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On the Eisenstein Series of Hilbert Modular Groups

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Introduction

Throughout the paper, we let F denote a totally real algebraic number field of degree n, and $\mathbf{a}$ the set of all archimedean primes of F. Given a set X, we denote by $X^{\mathbf{a}}$ the product of $\mathbf{a}$ copies of X, that is, the set of all indexed elements $(x_v)_{v \in \mathbf{a}}$ with $x_v \in X$. If $y \in X^{\mathbf{a}}$, y_v will detnote its v-component. Putting $H = \{z \in \mathbf{C} | \mathrm{Im}(z) > 0\}$, we let $SL_2(F)$ act on $H^{\mathbf{a}}$ through the injection of $SL_2(F)$ into $SL_2(\mathbf{R})^{\mathbf{a}}$. For $\sigma \in \mathbf{Z}^{\mathbf{a}}$ and $v \in \mathbf{a}$, we define a differential operator L_v^σ on $H^{\mathbf{a}}$ by

$$L_v^\sigma = -4y_v^{2-\sigma_v}(\partial/\partial z_v)y_v^{\sigma_v}(\partial/\partial \bar{z}_v),$$

where z_v is the variable on the v-factor of $H^{\mathbf{a}}$ and $y_v = \mathrm{Im}(z_v)$.

Given a congruence subgroup Γ of $SL_2(F)$ and $\lambda \in \mathbf{C}^{\mathbf{a}}$, we denote by $\mathcal{A}(\sigma, \lambda, \Gamma)$ the set of all C^∞-functions f on $H^{\mathbf{a}}$ such that

i) $f(\gamma(z)) = \prod_{v \in \mathbf{a}}(c_v z_v + d_v)^{\sigma_v} f(z)$ for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$,
ii) $L_v^\sigma f = \lambda_v f$ for every $v \in \mathbf{a}$,
iii) f is slowly increasing at every cusp.

Further we let $\mathcal{S}(\sigma, \lambda, \Gamma)$ denote the set of all cusp forms, defined as usual, belonging to $\mathcal{A}(\sigma, \lambda, \Gamma)$, and $\mathcal{N}(\sigma, \lambda, \Gamma)$ the orthogonal complement of $\mathcal{S}(\sigma, \lambda, \Gamma)$ in $\mathcal{A}(\sigma, \lambda, \Gamma)$.

Now the main purpose of this paper is to show that $\mathcal{N}(\sigma, \lambda, \Gamma)$ can be spanned, in most cases, by certain Eisenstein series, which are functions $E(z, s; \rho)$ of the variable z on $H^{\mathbf{a}}$, a complex parameter s, and another discrete parameter $\rho \in \mathbf{C}^{\mathbf{a}}$. Namely, given σ, λ, and Γ, we can choose s_0 and ρ so that a suitable finite set of $E(z, s_0; \rho)$ spans $\mathfrak{N}(\sigma, \lambda, \Gamma)$ (Theorem 7.3). If $F = \mathbf{Q}$, the parameter ρ indicates nothing but the weight σ, but if $F \neq \mathbf{Q}$, ρ involves a variable in $\mathbf{R}^{\mathbf{a}}$ which parametrizes the archimedean factors of Hecke (Grössen-) characters of F. There are two cases in which Eisenstein series by themselves cannot generate $\mathfrak{N}(\sigma, \lambda, \Gamma)$. In fact, if $4\lambda_v = (1 - \sigma_v)^2$ for every $v \in \mathbf{a}$, we need $\partial E/\partial s$ (Theorem 7.8); in the other case, we need the residues of the $E(z, s)$ (Theorem 7.9). These theorems are valid also for eigenforms of half-integral weight, which can be defined by making suitable modifications in the above definition. Our results are not complete in the sense that we have to exclude the case of «multiple» λ, which occurs only when $F \neq \mathbf{Q}$, $(1 - \sigma_v)^2 \leqslant 4\lambda_v \in \mathbf{R}$ for all $v \in \mathbf{a}$, and $(1 - \sigma_v)^2 < 4\lambda_v$ for at least one v. We believe, however, that our technique is applicable even to multiple λ, and therefore no serious difficulties are expected in the task of extending our results to the most general case.

As an application, we shall show that every holomorphic Hilbet modular form is a sum of a holomorphic cusp form and a holomorphic Eisenstein series. This holds for all integral and half-integral weights $\geqslant \frac{1}{2}$ (Theorems 8.3, 8.4, and formula (8.3)). The explicit Fourier expansions of certain Eisenstein series obtained in our previous papers [12] and [13] play an essential role in the proof of this result as well as in that of the theorems on $\mathfrak{N}(\sigma, \lambda, \Gamma)$.

Another application concerns an interpretation of the zeros of L-functions of F in the critical strip. To explain the idea, let us assume $F = \mathbf{Q}$ for simplicity. Given $\xi = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in SL_2(\mathbf{Q})$ and $f \in \mathfrak{N}(\sigma, \lambda, \Gamma)$ with $\lambda \in \mathbf{C}$, $\sigma \in \mathbf{Z}$ and $\Gamma \subset SL_2(\mathbf{Z})$, we can speak of a Fourier expansion of f at the cusp $\xi(\infty)$, which has the form

$$(cz + d)^{-\sigma} f(\xi(z)) = a_\xi y^{s_0 - \sigma/2} + a'_\xi y^{1 - s_0 - \sigma/2} + \sum_{n \in \mathbf{Z}} b_\xi(n, y) e^{2\pi i n x/N}$$

with $0 < N \in \mathbf{Z}$, constants a_ξ and a'_ξ, and a complex number s_0 such that $\lambda = (s_0 - \sigma/2)(1 - s_0 - \sigma/2)$. Now we call f a *cyclopean form of exponent* $1 - s_0 - \sigma/2$ if $0 < \mathrm{Re}(s_0) < \frac{1}{2}$ and $a_\xi = 0$ for every $\xi \in SL_2(\mathbf{Q})$. Then we shall show that a nonzero cyclopean form exists if and only if there exists a Dirichlet character ψ such that $L(2s_0, \psi) = 0$ and $\psi(-1) = (-1)^\sigma$. The same type of assertion can be made also for $F \neq \mathbf{Q}$ (Theorem 9.1). This result is tautological if $\Gamma = SL_2(\mathbf{Z})$, in the sense that it follows immediately from the well-known Fourier expansion of the Eisenstein series of $SL_2(\mathbf{Z})$. The assertion in the general case, however, is nontrivial, even when $F = \mathbf{Q}$. In fact, the L-functions involve Euler-products and Dirichlet (or Hecke) characters ψ while our definition of cyclopean forms does not require any such multiplicative structure at least on the surface, which is why we think that the fact deserves a statement as we present here.

Let us conclude the introduction by mentioning the previous investigations. The eigenforms were first studied by Maass in [3] and [4] for the congruence subgroups of $SL_2(\mathbf{Z})$. In particular, he proved a certain bilinear relation of the coefficients of the constant terms of eigenforms and showed that $\mathfrak{N}(\sigma, \lambda, \Gamma)$ can be spanned by Eisenstein series when $\sigma = 0$, $\lambda \geqslant \frac{1}{4}$ and $\Gamma \subset SL_2(\mathbf{Z})$. In [7], Roelcke generalized these to the eigenforms of an arbitrary weight with respect to an arbitrary Fuchsian group. The present paper owes much to their ideas in those papers; in fact, one of the key points in our treatment is a generalization of their bilinear relations.

In the holomorphic case, the fact that an elliptic modular form of integral weight is the sum of a cusp form and an Eisenstein series was proved by Hecke [1]. This was extended by Kloosterman [2] to the Hilbert modular forms of weight $\geqslant 2$. The case of weight 1 was proved recently by Shimizu [8]. As for the forms of half-integral weight, Petersson [6] obtained a corresponding result for weight $\geqslant \frac{5}{2}$ when $F = \mathbf{Q}$. Recently the case of weight $\frac{3}{2}$ with $F = \mathbf{Q}$ was settled by Pei [5].

1. Congruence subgroups and factors of automorphy

The symbols F, n, $\mathbf{a}$, $X^{\mathbf{a}}$, and H we used in the introduction will have the same meaning throughout the paper. In addition, we let $\mathbf{f}$ denote the set of all nonarchimedean primes of F, $\mathfrak{g}$ the maximal order of F, $\mathfrak{g}^{\times}$ the group of all units of F, and $\mathfrak{d}$ the different of F. Each element of $\mathbf{a}$ will be viewed as an injection of F into $\mathbf{R}$. Then $F \otimes_{\mathbf{Q}} \mathbf{R}$ and $F \otimes_{\mathbf{Q}} \mathbf{C}$ can be identified naturally with $\mathbf{R}^{\mathbf{a}}$ and $\mathbf{C}^{\mathbf{a}}$, respectively, through the map $a \otimes b \mapsto (a_v b)_{v \in \mathbf{a}}$ for $a \in F$ and $b \in \mathbf{R}$ (or $\mathbf{C}$), where a_v denotes the image of a under v. We write $a \gg 0$ for $a \in \mathbf{R}^{\mathbf{a}}$ if $a_v > 0$ for all v. For two elements c and x of $\mathbf{C}^{\mathbf{a}}$, we put

$$c^x = \prod_{v \in \mathbf{a}} c_v^{x_v} \tag{1.1}$$

whenever each factor is well-defined (according to the context). We denote by u the identity element of the ring $\mathbf{C}^{\mathbf{a}}$. We have then

$$c^{su} = \prod_{v \in \mathbf{a}} c_v^s \quad \text{for} \quad s \in \mathbf{C}. \tag{1.2}$$

Given an associative ring R with identity element, we denote by $R^{\times}$ the group of all invertible elements of R, and by $M_2(R)$ the ring of all 2×2-matrices with entries in R, and put $SL_2(R) = \{\xi \in M_2(R) | \det(\xi) = 1\}$ when R is commutative. For $\xi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R)$, we write $a = a_\xi$, $b = b_\xi$, $c = c_\xi$, and $d = d_\xi$. For $\alpha \in SL_2(\mathbf{R})$ and $z \in \mathbf{C}$, we put

$$\alpha(z) = (a_\alpha z + b_\alpha)/(c_\alpha z + d_\alpha), \quad j(\alpha, z) = c_\alpha z + d_\alpha. \tag{1.3}$$

Further, for $\alpha = (\alpha_v)_{v \in \mathbf{a}} \in SL_2(\mathbf{R})^{\mathbf{a}}$ and $z = (z_v)_{v \in \mathbf{a}} \in \mathbf{C}^{\mathbf{a}}$, we put

$$(1.4a)\qquad \alpha(z) = (\alpha_v(z_v))_{v\in\mathbf{a}}, \quad j_v(\alpha, z) = j(\alpha_v, z_v),$$

$$(1.4b)\qquad j_\alpha(z) = j(\alpha, z) = (j_v(\alpha, z))_{v\in\mathbf{a}} \quad (\in \mathbf{C}^{\mathbf{a}}).$$

With u as in (1.2), we have $x^u = N_{F/\mathbf{Q}}(x)$ for $x \in F$ and also

$$j_\alpha(z)^u = \prod_{v\in\mathbf{a}} j_v(\alpha, z).$$

We define our basic group G and its parabolic subgroup P by

$$(1.5)\qquad G = SL_2(F), \qquad P = \{\alpha \in G \mid c_\alpha = 0\}.$$

We identify $M_2(F) \otimes_{\mathbf{Q}} \mathbf{R}$ with $M_2(\mathbf{R})^{\mathbf{a}}$ and embed $M_2(F)$ and G into $M_2(\mathbf{R})^{\mathbf{a}}$ and $SL_2(\mathbf{R})^{\mathbf{a}}$; then we let G act on $H^{\mathbf{a}}$ (or even on $\mathbf{C}^{\mathbf{a}}$) through this embedding.

Given an integral ideal $\mathfrak{z}$ and fractional ideals $\mathfrak{x}$ and $\mathfrak{y}$ in F such that $\mathfrak{x}\mathfrak{y}$ is integral, we define a subring $\mathfrak{o}[\mathfrak{x}, \mathfrak{y}]$ of $M_2(F)$ and subgroups $\Gamma[\mathfrak{x}, \mathfrak{y}]$ and $\Gamma[\mathfrak{z}]$ of G by

$$(1.6)\qquad \mathfrak{o}[\mathfrak{x}, \mathfrak{y}] = \{\alpha \in M_2(F) \mid a_\alpha \in \mathfrak{g}, d_\alpha \in \mathfrak{g}, b_\alpha \in \mathfrak{x}, c_\alpha \in \mathfrak{y}\},$$

$$(1.7a)\qquad \Gamma[\mathfrak{x}, \mathfrak{y}] = \mathfrak{o}[\mathfrak{x}, \mathfrak{y}] \cap G,$$

$$(1.7b)\qquad \Gamma[\mathfrak{z}] = \{\alpha \in G \mid a_\alpha \equiv d_\alpha \equiv 1, b_\alpha \equiv c_\alpha \equiv 0 \pmod{\mathfrak{z}}\}.$$

A subgroup Γ of G is called a *congruence subgroup* of G if it contains $\Gamma[\mathfrak{z}]$ as a subgroup of finite index for some $\mathfrak{z}$.

