Therefore we have denoted by $\alpha = \int_{(x_N > 0)} |\nabla u|^2 = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx$

$$\begin{aligned} I &= \theta^2 \frac{\alpha}{2} - \theta^{q_1} \frac{\alpha}{q_1} - \frac{1}{q_2} (\theta^2 \alpha - \theta^{q_1} \alpha) \\ &= \theta^2 \alpha \left(\frac{1}{2} - \frac{1}{q_2} \right) + \theta^{q_1} \alpha \left(\frac{1}{q_2} - \frac{1}{q_1} \right) \leqslant \alpha \left(\frac{1}{2} - \frac{1}{q_1} \right) = \frac{1}{2} I^{\infty,1}. \end{aligned}$$

Thus, by a convenient choice of the scaling and of a translation, we may assume that any minimizing sequence satisfies

$\rho_n = |\nabla u_n|^2 + \gamma_1 |u_n|^{q_1} + \gamma_2 |u_n|^{q_2} \otimes \delta_0(x_N)$ is tight and $\mathrm{Sup}_{y \in \mathbb{R}^N} \int_{y + B_1} d\rho_n = L$ where $L < \mathrm{Inf}_n \int_{\mathbb{R}^N} d\rho_n$. Indeed vanishing is ruled out by the scaling, dichotomy as in [55] and the tightness cannot be obtained through a sequence (y^n) such that y_N^n is unbounded because of the strict inequality: $I < I^{\infty,1}$. Assuming that u_n converges weakly and a.e. to u, we have to show that $u \not\equiv 0$: If this is the case, we conclude easily adapting arguments given before (or in Part 1) and in [56]. Now if $u \equiv 0$, we may assume that $|\nabla u_n|^2$, $\gamma_1 |u_n|^{q_1}$, $\gamma_2 |u_n|^{q_2} \otimes \delta_0(x_N)$ converge weakly to some measures μ, ν_1, ν_2: we already know that ν_1, ν_2 are given by countable sums of Dirac masses and that μ charges any point charged by $\nu_1 + \nu_2$. We claim that μ, ν_1, ν_2 are given by *one* Dirac mass contradicting thus the constraint on ρ_n.

Indeed if $x^0 \in \{x_N \geqslant 0\}$ is such that

$$\mu(\{x^0\}) - \nu^1(\{x^0\}) - \nu^2(\{x^0\}) < 0$$

then we may find $\xi \in \mathcal{D}(\mathbb{R}^N)$ supported in a small enough ball centered at x^0 such that

$$\begin{cases} J(\xi u_n) \to -\alpha < 0 \\ \left(\frac{1}{2} - \frac{1}{q_1}\right)\gamma_1 \int_{(x_N > 0)} |\xi u_n|^{q_1}\, dx + \left(\frac{1}{2} - \frac{1}{q_2}\right)\gamma_2 \int_{(x_N = 0)} |\xi u_n|^{q_2}\, dx' \to \beta < I \end{cases}$$

Indeed observe that

$$I = \left(\frac{1}{2} - \frac{1}{q_1}\right) \int d\nu^1 + \left(\frac{1}{2} - \frac{1}{q_2}\right) \int d\nu^2.$$

It is then easy to reach a contradiction as in [56]. Therefore for each point x^j in the support of $\nu_1 + \nu_2$, we find

$$\mu(\{x^j\}) - \nu^1(\{x^j\}) - \nu^2(\{x^j\}) \geqslant 0$$

and thus

$$0 \leqslant \sum_{j \in J} \mu(\{x^j\}) - \nu^1(\{x^j\}) - \nu^2(\{x^j\}) \leqslant \int d\mu - \int d\nu^1 - \int d\nu^2 = 0.$$

Hence

$$\mu = \sum_{j \in J} \mu_j \delta_{x^j}, \qquad \nu^1 = \sum_{j \in J} \nu_j^1 \delta_{x^j}, \qquad \nu^2 = \sum_{j \in J} \nu_j^2 \delta_{x^j}$$

and

$$\mu_j = \nu_j^1 + \nu_j^2 > 0, \quad x^j \in \{x_N \geqslant 0\}, \quad \mu_j \geqslant c_1(\nu_j^1)^{1/q_1} + c_2(\nu_j^2)^{2/q_2}.$$

This last inequality yields that J is finite (since $\mu_j > 0$, $\forall j \in J$). In addition choosing for each $j \in J$ a cut-off function ξ supported in a small ball centered at x^j, we see that

$$I \leqslant \left(\frac{1}{2} - \frac{1}{q_1}\right)\nu_j^1 + \left(\frac{1}{2} - \frac{1}{q_2}\right)\nu_j^2, \qquad I = \sum_{j \in J} \left(\frac{1}{2} - \frac{1}{q_1}\right)\nu_j^1 + \left(\frac{1}{2} - \frac{1}{q_2}\right)\nu_j^2$$

and this only possible if J is singleton. Hence: $\mu = \mu_0 \delta_{x^0}$, $\nu^1 = \nu_0^1 \delta_{x^0}$, $\nu^2 = \nu_0^2 \delta_{x^0}$ where $\mu_0 > 0$, $\nu_0^1 + \nu_0^2 = \mu_0$, $\nu_0^1 \geqslant 0$, $\nu_0^2 \geqslant 0$, $x^0 \in \{x_N \geqslant 0\}$. (If $x_N^0 > 0$, $\nu_0^2 = 0$, $\nu_0^1 = \mu_0$ and we would have: $I = I^{\infty,}$. Therefore $x_N^0 = 0$, $\nu_0^2 > 0$, $\nu_0^1 > 0$).

In the general case, we apply the arguments of P. L. Lions [55], [56] to deduce that $\rho_n = |\nabla u_n|^2 + |u_n|^{q_1} + m u_n^2 + (|u_n|^{q_2} + \mu u_n^2) \otimes \delta_0(x_N)$ —if (32) holds; if (33) holds we consider $|\nabla u_n|^2 + |u_n|^{q_1} + m u_n^2$— is tight: in particular we use the strict inequality $I < I^{\infty, 1}$ to obtain that if $\rho_n(\cdot + y^n)$ is tight then y_N^n is bounded if (32) holds, or y^n is bounded if (33) holds. Then if u_n converges weakly and a.e. to u, we have to check that $u \not\equiv 0$ and the remainder of the proof is then a combination of arguments of P. L. Lions [56] and of those given in Part 1 and above.

Let us check that $u \not\equiv 0$: if $u \equiv 0$, we observe

$$I = \lim \epsilon(u_n) = \lim \int_\Omega \frac{1}{2} f(u_n)u_n - F(u_n)dx + \int_{\partial\Omega} \frac{1}{2} g(u_n)u_n - G(u_n)ds.$$

And since ρ_n is tight, we deduce easily

$$\left| \int_\Omega \frac{1}{2} f(u_n)u_n - F(u_n)dx - \left(\frac{1}{2} - \frac{1}{q_1}\right) \int_\Omega \beta_1 |u_n|^{q_1} dx \right| \to 0$$

$$\left| \int_{\partial\Omega} \frac{1}{2} g(u_n)u_n - G(u_n)dx - \left(\frac{1}{2} - \frac{1}{q_2}\right) \int_{\partial\Omega} \beta_2 |u_n|^{q_2} dx \right| \to 0.$$

Similarly, we have

$$J(u_n) - J^{\infty,2}(u_n) \to 0, \quad \text{thus} \quad J^{\infty,2}(u_n) \to 0.$$

This shows that

$$I \geqslant \inf\left\{ \int_\Omega \left(\frac{1}{2} - \frac{1}{q_1}\right)\beta_1 |u|^{q_1} dx + \int_{\partial\Omega} \left(\frac{1}{2} - \frac{1}{q_2}\right)\beta_2 |u|^{q_2} ds / \right. \\ \left. u \in \mathfrak{D}^{1,2}(\Omega), J^{\infty,2}(u) = 0 \right\} = I^{\infty,2}$$

and this contradicts (16).

3. The General Principle

3.1 *Heuristic derivation*

In this section, we want to explain the common features of the problems and methods introduced in Part 1 and here exactly as we did in the locally compact case in P. L. Lions [58], [55], [56]. By no means, the claims below concerning the equivalence between certain compactness results and the subadditivity inequalities (S.1), (S.2) are to be understood as rigorous results: they are indications on what are the crucial inequalities to be checked and on a general scheme of proof.

We begin with the general case (the case of invariance by translations or dilations being treated below) and we keep the setting used in [58], [55]: let

H be a functions space on $\mathbb{R}^N$ (more general situations are considered below) and let J, $\mathcal{E}$ be functionals defined on H (or on a subdomain of H) of the following type

$$\mathcal{E}(u) = \int_{\mathbb{R}^N} e(x, Au(x))dx; \quad J(u) = \int_{\mathbb{R}^N} j(x, Bu(x))dx$$

where $e(x,p)$, $j(x,q)$ are real-valued functions defined respectively on $\mathbb{R}^N \times \mathbb{R}^m$, $\mathbb{R}^N \times \mathbb{R}^n$ and j is nonnegative; A, B are operators (possibly nonlinear) from H into E, F (functions spaces defined on $\mathbb{R}^N$ with values in $\mathbb{R}^m$, $\mathbb{R}^n$) which commute with the translations group of $\mathbb{R}^N$. We assume $\mathcal{E}(0) = J(0) = 0$. We want to study the following minimization problem

$$I = \operatorname{Inf}\{\mathcal{E}(u)/u \in H, \quad J(u) = 1\} \tag{3.46}$$

and we embed this problem in a one parameter family of problems

$$I = \operatorname{Inf}\{\mathcal{E}(u)/u \in H, \quad J(u) = \lambda\} \tag{3.47}$$

where $\lambda > 0$; of course $I = I_1$.

As we saw in the examples we have treated in sections 1 and 2, we have to evaluate the effects of the non-compactness of the translations and dilations group. This is why (to simplify) we assume

$$e(x,p) \to e_\infty^\infty(p), \quad j(x,q) \to j_\infty^\infty(q) \quad \text{as} \quad |x| \to \infty \tag{3.48}$$

(the precise meaning of the above convergence has to be worked out in all examples) and we set

$$I_\lambda^{\infty,\infty} = \operatorname{Inf}\{\mathcal{E}^{\infty,\infty}(u)/u \in H, J^{\infty,\infty}(u) = \lambda\} \tag{3.49}$$

where

$$\mathcal{E}^{\infty,\infty}(u) = \int_{\mathbb{R}^N} e_\infty^\infty(Au)dx, \quad J^{\infty,\infty}(u) = \int_{\mathbb{R}^N} j_\infty^\infty(Bu)\, dx.$$

Next, to take care of the dilations group, we assume to simplify that there exists a critica power $\alpha > 0$ such that $T_{\sigma,y}u = \sigma^{-\alpha}u((\cdot - y)/\sigma) \in H$ if $u \in H$ and if $y \in \mathbb{R}^N$, and we assume

$$\mathcal{E}(T_{\sigma,0}u) \to \tilde{\mathcal{E}}^{\infty,\infty}(u), \quad J(T_{\sigma,0}u) \to \tilde{J}^{\infty,\infty}(u) \quad \text{if} \quad \sigma \to +\infty \tag{3.50}$$

$$\mathcal{E}(T_{\sigma,y}u) \to \tilde{\mathcal{E}}^{\infty,y}(u), \quad J(T_{\sigma,y}u) \to \tilde{J}^{\infty,y}(u) \quad \text{if} \quad \sigma \to 0_+ \tag{3.51}$$

and we introduce for all $y \in \mathbb{R}^N$

$$\tilde{I}_\lambda^{\infty,\infty} = \operatorname{Inf}\{\tilde{\mathcal{E}}^{\infty,\infty}(u)/u \in H, \tilde{J}^{\infty,\infty}(u) = \lambda\} \tag{3.52}$$

$$I_\lambda^\infty = \operatorname{Min}\left(I_\lambda^{\infty,\infty}, \tilde{I}_\lambda^{\infty,\infty}, \operatorname*{Inf}_{y \in \mathbb{R}^N} I_\lambda^{\infty,y}\right). \tag{3.54}$$

We may now state a heuristic principle (which holds in all the examples treated before and below) that we call the concentrantion-compactness principle. To be rigorous, the following claims need many structure conditions (a priori bounds on minimizing sequences which insures in particular the finiteness of I_λ, the continuity of I_λ with respect to $\lambda \ldots$; convexity or weak l.s.c. properties of the «main» terms in $\mathcal{E}, J \ldots$) and it seems very difficult to give a unique framework covering the variety of the examples we treat.

We first claim (this part being easy to justify by the very way I_λ^∞ was defined and the arguments of [55]) that we *always have the large subadditivity inequalities*

$$I_\lambda \leqslant I_\alpha + I^\infty{}_{\lambda-\alpha}, \qquad \forall \alpha \in [0, \lambda]. \tag{3.55}$$

Next, we «claim» that, for a fixed $\lambda > 0$, all *minimizing sequences* of (47) *are compact if and only if*

$$I_\lambda < I_\alpha + I^\infty{}_{\lambda-\alpha}, \qquad \forall \alpha \in [0, \lambda[. \tag{S.1}$$

Indeed we first «prove» the «tightness» of any minimizing seqeunce $(u_n)_n$ by applying the first concentration-compactness lemma (see [58], [55]): since (S.1) implies

$$\begin{cases} I_\lambda < I_\alpha + I_{\lambda-\alpha}^{\infty,\infty} & \forall \alpha \in]0, \lambda[\\ I_\lambda < \operatorname{Min}(I_\lambda^{\infty,\infty}, \tilde{I}_\lambda^{\infty,\infty}) \end{cases}$$

dichotomy, vanishing and tightness up to an unbounded translation cannot happen. Next if $(u_n)_n$ «converges weakly» to some $u \in H$ we claim that $u \not\equiv 0$: if u were 0, then the effect of the «almost dilations invariance» ($u \to T_{\sigma,0}u$) would be that u_n concentrates around at most a countable number of points. But since $(u_n)_n$ is a minimizing sequence, we claim that u_n concentrates around a *single point* (up to subsequences); one way to understand this claim is to argue as follows, isolate one concentration point x^0 and split u_n into two parts: the part concentrating at x^0 and the part concentrating around the other points. If this were to happen, we would have for some $\alpha \in]0, \lambda[$

$$I_\lambda \geqslant I_\alpha + \operatorname*{Inf}_{y \in \mathbb{R}^N} I_{\lambda-\alpha}^{\infty,y} \geqslant I_a + I_{\lambda-\alpha}^\infty$$

contradicting (S.1). Hence, (u_n) concentrates at a single point x^0 and we deduce

$$I_\lambda \geqslant I_\lambda^{\infty,x^0} \geqslant \inf_{y \in \mathbb{R}^N} I_\lambda^{\infty,y} \geqslant I_\lambda^\infty$$

again contradicting (S.1). Therefore $u \not\equiv 0$. Finally if $J(u) = \alpha \in]0, \lambda[$ we split u_n into two parts: basically u and $u_n - u$ (this is only a rough idea — cf.

precise arguments in sections 1-2). Again $u_n - u$ concentrates at a countable number of points and we deduce

$$I_\lambda \geqslant I_\alpha + \inf_{y \in \mathbb{R}^N} I^{\infty,y}_{\lambda-\alpha} \geqslant I_\alpha + I^{\infty}_{\lambda-\alpha}$$

The contradiction shows that u_n converges to u, a minimun of (47).

This heuristic argument not only shows that (S.1) is *a necessary and sufficient condition* for *the compactness of all minimizing sequences* of (47) but also enables us to analyse what are the *possible losses of compactness if* (S.1) *fails*. For example if we know that

$$I_\lambda < I_\alpha + I^{\infty}_{\lambda-\alpha}, \qquad \forall \alpha \in]0, \lambda[\tag{3.56}$$

then (S.1) is equivalent to

$$I_\lambda < I^{\infty}{}_{\lambda}. \tag{16}$$

And if $I_\lambda = I^{\infty,\infty}_\lambda < \mathrm{Min}(\tilde{I}^{\infty,\infty}_\lambda, I^{\infty,y}_\lambda)$, $\forall y \in \mathbb{R}^N$, we obtain that any noncompact minimizing sequence is compact up to a translation y_n such that $|y_n| \to \infty$. Similarly if $I_\lambda = I^{\infty,y}_\lambda < \mathrm{Min}(I^{\infty,\infty}_\lambda, \tilde{I}^{\infty,\infty}_\lambda)$ for some $y \in \mathbb{R}^N$; then any noncompact minimizing sequence concentrates at an infimum point y^0 of $\mathrm{Inf}_{y \in \mathbb{R}^N} I^{\infty,y}_\lambda$ (and conveniently rescaled is compact, converging to a minimun point of $I^{\infty,y}_\lambda$ if $I^{\infty,y}_\lambda$ satisfies (S.2) below !)...