We are going to consider automorphic forms of integral and half-integral weights with respect to congruence subgroups of G. A *weight* will be an element σ of $(1/2)\mathbf{Z}^{\mathbf{a}}$ such that $2\sigma_v \pmod 2$ is independent of v. Our treatment will be divided into two cases according to the parity: Case I for $\sigma \in \mathbf{Z}^{\mathbf{a}}$ (*integral weight*) and Case II for $\sigma \notin \mathbf{Z}^{\mathbf{a}}$ (*half-integral weight*). We consider the group $\mathfrak{G}_\sigma$ consisting of all couples (α, l) formed by $\alpha \in G$ and a holomorphic function l on $H^{\mathbf{a}}$ such that $l(z)^2 = t j_\alpha(z)^{2\sigma}$ with a root of unity t, the group-law being defined by

$$(1.8)\qquad (\alpha, l)(\alpha', l') = (\alpha\alpha', l(\alpha'(z))l'(z)).$$

In Case I, $\mathfrak{G}_\sigma$ is obviously isomorphic to the direct product of G and the group of all roots of unity. For $\xi = (\alpha, l) \in \mathfrak{G}_\sigma$, we write $\alpha = \mathrm{pr}(\xi)$, $l = l_\xi$, $a_\xi = a_\alpha$, $b_\xi = b_\alpha$, $c_\xi = c_\alpha$, $d_\xi = d_\alpha$, and put $\xi(z) = \alpha(z)$ for $z \in H^{\mathbf{a}}$. The group $\mathfrak{G}_\sigma$ is introduced for the purpose of dealing with Case II. We consider it even in Case I, simply in order to make our exposition uniform.

Let F_v denote the v-completion of F for each $v \in \mathbf{a} \cup \mathbf{f}$. If $\mathfrak{x}$ is a fractional ideal in F and $v \in \mathbf{f}$, we denote by $\mathfrak{x}_v$ its closure in F_v. We put $G_v = SL_2(F_v)$ and define the adelization $G_{\mathbf{A}}$ and $P_{\mathbf{A}}$ of G and P as usual. We denote by $G_{\mathbf{a}}$ and $G_{\mathbf{f}}$ the archimedean and nonarchimedean factors of $G_{\mathbf{A}}$; we identify G with its diagonal embedding into $G_{\mathbf{A}}$, and $SL_2(\mathbf{R})^{\mathbf{a}}$ with $G_{\mathbf{a}}$. For $\mathfrak{x}$ and $\mathfrak{y}$ as in (1.6), we put

(1.9a) $$D[\mathfrak{x}, \mathfrak{y}] = \prod_{v \in \mathbf{f} \cup \mathbf{a}} D_v[\mathfrak{x}, \mathfrak{y}],$$

(1.9b) $$D_v[\mathfrak{x}, \mathfrak{y}] = \begin{cases} SO(2) = \{x \in G_v \mid {}^t xx = 1\} & (v \in \mathbf{a}), \\ \mathfrak{o}[\mathfrak{x}, \mathfrak{y}]_v \cap G_v & (v \in \mathbf{f}), \end{cases}$$

where $\mathfrak{o}[\mathfrak{x}, \mathfrak{y}]_v$ is the closure of $\mathfrak{o}[\mathfrak{x}, \mathfrak{y}]$ in $M_2(F_v)$. We observe that $\Gamma[\mathfrak{x}, \mathfrak{y}] = G \cap D[\mathfrak{x}, \mathfrak{y}]G_{\mathbf{a}}$. There is another important subset

(1.10) $$W = G \cap P_{\mathbf{A}} \cdot D[2\mathfrak{d}^{-1}, 2\mathfrak{d}]$$

of $G_{\mathbf{A}}$. Obviously $P \cdot W \cdot \Gamma[2\mathfrak{d}^{-1}, 2\mathfrak{d}] = W$. In [13, Proposition 3.2], we assigned, to each $\beta \in W$, a holomorphic function h_β on $H^{\mathbf{a}}$ that satisfies the following conditions:

(1.11a) $h_\beta(z)^4 = j_\beta(z)^{2u}$ (and hence $(\beta, h_\beta) \in \mathcal{G}_{u/2}$);

(1.11b) $h_{\alpha\beta\gamma}(z) = h_\alpha(z) h_\beta(\gamma(z)) h_\gamma(z)$ if $\alpha \in P$, $\beta \in W$, and $\gamma \in \Gamma[2\mathfrak{d}^{-1}, 2\mathfrak{d}]$;

(1.11c) $h_\alpha(z) = |d_\alpha|^{u/2}$ if $\alpha \in P$;

(1.11d) $h_\gamma^2 / j_\gamma^u = (d_\gamma / |d_\gamma|)^u \left(\dfrac{F(\sqrt{-1})/F}{d_\gamma \mathfrak{g}} \right)$ if $\gamma \in \Gamma[2\mathfrak{d}^{-1}, 2\mathfrak{d}]$.

As for the last two properties, see [13, Proposition 1.2 and (3.13)]. We then define, in Case II, a map $\Lambda_\sigma^k \colon W \to \mathcal{G}_\sigma$ for each odd integer k by

(1.12) $$\Lambda_\sigma^k(\beta) = (\beta, h_\beta^k j_\beta^{g - (k/2)u}) \qquad (\beta \in W).$$

Then (1.11b) implies that

(1.13) $$\Lambda_\sigma^k(\alpha\beta\gamma) = \Lambda_\sigma^k(\alpha)\Lambda_\sigma^k(\beta)\Lambda_\sigma^k(\gamma) \quad \text{for} \quad \alpha, \beta, \gamma \quad \text{as in (1.11b).}$$

In Case I, we define an injection $\Lambda_\sigma^0 \colon G \to \mathcal{G}_\sigma$ by

(1.14) $$\Lambda_\sigma^0(\beta) = (\beta, j_\beta^g) \qquad (\beta \in G).$$

Given an integral ideal $\mathfrak{c}$, we put

(1.15) $$\Delta_\sigma^k[\mathfrak{c}] = \begin{cases} \Lambda_\sigma^0(\Gamma[\mathfrak{c}]) & \text{(Case I)}, \\ \Lambda_\sigma^k(\{\beta \in \Gamma[2\mathfrak{c}\mathfrak{d}^{-1}, 2^{-1}\mathfrak{c}\mathfrak{d}] \mid a_\beta - 1 \in \mathfrak{c}\}) & \text{(Case II)}, \end{cases}$$

assuming that $\mathfrak{c} \subset 4\mathfrak{g}$ in Case II. Here and henceforward, we understand that $k = 0$ in Case I. There is a congruence subgroup Γ of G such that

(1.16) $$\Lambda_\sigma^k(\gamma) = \Lambda_\sigma^1(\gamma) \quad \text{for every} \quad \gamma \in \Gamma.$$

This follows from (1.11d) and [10, Lemma 7.4].

Now, by a *congruence subgroup of* $\mathfrak{G}_\sigma$, we understand a subgroup Δ of $\mathfrak{G}_\sigma$ satisfying the following two conditions:

(1.17*a*) pr *gives an isomorphism of* Δ *onto a subgroup of* G;

(1.17*b*) Δ *contains* $\Delta_\sigma^k[\mathfrak{a}]$ *as a subgroup of finite index for some* $\mathfrak{a}$ *and* k, *where* k *should be* 0 *in Case I.*

If Δ is a congruence subgroup of $\mathfrak{G}_\sigma$, then so is $\xi\Delta\xi^{-1}$ for every $\xi \in \mathfrak{G}_\sigma$. This is trivial in Case I, and is proved in [13, Proposition 1.3] in Case II.

2. Automorphic eigenforms

For a function f: $H^{\mathbf{a}} \to \mathbf{C}$ and $\alpha \in \mathfrak{G}_\sigma$, we define $f \| \alpha$: $H^{\mathbf{a}} \to \mathbf{C}$ by

$$(f \| \alpha)(z) = l_\alpha(z)^{-1} f(\alpha(z)). \tag{2.1}$$

From now on, we always put $y_v = \mathrm{Im}(z_v)$, $y = (y_v)_{v \in \mathbf{a}}$, and view y as an $\mathbf{R}^{\mathbf{a}}$-valued function on $H^{\mathbf{a}}$. Then we have

$$y^p \| \alpha = l_\alpha^{-1} |j_\alpha|^{-2p} y^p \qquad (p \in \mathbf{R}^{\mathbf{a}}, \alpha \in \mathfrak{G}_\sigma). \tag{2.2}$$

For $v \in \mathbf{a}$ and $\sigma \in \mathbf{R}^{\mathbf{a}}$, we define differential operators ϵ_v, δ_v^σ, and L_v^σ acting on C^∞-functions f on $H^{\mathbf{a}}$ by

$$\epsilon_v f = -y_v^2 \cdot \partial f / \partial \bar{z}_v, \tag{2.3a}$$

$$\delta_v^\sigma f = y_v^{-\sigma_v} \cdot \partial (y_v^{\sigma_v} f) / \partial z_v, \tag{2.3b}$$

$$L_v^\sigma f = 4 \delta_v^{\sigma'} \epsilon_v, \qquad \sigma_v' = \sigma_v - 2. \tag{2.3c}$$

We have then

$$\begin{aligned} L_v^\sigma &= -4 y_v^2 \partial^2 / \partial z_v \partial \bar{z}_v + 2 i \sigma_v y_v \partial / \partial \bar{z}_v \\ &= -\sigma_v + 4 \epsilon_v \delta_v^\sigma. \end{aligned} \tag{2.4}$$

It can easily be seen, for every $\xi \in \mathfrak{G}_\sigma$, that

$$\delta_v^\sigma (f \| \xi) = (\delta_v^\sigma f) \| \xi^* \quad \text{with} \quad \xi^* = (\mathrm{pr}(\xi), j_\xi^{2v} l_\xi), \tag{2.5a}$$

$$\epsilon_v (f \| \xi) = (\epsilon_v f) \| \xi_* \quad \text{with} \quad \xi_* = (\mathrm{pr}(\xi), j_\xi^{-2v} l_\xi), \tag{2.5b}$$

$$L_v^\sigma (f \| \xi) = (L_v^\sigma f) \| \xi. \tag{2.5c}$$

Furthermore, if $p \in \mathbf{R}^{\mathbf{a}}$, we have

$$L_v^\sigma y^p = p_v (1 - \sigma_v - p_v) y^p. \tag{2.6}$$

Let Δ be a congruence subgroup of $\mathcal{G}_\sigma$. By an *automorphic eigenform* with respect to Δ, we understand a real-analytic function f on $H^{\mathbf{a}}$ satisfying the following three conditions:

(2.7*a*) $f \| \alpha = f$ *for every* $\alpha \in \Delta$;

(2.7*b*) $L_v^\sigma f = \lambda_v f$ *with* $\lambda_v \in \mathbf{C}$ *for every* $v \in \mathbf{a}$;

(2.7*c*) *for every* $\xi \in \mathcal{G}_\sigma$, *there exist positive numbers* A, B. *and* c (*depending on* f *and* ξ) *such that* $y^{\sigma/2}\, |(f \| \xi)(x+iy)| \leqslant Ay^{cu}$ *if* $y^u > B$.

We denote by $\mathfrak{A}(\sigma, \lambda, \Delta)$ the set of all such f, and by $\mathfrak{A}(\sigma, \lambda)$ the union of $\mathfrak{A}(\sigma, \lambda, \Delta)$ for all congruence subgroups Δ of $\mathcal{G}_\sigma$. Condition (2.7*c*) concerns, in essence, only finitely many elements ξ of $\mathcal{G}_\sigma$. In fact, put

$$\mathcal{P}_\sigma = \{\xi \in \mathcal{G}_\sigma \mid \mathrm{pr}(\xi) \in P\}. \tag{2.8}$$

Then $\Delta \backslash \mathcal{G}_\sigma / \mathcal{P}_\sigma$ is a finite set as will be seen in Section 3. The inequality of (2.7*c*) is true for all $\xi \in \mathcal{G}_\sigma$ if it is true for the members of a complete set of representatives of $\Delta \backslash \mathcal{G}_\sigma / \mathcal{P}_\sigma$.