Next, we explain that the above ideas still carry out to cover more general situations where *j is not nonnegative*, or $\mathbb{R}^N$ *is replaced by* and unbounded region Ω such that

$$\forall R < \infty, \quad \exists y \in \Omega, \quad y + B_R \subset \Omega.$$

Indeed if j is negative somewhere, in general we still have to consider only $\alpha \in [0, \lambda[$: this is basically due to the fact that $J(0) = \mathcal{E}(0) = 0$ and with the above notations if $\mu = J(u) > \lambda$, we would have: $I_\mu \leqslant I_\lambda$ and this is not possible in general.

And when $\mathbb{R}^N$ is replaced by Ω, we assume (48) for $|x| \to \infty$, $x \in \bar{\Omega}$ and we replace in (54) the infimum over $y \in \mathbb{R}^N$ by the infimum over $y \in \bar{\Omega}$.

We now turn to problems with a complete or partial invariance: first of all we consider problems which are invariant by the changes $I_{\sigma,y}$ for all $\sigma > 0$, $y \in \mathbb{R}^N$ i.e.: $\mathcal{E}(T_{\sigma,y}u) = \mathcal{E}(u)$, $J(T_{\sigma,y}u) = J(u) \quad \forall u \in H$, $\forall \sigma > 0$, $\forall y \in \mathbb{R}^N$. In this case by similar arguments to the ones given above *any minimizing sequence $(u_n)_n$ is relatively compact up to a change T_{σ_n,y_n} if and only if (S.2) holds*: in particular if (S.2) holds, then there exist $(\sigma_n)_n$ in $]0, \infty[$, $(y_n)_n$ in $\mathbb{R}^N$ such that $T_{\sigma_n,y_n}u_n$ is relatively compact.

Next, we may consider *problems* which are *invariant by translations but not by dilations*: in this case we set

$$I_\lambda^\infty = \mathrm{Min}(\tilde{I}_\lambda^{\infty,\infty}, I_\lambda^{\infty,0}$$

(observe that $I_\lambda^{\infty,y} = I_\lambda^{\infty,0}$, $\forall y \in \mathbb{R}^N$); and *any minimizing sequence is compact up to a translation if and only if* (S.1) *holds*. Conversely, we may have to solve *problems invariant by dilations but not by all translations*: for example problems in a half space $\{x_N \geqslant 0\}$ invariant by dilations and by translations of the form $(y', 0)$. In this situation, we define $I_\lambda^{\infty,\infty}$ as before by considering

$$\mathcal{E}^{\infty,\infty}(u) = \lim_{|y|\to\infty} \mathcal{E}(u(\cdot + y)), \quad J^{\infty,\infty}(u) = \lim \mathcal{E}(u(\cdot + y))$$

(in this example above we only take: $\lim_{y_N \to +\infty}$) and we set $I_\lambda^\infty = I_\lambda^{\infty,\infty}$. And all *minimizing sequences are compact up to a dilation* (and a translation of the form $(y_n', 0)$ in the example) if and only if (S.1) holds. Similar variants exist if the problem —or the domain— has only a *restricted* number of *translation invariance* (exs.: strips, half-spaces...).

We now turn to the important particular case of a *compact region* Ω of $\mathbb{R}^N$ (or a *N-dimensionnal compact Riemannian manifold*). If the problem is «set in Ω», it is clear that the translations do not play anymore any role and similarly for $T_{\sigma,y}u$ when $\sigma \to +\infty$. Thus the only «*non-compactness*» remaining concerns the *action* of $T_{\sigma,y}u$ as $\sigma \to 0_+$ for any $y \in \bar{\Omega}$; hence we just assume (51) for $y \in \bar{\Omega}$ and we set for all $\lambda > 0$

$$I_\lambda^\infty = \operatorname*{Inf}_{y \in \bar{\Omega}} I_\lambda^{\infty,y}. \tag{3.57}$$

In this very particular case, the above principle reduces to the following ideas: *(S.1) is a necessary and sufficient condition for the compactness of all minimizing sequences*. In addition if (56) holds and thus (S.1) *is equivalent to* (16), then we have:

i) *if* (16) *holds, any minimizing sequence is compact*, ii) *if* (16) *does not hold i.e.* $I_\lambda = I_\lambda^\infty$ then *there exists a noncompact minimizing sequence* and *any such sequence converges weakly to* 0, *concentrating at a minimum point* y_0 *of* (57) (up to subsequences). In addition *if* I_λ^∞ *satisfies* (S.2), *there exist* $(\sigma_n)_n$ *in* $]0, \infty[$, (y_n) *in* $\mathbb{R}^N$ *such that* $T_{\sigma_n,y_n} u_n$ *is compact and converges to a minimum of* I_λ^{∞,y_0}, and $\sigma_n \to \infty$, $-y_n/\sigma_n \to y$ up to (subsequences). Let us also point out that when Ω is a compact manifold, the action $T_{\sigma,y}$ is not well defined but since we want to concentrate u at the point y only the local properties of Ω near y matter and via local charts and the tangent space $T_y\Omega$, we may still define $I_\lambda^{\infty,y}$ as a problem on the tangent space i.e. $\mathbb{R}^N$ if Ω is N-dimensional.

We next want to make several remarks: i) we may treat as well problems with *multiple constraints*

$$I(\lambda_1, \ldots, \lambda_m) = \operatorname{Inf}\{\mathcal{E}(u)/u \in H, J_i(u) = \lambda_i\}$$

then one defines exactly as we did $I^\infty(\lambda_1, \ldots, \lambda_m)$ and (S.1) is to be replaced by

$$I(\lambda_1, \ldots, \lambda_m) < I(\alpha_1, \ldots, \alpha_m) + I^\infty(\lambda_1 - \alpha_1, \ldots, \lambda_m - \alpha_m)$$

for all $\alpha_i \in [0, \lambda_i]$, $\sum_i \alpha_i < \sum \lambda_i$.

ii) It may be important to treat the following type of constraints

$$I = \operatorname{Inf}\{\mathcal{E}(u)/u \in H, J(u) = 0, u \not\equiv 0\}$$

—see [56] and section II—. Then denoting by I_λ the infimum corresponding to $J(u) = \lambda$ for $\lambda \in \mathbb{R}$, (S.1) is to be replaced by

$$I < I_\lambda + I^\infty_{-\lambda} \quad \forall \lambda \neq 0, \quad I < I^\infty$$

where I^∞ is defined as before. Very often the first series of inequalities hold easily (notice also that I is not modified if we replace $\mathcal{E}$ by $\mathcal{E} + \mu J \ldots$).

iii) In the locally compact case (cf. P. L. Lions [55], [56]) we refind the fact that the action of $T_{\sigma,y}$ does not play any role observing that $\tilde{\mathcal{E}}^{\infty,\infty}$ or $\tilde{J}^{\infty,\infty}$ and $\mathcal{E}^{\infty,y}$ or $J^{\infty,y}$ are trivial in this case and thus I^∞_λ reduces to $I^{\infty,\infty}_\lambda$.

iv) If J has a completely indefinite sign, it may happen that (S.1) has to be replaced by

$$I_\lambda < I_\alpha + I^\infty_{\lambda-\alpha}, \quad \forall \alpha \in \mathbb{R} - \{\lambda\} \qquad \text{(S.1')}$$

Remark III.1. In order to illustrate (at last !) the above discussion we wish to indicate briefly a list of the various types of problems encountered and the corresponding results in Part 1 and here:

1. Invariance by dilations and translations: Theorem I.1; Corollary I.2; Problem (1.35); Theorem II.1.
2. Invariance by translations, not by dilations: Problem (1.30); Theorem I.5; Theorem II.2; Theorem II.7 ii).
3. Invariance by dilations, not by translations: Theorem I.3; Theorem II.3; Theorem II.4; Theorem II.7 i).
4. Restricted invariance by translations: Theorem II.3; Theorem II.5; Theorem II.6; Theorem II.7.
5. Nonisotropic dilation invariance: Corollary I.3.
6. General situations: Theorem I.2; Theorem I.4; Theorem II.6; Theorem II.8.

7. Multiple constraints: Problem (1.33)
8. Constraint $J(u) = 0$: Theorem II.7; Theorem II.8.
9. Problems in compact domains: section IV.

We next would like to explain what we mean by concentrates around a point: this means that the densities of the functionals or of related norms — which are bounded L^1_+ functions— converge weakly to Dirac masses (cf. Lemma I.1; sections I.4 ii), iii), iv), vi), vii); Lemma I.4; Theorem I.6; Lemma II.1; Lemma II.2; Lemma II.3; lemma II.4...).

Finally, we want to conclude this section by emphasizing that (S.1), (S.2) are necessary and sufficient conditions for the compactness of all minimizing sequences but that there might exist a minimum even if (S.1) (or (S.2) fails) see P. L. Lions [ ] for such an example in the locally compact case. In addition (S.1), (S.2) may be difficult to check (but anyway one has to check them!): in particular when (S.1) reduces to (16) and $I^\infty_\lambda = \text{Inf}_y I^{\infty,y}_\lambda$, in order to check (16), it is natural to try as a test function: $\bar{u}_\varepsilon = T_{\varepsilon,y_0}\bar{u}$ where y_0 is a minimum point of (57), $\bar{u}$ a minimum of I^{∞,y_0}_λ and ε goes to 0. Indeed observe that any noncompact minimizing sequence will be very similar to $\bar{u}_\varepsilon$ (if I^∞_λ satisfies (S.2)) if $I_\lambda = I^\infty_\lambda$. This motivates the choice of $\bar{u}_\varepsilon$ in order to analyse (16): this choice was first considered by T. Aubin [6], see also H. Brézis and L. Nirenberg [23], H. Brézis and J. M. Coron [19], [20], P. L. Lions [65].

We make two final remaks on (S.1) and (S.2) —that will be developed further elsewhere—: first of all, if $I_\lambda < I^\infty_\lambda$, I^∞_μ satisfies (S.2) for all $\mu \in]0, \lambda]$ and (S.1) does not hold there exists $\alpha \in]0, \lambda[$ such that

$$I_\lambda = I_\alpha + I^\infty_{\lambda-\alpha}$$

We then claim that I_α satisfies (S.1): indeed if we had

$$I_\alpha = I_\beta + I^\infty_{\alpha-\beta} \quad \text{with} \quad \beta \in [0, \alpha[$$

this would imply

$$I_\lambda = I_\beta + I^\infty_{\alpha-\beta} + I^\infty_{\lambda-\alpha} > I_\beta + I^\infty_{\lambda-\beta} \geqslant I_\lambda;$$

a contradiction. In addition $\{\alpha \in]0, \lambda], \text{ (S.1) holds for } I_\alpha\}$ is open if $I^\infty_\mu \mu^{-1} \to +\infty$ when $\mu \to 0_+$: indeed if (S.1) holds for I_{α_0}, then for α near α_0, $I_\alpha < I^\infty_\alpha$ and if (S.1) fails for I_α there exists $\beta \in]0, \alpha[$ such that

$$I_\beta + I^\infty_{\alpha-\beta} = I_\alpha, \quad \beta \to \alpha_0 \quad \text{as} \quad \alpha \to \alpha_0.$$

But there exists a minimum for I_β (cf. the argument above) which converges to a minimum of I_{α_0} as $\beta \to \alpha_0$. Hence it is easy to show that: $I_\alpha \leqslant I_\beta + C(\alpha - \beta)$ for α near α_0; in other words $I^\infty_{\alpha-\beta} \leqslant C(\alpha - \beta)$ for α near α_0, if (S.1) fails. Thus (S.1) holds in a neighborhood of α_0.

3.2. *The role of symmetries*

In this section, we want to explain how *the invariance of functionnals by symmetries* (orthogonal transformations of $\mathbb{R}^N$) fits in the general picture of minimization problems and the concentration-compactness principle. To motivate what follows let us recall that it was observed in W. Strauss [75] (and developped in H. Berestycki and P. L. Lions [13]) that the embedding from $H^1(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ for $2 < p < 2N/(N-2)$ $(N \geqslant 3)$ is compact when restricted to spherically symmetric functions. This was used in [75], [13] to solve various minimization problems by restricting a priori (or a posteriori via symmetrization) the infimum to spherically symmetric functions (see also P. L. Lions [62]). Such compactness arguments are extended to more general symmetries in P. L. Lions [62], [63]. In addition in those compactness results one proves that if $H_r^1(\mathbb{R}^N)$ is the subspace of $H^1(\mathbb{R}^N)$ consisting of spherically symmetric functions then $H_r^1(\mathbb{R}^N) \hookrightarrow L^\infty(|x| > \delta)$ for any $\delta > 0$ (see Appendix 2 for more general results of this type) hence on the domain $(|x| > \delta)$ the limit exponent $2N/(N-2)$ is meaningless for $H_r^1(\mathbb{R}^N)$ and compactness is available (see M. J. Esteban and P. L. Lions [35] for an application of this fact).

We want here to explain these observations by the help of an extension of the *concentration-compactness principle*, taking into account the *invariance of the functionals by a group of orthogonal transformations* of $\mathbb{R}^N$. Let us also mention that we were led to the heuristic principle which follows by the study due to C.V. Coffman and Markus [28] and that the analysis below will be developped further elsewhere.

We still consider the general setting of the preceding section where $\mathbb{R}^N$ is replaced by a domain Ω. We assume that Ω, $\mathcal{E}$, J are *invariant* under the *action of a group G* of orthogonal transformations of $\mathbb{R}^N$ (of course if Ω is a compact N-dimensionnal manifold we adapt the notion of such a group in a straightforward way...) and we consider for $\lambda > 0$

$$\bar{I}_\lambda = \operatorname{Inf}\{\,\mathcal{E}(u)/u \in H, \quad u \text{ is } \quad G\text{-invariant}, J(u) = \lambda\,\}$$

where G-invariant means: $u(x) = u(g \cdot x)$, $\forall x \in \bar{\Omega}$, $\forall g \in G$.

We need now to define *the problems at infinity*: first of all we define $\bar{I}_\lambda^{\infty,\infty}$ exactly as before adding to the set of minimizers the constraint that u is G-invariant.

Next, observing that if «u is concentrated at y» and if u is G-invariant u is also concentrated at every point $z = g \cdot y$ for some $g \in G$, we consider the *equivalence class*: $\omega(y) = \{z = g \cdot y/g \in G\}$, and we denote by $s(y) = \#\,\omega(y)$. If $s(y) < \infty$, we define

$$\bar{I}_\lambda^{\infty,y} = \sum_{z\in\omega(y)} I^{\infty,z}_{\lambda/s(y)} = s(y) I^{\infty,y}_{\lambda/s(y)}; \tag{59}$$

(Of course if $0 \in \bar{\Omega}$, $\omega(0) = \{0\}$ and $s(0) = 1$) $\bar{I}_\lambda^{\infty,y}$ does not really depend of y but on its equivalence class $\omega(y)$. Next if $s(y) = +\infty$, we set: $\bar{I}_\lambda^{\infty,y} = \lim_{n\to\infty} n I^{\infty,y}_{\lambda/n} \in]-\infty, +\infty]$ (The fact that the limit exists is an easy exercise, since the function $\varphi(t) = \bar{I}_t^{\infty,y}$ is subadditive on $[0, \lambda]$). In many cases this limit is trivial (either 0 or $+\infty$).

Finally, to take into account the effect of the translations (if Ω satisfies: $\forall R < \infty$, $\exists y \in \Omega$, $B(y, R) \subset \Omega$) we consider

$$s_R = \inf\left[\#\,\omega(x)/|x| \geqslant R, \quad x \in \bar{\Omega}\right] \leqslant +\infty.$$

For R large s_R is constant and we denote by s its value. We then set

$$\begin{cases} \bar{I}_\lambda^{\infty,\infty} = s I^{\infty,\infty}_{\lambda/s} & \text{si} \quad s < \infty \\ \bar{I}_\lambda^{\infty,\infty} = \lim_{n\to\infty} n I^{\infty,e}_{\lambda/n} & \text{if} \quad s = +\infty \end{cases} \tag{60}$$

The same heuristic considerations of the preceding section show that *the strict sub-additivity inequality.*

$$\bar{I}_\lambda < \bar{I}_\alpha + \bar{I}^\infty_{\lambda-\alpha}, \qquad \forall \alpha \in [0, \lambda[\tag{S.3}$$

is still *necessary and sufficient for the compactness of all minimizing sequences* of (58). And we have the same adaptations, extensions, variations as before for problems invariant by dilations, (some) translations. Furthermore if (S.3) fails, we know how compactness is lost on noncompact minimizing sequences.

In particular, in *the locally compact case*, $\bar{I}_\mu^\infty$ reduces to $I_\mu^{\infty,\infty}$ defined by (60); while if $\bar{\Omega}$ *is compact*, $\bar{I}_\lambda^{\infty,\infty}$, $\bar{I}_\lambda^{\infty,\infty}$ disappear and $\bar{I}_\lambda^\infty$ reduces to $\operatorname{Inf}_{y\in\bar{\Omega}} \bar{I}_\lambda^{\infty,y}$.

Before giving briefly two examples below (more may be found in section IV and in a future study), we would like to point out that in some vague sense symmetries may help to find a solution of the Euler equation associated with (58) or equivalently with (47) (if $\mathcal{E}$, J are $C^1, \ldots$) since $n\varphi\left(\frac{\lambda}{n}\right) \geqslant \varphi(\lambda)$ if φ is subadditive and since $n\varphi\left(\frac{\lambda}{n}\right)$ is «essentially nondecreasing» with respect to n (at least along multiples...) therefore $\bar{I}_\lambda^\infty$ essentially increases if s, $\inf_{y\in\bar{\Omega}} s(y)$ increase.

Another way to see this improvement of the conditions (S.1) – (S.2) is to observe that if $s = +\infty$, the first concentration-compactness lemma yields that we have either vanishing, or compactness. And recalling that if $\rho_n = u_n^2 + |\nabla u_n|^2$ vanishes then $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2N/(N-2)$ (cf. P. L. Lions [55], [56]), we find back the compactness results of W. Strauss [75], P. L. Lions [63] in a very direct way. Similarly if $\operatorname{Inf}\{s(y) \mid y \in \bar{Q}\} = +\infty$, *no Dirac masses may form since* J would contain an infinite set $\omega(y)$ on which each Dirac mass δ_z has a fixed intensity thus contradicting the summability of the measure!

Example 3.1. Let $N \geqslant 3$, consider the functionals on $H^1(\mathbb{R}^N)$

$$\mathcal{E}(u) = \int_{\mathbb{R}^N} \frac{1}{2}|\nabla u|^2 + \frac{1}{2} V(|x|)u^2 \, dx - \frac{1}{4} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{u^2(x)u^2(y)}{|x-y|} \, dx \, dy$$

$$J(u) = \int_{\mathbb{R}^N} u^2 \, dx;$$

where $V \in L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ with $\frac{N}{2} \leqslant p, q < \infty$. If we do not use the spherical symmetry of V i.e. we only consider

$$I = \operatorname{Inf}\{\mathcal{E}(u)/J(u) = 1, \quad u \in H^1(\mathbb{R}^N)\}$$

then —cf. P. L. Lions [55], [59]— all minimizing sequences are relatively compact if and only if

$$I < I^\infty = \operatorname{Inf}\left\{\int_{\mathbb{R}^N} \frac{1}{2}|\nabla u|^2 \, dx - \frac{1}{4} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{u^2(x)u^2(y)}{|x-y|} \, dx \, dy / J(u) = 1\right\}.$$

And if $V \geqslant 0$, $V \not\equiv 0$, there is *no* minimum.

On the other hand (this was observed in P. L. Lions [57] and it is clear in view of the above arguments) if we consider for $\lambda > 0$

$$\bar{I}_\lambda = \operatorname{Inf}\{\mathcal{E}(u)/J(u) = \lambda, \quad u \in H^1(\mathbb{R}^N), \quad u \text{ spherically symmetric}\},$$

then $\bar{I}_\lambda < 0$ and all minimizing sequences are compact and a minimum exists (thus $\bar{I}_1 > I$!). This is also clear in view of our arguments above: since (we are in the locally compact case) $\bar{\bar{I}}_\lambda^{\infty,\infty} = 0$, $\bar{I}_\lambda^{\infty,y} = 0$, $\forall y$ and $\bar{I}_\lambda^{\infty,\infty} = \lim n I_{\lambda/n}^\infty = 0$ and thus (S.3) is equivalent to $\bar{I}_\lambda < \bar{I}_\alpha$, $\forall \alpha \in]0, \lambda[$, and this is easily checked since $\bar{I}_\lambda < 0$.

Example 3.2. Let $N \geqslant 3$, consider the functionals defined on $\mathcal{D}^{1,2}(\Omega)$ —where $\Omega = \{x \in \mathbb{R}^N, \quad |x| > 1\}$— by

$$\mathcal{E}(u) = \int_\Omega a(|x|)|\nabla u|^2 \, dx, \qquad J(u) = \int_\Omega K(|x|)|u|^{2N/(N-2)} \, dx$$

where a, K are positive continuous, $a, K \to a^\infty, K^\infty > 0$ as $|x| \to \infty$. We then consider

$$\bar{I}_\lambda = \operatorname{Inf}|\mathcal{E}(u)/u \in D^{1,2}(\Omega), \quad J(u) = \lambda, \quad u \text{ spherically symmetric}\}.$$

We compute easily: $\bar{I}_\lambda^{\infty,\infty} = +\infty$, $\bar{I}_\lambda^{\infty,y} = +\infty$, $\forall y \in \bar{\Omega}$ and

$$\bar{I}_\lambda^\infty = \operatorname{Inf}\left\{a^\infty \int_{\mathbb{R}^N} |\nabla u|^2\, dx / u \in \mathcal{D}^{1,2}(\mathbb{R}^N),\quad u \text{ spherically symmetric}, \right.$$
$$\left. K^\infty \int_{\mathbb{R}^N} |u|^{2N/(N-2)}\, dx = \lambda\right\}.$$

And (S.3) reduces (since $\bar{I}_\lambda = \lambda^{(N-2)/N}\bar{I}_1$, $\bar{I}_\lambda^\infty = \lambda^{(N-2)/N}\bar{I}_1^\infty$) to

$$I_\lambda{}^\infty < \bar{I}_\lambda^\infty$$

If this condition holds, all minimizing sequences are compact and a minimum exists, while if $I_\lambda{}^\infty = \bar{I}_\lambda^\infty$ there exists a minimizing sequence which is not compact and any such sequence $(u_n)_n$ satisfies: $\exists(\sigma_n)_n \in]0, \infty[$ such that $\sigma_n \to 0$, $\sigma_n^{-(N-2)/2} u_n\left(\frac{\cdot}{\sigma_n}\right)$ is relatively compact and its limit points are minima of $\bar{I}_\lambda^\infty$.

4. Problems in compact regions

4.1 *Yamabe problem*

Our main goal in this section is to explain T. Aubin's results on Yamabe problem in the light of our general arguments. We first recall the nature of the problem.

Let (M, g) be a C^∞ N-dimensionnal Riemannian manifold. We denote by Δ the Laplace-Beltrami operator on (M, g); in local coordinates this operator is given by

$$\frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial x_i}\left(\sqrt{g}\, g^{ij} \frac{\partial}{\partial x_j}\right)$$

where $\sum_{i,j} g_{ij}\, dx^i\, dx^j$ is the metric, $g^{ij} = (g_{ij})^{-1}$, $g = \det(g_{ij})$.

Let k be the scalar curvature of (M, g). One is interested in the determination of all functions K which can be realized as the scalar curvature of a metric which is pointwise conformal to g i.e. of a metric $\tilde{g}$ obtained by multiplying g by a positive function on M. Now if we introduce the unknown function u (positive on M) such that

$$\tilde{g} = u^{4/(N-2)} g,$$

the above condition on K is equivalent to the so-called Yamabe equation (see H. Yamabe [84], T. Aubin [9]; H. Eliasson [33] for the detailed computations)

$$-4\frac{N-1}{N-2}\Delta u + ku = Ku^{(N+2)/(N-2)} \text{ in } M, \qquad u > 0 \text{ in } M \tag{Y}$$

—where of course $N \geqslant 3$. In fact, H. Yamabe considered in [84] only the case when K is constant and claimed that in this case the problem could always be solved. As it was remarked by N. Trudinger [79], the argument in [84] was not complete and the case when $K \equiv 1$ is still an open question (at least for $3 \leqslant N \leqslant 5$).

Let us mention at this point that related questions concerning scalar curvature and deformations on variations of metrics are considered in J. L. Kazdan and F. W. Warner [(47], [48], [49], [50], J. L. Kazdan [46]; A. E. Fischer and J. E. Marsden [37]; J. P. Bourguignon and J. P. Ezin [17]; J. P. Bourguignon [16].

Let λ_1 denote the first eigenvalue of the operator

$$-4\frac{N-1}{N-2}\Delta + k$$

on $H^1(M)$; it is easily seen that:

i) if $\lambda_1 > 0$ and $K \leqslant 0$, no solution of (Y) exists;
ii) if $\lambda_1 = 0$: no solutions exists if $K \not\equiv 0$, $K \leqslant 0$ or $K \geqslant 0$; trivial solutions exists if $K \equiv 0$ (and are unique up to a multiplicative constant);
iii) if $\lambda_1 < 0$: no solution exists if $K \geqslant 0$ while if $K \leqslant 0$, $K \not\equiv 0$ it is a standard exercize on semilinear elliptic equations to show that (Y) has a unique positive solution (one can also make a few remarks of the same spirit if K has both signs). We refer to T. Aubin [9] for a brief exposition of these facts.

In view of these remarks, it is natural to assume

$$\lambda_1 > 0; \qquad \operatorname*{Max}_M K > 0. \tag{61}$$

In this case, one way of finding (possibly) solutions of (Y) is to look at the following minimization problem

$$I = \operatorname{Inf}\{\mathcal{E}(u)/u \in H^1(M), \quad J(u) = 1\} \tag{62}$$

where

$$\mathcal{E}(u) = \int_M 4\frac{N-1}{N-2}|\nabla u|^2 + ku^2, \qquad J(u) = \int_M K|u|^{2N/(N-2)}.$$

Then any minimum of (62) is, up to a change of sign and multiplication by a positive constant, a solution of (Y). Let us emphasize that the converse may be false! Let us also mention that, as long as (62) is concerned, it is not necessary to consider only a function k which is the scalar curvature and in what follows, k, K are *arbitrary functions* in $C(M)$ *satisfying* (61).

At this stage, we want to explicit the condition (S.1) (which, as it should be, will be the critical condition needed to solve (62)): first of all since

$I_\lambda = \lambda^{(N-2)/N} I > 0$ (where I_λ is the same infimum as in (62) with $J(u) = 1$ replaced by $J(u) = \lambda$), we see that (62) reduces to

$$I < I^\infty \tag{16}$$

and we have to compute

$$I^\infty = \operatorname*{Inf}_{y \in M} I^{\infty,y};$$

$$I^{\infty,y} = \operatorname{Inf}\{\mathcal{E}^{\infty,y}(u)/u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \quad J^{\infty,y}(u) = 1\}$$

$\mathcal{E}^{\infty,y}(u)$, $J^{\infty,y}(u)$ being obtained by concentrating u in $\mathbb{R}^N$ at 0 by the dilations $\left(\sigma^{-(N-2)/2} u\left(\frac{\cdot}{\sigma}\right), \ \varphi \to 0_+\right)$ and bringing it back at y on M by a local chart. Remarking that if $a_{ij} = a_{ij} > 0$

$$I(a,\lambda) = \operatorname{Inf}\left\{\int_{\mathbb{R}^N} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx/u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} |u|^{2N/(N-2)} dx = \lambda\right\}$$
$$= \det(a_{ij})^{1/N} \lambda^{N/(N-2)} I_0$$

where

$$I_0 = \operatorname{Inf}\left\{\int_{\mathbb{R}^N} |\nabla u|^2 dx/u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} |u|^{2N/(N-2)} dx = 1\right\}$$

(i.e. $I_0^{-1/2}$ is the best constant for Sobolev inequalities, cf. Part 1 [65]); we deduce easily that

$$I^{\infty,y} = 4\frac{N-1}{N-2} K(y)^{-(N-2)/N} I_0 \qquad \text{if } K(y) > 0$$
$$= +\infty \qquad \text{if } K(y) \leqslant 0.$$

Therefore we have

$$I^\infty = 4\frac{N-1}{N-2}\left(\max_M K\right)^{-(N-2)/N} I_0; \tag{63}$$

and we already know that $I \leqslant I^\infty$.

Theorem 4.1. *We assume* (61). *If $I \leqslant I^\infty$, any minimizing sequence of* (62) *is relatively compact in $H^1(M)$ and a minimum exists. If $I = I^\infty$, there exist minimizing sequences which are not compact and any such sequence $(u_n)_n$ satisfies* (*up to subsequences*)

$$\begin{cases} u_n \to 0 \quad \text{weakly in } H^1(M), \\ |u_n|^{2N/(N-2)} \to \left(\max_M K\right)^{-1} \delta_{x_0}, \quad |\nabla u_n|^2 \to I^\infty \delta_{x_0} \qquad (\text{in } \mathcal{D}'(M)) \\ K(x_0) = \max_M K. \end{cases} \tag{64}$$

Let us immediately mention that this result is a small extension of a *result due to T. Aubin* [6] (see also [9], [11], [12]) where it is proved that, if $I < I^\infty$, a minimum exists by a different method. In addition in T. Aubin [6], (16) is discussed in details and in particular if k is the scalar curvature, $K \equiv 1$ (*Yamabe original problem*) it is proved that (16) *holds for «most» manifolds M if $N \geq 6$.*

Remark 4.1 In addition to (64), we may prove that one can find cut-off functions $\xi_n \in C^\infty(M)$ supported in $B(x_0, \epsilon_n)$ with $\epsilon_n \to 0$ such that $v_n = \xi_n u_n \to 0$ in $H^1(M)$ strongly, and if w_n is the sequence in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ obtained from v_n by local charts, there exists $(\sigma_n)_n$ in $]0, \infty[$, $(y_n)_n$ in $\mathbb{R}^N$ such that

$$\begin{cases} \sigma_n \to \infty, \quad y_n/\sigma_n \to 0, \quad \sigma_n^{-(N-2)/2} w_n((\cdot + y_n)/\sigma_n) \text{ is relatively} \\ \text{compact in } \mathcal{D}^{1,2}(\mathbb{R}^N) \quad \text{and its limit points are minima of } I^\infty. \end{cases}$$

The proof of Theorem 4.1 is an easy adaptation of arguments given before (in particular Theorem 1.2 in Part 1 [65]): we just observe that M being compact the tightness of $\rho_n = |\nabla u_n|^2 + |u_n|^{2N/(N-2)}$ is automatic and that we have the:

Lemma 4.1. *Let $(u_n)_n$ converge weakly to u in $H^1(M)$. We may assume that $|\nabla u_n|^2$, $|u_n|^{2N/(N-2)}$ converge weakly to some measures μ, ν. Then we have*

$$\nu = |u|^{2N/(N-2)} + \sum_{j \in J} \nu_j \delta_{x_j} \tag{65}$$

$$\mu \geq |\nabla u|^2 + I_0 \sum_{j \in J} \nu_j^{(N-2)/N} \delta_{x_j} \tag{66}$$

for some at most countable family J, and where $\nu_j > 0$, x_j are distinct points of M.

We skip the proof of this lemma since it is totally similar to the proof of Lemma 1.1: let us only point out that I_0 in (66) corresponds to $\mathbb{R}^N$ ($\simeq T_{x_j}M$) and that this comes from the localization procedure around x_j used in the proof of Lemma 1.1.

Remark 4.2. Of course similar results hold for $(u_n)_n$ bounded sequence in $W^{m,p}(M)$.

Remark 4.3. The proof of T. Aubin [6] concerning the existence of a minimum if (16) holds uses heavily the «best constant» C_λ of the Sobolev inequality on M

$$\left(\int_M |u|^{2N/(N-2)}\right)^{(N-2)/2N} \leq C_\lambda \left(\int_M |\nabla u|^2 + \lambda u^2\right)^{1/2}$$

and the fact proved in T. Aubin [7] that $C_\lambda \downarrow C_0$ as $\lambda \uparrow \infty$ where C_0 is the best constant of the Sobolev inequality in $\mathbb{R}^N$. In fact our methods also prove this elementary fact easily: indeed consider

$$I^\lambda = \operatorname{Inf}\left\{ \int_M |\Delta u|^2 + \lambda u^2 \Big/ \int_M |u|^{2N/(N-2)} = 1, u \in H^1(M) \right\}.$$

Applying Theorem IV.1 we see that $I^\lambda \leqslant I_0$, $\forall \lambda > 0$, $I^\lambda \downarrow$ as $\lambda \uparrow$, I^λ is achieved if $I^\lambda < I_0$ (and this happens for λ small since $I^\lambda \downarrow 0$ as $\lambda \downarrow 0_+$). Therefore either $I^\lambda = I_0$ for $\lambda \leqslant \lambda_0 > 0$ (and this is a very interesting situation where the best constant C_0 is achieved on M!), or $I^\lambda < I_0$. We claim that in this case $I^\lambda \uparrow I_0$ as $\lambda \uparrow +\infty$. If this is not the case, denoting by u_λ the minimum corresponding to I^λ, we have

$$\int_M |\nabla u_\lambda|^2 \underset{\lambda \to \infty}{\longrightarrow} \alpha I_0, \qquad u_\lambda \underset{\lambda \to \infty}{\longrightarrow} 0 \quad \text{in} \quad L^2(M).$$

Thus

$$|\nabla u_\lambda|^2 \to \mu \geqslant I_0 \sum_{j \in J} \nu_j^{(N-2)/N} \delta_{x_j},$$

$$|u_j|^{2N/(N-2)} \to \nu = \sum_{j \in J} \nu_j \delta_{x_j},$$

and this would give

$$I_0 > \alpha \geqslant I_0 \sum_{j \in J} \nu_j^{(N-2)/N} \geqslant I_0 \Big(\sum_{j \in J} \nu_j \Big)^{(N-2)/N} = I_0.$$

Therefore we have proved not only that $I^\lambda \uparrow I_0$ as $\lambda \uparrow \infty$ but also that either $I^\lambda = I_0$ for λ large or I^λ is achieved and any corresponding minimum u_λ satisfies (up to subsequences)

$$\int_M |\nabla u_\lambda|^2 \underset{\lambda}{\to} I_0, \qquad \int_M u_\lambda^2 \underset{\lambda}{\to} 0$$

$$|\nabla u_\lambda|^2 \underset{\lambda}{\to} I_0 \delta_{x_0}, \qquad |u_\lambda|^{2N/(N-2)} \underset{\lambda}{\to} \delta_{x_0}$$

for some $x_0 \in M$.