Given two continuous functions f and g satisfying (2.7*a*), we define their inner product $\langle f, g \rangle$ by

$$\langle f, g \rangle = \mu(\Phi)^{-1} \int_\Phi \bar{f} g y^\sigma \, d\mu(z) \qquad (\Phi = \Delta \backslash H^{\mathbf{a}}), \tag{2.9}$$

where

$$\mu(\Phi) = \int_\Phi d\mu(z) \quad \text{and} \quad d\mu(z) = y^{-2u} \prod_{v \in \mathbf{a}} dx_v \, dy_v.$$

This does not depend on the choice of Δ. We see easily that

$$\langle f, g \rangle = \langle f \| \alpha, g \| \alpha \rangle \quad \text{for every} \quad \alpha \in \mathcal{G}_\sigma. \tag{2.10}$$

To study the Fourier expansion of an eigenform, let us put

$$\mathbf{e}(w) = e^{2\pi i w} \qquad \text{for} \quad w \in \mathbf{C}, \tag{2.11a}$$

$$\mathbf{e}_{\mathbf{a}}(z) = \mathbf{e}\Big(\sum_{v \in \mathbf{a}} z_v\Big) \quad \text{for} \quad z \in \mathbf{C}^{\mathbf{a}}. \tag{2.11b}$$

If $f \in \mathfrak{A}(\sigma, \lambda, \Delta)$, f has a Fourier expansion of the form

$$f(x+iy) = \sum_{h \in \mathfrak{m}} b(h, y) \mathbf{e}_{\mathbf{a}}(hx) \tag{2.12}$$

with a lattice $\mathfrak{m}$ in F, where $hx = (h_v x_v)_{v \in \mathbf{a}}$ (i.e., the product in the algebra $\mathbf{C}^{\mathbf{a}}$). We can find a subgroup U of $\mathfrak{g}^\times$ of finite index such that

$$\Lambda_\sigma^k(\mathrm{diag}[a, a^{-1}]) \in \Delta \quad \text{for all} \quad a \in U.$$

Then

$$f(a^2z) = |a|^{-(k/2)u}a^{-\sigma+(k/2)u}f(z) \quad \text{for every} \quad a \in U, \tag{2.13a}$$

$$b(a^2h, y) = |a|^{(k/2)u}a^{\sigma-(k/2)u}b(h, a^2y) \quad \text{for every} \quad a \in U. \tag{2.13b}$$

Now (2.7*b*) implies that $b(h, y)$ as a function in y_v satisfies

$$(y_v^2\partial^2/\partial y_v^2 + \sigma_v y_v \partial/\partial y_v - 4\pi^2 h_v^2 y_v^2 + 2\pi h_v \sigma_v y_v + \lambda_v)b = 0. \tag{2.14}$$

If $h_v \neq 0$, the solutions are given by Whittaker functions. To present them in a normalized form, we introduce a function $V(g; \alpha, \beta)$, defined for $0 < g \in \mathbf{R}$ and $(\alpha, \beta) \in \mathbf{C}^2$, which has an expression

$$V(g; \alpha, \beta) = e^{-g/2}g^{\beta}\Gamma(\beta)^{-1}\int_0^{\infty} e^{-tg}(1+t)^{\alpha-1}t^{\beta-1}\,dt \tag{2.15}$$

for $\mathrm{Re}(\beta) > 0$. This can be continued as a holomorphic function in (α, β) (and real-analytic in (g, α, β)) to the whole $\mathbf{C}^2$, and satisfies

$$V(g; 1-\beta, 1-\alpha) = V(g; \alpha, \beta), \tag{2.16}$$

$$\lim_{g\to\infty} e^{g/2}V(g; \alpha, \beta) = 1. \tag{2.17}$$

These facts are well known. For the reader's convenience, we give in Section 10 a self-contained treatment of Whittaker functions of this type including the proofs of these and other properties of V.

Now, given $\sigma \in \mathbf{C}^{\mathbf{a}}$ and $\lambda \in \mathbf{C}^{\mathbf{a}}$, we take α_v and β_v so that

$$\sigma_v = \alpha_v - \beta_v, \qquad \lambda_v = \beta_v(1-\alpha_v), \tag{2.18}$$

and define a function W_v for $t \in \mathbf{R}^{\times}$ by

$$W_v(t; \sigma, \lambda) = \begin{cases} V(4\pi t; \alpha_v, \beta_v) & \text{if} \quad t > 0, \\ |4\pi t|^{-\sigma_v}V(-4\pi t; \beta_v, \alpha_v) & \text{if} \quad t < 0. \end{cases} \tag{2.19}$$

If (α_v, β_v) is a solution of (2.18), then the other solution is $(1-\beta_v, 1-\alpha_v)$ (which may be equal to (α_v, β_v)). In view of (2.16), W_v is well-defined. Now it can be verified that $W_v(h_v y_v; \sigma, \lambda)$ as a function of y_v satisfies (2.14); it is $O(y_v^c)$ with $c \in \mathbf{R}$ when $y_v \to \infty$, as can be seen from (2.17); moreover, such a solution of (2.14) is unique up to constant factors (see Proposition 10.1).

We now put, for $t \in (\mathbf{R}^{\times})^{\mathbf{a}}$, $\sigma \in \mathbf{R}^{\mathbf{a}}$, and $\lambda \in \mathbf{C}^{\mathbf{a}}$,

$$W(t; \sigma, \lambda) = \prod_{v \in \mathbf{a}} W_v(t_v; \sigma, \lambda). \tag{2.20}$$

Then conditions (2.7*b*, *c*) imply that $b(h, y)$ is a constant multiple of $W(hy; \sigma, \lambda)$, and hence

(2.21) $$f(x+iy) = b_0(y) + \sum_{0 \neq h \in \mathfrak{m}} b_h W(hy; \sigma, \lambda) \mathbf{e}_{\mathbf{a}}(hx)$$

with a function $b_0(y)$ and $b_h \in \mathbf{C}$. The nature of b_0 will be examined in the next section. We call b_0 *the constant term of* f, and understand, by a *cusp form*, an element f of $\mathfrak{A}(\sigma, \lambda)$ such that the constant term of $f \| \xi$ is 0 for every $\xi \in \mathcal{G}_\sigma$. We denote by $\mathcal{S}(\sigma, \lambda)$ the set of all such forms and put $\mathcal{S}(\sigma, \lambda, \Delta) = \mathfrak{A}(\sigma, \lambda, \Delta) \cap \mathcal{S}(\sigma, \lambda)$.

Proposition 2.1. *Let* f *and* b_h *be as in* (2.21). *Then*:

(1) *There exist constants* $p > 0$ *and* $q \geqslant 0$ *such that* $|b_h| \leqslant p|h|^{qu + \sigma/2}$ *for all* $h \in \mathfrak{m}$, $\neq 0$.

(2) *There exist positive constants* A, B, *and* C *such that*

$$y^{\sigma/2} \sum_{0 \neq h \in \mathfrak{m}} |b_h W(hy; \sigma, \lambda)| \leqslant A \cdot \exp(-By^{u/n}) \quad \text{if} \quad y^u \geqslant C.$$

(3) $\langle f, g \rangle$ *is meaningful if either* f *or* g *is a cusp form.*

(4) *If* f *is a cusp form, we can take* $q = 0$ *in* (1).

Proposition 2.2. *Put* $\Delta^* = \{\xi^* \mid \xi \in \Delta\}$, $\Delta_* = \{\xi_* \mid \xi \in \Delta\}$ *with* ξ^* *and* ξ_* *of* (2.5*a*, *b*). *Then*

$$\epsilon_v \mathfrak{A}(\sigma, \lambda, \Delta) \subset \mathfrak{A}(\sigma - 2v, \lambda - (\sigma_v - 2)v, \Delta_*),$$

$$\delta_v^\sigma \mathfrak{A}(\sigma, \lambda, \Delta) \subset \mathfrak{A}(\sigma + 2v, \lambda + \sigma_v v, \Delta^*),$$

where v *is viewed as the element of* $\mathbf{C}^{\mathbf{a}}$ *of which the* v-*component is* 1 *and all other components are* 0.

Proposition 2.3. $\mathfrak{A}(\sigma, \lambda, \Delta)$ *is finite-dimensional over* $\mathbf{C}$.

These propositions will be proved in Section 11.

Define subsets $\mathfrak{N}(\sigma, \lambda)$ and $\mathfrak{N}(\sigma, \lambda, \Delta)$ of $\mathfrak{A}(\sigma, \lambda)$ by

(2.22*a*) $$\mathfrak{N}(\sigma, \lambda) = \{g \in \mathfrak{A}(\sigma, \lambda) \mid \langle f, g \rangle = 0 \quad \text{for all} \quad f \in \mathcal{S}(\sigma, \lambda)\},$$

(2.22*b*) $$\mathfrak{N}(\sigma, \lambda, \Delta) = \{g \in \mathfrak{A}(\sigma, \lambda, \Delta) \mid \langle f, g \rangle = 0 \quad \text{for all} \quad f \in \mathcal{S}(\sigma, \lambda, \Delta)\}.$$

Then we see easily that

(2.23*a*) $$\mathfrak{A}(\sigma, \lambda, \Delta) = \mathcal{S}(\sigma, \lambda, \Delta) \oplus \mathfrak{N}(\sigma, \lambda, \Delta),$$

(2.23*b*) $$\mathfrak{A}(\sigma, \lambda) = \mathcal{S}(\sigma, \lambda) \oplus \mathfrak{N}(\sigma, \lambda),$$

(2.24) $$\mathfrak{N}(\sigma, \lambda, \Delta) = \mathfrak{N}(\sigma, \lambda) \cap \mathfrak{A}(\sigma, \lambda, \Delta).$$

The inclusion $\mathfrak{N}(\sigma, \lambda, \Delta) \subset \mathfrak{N}(\sigma, \lambda)$ is not completely trivial. To see this, let $g \in \mathfrak{N}(\sigma, \lambda, \Delta)$; take any normal subgroup Δ' of Δ. Then $g = f + h$ with $f \in$ $\in \mathcal{S}(\sigma, \lambda, \Delta')$ and $h \in \mathfrak{N}(\sigma, \lambda, \Delta')$. For $\gamma \in \Delta$, we have $g = f\|\gamma + h\|\gamma$. Observing that $f\|\gamma \in \mathcal{S}(\sigma, \lambda, \Delta')$ and $h\|\gamma \in \mathfrak{N}(\sigma, \lambda, \Delta')$. We obtain $f\|\gamma = f$ and $h\|\gamma = h$ and hence $f \in \mathcal{S}(\sigma, \lambda, \Delta)$ and $h \in \mathfrak{N}(\sigma, \lambda, \Delta)$. Therefore $g = h \in \mathfrak{N}(\sigma, \lambda, \Delta')$, which shows that $g \in \mathfrak{N}(\sigma, \lambda)$.

Proposition 2.4. *For each* $v \in \mathbf{a}$, *we have*

$$\langle \epsilon_v f, g \rangle = \langle f, \delta_v^\sigma g \rangle \quad for \quad f \in \mathcal{A}(\sigma + 2v, \lambda), \qquad g \in \mathcal{A}(\sigma, \lambda'), \tag{2.25}$$

if either f or g is a cusp form;

$$\langle L_v f, g \rangle = \langle f, L_v g \rangle \quad for \quad f \in \mathcal{S}(\sigma, \lambda), \qquad g \in \mathcal{A}(\sigma, \lambda'). \tag{2.26}$$

These formulas are actually true for C^∞-functions f and g satisfying only (2.7*a*, *c*), under a suitable condition on the convergence, as will be shown in Section 6; (2.26) follows from (2.25), since

$$\langle L_v f, g \rangle = 4\langle \delta_v^{\sigma - 2v} \epsilon_v f, g \rangle = 4\langle \epsilon_v f, \epsilon_v g \rangle = \langle f, L_v g \rangle.$$

(Formula (2.25) was proved also in [14, Lemma 2.3].) This shows also that $\langle f, L_v f \rangle = 4\langle \epsilon_v f, \epsilon_v f \rangle \geqslant 0$. Therefore $\mathcal{S}(\sigma, \lambda) \neq \{0\}$ only if $0 \leqslant \lambda_v \in \mathbf{R}$ for every $v \in \mathbf{a}$.

Proposition 2.5. *Every holomorphic function on* $H^{\mathbf{a}}$ *satisfying* (2.7*a*, *c*) *belongs to* $\mathcal{A}(\sigma, 0, \Delta)$. *Moreover, every element of* $\mathcal{S}(\sigma, 0)$ *is holomorphic on* $H^{\mathbf{a}}$.

PROOF. A function f on $H^{\mathbf{a}}$ is holomorphic if and only if $\epsilon_v f = 0$ for every $v \in \mathbf{a}$. Thus the first assertion is obvious. If $f \in \mathcal{S}(\sigma, 0)$, we have $4\langle \epsilon_v f, \epsilon_v f \rangle =$ $= \langle f, L_v f \rangle = 0$, so that $\epsilon_v f = 0$, which proves the second assertion.

3. The constant term of an eigenform

Let U be a subgroup of $\mathfrak{g}^\times$ of finite index. We call an element τ of $\mathbf{R}^{\mathbf{a}}$ *U-admissible* if

$$|x|^{i\tau} = 1 \quad for\ all \quad x \in U \quad and \quad \sum_{v \in \mathbf{a}} \tau_v = 0, \tag{3.1}$$

and denote by T_U the set of all U-admissible τ. We call τ *admissible* if it is U-admissible for some U. We can easily prove

(3.2) $$\{p \in \mathbf{C}^{\mathbf{a}} \mid |x|^p = 1 \quad \text{for all} \quad x \in U\} = iT_U \oplus \mathbf{C}u.$$

Now let $b(y)$ be the constant term of an element of $\mathfrak{A}(\sigma, \lambda)$. Putting $h = 0$ in (2.14), we have

(3.3) $$(y_v^2 \partial^2/\partial y_v^2 + \sigma_v y_v \partial/\partial y_v + \lambda_v) b = 0.$$

A pair of independent solutions of this equation can be given as follows:

(3.4*a*) y_v^p and y_v^q with the roots p and q of $X^2 - (1 - \sigma_v)X + \lambda_v$ if $4\lambda_v \neq (1 - \sigma_v)^2$,

(3.4*b*) y_v^q and $y_v^q \log y_v$ with $q = (1 - \sigma_v)/2$ if $4\lambda_v = (1 - \sigma_v)^2$.