We next present some new existence results concerning (Y) using «symmetries». Assume that (M, g) is embedded in $\mathbb{R}^p$ (for some $p > N$) and that (M, g), $\mathcal{E}$, J are invariant under the action of a group G of one to one transformations of $\mathbb{R}^p$. In particular we have: $\forall h \in G$

$$\mathcal{E}(u(h\cdot)) = \mathcal{E}(u(\cdot)), \quad J(u(h\cdot)) = J(u), \qquad \forall u \in C^\infty(M).$$

Our typical example is S^N with the usual metric, then we may take for G any subgroup of the group of orthogonal transformations $O(p)$ and our assumption just means that K is invariant by G.

Any minimum of the following minimization problem is still a solution of (Y) (invariant by G, giving a new metric invariant by G)

$$\bar{I} = \operatorname{Inf}\{\mathcal{E}(u)/J(u) = 1, \quad u \in H^1(M), \quad u \text{ is } G\text{-invariant}\}. \tag{65}$$

We denote by: $\omega(y) = \{h \cdot y / h \in G\}$, $s(y) = \#\omega(y)$ and we set

$$\begin{cases} \bar{I}^{\infty,y} = K(y)^{-(N-2)/N} s(y)^{2/N} I_0 & \text{if } K(y) > 0, \quad s(y) < \infty \\ \bar{I}^{\infty,y} = +\infty & \text{if } K(y) \leqslant 0, \quad s(y) = +\infty. \end{cases} \tag{66}$$

$$\bar{I} = \operatorname*{Inf}_{y \in M} \bar{I}^{\infty,y} \tag{67}$$

(notice that $s(y)$, $\bar{I}^{\infty,y}$ are not, in general, continuous on M).

We have immediately the following:

Theorem 4.2. *We assume* (61). *Then* $\bar{I} \leqslant \bar{I}^\infty$ *and if* $\bar{I} < \bar{I}^\infty$ *all minimizing sequences of* (65) *are relatively compact in* $H^1(M)$ *and there exists a minimum. While if* $\bar{I} = \bar{I}^\infty$, *there exists a minimizing sequence which is not relatively compact and any such sequence* $(u_n)_n$ *satisfies*

$$u_n \to 0 \quad \text{weakly in} \quad H^1(M); \qquad |\nabla u_n|^2 \to \frac{\bar{I}^\infty}{s(x_0)} \sum_{z \in \omega(x_0)} \delta_z;$$
$$|u_n|^{2N/(N-2)} \to K(x_0)^{-1} \frac{1}{s(x_0)} \sum_{z \in \omega(x_0)} \delta_z$$

for some x_0 *satisfying*

$$K(x_0) > 0, \quad s(x_0) < \infty, \qquad K(x_0)^{-(N-2)/N} s(x_0)^{2/N} = \operatorname*{Inf}_{y \in M} K(y)^{-(N-2)/N} s(y)^{2/N}$$

Remark 4.4. If $K \equiv 1$ and $\operatorname*{Inf}_{y \in M} s(y) = p > 1$ then for p large $\bar{I} < \bar{I}^\infty$: indeed take $u \equiv 1$ in (65), $\bar{I} \leqslant \int_M k < p^{2/M} I_0$ for p large.

Remark 4.5. If $M = S^N$ and K is invariant by a subgroup G of $O(N+1)$ such that $\operatorname*{Inf}_{y \in M} s(y) = p$. Then

$$\bar{I}^\infty \geqslant p^{2/N} (\max K)^{-(N-2)/N} I_0.$$

If (for example) $M = S^2$, if $K(x_1, x_2, x_3) = K(x_3)$ (for $x = (x_1, x_2, x_3) \in S^2$) and $K(\pm 1) \leqslant 0$ or if $K(x_1, x_2, x_3) = K(x_2, x_3)$ and $K \leqslant 0$ for $x_2^2 + x_3^2 = 1$, then $\bar{I}^{\infty,y} = +\infty$, $\forall y \in M$ and a minimum exists. These last examples may also be obtained by symmetrization and results corresponding to Appendix 2.

4.2 *Related problems*

In this section, we consider various problems strongly related to Yamabe equation: the first one concerns Yamabe equation in a bounded open set Ω of $\mathbb{R}^N$ with Dirichlet boundary conditions and we will present some variants of results due to H. Brézis and L Nirenberg [23] and also some new results when symmetries occur. We will also briefly consider the case of Newmann conditions. The second class of problems we consider is the problem introduced in Cherrier [25] which extends the Yamabe problem to manifolds with boundary.

We thus begin with the following problem: let Ω be a bounded open set of $\mathbb{R}^N$ with $N \geqslant 3$, let $a_{ij}(x)$, $c(x)$, $K(x)$ be continuous functions of $\bar{\Omega}$ satisfying

$$(a_{ij}(x)) = (a_{ji}(x)) \geqslant \nu I_N, \qquad \forall x \in \bar{\Omega}, \quad \text{for some} \quad \nu > 0 \tag{68}$$

$$\forall \varphi \in \mathfrak{D}(\Omega), \qquad \int_\Omega \sum_{i,j} a_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} + c\varphi^2 \, dx \geqslant \alpha \|\varphi\|^2_{H^1_0(\Omega)}$$

for some $\alpha > 0$

$$\max_\Omega K > 0. \tag{70}$$

We want to solve the following equation

$$-\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + cu = Ku^{(N+2)/(N-2)} \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \quad u > 0, \quad \text{in } \Omega, \tag{71}$$

and we thus consider

$$I = \operatorname{Inf}\{\mathcal{E}(u)/u \in H^1_0(\Omega), \quad J(u) = 1\}. \tag{72}$$

where

$$\mathcal{E}(u) = \int_\Omega \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + cu^2 \, dx, \qquad J(u) = \int_\Omega K|u|^{2N/(N-2)} \, dx.$$

In view of the homogeneity of the problem ($I_\lambda = \lambda^{(N-2)/N} I$) (S.1) once more reduces to

$$I < I^\infty \tag{16}$$

where $I^\infty = \operatorname{Min}_{y \in \Omega} I^{\infty,y}$ and

$$I^{\infty,y} = \operatorname{Min}\left\{ \int_{\mathbb{R}^N} \sum_{i,j} a_{ij}(y) \frac{\partial u}{\partial x_i}(x) \frac{\partial u}{\partial x_j}(x) \, dx / u \in \mathfrak{D}^{1,2}, \right.$$

$$\left. \int_{\mathbb{R}^N} K(y)|u|^{2N/(N-2)} \, dx = 1 \right\}$$

$$= \det(a_{ij}(y))^{1/N} K(y)^{-(N-2)/N} I_0, \quad \text{if } K(y) > 0;$$

$$I^{\infty,y} = +\infty \qquad \text{if } K(y) \leqslant 0.$$

The strict analogue of Theorem 4.1 is the following result (and we skip its proof):

Corollary 4.1. *We assume* (68) – (70). *If* (16) *holds, any minimizing sequence is relatively compact in* $H_0^1(\Omega)$ *and there exists a minimum of* (72) *and a solution of* (71). *If* (16) *does not hold i.e.* $I = I^\infty$, *there exist non-compact minimizing sequences and any such sequence* $(u_n)_n$ *satisfies (up to subsequences)*

$$\begin{cases} u_n \to 0 \quad \textit{weakly in} \quad H_0^1(\Omega); \quad \sum_{i,j} a_{ij}\dfrac{\partial u_n}{\partial x_i}\dfrac{\partial u_n}{\partial x_j} \to (\det a_{ij}(x^0))^{1/N} I_0 \delta_{x_0}, \\ |u_n|^{2N/(N-2)} \to K(x_0)^{-1}\delta_{x_0}, \quad \textit{for some} \quad x^0 \in \bar{\Omega} \textit{ satisfying} \\ (\det a_{ij}(x^0))^{1/N} K(x^0)^{-(N-2)/N} = \operatorname*{Min}_{x\in\Omega} (\det a_{ij}(x))^{1/N} K(x)^{-(N-2)/N} \\ \sigma_n \to +\infty, \quad \exists y_n \in \mathbb{R}^N, \quad y_n/\sigma_n \to x^0, \quad \sigma_n^{-(N-2)/N} u_n((\cdot + y_n)/\sigma_n) \to \bar{u} \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{minimum of } I^{\infty,x^0} \end{cases} \tag{73}$$

Of course we know explicitly the minima of I^{∞,x_0} (cf. Part 1 [65]).

Remark 4.6. In H Brézis and L. Nirenberg [23], the case when $a_{ij}(x)$, c, K are independent of x is treated: not only the fact that (16) implies the compactness of minimizing sequences is proved but also discussed in details (following their argument we discuss below (16)). But it is worth pointing out that the method used in [23] to pass to the limit on minimizing sequences cannot work as such in our general setting: indeed the main point is to avoid the weak convergence of $(u_n)_n$ to 0. In [23], one simply says that if $u_n \to 0$ then

$$\mathcal{E}(u_n) - \int_{\mathbb{R}^N} \sum_{i,j} a_{ij}\frac{\partial u_n}{\partial x_i}\frac{\partial u_n}{\partial x_j}\,dx \to 0$$

but this is not enough to use (16)! The loss may be seen on the fact that we give criteria below which show that (16) holds for $c(x) \equiv \lambda \geqslant 0$ under appropiate conditions on a_{ij}, K.

Let us also mention that the difference on the methods may be seen on the following («artificial») problem

$$I = \operatorname{Inf}\left\{\int_\Omega |\nabla u|^p\,dx - \lambda\int_\Omega |u|^p\,dx / u \in W_0^{1,p}(\Omega), \quad \int_\Omega |u|^q\,dx = 1\right\} \tag{74}$$

where $1 < p < N$, $q = Np(N-p)^{-1}$, $0 < \lambda < \lambda_1^p$ (λ_1^p is the largest constant $\mu > 0$ such that: $\mu\int_\Omega |u|^p\,dx \leqslant \int_\Omega |\nabla u|^p\,dx$). Then if $p \neq 2$, the arguments of H. Brézis and L. Nirenberg [23] do not apply anymore while we may still prove that if $I < I^\infty$, (74) is «well-posed».

Remark 4.7. In order to analyse (16), we may follow the choice of T. Aubin [6], H. Brézis and L. Nirenberg [23] (this choice was explained in section III) and we find, if $N \geqslant 5$ for example, that (16) holds provided there exists a minimum point x^0 of $(\det(a_{ij}))^{1/N} K^{-(N-2)/N}$ on $\bar{\Omega}$ which lies in Ω and such that

$$\int \sum_{i,j,k,l} a_{ij,kl}(x^0) y_i y_j y_k y_l (1 + |y|^2)^{-N} dy$$
$$< C_N^1 \lambda + C_N^2 \frac{(\det a_{ij}(x^0))^{1/N}}{K(x^0)} \cdot \int \sum_{i,j} K_{ij}(x^0) y_i y_j (1 + |y|^2)^{-N} dy$$

for some explicit positive constants C_N^1, C_N^2. Here we took $c \equiv -\lambda$. Observe that a_{ij}, K are independent of x, this condition holds automatically and it may hold even if $\lambda \leqslant 0$.

We now turn to a problem which is somewhat similar to the previous ones: we assume that Ω is smooth, (68), (70) and

$$\forall \varphi \in H^1(\Omega), \qquad \mathcal{E}(\varphi) \geqslant \alpha \|\varphi\|_{H'}^2 \tag{69'}$$

for some $\alpha > 0$. And we consider

$$I = \operatorname{Inf}\{\mathcal{E}(u)/u \in H^1(\Omega), \quad J(u) = 1\} \tag{74}$$

Again (16) is the key assumption and we have to compute I^∞ i.e. $I^{\infty,y}$

$$\begin{cases} \text{if} \quad y \in \bar{\Omega}, & K(y) \leqslant 0, \quad I^{\infty,y} = +\infty \\ \text{if} \quad y \in \Omega, & K(y) > 0, \quad I^{\infty,y} = (\det a_{ij}(y))^{1/N} K(y)^{-(N-2)/N} I_0 \\ \text{if} \quad y \in \partial\Omega, & K(y) > 0, \quad I^{\infty,y} = 2^{-2/N} (\det a_{ij}(y))^{1/N} K(y)^{-(N-2)/N} I_0 \end{cases}$$

(observe that K is l.s.c. on $\bar{\Omega}$), and $I^\infty = \operatorname{Min}_{y \in \Omega} I^{\infty,y}$. The value of $I^{\infty,y}$ if $y \in \partial\Omega$, $K(y) > 0$ comes from the fact that, if we concentrate u at y on $\partial\Omega$, extending u evenly across $\partial\Omega$, we obtain at the limit, since $\partial\Omega$ is smooth, the above value of $I^{\infty,y}$ that is

$$I^{\infty,y} = \frac{1}{2} \operatorname{Inf}\left\{\int_{\mathbb{R}^N} \sum_{i,j} a_{ij}(y) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx / u \in \mathcal{D}^{1,2}, \int_{\mathbb{R}^N} K(y) |u|^{2N/(N-2)} dx = 2\right\}.$$

And exactly as before we have:

Corollary 4.2. *We assume* (68), (69'), (70) *and that* Ω *is smooth. If* (16) *holds, any minimizing sequence is relatively compact in* $H^1(\Omega)$ *and there exists a minhere exists a minimum of* (74).

Remark 4.7. If $I = I^\infty$, we may analyze (as in (73) what happens for the non-compact minimizing sequences. Let us also mention that we could treat as well problems where i) the nonlinearity $K(x)|u|^{2N/(N-2)}$ is replaced by a general nonlinearity $F(x, t)$ such that

$$\lim_{|t|\to\infty} F^+(x,t)|t|^{-2N/(N-2)} = K_+(x), \text{ uniformly for } x \in \bar{\Omega};$$

ii) we consider nonlinear terms on the boundary for example

$$J(u) = \int_{\partial\Omega} G(x, u)\, dS$$

with $\lim_{|t|\to\infty} G^+(x,t)|t|^{-2(N-1)((N-2)} = K'^+(x)$, uniformly for $x \in \partial\Omega$.

Example 4.1. Let $a_{ij}(x) \equiv \delta_{ij}$, $K(x) \equiv 1$, $c(x) \equiv \lambda$; (69′) is equivalent to $\lambda > 0$. Choosing $\bar{u} \equiv \text{meas}(\Omega)^{-(N-2)/2N}$, we see that (16) holds if $\lambda \in]0, \lambda_0[$ with $\lambda_0 = I^\infty \text{meas}(\Omega)^{-2/N} = 2^{-2/N} I_0 \text{meas}(\Omega)^{-2/N}$. In addition for λ small the minimum is unique and is $\bar{u}$ (easy consequence of the implicit function theorem). We do not know much more information except that if $\lambda_2 - - \frac{4}{N-2}\lambda_0 < 0$ —where λ_2 is the second eigenvalue of $-\Delta$ on $H^1(\Omega)$— then for λ in $]0, \lambda_0 + \delta[$ where $\delta > 0$, (16) holds and any minimum is constant.

Indeed, if for $\lambda = \lambda_0$, $\bar{u}$ were a minimum, writing the second-order condition for the minimality of $\bar{u}$, we would find

$$\forall \varphi \in H^1(\Omega), \qquad \int_\Omega |\nabla\varphi|^2 - \frac{4\lambda_0}{N-2}\varphi^2\, dx + \frac{4\lambda_0}{(N-2)}\text{meas}(\Omega)^{-1}\left(\int_\Omega \varphi\right)^2 \geqslant 0$$

and this is impossible if $\lambda_2 - \frac{4}{N-2}\lambda_0 < 0$. Let us mention that it is easy to find examples of sets Ω for which this inequality is true.

Remark 4.7. It is interesting to observe that, for $y \in \partial\Omega$, the quantity $I^{\infty, y}$ depends on the regularity of $\partial\Omega$ at y. For example if Ω is given by

$$\Omega = \Omega_1 \times, \dots, \times \Omega_m, \quad \Omega_i \ \text{ smooth region of } \mathbb{R}^{n_i}$$

then

$$\text{for} \quad y \in \Omega_1 \times \Omega_2 \times \dots \times \Omega_m, \; I_y^\infty = (\det a_{ij}(y))^{1/N} K^+(y)^{-(N-2)/N} I_0$$

while

$$\text{if} \quad y \in \partial\Omega_1 \times \dots \times \Omega_m, \quad I_y^\infty = m^{-2/N}(\det a_{ij}(y))^{1/N} K^+(y)^{-(N-2)/N} I_0$$

i.e.

$$I_y^\infty = \frac{1}{m}\text{Inf}\left\{\int_{\mathbb{R}^N} \sum_{i,j} a_{ij}(y)\frac{\partial u}{\partial x_i}\frac{u}{\partial x_j}dx \Big/ \int_{\mathbb{R}^N} K^+(y)|u|^{2N/(N-2)}\, dx = m\right\}$$

We conclude these considerations on Yamabe-type problems by considering the effect of symmetries on the existence of solutions to

$$-\Delta u = u^{(N+2)/(N-2)} \text{ in } \Omega, \quad u|_{\partial\Omega} = 0, \quad u > 0 \quad \text{in } \Omega. \tag{75}$$

It is a well-known result due to S. Pohozaev [70] that if Ω is star-shaped with respect to, say, 0 then (75) has no solution. On the other hand if Ω is an annulus ($\Omega = \{x \in \mathbb{R}^N,\ r < |x| < R\}$, for some $r, R > 0$), it is well-known that (75) has a radial solution (see for example Kazdan and Warner [51]). If we wish the understand the implications of these two observations, we need to introduce the following setting: assume that $0 \notin \bar{\Omega}$, that Ω is invariant by a group of orthogonal transformations of $\mathbb{R}^N$ and set

$$\bar{I} = \operatorname{Inf}\left\{\int_\Omega |\nabla u|^2\,dx \Big/ \int_\Omega |u|^{2N/(N-2)}\,dx = 1, \quad u \in H_0^1(\Omega), \quad u \text{ is } G\text{-invariant}\right\};$$

$$s = \operatorname*{Inf}_{y \in \bar{\Omega}} s(y), \quad s(y) = \#\{a = g \cdot y, \quad g \in G\}. \tag{76}$$

Our methods immediately yield:

Corollary 4.3. *Let* $\bar{I}^\infty = s^{2/N} I_0$ *if* $s < \infty$, $\bar{I}^\infty = +\infty$ *if* $s = +\infty$. *Then if* $\bar{I} < \bar{I}^\infty$, *all minimizing sequences of* (76) *are relatively compact in* $H_0^1(\Omega)$ *and there exists a minimum of* (76) *and a solution of* (75).

Remark 4.8. if $\Omega = \Omega_1 \times \Omega_2$ with Ω_1 arbitrary in $\mathbb{R}^{N_1}$ ($N_1 \geqslant 0$) and $\Omega_2 = \{x_2 \in \mathbb{R}^{N-N_1},\ r < |x_2| < R\}$ for some $r, R > 0$, then $s = +\infty$ and (75) has a solution (this can also be seen on the results of Appendix 2). If $0 \notin \bar{\Omega}$ and $s \geqslant 2$ ($s \geqslant 2$ if Ω is symmetric with respect to 0!), and if $T_r^R = \{x \in \mathbb{R}^N / r < |x| < R\} \subset \Omega$ for some $0 < r < R$, then clearly

$$\bar{I} \leqslant I(r, R) = \operatorname{Min}\left\{\int_{T_r^R} |\nabla u|^2\,dx \Big/ \int_{T_r^R} |u|^{2N/(N-2)}\,dx = 1,\right.$$
$$\left. u \in H_0^1(T_r^R), \quad u \text{ is spherically symmetric}\right\}.$$

By the dilation invariance $I(r, R) = I\left(\frac{r}{R}, 1\right) = \lambda\left(\frac{r}{R}\right)$ and λ is a continuous, increasing function on $]0, 1[$ such that

$$\lim_{t \to 0_+} \lambda(t) = I_0, \qquad \lim_{t \to 1_-} \lambda(t) = +\infty.$$

Hence there exists a unique d_s such that

$$\lambda(d_s) = s^{2/N} I_0, \qquad d_s \to 1 \quad \text{if} \quad \to +\infty$$

and *thus if* $\frac{r}{R} < d_s$, *the condition* $\bar{I} < \bar{I}^\infty$ *is satisfied.*

Recently, by a critical point argument (instead of a minimization argument) J. M. Coron [30] —using arguments of section 4.6— was able to prove the existence of a solution of (75) without using symmetries if we assume $T_r^R \subset \Omega$ and $\frac{r}{R} < d_2$.

We conclude this section by a last example of applications of our arguments namely the problem introduced in P. Cherrier [25]: let $(\bar{M}, g)$ be some N dimensionnal compact Riemannian manifold with boundary, let N be its boundary endowed with the Riemannian structure induced by $\bar{M}$ and let $M = \bar{M} \backslash N$, we assume that $\bar{M}$ is orientable and we denote by $\frac{d}{dn}$ the derivation with respect to the vector field of outward unitary vectors (for the metric g) normal to N (in $\bar{M}$). If we look for a new metric $\tilde{g}$ pointwise conformal to g: $\tilde{g} = u^{4/(N-2)} g$ for some $u > 0$ on $\bar{M}$ such that the new scalar curvature and the new mean curvature are prescribed functions K, K', we are led to the following equation —see P. Cherrier [25]

$$\begin{cases} -\Delta u + ku = K u^{(N+2)/(N-2)} & \text{in } M \\ \dfrac{\partial u}{\partial n} + k' u = K' u^{N/(N-2)} & \text{on } M \end{cases} \tag{77}$$

where $k, k', K, K' \in C(\bar{M})$ and where we assume (to simplify)

$$K, K' \geqslant 0 \quad \text{on} \quad \bar{M}; \quad \max_{\bar{M}} (K + K') > 0; \quad \lambda_1 > 0 \tag{78}$$

where λ_1 is the first eigenvalue of the operator $(-\Delta + k)$ on $H^1(M)$ with the boundary condition $\left(\frac{\partial u}{\partial n} + k'u = 0\right)$ i.e.

$$\lambda_1 = \operatorname{Min}\left\{ \int_M |\nabla u|^2 + ku^2 + \int_N k' u^2 / u \in H^1 M), \quad \int_M u^2 = 1 \right\}.$$

In [25], conditions are given for a solvability of (77) with K, K' replaced by $\theta K, \theta K'$ qhere θ is some Lagrange multiplier. Using our method we may extend the results of [25] (for $N \geqslant 3$) but we prefer to solve directly the exact problem. To this end, we consider the artificial constraint method i.e.

$$I = \operatorname{Inf}\{ \mathcal{E}(u) / u \in H^1(M), \quad J(u) = 0, \quad u \not\equiv 0 \} \tag{79}$$

where

$$\mathcal{E}(u) = \int_M |\nabla u|^2 + ku^2 - \frac{N-2}{N} K |u|^{2N/(N-2)} + \\ + \int_N k' u^2 - \frac{N-1}{N} K' |u|^{2(N-1)/(N-2)},$$

$$J(u) = \langle \mathcal{E}'(u), u \rangle = \int_M |\nabla u|^2 + ku^2 - K|u|^{2N/(N-2)} + \\ + \int_N k' u^2 - K' |u|^{2(N-1)/(N-2)}.$$

We next compute $I^{\infty,y}$:

i) if $y \in \bar{M}$, $K(y) = 0$ or if $y \in N$, $K(y) = K'(y) = 0$ then $I^{\infty,y} = +\infty$

ii) if $y \in M$, $K(y) > 0$, $I^{\infty,y}$ is given by

$$I^{\infty,y} = \operatorname{Min}\left\{ \int_{\mathbb{R}^N} |\nabla u|^2 - \frac{N-2}{N} K(y) |u|^{2N/(N-2)}\, dx / u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \right.$$
$$\left. u \not\equiv 0, \quad \int_{\mathbb{R}^N} |\nabla u|^2\, dx = \int_{\mathbb{R}^N} K(y)|u|^{2N/(N-2)}\, dx \right\}$$
$$= \frac{2}{N} I_0^{N/2} K(y)^{-(N-2)/2}$$

iii) if $y \in N$, $K(y) + K'(y) > 0$, $I^{\infty,y}$ is given by

$$I^{\infty,y} = \operatorname{Min}\left\{ \int_{(x_N > 0)} |\nabla u|^2 - \frac{N-2}{N} K(y) |u|^{2N/(N-2)}\, dx + \right.$$
$$+ \frac{N-2}{N-1} \int_{(x_N = 0)} K'(y)|u|^{2(N-1)/(N-2)}\, dx' /$$
$$/ u \in \mathcal{D}^{1,2}(x_N > 0),\ u \not\equiv 0, \quad \int_{(x_N > 0)} |\nabla u|^2 - K(y)|u|^{2N/(N-2)}\, \mathrm{dx} =$$
$$\left. = \int_{(x_N = 0)} K'(y)|u|^{2(N-1)/(N-2)}\, dx' \right\}.$$

And we let $I^\infty = \operatorname{Min}_{y \in \bar{M}} I^{\infty,y}$. We obtain as before the

Theorem 4.3. *We assume* (78). *If $I < I^\infty$, any minimizing sequence is relatively compact in $H^1(M)$ and there exist a minimum of* (79) *and a solution of* (77). *If $I = I^\infty$, there exist non compact minimizing sequences and any such sequence $(u_n)_n$ satisfies* (*up to subsequences*)

$$\begin{cases} u_n \to 0 \quad \text{weakly in} \quad H^1(M); \quad |\nabla u_n|^2 \to \alpha\delta_{x^0}, \quad |u_n|^{2N/(N-2)} \to \beta\delta_{x^0} \\ \mu_n \to \gamma\delta_{x^0} \quad \text{for some} \quad x^0 \in \bar{M} \quad \text{which minimizes} \quad I^{\infty,x} \quad \text{on} \quad \bar{M}; \\ \text{if} \quad x^0 \in M, \quad \alpha = \frac{N}{2} I^\infty, \quad \beta = \frac{N}{2} K(x^0)^{-1} I^\infty, \quad \gamma = 0, \\ \text{if } x^0 \in N, \quad \alpha - \frac{N-2}{N} K(x^0)\beta - \frac{N-2}{N-1} K'(x^0)\gamma = I^\infty, \\ \qquad \alpha - K(x^0)\beta - K'(x^0)\gamma = 0 \end{cases}$$

where μ_n is the measure on $\bar{M}$, supported in N such that

$$\forall \varphi \in C(\bar{M}), \qquad \int \varphi \, d\mu_n = \int_N |u_n|^{2(N-1)/(N-2)} \varphi .$$

4.3. *An inequality for holomorphic functions*

In this section, we want to discuss some inequalities for holomorphic functions. Let Ω be a smooth domain of C: if $\Gamma = \partial\Omega$, we consider the space $E^p(\Omega)$ (for $p \geqslant 1$) of holomorphic functions in Ω with traces on Γ in L^p i.e. (for example) holomorphic functions f such that

$$\varlimsup_{\epsilon \to 0} \int_{\Gamma_\epsilon} |f(z)|^p \, |dz| < \infty$$

where $\Gamma_\epsilon = \{ G(z, z_0) = \epsilon \}$, if $z_0 \in \Omega$, $G(z, z_0)$ is the Green's function. The notation

$$\int_\Gamma |f(z)|^p |dx| = \varlimsup_{\epsilon \to 0} \int_{\Gamma_\epsilon} |f(z)|^p |dz|$$

will be used everywhere below. Then if $f \in E^p(\Omega)$, the following inequality holds

$$\left(\int_\Omega |f(z)|^{2p} \, dx \, dy \right)^{1/2p} \leqslant C_p \left(\int_\Gamma |f(z)|^p |dz| \right)^{1/p} \tag{80}$$

We will see below that if $p > 1$, this inequality is very easy to prove. If $p = 1$, it was proved by Carleman [24], Aronszajn [4] for simply connected domains Ω and by S. Jacobs [43] for arbitrary domains.

If $p > 1$, one just needs to observe that $|f(z)| = u(x, y)$ is subharmonic and thus by the maximum principle: $|f(z)| \leqslant w(x, y)$ where

$$-\Delta w = 0 \quad \text{in} \quad \Omega, \qquad w = |f| \quad \text{on} \quad \Gamma$$

this boundary value problem may be solved by duality one finds

$$\| w \|_{W^{1/p, p}(\Omega)} \leqslant C \| f \|_{L^p(\Gamma)}$$

and we obtain (80) using Sobolev inequalities. It is worth pointing out that such an argument is false for $p = 1$.

In S. Jacobs [4] the question of the best constant C_1 was solved for $p = 1$ for arbitrary domains. If Ω is simple connected $C_1 = (4\pi)^{-1/2}$ and this best constant is achieved for the Bergman kernel function. For a multiply connected domain, the problem to solve is

$$I = \operatorname{Inf} \left\{ \int_\Gamma |f|^p |dz| \Big/ f \in E^p(\Omega), \quad \int_\Omega |f|^{2p} \, dx \, dy = 1 \right\} \tag{81}$$

(if $p = 1$, Ω is simply connected, the result recalled above just says: $I = (4\pi)^{1/2}$).

The underlying dilatations invariance is

$$f \to \sigma^{-1/p} f\left(\frac{\cdot}{\sigma}\right)$$

Notice also that I is not changed if we replace Ω by Ω' provided Ω and Ω' are conformally equivalent, and that the functionals are preserved by conformal self maps of Ω.

The main result we want to discuss is the:

Theorem 4.4. (S. Jacobs [41]). *Let $p = 1$, Ω be multiply connected then $I < (4\pi)^{1/2}$ and any minimizing sequence of* (81) *is relatively compact in $L^2(\Omega)$. In particular there exists a minimum of* (81).

Our goal here is to interprete this result as an example of application of our method (thus providing a simpler existence proof). To this end we have to understand why $I^\infty = (4\pi)^{1/2}$: first of all it is clear that $I^{\infty,y} = +\infty$ if $y \in \Omega$ (local compactness of holomorphic functions...), next if $y \in \partial\Omega$ the concentration around y shows that we are led to the problem in small neighborhoods of 0 in an halfspace or by the conformal equivalence to the *some problem* but in the *unit disc* and thus $I^\infty = (4\pi)^{1/2}$ if $p = 1$; if $p > 1$

$$I^\infty = \operatorname{Inf}\left\{ \int_T |f|^p |dz| \Big/ \int_D |f|^{2p} |dz| = 1 \right\}.$$

where D is the unit disc and $T = \partial\Delta$.

Then the compactness of minimizing sequences in the above result in immediately deduced from the analogue of Lemma 1.1 (Lemma 2.1,...) that we give below. Notice that our method also yields that if $I < I^\infty$ (for any $p > 1$), then minimizing sequences are compact and the infimum is achieved.

Lemma 4.2. *Let $(f_n)_n$ be bounded in $E^p(\Omega)$, assume that f_n converges weakly in $L^{2p}(\Omega)$ to some f, where $p \in [1, \infty[$. We may assume that $|f_n|^{2p}$ converges weakly to a measure ν on $\bar{\Omega}$ and that the measure μ_n given by*

$$\forall \varphi \in C(\bar{\Omega}), \qquad \int \varphi \, d\mu_n = \int_\Gamma \varphi |f_n|^p |dz|$$

converges to some measure μ. Then we have

i) $\nu = |f|^{2p} + \sum_{j \in J} \nu_j \delta_{z_j}$

ii) $\mu \geqslant \mu_1 + \sum_{j \in J} \nu_j^{1/2} I^\infty \delta_{z_j}$

for some at most countable family J, constants ν_j in $]0, \infty[$, point z_j on Γ.

The proof of this lemma is again a repetition of arguments given several times before: if $f \equiv 0$, then clearly ν is supported on Γ and we find that for all φ holomorphic in Ω, continuous on $\bar{\Omega}$

$$\left(\int_\Gamma |\varphi|^{2p}\,d\nu\right)^{1/2p} \leqslant C\int_\Gamma |\varphi|^p\,d\mu$$

By a density result, this inequality actually holds for all $\varphi \in C(\Gamma)$ and we deduce that ν is an at most countable sum of Dirac masses (and $\sum_j \nu_j^{1/2} < \infty$) —cf. Lemma 1.2 of Part 1 [63]—.

Indeed we claim that $\{|\varphi|_{|\Gamma},\ \varphi$ holomorphic in Ω, continuous on $\bar{\Omega}\}$ is dense in $C_+(\Gamma)$: we need to prove this claim only when Ω is $(|z| < 1)$.

A short proof of this claim (which was indicated to us by J. M. Lasry) is as follows: any $\varphi \in C_+(\Gamma)$ may be approximated by a real nonnegative trigonometric polynomial

$$P = \sum_{n=-N}^{+N} a_n e^{in\theta},$$

then we consider the following holomorphic function (on $\mathbb{C}$) $\tilde{\varphi}$

$$\tilde{\varphi}(z) = \sum_{n=0}^{2N} a_{n-N} z^n$$

son that $\tilde{\varphi}(e^{i\theta}) = e^{iN\theta}P$ and $|\tilde{\varphi}|_{|\Gamma} = \mathrm{P}$.

The general representation of ν (part i) is then deduced as in the previous cases from the a.e. convergence of f_n to f (cf. Lemma 1.1).

Finally part ii) is obtained as before (usign the same density result as above) observing that if z_j is fixed, we may always find a simple connected domain $\omega \subset \Omega$ such that points of $\partial\Omega$ near z_j belong to $\partial\omega$ and essentially work in ω instead of Ω.