Therefore $b(y)$ is a linear combination of the products of these functions for all $v \in \mathbf{a}$. However, not every product can appear. In fact, in view of (2.13*b*), we can find a subgroup U of $\mathfrak{g}^{\times}$ of finite index whose elements are all totally positive and such that

(3.5) $$b(a^2 y) = a^{-\sigma} b(y) \quad \text{for every} \quad a \in U.$$

This imposes a nontrivial condition on the combination of the solutions of (3.4*a*, *b*). To be precise, we have:

Proposition 3.1. *The constant term $b(y)$ of an element of $\mathfrak{A}(\sigma, \lambda)$ has one of the following forms*:

(i) *If $4\lambda_v = (1 - \sigma_v)^2$ for all $v \in \mathbf{a}$, then $b(y) = a_1 y^q + a_2 y^q \log y^u$ with $a_i \in \mathbf{C}$ and $q = (u - \sigma)/2$.*

(ii) *If $4\lambda_v \neq (1 - \sigma_v)^2$ for some $v \in \mathbf{a}$, then $b(y) = \sum_p a_p y^p$ with $a_p \in \mathbf{C}$ and $p \in \mathbf{C}^{\mathbf{a}}$. Each p must satisfy the following two conditions*:

(3.6*a*) $$\lambda_v = p_v(1 - \sigma_v - p_v) \quad \text{for all} \quad v \in \mathbf{a};$$

(3.6*b*) $$p = su - (\sigma - i\tau)/2 \ \text{with} \ s \in \mathbf{C} \ \text{and} \ \tau \in T_U.$$

PROOF. Put $\mathbf{b} = \{v \in \mathbf{a} \mid 4\lambda_v = (1 - \sigma_v)^2\}$ and $(\log y)^{\mathbf{d}} = \prod_{v \in \mathbf{d}} \log y_v$ for $\mathbf{d} \subset \mathbf{b}$. Then $b(y) = \sum_{\mathbf{d} \subset \mathbf{b}} \sum_p A_{p,\mathbf{d}} y^p (\log y)^{\mathbf{d}}$ with constants $A_{p,\mathbf{d}}$ and p_v as in (3.6*a*). Take a maximal subset $\mathbf{d}$ of $\mathbf{b}$ such that $A_{p,\mathbf{d}} \neq 0$ for some p. Fix such a p. Then (3.5) implies that $a^{\sigma + 2p} = 1$ for $a \in U$; hence $\sigma + 2p = i\tau + 2su$ with $s \in \mathbf{C}$ and $\tau \in T_U$ by (3.2). Thus p must be as in (3.6*b*). Suppose $\mathbf{d} \neq \varnothing$ and let $\mathbf{d} = \mathbf{e} \cup \{w\}$ with an arbitrarily fixed w. Then (3.5) implies that $\sum A_{p,\mathbf{x}} \log a_v^2 = 0$ for every $a \in U$, where the sum is taken over all $(v, \mathbf{x})$ such that $\{v\} \cup \mathbf{e} = \mathbf{x} \subset \mathbf{b}$. Since $A_{p,\mathbf{d}} \neq 0$, this can happen only when $\mathbf{e} = \varnothing$ and $\mathbf{b} = \mathbf{a}$. Then $p = (u - \sigma)/2$, $b(y) = Ay^p + \sum_{v \in \mathbf{a}} A_v y^p \log y_v$, and $\sum_v A_v \log a_v^2 = 0$ for all $a \in U$, and hence we obtain (i). If $A_{p,\mathbf{d}} = 0$ for all $\mathbf{d} \neq \varnothing$, then we obtain (ii).

Thus, given σ and λ, $b(y)$ belongs to a two-dimensional space if $4\lambda_v = (1 - \sigma_v)^2$ for every $v \in \mathbf{a}$, and to a 2^n-dimensional space otherwise. The latter space can actually be reduced to a 2-dimensional space in most cases. In fact, take p, τ, and s as in (3.6a, b). Let $q = u - \sigma - p$. Then $q = (1 - s)u - (\sigma + i\tau)/2$, and hence q, $-\tau$, and $1 - s$ satisfy (3.6a, b). Since $4\lambda_v \neq (1 - \sigma_v)^2$ for some v, we have $p \neq q$, and therefore y^p and y^q form a two-dimensional vector space. Now our question is whether y^r, with r different from p and q, can occur. Suppose it can, and let $r = tu - (\sigma - i\varkappa)/2$ with $t \in \mathbf{C}$ and $\varkappa \in T_U$. Then, for each v, r_v must coincide with p_v or q_v. Decompose $\mathbf{a}$ into the disjoint union of three subsets $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ so that $r_v = p_v = q_v$ for $v \in \mathbf{b}$, $r_v = p_v \neq q_v$ for $v \in \mathbf{c}$, and $r_v = q_v \neq p_v$ for $v \in \mathbf{d}$. Then $\mathbf{c} \neq \varnothing$ and $\mathbf{d} \neq \varnothing$; $t + i\varkappa_v/2 = s + i\tau_v/2$ for $v \in \mathbf{b} \cup \mathbf{c}$ and $t + i\varkappa_v/2 = 1 - s - i\tau_v/2$ for $v \in \mathbf{b} \cup \mathbf{d}$. Hence $\mathrm{Re}(s) = \mathrm{Re}(t) = \mathrm{Re}(1 - s)$, so that $\mathrm{Re}(s) = 1/2$. Therefore $\mathrm{Re}(p) = (u - \sigma)/2$. Observing that $(1 - \sigma_v)^2 \leqslant 4\lambda_v \in \mathbf{R}$ if and only if $\mathrm{Re}(p_v) = (1 - \sigma_v)/2$, we obtain

Proposition 3.2. *The constant term $b(y)$ of an element of $\mathfrak{A}(\sigma, \lambda)$, for fixed σ and λ, belongs to a two-dimensional vector space unless the following condition is satisfied*:

(3.7) $F \neq \mathbf{Q}$, $(1 - \sigma_v)^2 \leqslant 4\lambda_v \in \mathbf{R}$ *for all* $v \in \mathbf{a}$, *and* $(1 - \sigma_v)^2 < 4\lambda_v$ *for at least one* v.

If this is satisfied and if p is as in (3.6a, b), *then* $\mathrm{Re}(p) = (u - \sigma)/2$.

We call λ *critical* if $4\lambda_v = (1 - \sigma_v)^2$ for all $v \in \mathbf{a}$; otherwise we call λ *noncritical*. We call λ *simple* if either λ is critical or λ is noncritical and there are only two p's satisfying (3.6a, b). In the latter case, if p is one, the other is $u - \sigma - p$. We call λ *multiple* if it is not simple. Any p as in (3.6a, b) is called an *exponent attached to* λ. In Remark 5.5 below, we shall give an example of multiple λ.

Hereafter we fix a weight σ and write simply $\mathcal{G}$ and $\mathcal{P}$ for $\mathcal{G}_\sigma$ and $\mathcal{P}_\sigma$, where $\mathcal{P}_\sigma$ is defined by (2.8). Given an admissible τ, we put, throughout the rest of the paper,

$$\rho = (\sigma - i\tau)/2 \quad (\in \mathbf{C}^{\mathbf{a}}). \tag{3.8}$$

Then, for λ and p of (3.6a, b), we have

$$\lambda_v = (s - \rho_v)(1 - s - \bar{\rho}_v), \tag{3.9a}$$

$$p = su - \rho, \qquad u - \sigma - p = (1 - s)u - \bar{\rho}. \tag{3.9b}$$

Let Δ be a congruence subgroup of $\mathcal{G}$ and let $\Gamma = \mathrm{pr}(\Delta)$. Then pr gives a bijective map of $\mathcal{P}\backslash\mathcal{G}/\Delta$ onto $P\backslash G/\Gamma$, which is a finite set corresponding bijectively to $\Gamma\backslash(F\cup\{\infty\})$ via the map $\alpha \mapsto \alpha^{-1}(\infty)$. Therefore we call a coset $\mathcal{P}\xi\Delta$ with $\xi \in \mathcal{G}$ a *cusp-class* of Δ. Given a cusp-class $\mathcal{P}\xi\Delta$, we call it *ρ-regular* if

$$(3.10a) \qquad y^{-\rho}\|\gamma = y^{-\rho} \quad \text{for every} \quad \gamma \in \mathcal{P}\cap\xi\Delta\xi^{-1}.$$

or equivalently,

$$(3.10b) \qquad l_\gamma^{-1}|d_\gamma|^{\sigma-i\tau} = 1 \quad \text{for every} \quad \gamma \in \mathcal{P}\cap\xi\Delta\xi^{-1}.$$

It can easily be seen that (3.10*a*) is in fact a condition on $\mathcal{P}\xi\Delta$, independent of the choice of a representative ξ. The meaning of this condition is explained by:

Proposition 3.3. *Let ρ, λ, and p be as above; let $f \in \mathcal{C}(\sigma, \lambda, \Delta)$ and $\xi \in \mathcal{G}$. Then y^p or $y^p \log y^u$ can appear nontrivially in the constant term of $f\|\xi^{-1}$ only if $\mathcal{P}\xi\Delta$ is ρ-regular.*

This follows immediately from our definition.

If all cusp-classes of Δ are ρ-regular, then the same is true for every congruence subgroup of Δ. Such a Δ indeed exists because of

Proposition 3.4. *Given $\rho = (\sigma - i\tau)/2$ as above, there exists an integral ideal $\mathfrak{a}$ such that all cusp-classes of $\Delta_\sigma^k[\mathfrak{a}]$ are ρ-regular.*

PROOF. Take an integral ideal $\mathfrak{b}$ so that $x \gg 0$ and $x^{i\tau} = 1$ if $x \in \mathfrak{g}^\times$ and $x - 1 \in \mathfrak{b}$. In Case II, choose $\mathfrak{b}$ so that $\mathfrak{b} \subset 4\mathfrak{g}$. Write simply $\Delta[\mathfrak{a}]$ for $\Delta_\sigma^k[\mathfrak{a}]$. In either case, we have $\mathcal{G} = \mathcal{P}Z\Delta[\mathfrak{b}]$ with a finite subset Z of $\mathcal{G}$. We can find an integral ideal $\mathfrak{a} \subset \mathfrak{b}$ such that $\Delta[\mathfrak{a}] \subset \bigcap_{\zeta\in Z}\zeta^{-1}\Delta[\mathfrak{b}]\zeta$. Obviously $\mathcal{G} = \mathcal{P}X\Delta[\mathfrak{a}]$ with a suitable subset X of $Z\Delta[\mathfrak{b}]$. If $\xi \in \zeta\Delta[\mathfrak{b}]$ with $\zeta \in Z$, then $\xi\Delta[\mathfrak{a}]\xi^{-1} = \zeta\Delta[\mathfrak{a}]\zeta^{-1} \subset \Delta[\mathfrak{b}]$. Let $\gamma \in \mathcal{P}\cap\Delta[\mathfrak{b}]$. Then, in Case II, we have $l_\gamma = |d_\gamma|^{ku/2}d_\gamma^{\sigma-(k/2)u}$ by (1.11*c*) and (1.12). Hence our choice of $\mathfrak{b}$ implies (3.10*b*) for $\Delta = \Delta[\mathfrak{a}]$. The same can be verified in Case I in a similar way.

4. Eisenstein series

Given a congruence subgroup Δ of $\mathcal{G}$, we define its Eisenstein series by

$$(4.1) \qquad E(z,s) = E(z,s;\rho,\Delta) = \sum_{\alpha\in(\mathcal{P}\cap\Delta)\backslash\Delta} y^{su-\rho}\|\alpha.$$

Here $s \in \mathbf{C}$ and $\rho = (\sigma - i\tau)/2$ with an admissible τ. To make the sum meaningful, we have to assume that $y^{-\rho} \| \gamma = y^{-\rho}$ for every $\gamma \in \mathcal{P} \cap \Delta$, that is, $\mathcal{P}\Delta$ is ρ-regular. The series is covergent for $\mathrm{Re}(s) > 1$, and can be continued as a meromorphic function in s to the whole s-plane. (See Theorem 4.2 below for a precise statement.) Assuming this result, we have obviously $E(z, s) \| \gamma = E(z, s)$ for every $\gamma \in \Delta$, and moreover, by (2.6) and (2.5*c*),

$$L_v^\sigma E(z, s) = \lambda_v E(z, s) \quad \text{with} \quad \lambda_v = (s - \rho_v)(1 - s - \bar{\rho}_v) \tag{4.2}$$

for every $v \in \mathbf{a}$. Therefore, if $E(z, s)$ is finite at s, it satisfies (2.7*a*, *b*) as a function in z, and in fact belongs to $\mathcal{A}(\sigma, \lambda, \Delta)$ as (2.7*c*) can be shown in our later discussion.