We conclude observing that i) and ii) imply

$$\varliminf_n \left\{\int_\Gamma |f_n|^p |dz| - I^\infty\left(\int_\Omega |f_n - f|^{2p}\right)^{1/2}\right\} \geqslant \int_\Gamma |f|^p |dz|$$

and if $p = 1$, this was the crucial lemma in [41] for the proof of the existence of a minimum.

4.4 *A remark on some isoperimetric inequalities*

We want to discuss here some properties of the following isopermetric inequality

$$|Q(v)|^{2/3} \leqslant C_0 \int_{\mathbb{R}^2} |\nabla v|^2\,dx\,dy, \qquad \forall v \in \mathcal{D}(\mathbb{R}^2; \mathbb{R}^3) \tag{82}$$

where $C_0 = (32\pi)^{-1/3}$, and Q is the functional defined by

$$Q(v) = \int_{\mathbb{R}^2} v \cdot (v_x \wedge v_y)\, dx\, dy. \tag{83}$$

The proof of (82) —which is an isoperimetric inequality for the graph of v— may be found in Wente [83]. By density (82) still holds for $v \in H^1(\mathbb{R}^2; \mathbb{R}^3) \cap L^\infty(\mathbb{R}^2; \mathbb{R}^3)$ and C_0 is achieved for

$$\bar{v}(x,y) = (1 + x^2 + y^2)^{-1}(x, y, 1)$$

(see Wente [83], H. Brézis and J. M. Coron [18] for more details).

Finally let us mention that Q may be defined actually on functions in $H^1(\mathbb{R}^2; \mathbb{R}^3)$ with compact support and that $v_x \wedge v_y$ is not only meaningful in L^1 but also in H^{-1} for such functions V.

We next want to observe that (82) is invariant by translations and dilations and that Q, $\mathcal{E}$ are invariant by

$$v \to v\left(\frac{\cdot}{\sigma}\right), \qquad \forall \sigma > 0$$

where

$$\mathcal{E}(v) = \int_{\mathbb{R}^2} |\nabla v|^2\, dx\, dy.$$

Therefore to check if we may apply our method to such functionals we need to see if the analogues of Lemma 1.1 still hold true here: let us thus consider a sequence $(v^n)_n$ bounded in $H_0^1(\Omega; \mathbb{R}^3)$ where Ω is bounded in $\mathbb{R}^2$ (so that v^n extended by 0 is in $H^1(\mathbb{R}^2; \mathbb{R}^3)$). We next want to define a distribution T^n given by: $T^n = v^n \cdot (v_x^n \wedge v_y^n)$, T^n will be supported in $\bar{\Omega}$ and is defined by

$$\langle T^n, \varphi \rangle = \langle \varphi v^n, v_x^n \wedge v_y^n \rangle, \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}^2)$$

and thus T^n is (for example) bounded in $W^{-1,p'}(\mathbb{R}^2)$, $\forall p > 2$. In addition: $Q(v^n) = \langle T^n, \chi \rangle$, for any $\chi \in \mathcal{D}(\mathbb{R}^2)$, $\chi \equiv 1$ on $\bar{\Omega}$.

We may now state the

Lemma 4.3. *With the above observations, we may assume that* $|\nabla u_n|^2$, T_n *converge in* $\mathcal{D}'(\mathbb{R}^2)$ *to* μ, T *and that* v^n *converges weakly to* v. *Then we have*

i) $$T = T_0 + \sum_{j \in J} \nu_j \delta_{x_j}$$

where T_0 *is defined through* v *as* T_n *is defined through* v^n, J *is some at most countable set,* $(\nu_j)_{j \in J} \in \mathbb{R} - \{0\}$, $(x_j)_{j \in J}$ *are points of* $\bar{\Omega}$.

ii) $$\mu \geqslant |\nabla v|^2 + (1/C_0) \sum_{j \in J} |\sigma_j|^{2/3}\, \delta_{x_j}.$$

iii) *If* $u \in H_0^1(\Omega; \mathbb{R}^3)$ *and if we denote by* $\tilde{T}_n$ *the distributions associated with* $v^n + u$ *as* T_n *is associated to* v^n *the (up to subsequences)*

$$\begin{cases} |\nabla(v^n + u)|^2 \rightharpoonup \tilde{\mu}, & \tilde{\mu} - \mu \in L^1(\mathbb{R}^2) \\ \tilde{T}_n \rightharpoonup \tilde{T}_0 + \sum_{j\in J} \nu_j \delta_{x_j} \end{cases}$$

where $\tilde{T}_0$ *corresponds to* $u + v$.

iv) *In particular*

$$\underline{\lim}_n\, \mathcal{E}(v_n) - \frac{1}{C_0} |Q(v_n - v)|^{2/3} \geqslant \mathcal{E}(v).$$

The proof of lemma 4.3 is totally similar to the proofs of the corresponding results we proved before: considering first the case when $v \equiv 0$, we obtain for all $\varphi \in \mathcal{D}(\mathbb{R}^2)$

$$|\langle T^n, \varphi^3 \rangle| = |Q(\varphi v^n)| \leqslant C_0^{3/2} \Big(\int_{\mathbb{R}^2} |\nabla(\varphi v^n)|^2 \Big)^{3/2}$$

and thus passing to the limit

$$|\langle T, \varphi^3 \rangle|^{2/3} \leqslant C_0 \Big(\int \varphi^2 \, d\mu \Big), \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}^2).$$

But this implies easily that T is a signed measure on $\mathbb{R}^2$; and the reaminder of the proof is then totally similar to the proof of Lemma 1.1 (and of the other related results...).

This observation enables us to apply the general concentration-compactness arguments and we may now give in interpretation of the results of H. Brézis and J. M. Coron [18] on *the existence of a second solution to H-systems*: we refer the reader to [18] for the motivations of the introduction of the following *minimization problem* which, if solved, yields the *existence of a second solution* to the Dirichlet problem for H systems (a similar analysis works also for the Plateau problem). We thus consider

$$I = \operatorname{Inf}\{ \mathcal{E}(u) \,/\, v \in H_0^1(\Omega; \mathbb{R}^3), \quad Q(v) = 1 \}. \tag{84}$$

where Ω is bounded in $\mathbb{R}^2$, $\mathcal{E}$ is given by

$$\mathcal{E}(v) = \int_\Omega |\nabla v|^2 + 4H \int_\Omega u \cdot (v_x \wedge v_y)$$

and u is a given function in $H^1(\Omega; \mathbb{R}^3)$ —for example—, H is a given positive constant—, in [18], u is in fact the «first» solution of the H system, solution obtained by Hildebrandt [41], [42]. In order to have a non trival minimization problem, we assume

$$\exists \alpha > 0, \quad \forall v \in H_0^1(\Omega; \mathbb{R}^3), \qquad \mathcal{E}(v) \geqslant \alpha |\nabla v|_{L^2}^2. \tag{85}$$

Clearly enough we have for all $y \in \bar{\Omega}$

$$I^{\infty} = I^{\infty, y} = \operatorname{Inf}\{ \mathcal{E}(u) \,/\, u \in \mathcal{D}(\mathbb{R}^2; \mathbb{R}^3), \quad Q(u) = 1 \} = \frac{1}{C_0}; \tag{86}$$

and by homogeneity (81) reduces to

$$I < I^{\infty} \tag{16}$$

Therefore we deduce from our general arguments (and Lemma 4.3) that *if* (16) *holds, any minimizing sequence of* (84) *is relatively compact in* $H_0^1(\Omega; \mathbb{R}^3)$. To conclude our interpretation of the results of [17], we recall that in [17] it is proved that if $u \in C^2_{\mathrm{loc}} \cap L^{\infty}$ then (16) holds if and only if u is not constant on Ω. We emphasize the fact that we did not prove here any new result but we only show one *needs* to compare I with $\frac{1}{C_0} = (32\pi)^{1/2} (= I^{\infty})$ and this again is a consequence or our general method.

4.5 *Harmonic maps*

As in the preceding two sections, we will not prove any new result but we will just explain in the light of our systematic treatment the solution of some minimization problem associated with the question of harmonic maps. We will thus follow the presentation of H. Brézis and J. M. Coron [18] (see also J. Jost [43] for related results). By no means, the remakrs which follow pretend to cover the subjetct of harmonic maps and we refer the interested reader to the deep work of J. Sacks and K. Uhlenbeck [72], R. Schasen and S. T. Yau [73], Y. T. Siu and, S. T. Yau [75]. To simplify we will consider only harmonic maps from the unit ball Ω of $\mathbb{R}^2$ into S^2 with a precribed boundary condition

$$u = \gamma \quad \text{on} \quad \partial\Omega$$

(where γ is, of course, the restriction to $\partial\Omega$ —the trace— of some function v *in* $H^1(\Omega{:}\, S^2)$ i.e. $v \in H^1(\Omega; \mathbb{R}^3)$, $v \in S^2$ a.e. in Ω).

Harmonic maps from Ω into S^2 with the above boundary condition are critical points of the functional

$$\mathcal{E}(u) = \int_{\Omega} |\nabla u|^2$$

«restricted to the set» $A = \{ u \in H^1(\Omega; S^2);\ u = \gamma \text{ on } \partial\Omega \}$. Clearly $\mathcal{E}$ achieves its minimum on A: let u_0 be such a minimum.

Following [18], we consider for $u \in H^1(\Omega; \mathbb{R}^3) \cap L^{\infty}(\Omega; \mathbb{R}^3)$ the functional

$$Q(u) = \frac{1}{4\pi} \int_{\Omega} u \cdot (u_x \wedge u_y)$$

and we recall (see [18] for more details) that if $u_1, u_2 \in A$ then

$$Q(u_1) - Q(u_2) \in \mathbb{Z}$$

(identifying Ω with the northern hemisphere of S^2, and «reflecting» u_2 we may consider (u_1, u_2) as a map from S^2 to S^2 and $Q(u_1) - Q(u_2)$ is the degree of this map).

We set

$$J(u) = Q(u) - Q(u_0), \qquad \forall u \in A;$$

so that J is integer-value on A (and $J(A) = \mathbb{Z}$).

Then let $k \neq 0$, if we find a minimum of

$$I_k = \operatorname{Inf}\{\mathcal{E}(u) \,/\, u \in A, \quad \mathrm{J}(u) = k\} \tag{87}$$

then such a minimum will be a local minimum and thus a critical point of $\mathcal{E}$ on A.

We may now apply the concentration-compactness argument: we then need to define the *problem at infinity* (the underlying *scaling invariance* is: $u \to u\left(\frac{\cdot}{\sigma}\right)$ for $\sigma > 0$)

$$I_\mu^\infty = \operatorname{Inf}\left\{\int_{\mathbb{R}^2} |\nabla\varphi|^2 \,/\, \varphi \in C^\infty(\mathbb{R}^2; S^2), \quad \varphi \text{ constant near infinity}, \right.$$
$$\left. \int_{\mathbb{R}^2} \varphi \cdot (\varphi_x \wedge \varphi_y) = 4\pi\mu \right\}.$$

Using the above remark on the degree and the value of C_0 in the preceding constant, we find

$$\begin{cases} I_\mu^\infty = +\infty \quad \text{if} \quad \mu \notin \mathbb{Z}, \qquad I_\mu^\infty = |\mu| I_1^\infty \quad \text{if} \quad \mu \in \mathbb{Z} \\ I_1^\infty = 8\pi. \end{cases}$$

Then *in this setting,* (S.1) *reduces to*

$$I_k < I_l + I_{k-l}^\infty, \qquad \forall l \in \mathbb{Z} - \{k\}; \tag{88}$$

with, in fact,

$$I_{k-l}^\infty = 8\pi|k - l|; \qquad I_0 = \operatorname*{Inf}_A \mathcal{E} \leqslant I_k, \qquad \forall k \in \mathbb{Z} - \{0\}.$$

And it is now a straightforward application of our arguments to show that (88) *is a necessary and sufficient condition for the compactness of all minimizing sequences of* (87). But in addition the very especial form of I_μ^∞ enables us to make the following remarks: if $l \geqslant 2k$ and $k > 0$ (or $l \leqslant 2k$, $k < 0$)

$$I_l + I_{k-l}^\infty = I_l + 8\pi(l - k) \geqslant I_0 + 8\pi k = I_0 + I_k^\infty$$

therefore (88) *is equivalent to*

$$I_k < I_l + I_{k-l}^\infty = I_l + 2\pi|k-l|, \qquad \forall l \text{ between } 0 \text{ and } 2k, \quad l \neq k, \tag{89}$$

In particular *if* $k = \pm 1$, (89) *reduces to*

$$I_k < I_0 + 8\pi \tag{90}$$

And *we recover the crucial inequality* of [18] (inequality (3), Lemma 2) as a very particular case of (S.1); in [18], it is proved that if γ is not constant, (90) holds either for $k = 1$, or for $k = -1$. In both cases this yields the existence of a local minimum different from u_0. Of course (90) is the *major difficulty* in the proof of the existence of such a second critical point (let us just mention that the method folowed in [18] to check (90) follows the empirical rule given in section III) but our goal here is to show that (90) is natural and had to be expected!

4.6 *Morse theory*

We want to explain on the example of Yamabe type equations what informations the results such as Lemma 1.1 (and the related weak convergence results) imply on the possibility of using Morse theory on functionals associated with the preceding problems.

To simplify the presentation, we will only present our results in the case of Yamabe equations even if they apply to *all* the situations considered before (convolution, trace, H-systems, holomorphic functions, harmonic maps...). We will thus consider a sequence $(u_n)_n$ in $H_0^1(\Omega)$ —where Ω is a bounded open set in $\mathbb{R}^n$— satisfying

$$\begin{cases} -\Delta u_n = |u_n|^{4/(N-2)} u_n + f_n \quad \text{in} \quad \Omega, \qquad f \to f \quad \text{in} \quad H^{-1}(\Omega) \\ S_n(u_n) \to c \end{cases} \tag{91}$$

where $c \in \mathbb{R}$ is fixed,

$$S_n(v) = \int_\Omega \frac{1}{2}|\nabla v|^2 - \frac{N-2}{2N}|v|^{2N/(N-2)}\,dx - \langle f_n, v\rangle.$$

We denote by $S^\infty(v)$, $S(v)$ the functionals corresponding to $f_n = 0$, $f_n = f$. The reasons for considering (91) come from the (P.S.) condition which is the crucial condition for the application of critical point hteory. The following result is an obvious application of Lemma 1.1:

Corollary 4.4. *Assume that* $(u_n)_n$ *satisfies* (91), *then* u_n *is bounded in* $H_0^1(\Omega)$ *and assuming that* u_n, $|\nabla u_n|^2$ *converge weakly to* $u \in H_0^1(\Omega)$, μ *bounded nonnegative measure on* $\bar{\Omega}$ *we have:*

i) u *solves*:

$$-\Delta u = |u|^{4/(N-2)}u + f \quad \text{in} \quad \Omega, \qquad u \in H_0^1(\Omega) \tag{92}$$

ii)
$$\mu = |\nabla u|^2 + \sum_{i=1}^{m} \mu_i \delta_{x_i}$$

where $m \geqslant 0$; $x_1, \ldots, x_m$ *are* m *distinct points of* $\bar{\Omega}$ *and* $(\mu_i)_i$ *satisfies*

$$\mu_i \geqslant I_0^{N/2} \tag{93}$$

where

$$I_0 = \operatorname{Min}\left\{ \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \,/\, u \in \mathfrak{D}^{1,2}(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} |u|^{2N/(N-2)} \, dx = 1 \right\}.$$

iii) $|u_n|^{2N/(N-2)}$ *converges weakly to*: $|u|^{2N/(N-2)} + \sum_{i=1}^{m} \mu_i \delta_{x_i}$.

iv) $c = S(u) + \dfrac{1}{N} \sum_{i=1}^{m} \mu_i$.

Remark 4.9. The fact that compactness is lost at most at a finite number of points was first observed by J. Sacks and K Uhlenbeck [72] in the study of harmonic maps; see also Y. T. Siu and S. T. Yau [75], K. Uhlenbeck [85, 86], C. Tanbes [78].

Remark 4.10. Take $f = 0$, then (92) implies that $S(u) \geqslant 0$ and thus $c \geqslant \frac{m}{N} I_0^{N/2}$. Hence critical point theory (or Morse theory) may be applied on level sets below c: this was used in H. Brézis and L. Nirenberg [23]; see also C. Taubes [78] for related considerations.

Notice also that if $c < \frac{2I_0^{N/2}}{N}$, only one point (one Dirac mass) may occur; similarly if u_n is nonnegative and $c \in \left]\frac{1}{N} I_0^{N/2}, \frac{2}{N} I_0^{N/2}\right[$ no Dirac mass may appear and u_n converges in H^1 to u. This observation is used in J. M. Coron [30].

Remark 4.11. If one had a complete description of solutions in $\mathfrak{D}^{1,2}(\mathbb{R}^N)$ of

$$-\Delta u = |u|^{4/(N-2)} u \quad \text{in} \quad \mathbb{R}^N;$$

one would be able to obtain (in a straightforward way) a much more precise behavior of u_n nearby each point x_i. This programa was recently completed (in great details) by H. Brézis and J. M. Coron [20] in the case of H-systems; and that should be a general phenomenon.