From our definition, we can easily derive a relation

$$[\mathcal{P} \cap \Delta : \mathcal{P} \cap \Delta'] E(z, s; \rho, \Delta) = \sum_{\gamma \in \Delta' \backslash \Delta} E(z, s; \rho, \Delta') \| \gamma \tag{4.3}$$

for every congruence subgroup $\Delta' \subset \Delta$. For each ρ-regular cusp-class $\mathcal{P}\xi\Delta$ (see (3.10*a*, *b*)), we put

$$E(z, s; \rho, \xi, \Delta) = E(z, s; \rho, \xi\Delta\xi^{-1}) \| \xi. \tag{4.4}$$

Then we see easily that

(4.5) $E(z, s; \rho, \xi, \Delta) \| \gamma = E(z, s; \rho, \xi, \Delta)$ for every $\gamma \in \Delta$,

(4.6) $E(z, s; \rho, \alpha\xi\gamma, \Delta) = l_\alpha^{-1} |d_\alpha|^{2\rho - 2su} E(z, s; \rho, \xi, \Delta)$ if $\alpha \in \mathcal{P}$ and $\gamma \in \Delta$.

Thus, ignoring elementary factors, we associate with Δ exactly as many Eisenstein series as its ρ-regular cusp-classes.

We now introduce another type of Eisenstein series, which is attached to an integral ideal $\mathfrak{c}$ in F and a Hecke character ψ: $F_\mathbf{A}^\times / F^\times \to \mathbf{C}^\times$. We assume

(4.7*a*) $4\mathfrak{g} \supset \mathfrak{c}$ in Case II;

(4.7*b*) $|\psi| = 1$;

(4.7*c*) the finite part of the conductor of ψ divides $\mathfrak{c}$;

(4.7*d*) $\psi(x) = |x|^{i\tau} (x/|x|)^{\sigma'}$ for $x \in F_\mathbf{a}^\times$,

where

$$\sigma' = \begin{cases} \sigma & \text{(Case I)}, \\ \sigma - (k/2)u & \text{(Case II)}, \end{cases} \tag{4.8}$$

$F_\mathbf{A}^\times$ denotes the idele group of F, and $F_\mathbf{a}^\times$ its archimedean factor. We fix $\mathfrak{c}$, assume $\mathfrak{c} \subset 4\mathfrak{g}$ in Case II, and put

(4.9a) $$D = \begin{cases} D[\mathfrak{g}, \mathfrak{c}] & \text{(Case I)}, \\ D[2\mathfrak{d}^{-1}, 2^{-1}\mathfrak{c}\mathfrak{d}] & \text{(Case II)}, \end{cases}$$

(4.9b) $$\Gamma_0(\mathfrak{c}) = G \cap DG_{\mathbf{a}}.$$

Writing simply Γ for $\Gamma_0(\mathfrak{c})$, take a complete set of representatives B of $P\backslash(G\cap P_{\mathbf{A}}D)/\Gamma$. Take, for each $\beta \in B$, a complete set of representatives R_β of $(P\cap\beta\Gamma\beta^{-1})\backslash\beta\Gamma$. Then we put

(4.10) $$E_k(z, s; \rho, \psi, \mathfrak{c}) = \sum_{\beta\in B} N(\mathfrak{a}_\beta)^{2s} \sum_{\alpha\in R_\beta} \psi(d_\alpha\mathfrak{a}_\beta^{-1})\psi_{\mathbf{a}}(d_\alpha)y^{su-\rho}\|\Lambda_\sigma^k(\alpha),$$

where $\mathfrak{a}_\beta = c_\beta\mathfrak{g} + d_\beta\mathfrak{g}$ in Case I and $\mathfrak{a}_\beta = 2c_\beta\mathfrak{d}^{-1} + d_\beta\mathfrak{g}$ in Case II, and $\psi_{\mathbf{a}}$ is the archimedean part of ψ; we use the same letter ψ for the ideal character attached to ψ; we understand that $\psi(d_\alpha\mathfrak{a}_\beta^{-1})\psi_{\mathbf{a}}(d_\alpha) = \psi(\mathfrak{a}_\beta^{-1})$ if $\mathfrak{c} = \mathfrak{g}$. The right-hand side of (4.10) is convergent for $\mathrm{Re}(s) > 1$ and satisfies (4.2). In Case I, we have $k = 0$, and so we write simply E for E_k. As for the relation between the series of type (4.10) and that of (4.1), see (4.24) and Proposition 5.3 below.

Define the L-function $L(s, \psi)$ of ψ as usual and put

(4.11) $$L_{\mathfrak{c}}(s, \psi) = L(s, \psi) \prod_{\mathfrak{p}|\mathfrak{c}} [1 - \psi(\mathfrak{p})N(\mathfrak{p})^{-s}],$$

where $\mathfrak{p}$ denotes a prime ideal in F.

Theorem 4.1. *The series $E_k(z, s; \rho, \psi, \mathfrak{c})$ can be continued as a meromorphic function to the whole s-plane. More precisely, put*

$$D(z, s) = \begin{cases} \prod_{v\in\mathbf{a}} \Gamma(s + (|\sigma_v| + i\tau_v)/2)L_{\mathfrak{c}}(2s, \psi)E(z, s; \rho, \psi, \mathfrak{c}) & \text{(Case I)}, \\ \prod_{v\in\mathbf{a}} \Gamma_v(s + i\tau_v/2)L_{\mathfrak{c}}(4s - 1, \psi^2)E_k(z, s; \rho, \psi, \mathfrak{c}) & \text{(Case II)}, \end{cases}$$

where Γ_v in Case II is defined by

$$\Gamma_v(s) = \Gamma(s + (\theta_v/2) - (1/4)) \cdot \begin{cases} \Gamma(s + (\sigma_v/2)) & \text{if } 2\sigma_v \geqslant -1, \\ \Gamma(s - (\sigma_v/2)) & \text{if } 2\sigma_v < -1, \end{cases}$$

with the smallest nonnegative integer θ_v that is congruent modulo 2 to $\sigma_v - 1/2$ or $\sigma_v + 1/2$ according as $2\sigma_v \geqslant -1$ or $2\sigma_v < -1$. Then there is a real analytic function on $H^{\mathbf{a}} \times \mathbf{C}$ that is holomorphic in s and that coincides with $s(s-1)D(z, s)$ in Case I and with $(s - 3/4)D(z, s)$ in Case II for $\mathrm{Re}(s) > 1$. (*Thus we are able to speak of possible simple poles at $s =$* 0, 1, or 3/4.) *In Case I, the pole at $s = 0$ occurs if and only if $\mathfrak{c} = \mathfrak{g}$, $\psi = 1$, and $\sigma = \tau = 0$; the pole at $s = 1$ occurs if and only if $\psi = 1$ and $\sigma = \tau = 0$. In Case II, the pole at $s = 3/4$ occurs if and only if $\psi^2 = 1$ and, for every $v \in \mathbf{a}$, $\sigma_v - 1/2$ is either an even nonnegative integer or an odd negative integer.*

The result in Case II is merely a paraphrase of [13, Corollary 6.2]. In fact, the symbols k, ρ, μ, and τ there correspond to k, σ', τ, and θ here. If we denote by $E^*(z, s)$ the function $E(z, s; k/2, \rho, \psi, \mathfrak{c})$ there, then, comparison of (4.10) with [13, (4.7c)] shows that

$$E_k(z, s; \rho, \psi, \mathfrak{c}) = y^{(k/4)u-\rho}E^*(z, s-k/4), \tag{4.12}$$

and hence our assertion follows immediately from [13, Corollary 6.2].

The result in Case I can be obtained by modifying the formulation of [12]. To be more specific, take $m = 1$ in [12]; using the same notation, we define a function f on $G_{\mathbf{A}}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \notin P_{\mathbf{A}}D, \\ \psi_{\mathfrak{f}}(d_p)^{-1}\psi_{\mathfrak{c}}(d_w)^{-1}J(x, \mathbf{i})^{-1} & \text{if } x = pw \text{ with } p \in P_{\mathbf{A}} \text{ and } w \in D, \end{cases} \tag{4.13}$$

where

$$\psi_{\mathfrak{f}}(a) = \prod_{v \in \mathbf{f}} \psi(a_v), \qquad \psi_{\mathfrak{c}}(a) = \prod_{v | \mathfrak{c}} \psi(a_v),$$

and

$$J(x, \mathbf{i}) = \prod_{v \in \mathbf{a}} j(x_v, i)^{\sigma_v}|j(x_v, i)|^{i\tau_v - \sigma_v}.$$

We then define a series $E_{\mathbf{A}}$ on $G_{\mathbf{A}}$ by

$$E_{\mathbf{A}}(x, s) = \sum_{\alpha \in P \backslash G} f(\alpha x)\epsilon(\alpha x)^{-2s} \qquad (x \in G_{\mathbf{A}}, s \in \mathbf{C}) \tag{4.14}$$

with ϵ of [12, (2.11)]. We can easily verify that

$$E(z, s; \rho, \psi, \mathfrak{c}) = y^{-\rho}E_{\mathbf{A}}(x, s)J(x, \mathbf{i}) \tag{4.15}$$

if $x \in G_{\mathbf{a}}$ and $x_v(i) = z_v$ for $v \in \mathbf{a}$. Put

$$E'(z, s) = E(z, s; \rho, \psi, \mathfrak{c}) \| \Lambda_\sigma^0(\eta), \qquad \eta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4.16}$$

This has a Fourier expansion of the form

$$E'(z, s) = \sum_{h \in \mathfrak{b}} a(h, y, s)\mathbf{e}_{\mathbf{a}}(hx) \tag{4.17}$$

with $\mathfrak{b} = (\mathfrak{c}\mathfrak{d})^{-1}$. Applying the methods of [12] to E' with obvious modifications, we find that if $\mathfrak{c} \neq \mathfrak{g}$,

$$a(h, y, s) = N(\mathfrak{b})^{-1/2}N(\mathfrak{c})^{-1}a_{\mathfrak{f}}(h, s)y^{su-\rho}\prod_{v \in \mathbf{a}} \xi(y_v, h_v; s + \bar{\rho}_v, s - \rho_v), \tag{4.18a}$$

$$a_{\mathfrak{f}}(h, s) = \prod_{v \in \mathbf{f}, v \nmid \mathfrak{c}} \alpha(v, h, \psi(\pi_v)q_v^{-2s}), \tag{4.18b}$$

where ξ, α, π_v, and q_v are defined by [12, (3.18), (3.22), and (3.23)] (in the one-dimensional case). In particular

$$\xi(g, h; \alpha, \beta) = \int_{\mathbf{R}} \mathbf{e}(-hx)(x+ig)^{-\alpha}(x-ig)^{-\beta}\,dx. \tag{4.19}$$

As for $a_{\mathfrak{f}}$, we have

$$L_{\mathfrak{c}}(2s, \psi)a_{\mathfrak{f}}(h, s) = \begin{cases} L_{\mathfrak{c}}(2s-1, \psi) & \text{if} \quad h = 0, \\ \Sigma\psi(\mathfrak{a})N(\mathfrak{a})^{1-2s} & \text{if} \quad h \neq 0, \end{cases} \tag{4.20}$$

where $\mathfrak{a}$ runs over all integral ideals in F prime to $\mathfrak{c}$ and dividing $h\mathfrak{c}\mathfrak{d}$. This result for $h = 0$ is already given in [12, Theorem 7.7, I]. If $h \neq 0$, the Euler v-factor for $v \nmid h\mathfrak{c}\mathfrak{d}$ is determined by [12, Proposition 4.6]. The «bad factors» can be determined by the methods of [13, §6]. In fact, the present case is easier than [13, §6]. Thus after an easy calculation, we obtain a final result as given in (4.20). As for ξ of (4.19), we have

$$\text{(4.21}a\text{)} \quad \prod_{v \in \mathbf{a}} \xi(y_v, 0; \alpha_v, \beta_v) =$$

$$= i^{-\{\sigma\}}(2\pi)^n(2y)^{u-\alpha-\beta} \prod_{v \in \mathbf{a}} \Gamma(\alpha_v + \beta_v - 1)\Gamma(\alpha_v)^{-1}\Gamma(\beta_v)^{-1},$$

$$\text{(4.21}b\text{)} \quad y^{\beta} \prod_{v \in \mathbf{a}} \xi(y_v, h_v; \alpha_v, \beta_v) =$$

$$= (-2i)^{\{\sigma\}}\pi^{\{\alpha\}}|h|^{\alpha-u}W(hy; \sigma, \lambda) \prod_{v \in \mathbf{a}} \Gamma(\gamma_v)^{-1}.$$

where σ and λ are determined by (2.18), $\{\alpha\} = \sum_{v \in \mathbf{a}} \alpha_v$, and $\gamma_v = \alpha_v$ or β_v according as $h_v > 0$ or $h_v < 0$ (see [11, (1.31), (4.34K)]). Therefore we obtain our assertion on D in Case I by examining the local behavior of each Fourier coefficient of E', provided that $\mathfrak{c} \neq \mathfrak{g}$. To treat the case $\mathfrak{c} = \mathfrak{g}$, we first observe that if $\mathfrak{c} \supset \mathfrak{e}$, we have (in both cases $\mathfrak{c} \neq \mathfrak{g}$ and $\mathfrak{c} = \mathfrak{g}$)

$$E(z, s; \rho, \psi, \mathfrak{c}) = \sum_{\gamma \in \Gamma_0(\mathfrak{e}) \backslash \Gamma_0(\mathfrak{c})} \psi_{\mathfrak{c}}(d_\gamma)^{-1} j_\gamma^{-\sigma} E(\gamma(z), s; \rho, \psi, \mathfrak{e}), \tag{4.22}$$

which can be proved in the same manner as in [12, Proposition 2.4, (ii)]. Suppose $\mathfrak{c} = \mathfrak{g}$. Take an arbitrary $\mathfrak{e} \neq \mathfrak{g}$. Then our result on $E(z, s; \rho, \psi, \mathfrak{e})$ shows that the poles can occur only at $s = 0$ and $s = 1$; the pole at $s = 0$ is produced by the difference of $L(s, \psi)$ from $L_{\mathfrak{e}}(s, \psi)$. To see that these poles do occur when $\psi = 1$ and $\rho = 0$, we apply the method of [12] to $E(z, s)$ (instead of $E'(z, s)$) to find that

$$E(z, s; \rho, \psi, \mathfrak{c}) = y^{su-\rho} + \sum_{h \in \mathfrak{b}^{-1}} b(h, y, s)\mathbf{e}_{\mathbf{a}}(hx),$$

with Fourier coefficients b which are similar to but somewhat more complicated than the above $a(h, y, s)$. It is easy, however, to see that $b(h, y, s) = a(h, y, s)$ if $\mathfrak{c} = \mathfrak{g}$, and hence the poles at $s = 0$ and $s = 1$ occur if $\psi = 1$ and $\rho = 0$. This completes the proof of Theorem 4.1.