PROOF OF COROLLARY 4.4. If $u_n - u = v_n$ and $|\nabla v_n|^2$, $|v_n|^{2N/(N-2)}$ converge weakly to μ_0, ν_0, Corollary 4.4 will be proved if we show that $\mu_0 = \nu_0$. Indeed by Lemma 4.1, we know

$$\mu_0 \geqslant \sum_{j\in J} I_0 \nu_j^{(N-2)/N} \delta_{x_j}, \qquad \nu_0 = \sum_{j\in J} \nu_j \delta_{x_j}, \qquad \mu - \mu_0 \in L^1(\Omega)$$

nad $\mu_0 = \nu_0$ would imply that J is finite and parts ii) – iv) of Corollary 4.4 (part i) is an exercize on weak limits).

To prove that $\mu_0 = \nu_0$, we observe that for all $\varphi \in C^1(\bar{\Omega})$

$$\int_\Omega \varphi|\nabla v_n|^2 + v_n(\nabla\varphi, \nabla v_n)\, dx =$$
$$= \int_\Omega \{ |u + v_n|^{4/(N-2)}(u + v_n) - |u|^{4/(N-2)}u \} \cdot v_n \varphi\, dx + \langle f_n - f, v_n\varphi \rangle$$

and passing to the limit as n goes to ∞, we obtain

$$\int_\Omega \varphi\, d\mu_0 = \int_\Omega \varphi\, d\nu_0, \qquad \forall \varphi \in C^1(\bar{\Omega}).$$

Appendix 1. Existence of two solutions of the Yamabe problem in $\mathbb{R}^N$

We want here to present a few results concerning the existence of solutions of

$$-\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + ku = K u^{(N+2)/(N-2)} \quad \text{in } \mathbb{R}^N \tag{A.1}$$

$$u \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \cap C_b(\mathbb{R}^N); \qquad \exists c_0 > 0, \quad u \geqslant c_0 \quad \text{on } \mathbb{R}^N \tag{A.2}$$

where $k, K \in C_b(\mathbb{R}^N)$ (for example), $a_{ij} = a_{ji} \in C_b(\mathbb{R}^N)$ and

$$\forall R < \infty, \quad \exists \nu > 0, \quad \forall x \in \bar{B}_R, \qquad (a_{ij}) \geqslant \nu I_N.$$

We will first present some results due to W. M. Ni [68] (see also [69], Kenig and Ni [52]): the main assumption for the application of Ni's method is the following

$$\exists \varphi_1 \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \cap C_b(\mathbb{R}^N), \quad A\varphi_1 = 0 \quad \text{in } \mathbb{R}^N, \qquad \varphi_1 \geqslant c_1 > 0 \quad \text{on } \mathbb{R}^N; \tag{A.3}$$

where A is the linear operator given by the left side of (A.1).

Of course (A.3) holds if $a_{ij}(x) \equiv \delta_{ij}$, $k \equiv 0$ (this corresponds to the usual metric on $\mathbb{R}^N$) or under convenient decay assumptions at infinity (cf. Kenig and Ni [52]). We also gave in section 1.5 conditions which ensures that (A.3) holds (and they may be easily extended...). Notice that (A.3) implies that the first eigenvalue of A in $H^1_0(\Omega)$ is positive for any bounded open set Ω.

The result which follows is an adaptation of the method of Ni:

Theorem A.1. (cf. [68], [69]). *We assume* (A.3). *If we assume*

$$\exists \bar{u} \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \qquad A\bar{u} \geqslant |K| \quad \text{in } \mathbb{R}^N \tag{A.4}$$

then there exists a sequence of solutions $(u_n)_n$ *of* (A.1) – (A.2) *satisfying:* $\sup_{\mathbb{R}^N} u_n \to 0$. *In addition if K has a constant sign,* (A.4) *is necessary for the the existence of a solution of* (A.1) – (A.2).

PROOF. First, if $K \geqslant 0$, (A.4) is clearly necessary and if (A.4) holds we may assume that: $\inf_{\mathbb{R}^N} \bar{u} > 0$ (if it is not the case, we consider $\bar{u} + \mu\varphi_1$ for μ large). Then for $\delta > 0$ small, $\delta\bar{u}$ is a supersolution of (A.1) while $\lambda\varphi_1$ is a subsolution for all $\lambda \geqslant 0$; and the method of sub and supersolutions immediately yields the above results. Next, if $K \leqslant 0$ and if u solves (A.1) then we have

$$A(-u) = (-K)u^{(N+2)/(N-2)} \geqslant c_0^{(N+2)/(N-2)}|K|$$

and (A.4) holds. On the other hand if (A.4) holds, replacing, if necessary, $\bar{u}$ by $\bar{u} - \mu\varphi_1$ for μ large, we may assume: $\inf \bar{v} > 0$, where $\bar{v} = -\bar{u}$. Again, for δ small, $\delta\bar{v}$ is a subsolution of (A.1) while $\lambda\varphi_1$ is a supersolution for all $\lambda \geqslant 0$ and we conclude.

Finally for some arbitrary K, we consider u_n, v_n solutions of (A.1) – (A.2) with respectively $|K|$, $-|K|$ such that $\sup_{\mathbb{R}^N} u_n$, $\sup_{\mathbb{R}^N} v_m \to 0$; u_n is a supersolution and v_m is a subsolution. And we conclude.

Let us mention that the above proof actually shows the existence of an uncountable infinity of solution of (A.1) – (A.2).

In order to have a more precise description of the solutions of (A.1) – (A.2) at infinity, we will assume instead of (A.3).

$$\begin{gathered} \exists \varphi_1 \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N), \qquad A\varphi_1 = 0 \quad \text{in } \mathbb{R}^N, \\ \varphi_1 > 0 \quad \text{in } \mathbb{R}^N, \qquad \varphi_1 \to 1 \quad \text{as} \quad |x| \to \infty. \end{gathered} \tag{A.5}$$

Observe that if φ_1 exists, φ_1 is unique. Similarly, we will strengthen (A.4) to

$$\exists \bar{u} \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N), \qquad A\bar{u} \geqslant |K| \quad \text{in } \mathbb{R}^N; \tag{A.6}$$

(and this implies that $\bar{u} \geqslant 0$ in $\mathbb{R}^N$ and the existence of $\tilde{u}$ such that

$$A\tilde{u} = |K| \quad \text{in } \mathbb{R}^N, \qquad \tilde{u} \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N), \qquad \tilde{u} \geqslant 0 \quad \text{in } \mathbb{R}^N).$$

We may now replace (A.2) by

$$\exists u \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N), \qquad u \to \mu \quad \text{as} \quad |x| \to \infty \tag{A.7}$$

where $\mu > 0$ is given. And we have the:

Theorem A.2. *We assume* (A.5) – (A.6).

i) *If* $K \leqslant 0$, *for any* $\mu > 0$, *there exists a unique solution* u_μ *of* (A.1) – (A.7). *In addition:* $u_\mu \leqslant \mu\varphi_1$ *on* $\mathbb{R}^N$ *and* u_μ *is increasing in* μ.
ii) *If* $K \geqslant 0$, *there exists* $\mu_0 \in]0, \infty[$, $\mu_0 < \infty$ *if* $K \not\equiv 0$ *such that for* $\mu > \mu_0$, *there does not exist a solution of* (A.1) – (A.7) *and for* $\mu \in]0, \mu_0[$, *there exists a minimum solution* u_μ *of* (A.1) – (A.7). *In addition:* u_μ *is increasing in* μ, $u_\mu \geqslant \mu\varphi_1$. *Finally under the assumptions of Corollary* 1.4, $u_\mu - \mu \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ *and as* $\mu \uparrow \mu_0$, u_μ *increases to* u_{μ_0} *the minimum solution of* (A.1) – (A.7) (*for* $\mu = \mu_0$).
iii) *If* K *is arbitrary, there exists* $\mu_0 \in]0, \infty[$ *such that for* $\mu \in]0, \mu_0[$ *there exists a solution of* (A.1) – (A.7).

PROOF. *We first prove part* (*i*). We remark that $\mu\varphi_1$ is a supersolution of (A.1) which satisfies (A.7). Next if $\tilde{u}$ satisfies

$$A\tilde{u} = -K \quad \text{in } \mathbb{R}^N, \qquad \tilde{u} \in H^1_{\text{loc}}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N), \qquad \tilde{u} \geqslant 0 \quad \text{in } \mathbb{R}^N$$

we set $\underline{u}_\mu = (\mu\varphi_1 - \lambda\tilde{u})^+$ for some $\lambda > 0$. We then have by standard results

$$A\underline{u}_\mu \leqslant 1_{(\underline{u}_\mu \geqslant 0)}\lambda K \leqslant K\underline{u}_\mu^{(N+2)/(N-2)} \quad \text{on } \mathbb{R}^N$$

if λ is chosen such that

$$\underline{u}_\mu^{(N+2)/(N-2)} \leqslant (\mu\varphi_1)^{(N+2)/(N+2)} \leqslant \lambda.$$

Thus $\underline{u}_\mu$ is a subsolution of (A.1) satisfying (A.7) and the existence part is complete.

The various uniqueness and comparison results are deduced from the following claim: let $v, w \in H^1_{\text{loc}}(\mathbb{R}^N) \cap C_b(\mathbb{R}^N)$ satisfy

$$\begin{cases} Av + |K|v^{(N+2)/(N-2)} \leqslant 0 \quad \text{in } \mathbb{R}^N, \quad v \geqslant 0 \quad \text{in } \mathbb{R}^N, \quad \limsup_{|x| \to \infty} v \leqslant \mu \\ Aw + |K|w^{(N+2)/(N-2)} \geqslant 0 \quad \text{in } \mathbb{R}^N, \quad w \geqslant 0 \quad \text{in } \mathbb{R}^N, \quad \liminf_{|x| \to \infty} w \geqslant \mu. \end{cases}$$

then $v \leqslant w$ on $\mathbb{R}^N$. Indeed for all $\epsilon > 0$, we may find R large enough such that

$$v \leqslant (1 + \epsilon)w = w_\epsilon \quad \text{for } |x| \geqslant R.$$

since we have on B_R

$$A(w_\epsilon - v) - \frac{N+2}{N-2}|K|w^{4/(N-2)}(w_\epsilon - v) \geqslant$$

$$\geqslant A(w_\epsilon - v) + |K|(w_\epsilon^{(N+2)/(N-2)} - v^{(N+2)/(N-2)}) \geqslant 0$$

and since the first eigenvalue of A and thus of

$$A+\frac{N+2}{N-2}|K|w_\epsilon^{4/(N-2)}$$

is positive (on $H_0^1(B_R)$), we conclude: $w_\epsilon \geqslant v$ in $\mathbb{R}^N$.

Observe also that part iii) is easily deduced from parts i) and ii). *We finally prove part* ii) and the arguments which follow are very much the same than those used in the study of semilinear elliptic problems with convex increasing nonlinearities in bounded domains (see for example M. G. Crandall and P. H. Rabinowitz [31]; F. Mignot and J. P. Puel [66]: D. G. De Figueiredo, P. L. Lions and R. D. Nussbaum [36]; P. L. Lions [64]). By the proof of Theorem A.1 we already know that for μ small there exists a solution of (A.1) – (A.7). We then let $\mu_0 = \sup\{\mu > 0/\exists v$ supersolution of (A.1), v *satisfies* (A.7)$\}$; so that $\mu_0 \in]0, \infty]$. If $\mu \in]0, \mu_0[$, we set $u^0 = \mu\varphi_1$ and we define by induction u^n as follows

$$Au^n = K(u^{n-1})^{(N+2)/(N-2)} \text{ in } \mathbb{R}^N, \quad u^n \to \mu \text{ as } |x| \to \infty, \qquad u^n \in H^1_{\text{loc}} \cap C_b$$

then observing that $v \geqslant u^0$, we deduce that u^n increases (strictly if $K \not\equiv 0$) to u_μ solution of (A.1) – (A.7). By arguments similar to those used above, we also check that any supersolution v of (A.1) satisfying (A.7) actually satisfies: $v \geqslant \mu\varphi_1$; and thus u_μ is the minimum solution.

We next claim that if $K \not\equiv 0$, $\mu_0 < \infty$: indeed if u_μ solves (A.1) – (A.7), sience $u_\mu \geqslant \mu\varphi_1$, we have on a fixed ball B_R (such that $K \not\equiv 0$ on B_R)

$$Au_\mu \geqslant K(\mu\varphi_1)^{4/(N-2)}u_\mu \quad \text{in} \quad B_R, \qquad u_\mu > 0 \quad \text{on } \bar{B}_R$$

and thus the first eigenvalue of $A - K(\mu\varphi_1)^{4/(N-2)}$ is positive and this is not possible for μ large.

Next, we claim that if $\mu \in]0, \mu_0[$ the first eigenvalue of

$$A-\frac{N+2}{N-2}Ku_\mu^{4/(N-2)}$$

is positive on $H_0^1(B_R)$ for all $R < \infty$ and thus

$$\forall\varphi \in \mathcal{D}(\mathbb{R}^N), \qquad \int_{\mathbb{R}^N}\sum_{i,j} a_{ij}\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j} + k\varphi^2\,dx \geqslant \frac{N+2}{N-2}\int_{\mathbb{R}^N} Ku_\mu^{4/(N-2)}\varphi^2\,dx.$$

Indeed if we denote by $v^n = u_\mu - u^n$ (assuming that $K \not\equiv 0$), we have

$$\begin{cases} Av^N \geqslant \dfrac{N+2}{N-2}K(u^{n-1})^{4/(N-2)}v^{n-1} \geqslant \dfrac{N+2}{N-2}K(u^{n-1})^{4/(N-2)}v^n & \text{in } \mathbb{R}^N \\ v^n > 0 \quad \text{on } \mathbb{R}^N \end{cases}$$

and thus the above claim is proved (observe that if the first eigenvalue in B_R, is nonnegative, it is positive in B_R for $R < R'$).

Now if we assume the conditions of Corollary 1.4, observing that $u_\mu - \mu\varphi_1$ (with the above notations) belongs to $\mathfrak{D}^{1,2}(\mathbb{R}^N)$, we deduce multiplying (A.1) by $u_\mu - \mu\varphi_1$ that we have for some $\delta > 0$

$$\delta \int_{\mathbb{R}^N} |\nabla(u_\mu - \mu\varphi_1)|^2 \, dx + (1+\delta) \int_{\mathbb{R}^N} K u_\mu^{4/(N-2)} (u_\mu - \mu\varphi_1)^2 \, dx \leqslant$$
$$\leqslant \int_{\mathbb{R}^N} K u_\mu^{(N+2)/(N-2)} (u_\mu - \mu\varphi_1) \, dx.$$

and we conclude easily (using the properties of K) that $u_\mu - \mu\varphi_1$ is bounded in $\mathfrak{D}^{1,2}$.

The analogy we have used above of (A.1) – (A.7) with semilinear problems strongly suggests of seeking a second solution above u_μ for $\mu \in]0, \mu_0[$. This is what we prove below (under convenient assumptions). To simplify the presentation, we will assume from now on that $a_{ij} \equiv \delta_{ij}$, $k \equiv 0$, $K \geqslant 0$ (so that $\varphi_1 = 1$). Our main assumption on K will be

$$K \in L^p(\mathbb{R}^N) \cap C_b(\mathbb{R}^N), \quad \text{for some} \quad p \in [1, \tfrac{N}{2}[\tag{A.8}$$

(it is possible to extend this assumption by a careful inspection of the proof below).

Notice that this insures that (A.6) holds and thus there exists $\mu_0 > 0$ such that for $0 < \mu < \mu_0$, there exists a minimum solution u_μ of (A.1) – (A.7) (which is increasing in μ).