Now observe that $E(z, s; \rho, \Delta_\sigma^k[\mathfrak{c}])$ is meaningful if and only if

$$(4.23)\qquad |x|^{i\tau}(x/|x|)^{\sigma'} = 1 \quad \text{for every} \quad x \in \mathfrak{g}^\times \quad \text{such that} \quad x \equiv 1 \pmod{\mathfrak{c}}.$$

This holds if and only if a Hecke character ψ satisfying (4.7*b*, *c*, *d*) exists. Assuming (4.23), let Ψ be the set of all such characters ψ with fixed $\mathfrak{c}$ and ρ, and $|\Psi|$ the number of elements in Ψ. Then we have

$$(4.24)\qquad |\Psi| E(z, s; \rho, \Delta_\sigma^k[\mathfrak{c}]) = \sum_{\psi \in \Psi} E_k(z, s; \rho, \psi, \mathfrak{c})$$

for the same reason as in [12, Proposition 2.4].

Theorem 4.2. *$E(z, s; \rho, \Delta)$ can be continued as a meromorphic function in s to the whole plane in the sense that there exist a nonzero holomorphic function $A(s)$ and a real analytic function $B(z, s)$ on $H^{\mathbf{a}} \times \mathbf{C}$, holomorphic in s such that $A(s)E(z, s; \rho, \Delta) = B(z, s)$ for $Re(s) > 1$. Moreover, $E(z, s; \rho, \Delta)$ is holomorphic in s except at the following points:*

(1) $0 \leqslant \mathrm{Re}(s) < 1/2$ *in Case I and* $1/4 \leqslant \mathrm{Re}(s) < 1/2$ *in Case II;*
(2) *a possible simple pole at $s = 1$ in Case I, which occurs only if $\rho = 0$;*
(3) *a possible simple pole at $s = 3/4$ in Case II, which occurs only if $\tau = 0$ and σ is given as at the end of Theorem* 4.1;
(4) *possible poles at the roots of a polynomial $T_\rho(s)$ given by*

$$T_\rho(s) = \prod_{v \in \mathbf{a}} \Gamma(s + (|\sigma_v| + i\tau_v)/2)/\Gamma(s + (\delta_v + i\tau_v)/2) \qquad (\textit{Case I}),$$

$$T_\rho(s) = \prod_{v \in \mathbf{a}} \Gamma_v(s + i\tau_v/2)/\left[\Gamma\left(s + (i\tau_v/2) - \tfrac{1}{4}\right)\Gamma\left(s + (i\tau_v/2) + \tfrac{1}{4}\right)\right] \qquad (\textit{Case II}),$$

where $\delta_v = 0$ or 1 according as σ_v is even or odd, and Γ_v is as in Theorem 4.1.

PROOF. In view of (4.3), it is sufficient to prove our theorem when $\Delta = \Delta_\sigma^k[\mathfrak{c}]$. Let D_ψ denote the function D defined in Theorem 4.1, and put

$$R_\psi(s) = \begin{cases} \prod_{v \in \mathbf{a}} \Gamma(s + (\delta_v + i\tau_v)/2) L_\mathfrak{c}(2s, \psi) & \text{(Case I)}, \\ \prod_{v \in \mathbf{a}} \Gamma\left(s + (i\tau_v/2) - \tfrac{1}{4}\right)\Gamma\left(s + (i\tau_v/2) + \tfrac{1}{4}\right) L_\mathfrak{c}(4s - 1, \psi^2) & \text{(Case II)}. \end{cases}$$

Then $R_\psi(s) \neq 0$ except at the points of (1). By (4.24), we have

$$(4.25)\qquad |\Psi| E(z, s; \rho, \Delta_\sigma^k[\mathfrak{c}]) = \sum_{\psi \in \Psi} D_\psi(z, s)/[T_\rho(s) R_\psi(s)].$$

Observing that $T_\rho(s)$ is indeed a polynomial in s, we obtain our assertions from Theorem 4.1.

The polynomial $T_\rho(s)$ has no zero when $\mathrm{Re}(s) \geqslant 1/2$. Therefore $E(z, s; \rho, \Delta)$ *is holomorphic in s if* $\mathrm{Re}(s) \geqslant 1/2$ *except for a possible simple pole described in* (2) *or* (3) *of the above theorem.* The pole at $s = 1$ does occur if $\rho = 0$. In fact we have

Proposition 4.3. *For a congruence subgroup* Γ *of* G, *let* $r(\Gamma)$ *be the residue of* $E(z, s; 0, \Lambda_0^0(\Gamma))$ *at* $s = 1$. *Then* $r(\Gamma)$ *is a positive number with the following properties*:

(i) $r(\Gamma)/r(\Gamma') = [\Gamma:\Gamma']/[P\cap\Gamma:P\cap\Gamma']$ *if* $\Gamma' \subset \Gamma$;

(ii) $r(SL_2(\mathfrak{g})) = 2^{n-2}\pi^n D_F^{-1}\zeta_F(2)^{-1}R_F$, *where* D_F *is the discriminant of* F, ζ_F *is the zeta function of* F, *and* R_F *is the regulator of* F;

(iii) $r(\Gamma)\mu(\Gamma\backslash H^{\mathbf{a}}) = \mu_K(P\cap\Gamma)\backslash K)$, *where* $K = \{z \in H^{\mathbf{a}} \mid \mathrm{Im}(z)^u = 1\}$, *furnished with a certain invariant measure* μ_K (*see the proof below*).

PROOF. Assertion (ii) follows immediately from (4.24) and the explicit Fourier expansion given in the proof of Theorem 4.1. This together with (4.3) proves that $r(\Gamma)$ is a positive number satisfying (i). As for (iii), we give here only a sketch of the proof. Put $A = \{y \in \mathbf{R}^{\mathbf{a}} \mid y \gg 0\}$ and $B = \{y \in A \mid y^u = 1\}$. Then every $y \in A$ can be written uniquely $y = t^{1/n}y'$ with $0 < t \in \mathbf{R}$ and $y' \in B$. Let $d^\times y = y^{-u}\,dy$ with the Euclidean measure dy on $\mathbf{R}^{\mathbf{a}}$. Then $d^\times y = t^{-1}\,dt\,dy'$ with a Haar measure dy' on B. Since $K = \mathbf{R}^{\mathbf{a}} \times B$, we can determine a measure μ_K on K by $d\mu_K(x + iy) = dx\,dy'$. Now take $\Gamma = SL_2(\mathfrak{g})$ and put $U = \{a^2 \mid a \in \mathfrak{g}^\times\}$. By a well known principle, we have

$$\int_{A/U} \varphi(y^u)d^\times y = 2^{n-1}R_F\int_0^\infty \varphi(t)t^{-1}\,dt \tag{4.26}$$

for a continuous function φ on A. In particular, this implies

$$\mu_K((P\cap\Gamma)\backslash K) = 2^{n-1}D_F^{1/2}R_F. \tag{4.27}$$

Take $\varphi(t) = e^{-t}t^s$. Then

$$2^{n-1}D_F^{1/2}R_F\Gamma(s) = \int_{(P\cap\Gamma)\backslash H^{\mathbf{a}}} \exp(-y^u)y^{(s+1)u}\,d\mu(z) = \int_{\Gamma\backslash H^{\mathbf{a}}} M(z, s)\,d\mu(z),$$

where

$$M(z, s) = \sum_{\alpha \in (P\cap\Gamma)\backslash\Gamma} \exp(-\mathrm{Im}(\alpha(z))^u)\,\mathrm{Im}(\alpha(z))^{(s+1)u}.$$

Since $1 - t \leqslant e^{-t} \leqslant 1$, we have, for $1 < s \in \mathbf{R}$,

$$E(z, s + 1) - E(z, s + 2) \leqslant M(z, s) \leqslant E(z, s + 1),$$

where $E(z, s) = E(z, s; 0, \Lambda_0^0(\Gamma))$. Then we see that $\lim_{s \to 0} sM(z, s) = r(\Gamma)$, and hence $r(\Gamma)\mu(\Gamma\backslash H^{\mathbf{a}}) = 2^{n-1}D_F^{1/2}R_F$, which proves (iii) when $\Gamma = SL_2(\mathfrak{g})$. The general case can be proved in a similar way; alternatively, it follows from the special case by virtue of (i).

Combining (ii), (iii), and (4.27), we find that

$$\mu(SL_2(\mathfrak{g})\backslash H^{\mathbf{a}}) = 2\pi^{-n}D_F^{3/2}\zeta_F(2), \tag{4.28}$$

which is classical.

Proposition 4.4. *Let Q be a finite set of functions of the form $E(z, s)\|\alpha$ with $\alpha \in \mathcal{G}$ and E of type* (4.1) *or* (4.10), *and let $g(z, s) = \sum_{q \in Q} f_q(s)q(z, s)$ with meromorphic functions f_q on $\mathbf{C}$. Then, for every $s_0 \in \mathbf{C}$, there exists an integer m and a neighborhood V of s_0 such that $(s - s_0)^m g(z, s)$ is a real analytic function on $H^{\mathbf{a}} \times V$ that is holomorphic in s. If in particular, g is finite at $s = s_0$, then $g(z, s_0)$ is an element of $\mathfrak{A}(\sigma, \lambda)$ with $\lambda_v = (s_0 - \rho_v)(1 - s_0 - \bar{\rho}_v)$.*

This will be proved in Section 11.

5. The constant terms of Eisenstein series

Lemma 5.1. *Let $\mathfrak{x}$ be a lattice in F and $\mathfrak{y}$ its dual lattice defined by*

$$\mathfrak{y} = \{ b \in F \mid \mathrm{Tr}_{F/\mathbf{Q}}(b\mathfrak{x}) \subset \mathbf{Z} \}.$$

Further let

$$S(z, \mathfrak{x}; \alpha, \beta) = \sum_{a \in \mathfrak{x}} (z + a)^{-\alpha}(\bar{z} + a)^{-\beta} \qquad (z \in H^{\mathbf{a}}; \alpha, \beta \in \mathbf{C}^{\mathbf{a}}).$$

Then this is convergent and real analytic (*at least*) *on*

$$H^{\mathbf{a}} \times \{ (\alpha, \beta) \in \mathbf{C}^{\mathbf{a}} \times \mathbf{C}^{\mathbf{a}} \mid \mathrm{Re}(\alpha_v) > \tfrac{1}{2}, \mathrm{Re}(\beta_v) > \tfrac{1}{2} \quad \text{for every} \quad v \in \mathbf{a} \},$$

and has a Fourier expansion

$$\mu(\mathbf{R}^{\mathbf{a}}/\mathfrak{x})S(x + iy, \mathfrak{x}; \alpha, \beta) = \sum_{h \in \mathfrak{y}} \mathbf{e}_{\mathbf{a}}(hx)\xi(y, h; \alpha, \beta),$$

$$\xi(y, h; \alpha, \beta) = \prod_{v \in \mathbf{a}} \xi(y_v, h_v; \alpha_v, \beta_v)$$

with ξ of (4.19).

This can be proved in the same fashion as in [11, (1.32), Lemma 1.4] (see the last sentence of [11, §1]).

Proposition 5.2. *Let Y be a complete set of representatives of ρ-regular cusp-classes of* Δ in the sense that every ρ-regular cusp-class is gi*ven as* $\mathcal{P}\xi\Delta$ *with exactly one $\xi \in Y$. Let E_ξ, for $\xi \in Y$, denote $E(z, s; \rho, \xi, \Delta)$ with fixed ρ and Δ. Then, for $(\xi, \eta) \in Y \times Y$, we have*

$$E_\xi \| \eta^{-1} = \delta_{\xi\eta} y^{su-\rho} + f_{\xi\eta}(s) y^{u-su-\bar{\rho}} + \sum_{0 \neq h \in \mathfrak{y}} g_{\xi\eta}(h, s, y) \mathbf{e}_{\mathbf{a}}(hx),$$

where $f_{\xi\eta}$ and $g_{\xi\eta}$ are meromorphic in s, $\delta_{\xi\eta}$ is Kronecker's delta, and $\mathfrak{y}$ is a lattice in F.

Proof. The point of our assertion is merely in the shape of the constant term. Fix one ξ and put $\Delta_\xi = \xi\Delta\xi^{-1}$. Put $r(a) = \Lambda_\sigma^k\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ for $a \in F$. Then $r(F) \cap \Delta_\xi = r(\mathfrak{x})$ with a lattice $\mathfrak{x}$ in F. Take a subset Φ of Δ so that $1 \notin \Phi$ and $\{1\} \cup \Phi$ is a complete set of representatives of $(\mathcal{P} \cap \Delta_\xi) \backslash \Delta_\xi / r(\mathfrak{x})$. Then 1 and the elements $\varphi r(a)$ with $\varphi \in \Phi$ and $a \in \mathfrak{x}$ represent $(\mathcal{P} \cap \Delta_\xi) \backslash \Delta_\xi$ without overlap. Therefore

$$E_\xi \| \xi^{-1} = y^{su-\rho} + \sum_{\varphi \in \Phi} \sum_{a \in \mathfrak{x}} y^{su-\rho} \| \varphi r(a).$$

Fix one $\varphi \in \Phi$ and put $c = c_\varphi$, $d = d_\varphi$. Then $c \neq 0$, and

$$\sum_{a \in \mathfrak{x}} y^{su-\rho} \| \varphi r(a) = t y^{su-\rho} c^{-\alpha-\beta} \sum_{a \in \mathfrak{x}} (z + c^{-1}d + a)^{-\alpha} (\bar{z} + c^{-1}d + a)^{-\beta}$$

with $\alpha = su + \bar{\rho}$, $\beta = su - \rho$, and a constant t such that $|t| = 1$. By Lemma 5.1, this has an expansion of the form

$$t y^{su-\rho} c^{-\alpha-\beta} \mu(\mathbf{R}^{\mathbf{a}}/\mathfrak{x})^{-1} \sum_{h \in \mathfrak{y}} \mathbf{e}_{\mathbf{a}}(h(x + c^{-1}d)) \xi(y, h; \alpha, \beta).$$

By (4.21*a*), the term $h = 0$ produces a function of the form $tc^{-\alpha-\beta} p(s) y^{u-su-\bar{\rho}}$ with a meromorphic fuction p independent of φ. Taking the sum over all $\varphi \in \Phi$, we obtain the Fourier expansion of $E_\xi \| \xi^{-1}$ in the form stated in our proposition. The meromorphic continuation of $E_\xi \| \xi^{-1}$ implies that of $f_{\xi\xi}$ and $g_{\xi\xi}$ to the whole s-plane.

Next let $\xi \neq \eta \in Y$. Let Z be a complete set of representatives of $(\mathcal{P} \cap \Delta_\xi) \backslash \xi\Delta\eta^{-1} / [r(F) \cap \Delta_\eta]$. Then the elements $\zeta r(a)$ with $\zeta \in Z$ and $r(a) \in r(F) \cap \Delta_\eta$ represent $(\mathcal{P} \cap \Delta_\xi) \backslash \xi\Delta\eta^{-1}$ without overlap. Therefore the same argument as above establishes the Fourier expansion of $E_\xi \| \eta^{-1}$; the only difference is that $y^{su-\rho}$ doesn't appear this time.

Proposition 5.3. *Let* $\Gamma = \Gamma_0(\mathfrak{c})$, $\Gamma' = \{\gamma \in \Gamma \mid d_\gamma - 1 \in \mathfrak{c}\}$, $\Delta = \Lambda_\sigma^k(\Gamma)$, *and* $\Delta' = \Lambda_\sigma^k(\Gamma')$. *Assume* (4.23). *Let D, B,* and $\mathfrak{a}_\beta$ *be as in* (4.9*a*) *and* (4.10). *Then $\mathcal{P}\alpha\Delta'$ is ρ-regular for every $\alpha \in \Lambda_\sigma^k(G \cap P_{\mathbf{A}} D)$. Moreover, if T_β is a complete* *set of representatives of* $(P\cap\beta\Gamma\beta^{-1})\backslash\beta\Gamma/\Gamma'$, *then*

$$E_k(z, s; \rho, \psi, \mathfrak{c}) = \sum_{\beta\in B} N(\mathfrak{a}_\beta)^{2s} \sum_{\xi\in T_\beta} \psi(d_\xi \mathfrak{a}_\beta^{-1})\psi_{\mathfrak{a}}(d_\xi)\cdot E(z, s; \rho, \Lambda_\sigma^k(\xi), \Delta').$$

PROOF. Put $\tilde{\alpha} = \Lambda_\sigma^k(\alpha)$ for $\alpha \in G\cap P_{\mathbf{A}}D$. Observe that

$$\mathcal{P}\cap\tilde{\alpha}\Delta'\tilde{\alpha}^{-1} = \Lambda_\sigma^k(P\cap\alpha\Gamma'\alpha^{-1}).$$

Then the first assertion can easily be verified. Let S_ξ be a complete set of representatives of $(P\cap\xi\Gamma'\xi^{-1})\backslash\xi\Gamma'$. Then the S_ξ for all $\xi\in T_\beta$ form a disjoint union, which gives a complete set of representatives of $(P\cap\beta\Gamma\beta^{-1})\backslash\beta\Gamma$. Taking this union as R_β of (4.10), we obtain our formula.

Proposition 5.4. *Let* E_ψ *denote the function of* (4.10), *and let* $\zeta\in\mathcal{G}$. *Then the constant term of* $E_\psi\|\zeta^{-1}$ *contains* $y^{su-\rho}$ *nontrivially if and only if* $\mathrm{pr}(\zeta)\in$ $\in G\cap P_{\mathbf{A}}D$ *with* D *of* (4.9a). *Moreover, the term involving* $y^{su-\rho}$ *has the form* $ab^s y^{su-\rho}$ *with* $a\in\mathbf{C}$ *and* $0<b\in\mathbf{R}$.

PROOF. Let $\alpha = \mathrm{pr}(\zeta)$. By Propositions 3.3, 5.2, and 5.3, $y^{su-\rho}$ appears nontrivially in $E_\psi\|\zeta^{-1}$ only if $\alpha\in P\beta\Gamma$ for some $\beta\in B$, that is, only if $\alpha\in G\cap P_{\mathbf{A}}D$. Conversely, if $\alpha\in P\beta\Gamma$ with $\beta\in B$, such a β is unique, and $\alpha\in P\xi\Gamma'$ with a unique $\xi\in T_\beta$. Therefore, Propositions 5.2 and 5.3 show that $y^{su-\rho}$ appears nontrivially in $E_\psi\|\zeta^{-1}$ in the form as claimed.

Remark 5.5. To show that the exceptional case of Proposition 3.2 can happen, take $[F:\mathbf{Q}]=2$ and set $\mathbf{a}=\{v, w\}$. Take θ so that $T_U=\mathbf{Z}\theta$. Then $\theta_v = -\theta_w$. Let $p = su-(\sigma-im\theta)/2$ and $r = tu-(\sigma-in\theta)/2$ with $s=(1+in\theta_v)/2$, $t=(1+im\theta_v)/2$, and $m,\ n\in\mathbf{Z}$. Suppose $|m|\neq|n|$. Then y^p, $y^{u-\sigma-p}$, y^r, $y^{u-\sigma-r}$ belong to the same set of eigenvalues $\{\lambda_v, \lambda_w\}$, where $4\lambda_v=(1-\sigma_v)^2+$ $+(m+n)^2\theta_v^2$ and $4\lambda_w=(1-\sigma_w)^2+(m-n)^2\theta_v^2$. If we put $\rho=(\sigma-im\theta)/2$ and $\rho'=(\sigma-in\theta)/2$, then y^p, $y^{u-\sigma-p}$, y^r, and $y^{u-\sigma-r}$ can appear nontrivially in $E(z,s;\rho,\Delta)$, $E(z,1-s;\bar{\rho},\Delta)$, $E(z,s;\rho',\Delta)$, and $E(z,1-t;\bar{\rho}',\Delta)$, respectively, for a sufficiently small Δ.

6. Bilinear relations

Let Δ be a congruence subgroup of $\mathcal{G}$ and let $\Gamma=\mathrm{pr}(\Delta)$. Take a minimal finite subset X of $\mathcal{G}$ so that $\mathcal{G}=\bigcup_{\xi\in X}\mathcal{P}\xi\Delta$. For each $\xi\in X$, let $Q_\xi = P\cap\mathrm{pr}(\xi\Delta\xi^{-1})$. We consider a group

$$\Theta = \left\{\begin{bmatrix} a & b \\ 0 & a^{-1}\end{bmatrix} \,\middle|\, a\in U_1, b\in\mathfrak{m}\right\} \tag{6.1}$$

with a fractional ideal $\mathfrak{m}$ of F and a subgroup U_1 of $\mathfrak{g}^\times$ of finite index. We choose U_1 and $\mathfrak{m}$ so that $\Lambda_\sigma^k(\Theta) \subset \xi\Delta\xi^{-1}$ for all $\xi \in X$ and $a \gg 0$ for every $a \in U_1$.

Let $f \in \mathfrak{A}(\sigma, \lambda, \Delta)$ and $g \in \mathfrak{A}(\sigma, \lambda', \Delta)$. Assuming that both λ and λ' are noncritical, put, for each $\xi \in X$,

$$f \,\|\, \xi^{-1} = \sum_p a_{p,\xi} y^p + \dots, \tag{6.2a}$$

$$g \,\|\, \xi^{-1} = \sum_q b_{q,\xi} y^q + \dots \tag{6.2b}$$

with constants $a_{p,\xi}$, $b_{q,\xi}$ and $p \in \mathbf{C}^\mathbf{a}$ as given in Proposition 3.1, where... indicates the nonconstant terms of the Fourier expansions. If both λ and λ' are critical, we put

$$f \,\|\, \xi^{-1} = a_\xi y^p + a_\xi' y^p \log y^u + \dots, \tag{6.3a}$$

$$g \,\|\, \xi^{-1} = b_\xi y^p + b_\xi' y^p \log y^u + \dots \tag{6.3b}$$

with $p = (u - \sigma)/2$.

Theorem 6.1. *Suppose $\lambda' = \bar{\lambda}$ and λ is noncritical and simple. Fix one exponent p attached to λ and put $q = u - \sigma - p$. Then*

$$\sum_{\xi \in X} \nu_\xi (\bar{a}_{p,\xi} b_{\bar{q},\xi} - \bar{a}_{q,\xi} b_{\bar{p},\xi}) = 0,$$

where $\nu_\xi = [Q_\xi\{\pm 1\} : \Theta\{\pm 1\}]^{-1}$. If $\lambda = \lambda'$ and λ is critical, one has

$$\sum_{\xi \in X} \nu_\xi (\bar{a}_\xi b_\xi' - \bar{a}_\xi' b_\xi) = 0.$$

PROOF. For $0 < r \in \mathbf{R}$, put

$$T_r = \{z \in H^\mathbf{a} \mid y^u > r\}, \qquad M_r = \{z \in H^\mathbf{a} \mid y^u = r\}.$$

We can find an r such that the sets $\xi^{-1}(Q_\xi \backslash T_r)$ can be embedded into $\Gamma \backslash H^\mathbf{a}$ without overlap. For each ξ, take a positive number $r(\xi) > r$. Also take a union J of small neighborhoods of elliptic fixed points on $\Gamma \backslash H^\mathbf{a}$. Let K be the complement of $\bigcup_{\xi \in X} \xi^{-1}(Q_\xi \backslash T_{r(\xi)}) \cup J$ in $\Gamma \backslash H^\mathbf{a}$. Then K is a compact manifold with boundary, and

$$\partial K = \sum_{\xi \in X} \xi^{-1}(Q_\xi \backslash M_{r(\xi)}) - \partial J.$$

Let φ be a Γ-invariant C^∞-form on $H^\mathbf{a}$ of codegree 1. Then

$$\int_K d\varphi = \int_{\partial K} \varphi = \sum_{\xi \in X} \nu_\xi \int_{B_\xi} \varphi \cdot \xi^{-1} - \int_{\partial J} \varphi, \tag{6.4}$$

where $B_\xi = \Theta_\xi \backslash M_{r(\xi)}$ (with a natural orientation). We fix one $v \in \mathbf{a}$ and put

$$\omega = y^{-2u} \prod_{v \in \mathbf{a}} dx_v \Lambda\, dy_v,$$

$$\zeta_v = (i/2) y^{-2u}\, d\bar{z}_v \Lambda \prod_{w \neq v} dx_w \Lambda\, dy_w,$$

and $\varphi = \bar{f} h y^\sigma \zeta_v$ with two C^∞-functions f and h on $H^{\mathbf{a}}$ satisfying (2.7a) with Δ and Δ_* (of Proposition 2.2), respectively. Then it is easy to see that

$$d\varphi = \bar{f} \delta_v^\tau h y^\sigma \omega - \overline{\epsilon_v f} \cdot h y^\tau \omega \qquad (\tau = \sigma - 2v).$$