In order to find a second solution of (A.1) – (A.7) above u_μ we are going to apply the Mountain Path lemma of Ambrosetti and Rabinowitz as in [31], [36] on the translated problem

$$-\Delta v = f(x, v) \quad \text{in } \mathbb{R}^N, \qquad v \in \mathfrak{D}^{1,2}(\mathbb{R}^N) \tag{A.9}$$

where $f(x,t) = K(x)(u_\mu(x) + t^+)^{(N+2)/(N-2)} - K(x) u_\mu(x)^{(N+2)/(N-2)}$. Hence we consider the functional

$$\mathcal{E}(v) = \int_{\mathbb{R}^N} \frac{1}{2} |\nabla v|^2 - F(x, v) \, dx,$$

where

$$F(x,t) = \int_0^t f(x,s)\, ds = \frac{N-2}{2N} K(x)(u_\mu(x) + t^+)^{2N/(N-2)} -$$
$$- \frac{N-2}{2N} K(x)\, u_\mu(x)^{2N/(N-2)} - K(x) u_\mu(x)^{(N+2)/(N-2)} t^+ .$$

But two difficulties occur: first of all we have to prove that 0 is a «local strict minimum» of $\mathcal{E}$; next in order to apply the Mountain Path lemma, we need to check Palais-Smale condition which in view of [23] or section 4.6 holds provided we check that the tentative critical value is below $\frac{1}{N} I_0^{N/2}$. The first step is thus:

Theorem A.3. *Under assumption* (A.8), *either there exists a second solution* $\bar{u}_\mu$ *of* (A.1) – (A.7) *satisfying:* $\bar{u}_\mu > u_\mu$ *on* $\mathbb{R}^N$, *or there exists* $\delta_0 > 0$ *such that*

$$\mathcal{E}(v) \geqslant 0, \qquad \forall v \in \mathcal{D}^{1,2}(\mathbb{R}^N), \qquad \|v\| \leqslant \delta_0 \tag{A.10}$$

$$\operatorname{Inf}\{\mathcal{E}(v) / v \in \mathcal{D}^{1,2}(\mathbb{R}^N), \ \|v\| = \delta\} > 0, \qquad \forall \delta \in]0, \delta_0]. \tag{A.11}$$

PROOF. We first show (A.10) assuming that there does not exist a second solution of (A.1) – (A.7) such that $\bar{u}_\mu > u_\mu$ on $\mathbb{R}^N$. To this end we set $I_\delta = \operatorname{Inf}\{\mathcal{E}(v) / v \in \mathcal{D}^{1,2}(\mathbb{R}^N), \|v\| \leqslant \delta\}$. We argue by contradiction and we thus assume: $I_\delta < 0$. If we show that the infimum is achieved for $v = v_\delta$, our nonexistence assumption yields that $\|v_\delta\| = \delta$ for δ small and thus there exists $\theta_\delta > 0$ such that

$$-(1 + \theta_\delta) \Delta v_\delta = f(x, v_\delta) \quad \text{in } \mathbb{R}^N, \qquad v_\delta > 0 \quad \text{in } \mathbb{R}^N, \qquad \|v_\delta\|_{\mathcal{D}^{1,2}} = \delta.$$

Recalling that in the proof of Theorem A.2 we have proved

$$\forall \varphi \in \mathcal{D}^{1,2}(\mathbb{R}^N), \qquad \int_{\mathbb{R}^N} |\nabla\varphi|^2 - \frac{N+2}{N-2} K u_\mu^{(N+2)/(N-2)} \varphi^2 \, dx \geqslant 0; \tag{A.12}$$

it is easy to deduce that $\theta_\delta \to 0$ as $\delta \to 0_+$.

Then one shows by standard regularity results that $v_\delta \to 0$ in $L^\infty(\mathbb{R}^N)$ as $\delta \to 0_+$: hence for δ small $u_\mu + v_\delta < u_{\mu'}$, for some $\mu' \in]\mu, \mu_0[$. Observing that $u_\mu + v_\delta$ is a subsolution of (A.1) – (A.7), we deduce the existence of a second solution of (A.1) – (A.7) between $u_\mu + v_\delta$ and $u_{\mu'}$. The contradiction proves our claim.

There just remains to show that the infimum of I_δ is achieved for δ small if $I_\delta < 0$: we apply the concentration-compactness arguments and we set $\rho_n = |\nabla u_n|^2 + |u_n|^{2N/(N-2)}$ where $(u_n)_n$ is a minimizing sequence. If ρ_n vanishes, because of (A.8), $\lim \mathcal{E}(u_n) \geqslant 0$; while dichotomy or tightness up to an unbounded sequence cannot occur still because of (A.8) since, for instance, if ρ_n is tight up to y_n and $|y_n| \to \infty$.

$$\mathcal{E}(u_n) = \int_{\mathbb{R}^N} \frac{1}{2} |\nabla u_n|^2 \, dx - \int_{\mathbb{R}^N} F(x, u_n) \, dx \geqslant -\int_{\mathbb{R}^N} F(x, u_n) \, dx$$

$$0 \leqslant \int_{\mathbb{R}^N} F(x, u_n) \, dx \leqslant \epsilon + \int_{|x - y_n| \leq R_\epsilon} F(x, u_n) \, dx$$

and the last integral goes to 0 in view of (A.8) and since $|y_n| \to \infty$.

Now if ρ_n is tight and u_n converges weakly and a.e. to u, we observe that

$$\left| \int_{\mathbb{R}^N} F(x, u_n) - F(x, u) - \frac{N-2}{2N} K(x) |u_n - u|^{2N/(N-2)} \, dx \right| \to 0.$$

And this yield observing that $\|u\| \leqslant \delta$ and denoting by $v_n = (u_n - u)$

$$I_\delta \geqslant I_\delta + \underline{\lim}_n \int_{\mathbb{R}^N} \frac{1}{2} |\nabla v_n|^2 - \frac{N-2}{2N} |v_n|^{2N/(N-2)} \, dx$$

$$\geqslant I_\delta + \underline{\lim}_n \frac{1}{2} \|v_n\|^2 - c_N \|v_n\|^{2N/(N-2)}$$

and since $\|v\| \leqslant 2\delta$, we deduce that for δ small $v_n \to 0$ strongly and u is a minimum of I_δ.

To prove (A.11), we first observe that without loss of generality we may assume that: $\mathcal{E}(v) > 0$ for $\|v\|$ small, $v \not\equiv 0$. Next if for all $\delta > 0$

$$\bar{I}_\delta = \operatorname{Inf} \{ \mathcal{E}(v) \, / \, v \in \mathcal{D}^{1,2}(\mathbb{R}^N), \quad \|v\|^2 = \delta \} = 0$$

we may prove exactly as above that $\bar{I}_\delta$ is achieved and we reach a contradiction (notice that (S.1) holds since $\bar{I}_\delta^\infty = \frac{\delta'}{2}$, $\bar{I}_{\delta'} = 0$ for δ' small).

The second step is given by:

Theorem A.4. *Under assumption* (A.8)*, and if there exists a path* γ *i.e. a continuous map* γ *from* $[0, 1]$ *into* $\mathcal{D}^{1,2}(\mathbb{R}^N)$ *such that*

$$\max_{t \in [0,1]} \mathcal{E}(\gamma(t)) < \frac{1}{N} \left(\sup_{\mathbb{R}^N} K \right)^{-(N-2)/2} I_0^{N/2}, \qquad \mathcal{E}(\gamma(1)) \leqslant 0; \tag{A.13}$$

then there exists a second solution $\bar{u}_\mu$ *of* (A.1) – (A.7).

Of course (A.13) holds of there exists v_1 in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ such that

$$\max_{t \geq 0} \mathcal{E}(tv_1) < \frac{1}{N} \left(\sup_{\mathbb{R}^N} K \right)^{-(N-2)/2} I_0^{N/2}.$$

And this strict inequality may be checked with the method of H. Brézis and L. Nirenberg [23] and we find, for example, that *if there exists a maximum point* x^0 *of* K *such that:* $D^j K(x^0)$ for $1 \leqslant j \leqslant [(N-2)/2]$ (where $[x]$ denotes the integer part of x, and where of course K is assumed to be nearby x^0) *then* (A.13) *holds* for $N \geqslant 4$ (notice that if $N = 4{,}5$, this condition is automatically satisfied).

PROOF OF THEOREM A.4. We may assume that (A.10), (A.11) and (A.13) hold. We then set $u_1 = \gamma(1)$,

$$c = \inf_{\bar{\gamma}\in\Gamma} \max_{t\in[0,1]} \mathcal{E}(\bar{\gamma}(t))$$

where $\Gamma = \{\bar{\gamma} \in C([0,1], \mathcal{D}^{1,2}(\mathbb{R}^N)),\ \bar{\gamma}(0) = 0,\ \bar{\gamma}(1) = u_1\}$. In view of (A.10), (A.11): $c > 0$. We need to check Palais-Smale condition i.e. if $(u_n)_n$ satisfies

$$\begin{cases} -\Delta u_n = f(x, u_n) + \epsilon_n \ \text{ in } \mathcal{D}'(\mathbb{R}^N), \qquad \epsilon_n \to 0 \ \text{ in } \ (\mathcal{D}^{1,2}(\mathbb{R}^N))' \\ \mathcal{E}(u_n) \to c \end{cases}$$

we have to show that u_n is relatively compact in $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

First of all observe that for δ small (A.11) yields

$$0 < c \leqslant \mathcal{E}(v) \leqslant \int_{\mathbb{R}^N} \frac{1}{2}|\nabla v|^2 - \frac{N+2}{N-2} K u_\mu^{4/(N-2)} v^2 \, dx$$

if $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, $v \geqslant 0$, $\|v\|_{\mathcal{D}^{1,2}} = \delta$. Therefore we have in fact

$$\exists \nu > 0, \qquad \int_{\mathbb{R}^N} \frac{1}{2}|\nabla v|^2 - \frac{N+2}{N-2} K u_\mu^{4/(N-2)} v^2 \, dx \geqslant \nu \int_{\mathbb{R}^N} |\nabla v|^2 \, dx.$$

The enables us to prove that $(u_n)_n$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$: indeed we have

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx = \int_{\mathbb{R}^N} f(x, u_n) u_n \, dx + \langle \epsilon_n, u_n \rangle \geqslant$$

$$\geqslant \int_{\mathbb{R}^N} \frac{N+2}{N-2} K u_\mu^{4/(N-2)} (u_n^+)^2 \, dx +$$

$$+ \gamma \int_{\mathbb{R}^N} F(x, u_n) - \frac{1}{2}\frac{N+2}{N-2} K u_\mu^{4/(N-2)} (u_n^+)^2 \, dx - \|\epsilon_n\| \, \|u_n\|$$

where $\gamma \in \left]2, \frac{2N}{N-2}\right[$ is arbitrary. And usign (A.14) it is easy to show that $(u_n)_n$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

If u_n converges weakly to $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, using (A.8) one may prove that $(|\nabla u_n|^2 + |u_n|^{2N/(N-2)})$ is tight and thus $u_n - u$ (by Lemma 1.1) concentrates at some points $(x_j)_{j\in J}$ and we have (see section 4.6 for more details)

$$\begin{cases} -\Delta u = f(x, u) \ \text{ in } \ \mathbb{R}^N, \qquad u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \\ c = \mathcal{E}(u) + \sum_{j\in J} \frac{1}{N} \mu_j, \qquad \mu_j K(x_j) \geqslant I_0 \mu_j^{(N-2)/N} \end{cases}$$

hence

$$\mu_j \geqslant I_0^{N/2} K(x_j)^{-(N-2)/2} \geqslant I_0^{N/2} \left(\sup_{\mathbb{R}^N} K\right)^{-(N-2)/2}.$$

And we reach a contradiction with (A.13) since

$$\mathcal{E}(u) = \int_{\mathbb{R}^N} \frac{1}{2} |\nabla u|^2 - F(x,u)\,dx = \int_{\mathbb{R}^N} \frac{1}{2} |\nabla u|^2 - \frac{3}{2}\frac{N+2}{N-2} K u_\mu^{4/(N-2)} (u^+)^2\,dx +$$

$$- \int_{\mathbb{R}^N} F(x,u) - \frac{1}{2}\frac{N+2}{N-2} K u_\mu^{4/(N-2)} (u^+)^2\,dx \geqslant$$

$$\geqslant \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - \frac{N+2}{N-2} K u_\mu^{4/(N-2)} (u^+)^2\,dx +$$

$$- \frac{1}{\gamma} \int_{\mathbb{R}^N} f(x,u)u - \frac{N+2}{N-2} K u_\mu^{4/(N-2)} (u^+)^2\,dx \geqslant 0$$

since $\gamma > 2$.

Appendix 2. Improved Sobolev inequalities by symmetries

We want to collect here a few easy remarks on classes of functions in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ presenting i) symmetry properties, ii) support properties. Roughly speaking if those functions possess enough symmetries and if fixed points of the symmetries do not lie in their supports, Sobolev inequalities may be improved. The easiest example is the following: let H be the space of functions u in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ for some $1 \leqslant p < N$ such that i) u is spherically symmetric, ii) $\operatorname{Supp} u \subset \{|x| \geqslant \delta\}$ for some fixed $\delta > 0$. Then $H \hookrightarrow L^q(\mathbb{R}^N)$ for $Np/(N-p) \leqslant q \leqslant \infty$.

To simplify the presentation we will only treat the following situation: let Ω be an open set like $\Omega = \omega \times O_1 \times \ldots \times O_m$ where $m \geqslant 1$, ω is a bounded open set of $\mathbb{R}^{N_0}$ (possibly empty), $O_1, \ldots, O_m$ are given by

$$O_i = \{x_i \in \mathbb{R}^{N_i} / |x_i| \geqslant \delta_i\}, \qquad \forall i \in \{1, \ldots, m\};$$

where $N_i \geqslant 2$, $\delta_i > 0$. Clearly $N = \sum_{i=0}^m N_i$. We will denote by $x = (x_0, x_1, \ldots, x_m)$ a generic point of Ω. Let E be the subspace of $\mathcal{D}^{1,p}(\mathbb{R}^{N_0} \times \mathbb{R}^{N_1} \times \ldots \times \mathbb{R}^{N_m})$ consisting of functions which are spherically symmetric with respect to each $x_i \in \mathbb{R}^{N_i}$ and let $F = E \in \mathcal{D}_0^{1,p}(\Omega)$.

We begin with the case when $N_0 + m > p$: then let $u \in \mathcal{D}(\mathbb{R}^N) \cap E$ and let v be defined on $\mathbb{R}^{N_0} \times (]0, \infty[)^m$ by

$$v(x_0, t_1, \ldots, t_m) = \prod_{i=1}^m t_i^{(N_i - 1)/p} u(x_0, x_1, \ldots, x_m), \quad \text{with} \quad |x_i| = t_i.$$

Then $v \in \mathcal{D}_0^{1,p}(Q)$ with $Q = \mathbb{R}^{N_0} = (]0, \infty[)^m$ and if $N_i > p$ of all i

$$\|v\|_{\mathfrak{D}_0^{1,p}(Q)} \leqslant C\|u\|_E + C\sum_{i=1}^{m}\left\|\frac{u}{|x_i|}\right\|_{L^p} \leqslant C\|u\|_E + C\sum_{i=1}^{M}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^p}$$
$$\leqslant C\|u\|_E$$

therefore: $\|v\|_{L^{\bar{q}}(Q)} \leqslant C\|u\|_E$, where $\bar{q} = (N_0 + m)p/(N_0 + m - p)$.

And we find in conclusion that *if* $N_0 + m > p$, $p < N_i$, $\forall i \in \{1, \ldots, m\}$

$$\left(\int_{\mathbb{R}^N} P(x)|u|^{\bar{q}}\, dx\right)^{1/\bar{q}} \leqslant C\|u\|_{\mathfrak{D}^{1,p}(\mathbb{R}^N)}, \qquad \forall u \in E \tag{B.1}$$

where

$$P = \prod_{i=1}^{m} |x_i|^{\theta_i},$$

with $\theta_i = (N_i - 1)(\bar{q} - p)/p$. Using Sobolev and Hölder inequalities we also find if $q^* = (Np)/(N - p)$

$$\left(\int_{\mathbb{R}^N} P^{\theta}|u|^{q}\, dx\right)^{1/q} \leqslant C\|u\|_{\mathfrak{D}^{1,p}(\mathbb{R}^N)}, \qquad \forall u \in E \tag{B.2}$$

for all $q \in [q^*, \bar{q}]$, where $\theta = (\bar{q}/q)(q - q^*)(\bar{q} - q^*)^{-1}$. And thus we have

$$\|u\|_{L^q(\Omega)} \leqslant C\|u\|_{\mathfrak{D}^{1,p}(\mathbb{R}^N)}, \qquad \forall u \in F, \qquad \forall q \in [q^*, \bar{q}]. \tag{B.3}$$

Next, *if* $p > N_0 + m$, $p < N_i$, $\forall i \in \{1, \ldots, m\}$; the same proof shows

$$|u| \leqslant C\left\{\inf_{i \le j \le m} |x_j|^{1 - n/p}\right\}\left\{\prod_{i=1}^{m} |x_i|^{-(N_i - 1)/p}\right\}|\nabla u|_{L^p}, \qquad \forall u \in E \tag{B.4}$$

where $n = N_0 + m$. In particular, we find

$$\|u\|_{L^q(\Omega)} \leqslant C\|\nabla u\|_{L^p(\Omega)}, \qquad \forall u \in F \tag{B.5}$$

for all $q^* \leqslant q \leqslant \infty$. The same result holds if $N_0 = 0$, $m = 1$, $p = 1$.

Similar results may be obtained for all $u \in E$ in the remaining cases ($p < N_0 + m$, $\exists i$, $p \geqslant N_i$, or $p = N_0 + m$; or $p > N_0 + m$. $\exists i$, $p \geqslant N_i$) but we will skip them. Now for $u \in F$, we indicate that if $p < N_0 + m$ then (B.3) *still holds, while if* $p \geqslant N_0 + m$, (B.5) *holds for* $q \in [q^*, \infty]$ *if* $p > N_0 + m$, *for* $q \in [q^*, \infty[$ *if* $p = N_0 + m$.

Indeed the above proofs are easily adapted for $u \in F$.

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