Applying (6.4) to this and taking the limit when $r \to \infty$, we find

$$\langle f, \delta_v^\tau h \rangle = \langle \epsilon_v f, h \rangle \tag{6.5}$$

provided that these inner products are meaningful, and that f or h is rapidly decreasing in the sense that the inequality of (2.7c) holds for *every* $c \in \mathbf{R}$. This proves (2.25). Now take $h = \epsilon_v g$ with g of weight σ. Then

$$d\varphi = \tfrac{1}{4} \bar{f} L_v^\sigma g \cdot y^\sigma \cdot \omega - \overline{\epsilon_v f} \cdot \epsilon_v g \cdot y^\tau \omega.$$

Putting similarly $\varphi' = \bar{g} \epsilon_v f \cdot y^\sigma \zeta_v$, we find that

$$d\varphi - \overline{d\varphi'} = \tfrac{1}{4} (\bar{f} L_v^\sigma g - \overline{L_v^\sigma f} \cdot g) y^\sigma \omega.$$

Applying (6.4) to this form, we obtain

$$\tfrac{1}{4} \int_K (\bar{f} L_v^\sigma g - \overline{L_v^\sigma f} \cdot g) y^\sigma \omega = \sum_{\xi \in X} \nu_\xi \int_{B_\xi} (\varphi - \bar{\varphi}') \circ \xi^{-1} - \int_{\partial J} (\varphi - \bar{\varphi}').$$

We now assume that f and g are eigenfunctions with expansions as in (6.2a, b). Then

$$\varphi \circ \xi^{-1} = -\tfrac{i}{2} \sum_{p,q} q_v \bar{a}_{p,\xi} b_{q,\xi} y^{\bar{p} + q + v + \sigma} \zeta_v + \dots,$$

$$\bar{\varphi}' \circ \xi^{-1} = \tfrac{i}{2} \sum_{p,q} \bar{p}_v \bar{a}_{p,\xi} b_{q,\xi} y^{\bar{p} + q + v + \sigma} \bar{\zeta}_v + \dots.$$

Here the unwritten terms contain some contributions to the «constant terms» of the Fourier expansions, but they tend to zero in our later limit process. (This can easily be shown by virtue of (2) of Proposition 2.1.) Put $U = \{a^2 \mid a \in U_1\}$. Then $\Theta \backslash M_r$ may be viewed as the product of $\mathbf{R}^{\mathbf{a}}/\mathfrak{m}$ and $\{y \in \mathbf{R}^{\mathbf{a}} \mid y \gg 0, y^u = r\}/U$. Then we can easily prove

Lemma 6.2. *Let* $t = su + i\tau \in \mathbf{C}^{\mathbf{a}}$ *with* $s \in \mathbf{C}$ *and* τ *in the set* T_U *of* (3.1). *Then*

$$\int_{\Theta \backslash M_r} y^{t+v} \zeta_v = (-i/2) \mu(\mathbf{R}^{\mathbf{a}}/\mathfrak{m}) R_U r^{s-1},$$

where R_U is the regulator of U defined by $R_U = R_F[\mathfrak{g}^\times : U\{\pm\}]$ with the regulator R_F of F.

Applying this to the first terms of $(\varphi - \bar{\varphi}') \circ \xi^{-1}$, we find that

$$(\lambda'_v - \bar{\lambda}_v) \int_K \bar{f} g y^\sigma \omega + 4 \int_{\partial_J} (\varphi - \bar{\varphi}')$$
$$= \mu(\mathbf{R}^{\mathbf{a}}/\mathfrak{m}) R_U \sum_{\xi \in X} \nu_\xi \sum_{p,q} (\bar{p}_v - q_v) \bar{a}_{p,\xi} b_{q,\xi} r_\xi^{e(p,q)} + \ldots$$

where $e(p, q) = \sum_{w \in \mathbf{a}} (\bar{p}_w + q_w + \sigma_w - 1)/[F: \mathbf{Q}]$. Suppose that $\lambda' = \bar{\lambda}$ and λ is simple. Then, with one exponent p fixed, the sum $\sum_{p,q}$ can be written as

$$(2\bar{p}_v - 1 + \sigma_v)(\bar{a}_{p,\xi} b_{u-\sigma-\bar{p},\xi} - \bar{a}_{u-\sigma-p,\xi} b_{\bar{p},\xi}).$$

Since λ is not critical, $2\bar{p}_v - 1 + \sigma_v \neq 0$ for at least one v. Therefore, taking the limit when $J \to \emptyset$ and $r_\xi \to \infty$, we obtain the first assertion of Theorem 6.1. The second one can be proved in a similar way.

If λ is not simple, $e(p, q)$ can be a pure imaginary number which is not necessarily equal to 0. Therefore we obtain certain linear relations even for multiple λ, whose nature is somewhat different from that for simple λ.

7. Construction of $\mathfrak{N}(\sigma, \lambda, \Delta)$ by Eisenstein series

We are going to show that the space $\mathfrak{N}(\sigma, \lambda, \Delta)$ of (2.23*a*) is generated by the series of type (4.1), their derivatives, and their residues. Given σ and λ, we are interested in the case where $\mathcal{A}(\sigma, \lambda, \Delta) \neq \mathcal{S}(\sigma, \lambda, \Delta)$, that is, the case in which nontrivial constant terms appear. Then λ must be given as in Proposition 3.1. We assume throughout that λ is simple. Then

$$\lambda_v = (s_0 - \rho_v)(1 - s_0 - \bar{\rho}_v), \qquad \rho = (\sigma - i\tau)/2 \tag{7.1}$$

with $s_0 \in \mathbf{C}$ and an admissible $\tau \in \mathbf{R}^{\mathbf{a}}$. Notice that (s_0, ρ) may be changed for $(1 - s_0, \bar{\rho})$ without changing (σ, λ). Notice also that (7.1) includes critical λ as a special case. In fact λ is critical if and only if $s_0 = \frac{1}{2}$ and $\tau = 0$; then $\rho = \frac{\sigma}{2}$. This is so if and only if $s_0 u - \rho = u - s_0 u - \bar{\rho}$.

In this section, we fix a complete set of representatives X of $\mathcal{P}\backslash\mathcal{G}/\Delta$, and also a subset Y of X that represents all ρ-regular cusp-classes of Δ; we then denote by $\varkappa$ the number of elements of Y. Further we let $\mathcal{E}[\rho, \Delta]$ denote the complex vector space spanned by the functions $E(z, s; \rho, \xi, \Delta)$ for all $\xi \in Y$. For a complex number s_0, we denote by $\mathcal{E}[s_0, \rho, \Delta]$ the subspace of $\mathcal{E}[\rho, \Delta]$ consisting of all functions $g(z, s)$ that are finite at s_0, and by $\mathcal{E}(s_0, \rho, \Delta)$ the vector space consisting of $g(z, s_0)$ for all $g \in \mathcal{E}[s_0, \rho, \Delta]$. Similarly we denote by $\mathcal{E}^*[s_0, \rho, \Delta]$ the set of elements of $\mathcal{E}[\rho, \Delta]$ that have at most a simple pole at s_0 and by $\mathcal{E}^*(s_0, \rho, \Delta)$ the residues at s_0 of all elements of $\mathcal{E}^*[s_0, \rho, \Delta]$.

Proposition 7.1. *Both* $\mathcal{E}(s_0, \rho, \Delta)$ *and* $\mathcal{E}^*(s_0, \rho, \Delta)$ *are contained in* $\mathfrak{N}(\sigma, \lambda, \Delta)$ *with* λ *of* (7.1).

PROOF. The spaces in question are contained in $\mathfrak{A}(\sigma, \lambda, \Delta)$ by virtue of Proposition 4.4. To prove that they are orthogonal to cusp forms, take a congruence subgroup $\Delta' \subset \Delta$ so that

$$\mathcal{P} \cap \Delta' = \left\{ \Lambda_\sigma^k\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) \mid a \in U_1, b \in \mathfrak{b} \right\}$$

with an ideal $\mathfrak{b}$ and a subgroup U_1 of $\mathfrak{g}^\times$ of finite index consisting of totally positive units. Then $(\mathcal{P} \cap \Delta')\backslash H^{\mathbf{a}}$ can be represented by $B \times A$, with $B = \mathbf{R}^{\mathbf{a}}/\mathfrak{b}$ and $A = \{y \in \mathbf{R}^{\mathbf{a}} \mid y \gg 0\}/U$ where $U = \{a^2 \mid a \in U_1\}$. We now consider an integral

$$\int_A \int_B \overline{f(x+iy)}\, dx \cdot y^{(s-2)u + \bar{\rho}}\, dy$$

for $f \in \mathcal{S}(\sigma, \lambda', \Delta)$ with any fixed λ'. Since the constant term of f is 0, this is obviously 0. If Re(s) is sufficiently large and $\Phi = \Delta'\backslash H^{\mathbf{a}}$, the integral can be transformed to

$$\int_\Phi \left\{ \sum_{\gamma \in (\mathcal{P} \cap \Delta')\backslash \Delta'} (y^{su + \bar{\rho}} \bar{f}) \circ \gamma \right\} d\mu(z) = \mu(\Phi) \langle f, E(z, s; \rho, \Delta') \rangle .$$

Therefore $\langle f, E(z, s; \rho, \Delta') \rangle = 0$ for sufficiently large Re(s). The same holds with Δ instead of Δ', by virtue of (4.3). Then the desired orthogonality can easily be shown by analytic continuation.

Proposition 7.2. (i) $\dim \mathcal{E}[\rho, \Delta] = \varkappa$.

(ii) *The map* $g(z, s) \to g(z, s_0)$ *gives an isomorphism of* $\mathcal{E}[s_0, \rho, \Delta]$ *onto* $\mathcal{E}(s_0, \rho, \Delta)$ *provided that* λ *of* (7.1) *is noncritical.*

PROOF. Assertion (i) follows immediately from Proposition 5.2. Let $g = \sum_{\xi \in Y} a_\xi E(z, s; \rho, \xi, \Delta) \in \mathcal{E}[s_0, \rho, \Delta]$. Then we have

$$g(z, s_0) \| \eta^{-1} = a_\eta y^{s_0 u - \rho} + \Big(\sum_{\xi \in Y} a_\xi f_{\xi\eta} \Big)(s_0) y^{u - s_0 u - \bar{\rho}} + \dots \tag{7.2}$$

for every $\eta \in Y$. If λ is noncritical, we have $s_0 u - \rho \neq u - s_0 u - \bar{\rho}$, and therefore, if $g(z, s_0) = 0$, we have $a_\eta = 0$ for all $\eta \in Y$, so that $g = 0$. This proves (ii).

Theorem 7.3. *With* λ, s_0, *and* ρ *as in* (7.1), *suppose* λ *is noncritical and simple; suppose also that* $\mathcal{E}[\rho, \Delta] = \mathcal{E}[s_0, \rho, \Delta]$ *and* $\mathcal{E}[\bar{\rho}, \Delta] = \mathcal{E}[\bar{s}_0, \bar{\rho}, \Delta]$. *Then* $\mathfrak{N}(\sigma, \lambda, \Delta) = \mathcal{E}(s_0, \rho, \Delta)$, *and* $\mathfrak{A}(\sigma, \lambda, \Delta)$ *is the direct sum of* $\mathcal{S}(\sigma, \lambda, \Delta)$ *and* $\mathcal{E}(s_0, \rho, \Delta)$.

PROOF. Put $p = s_0u - \rho$ and $q = u - \sigma - p$. Let Y' be the set of all $\xi \in X$ such that $\mathcal{P}\xi\Delta$ is $\bar{\rho}$-regular, and $\varkappa'$ the number of elements of Y'. Given $f \in \mathfrak{A}(\sigma, \lambda, \Delta)$ and $g \in \mathfrak{A}(\sigma, \bar{\lambda}, \Delta)$, we consider expansions

$$f \| \xi^{-1} = a_\xi y^p + a'_\xi y^q + \dots \tag{7.3a}$$

$$g \| \xi^{-1} = b_\xi y^{\bar{p}} + b'_\xi y^{\bar{q}} + \dots \tag{7.3b}$$

for each $\xi \in X$. By Proposition 3.3 and Theorem 6.1, we have

$$\sum_{\xi \in Y} \nu_\xi \bar{a}_\xi b'_\xi - \sum_{\xi \in Y'} \nu_\xi \bar{a}'_\xi b_\xi = 0. \tag{7.4}$$

Moreover, the map

$$f \mapsto ((a_\xi)_{\xi \in Y}, (a'_\xi)_{\xi \in Y'}) \tag{7.5}$$

gives an injection of $\mathfrak{A}(\sigma, \lambda, \Delta)/\mathfrak{S}(\sigma, \lambda, \Delta)$ into $\mathbf{C}^\mu$ with $\mu = \varkappa + \varkappa'$; a similar statement holds with g and $\bar{\lambda}$ instead of f and λ. By Proposition 7.2 and our assumption, $\mathcal{E}(s_0, \rho, \Delta)$ is $\varkappa$-dimensional, and $\mathcal{E}(\bar{s}_0, \bar{\rho}, \Delta)$ is $\varkappa'$-dimensional. Each g in the latter space produces a linear relation of type (7.4), and hence the image of the map of (7.5) is at most $\varkappa$-dimensional. This combined with (2.23a) completes the proof.

Remark 7.4. (1) If $\mathrm{Re}(s_0) \geqslant \frac{1}{2}$, then, by Theorem 4.2, $\mathcal{E}[\rho, \Delta] = \mathcal{E}[s_0, \rho, \Delta]$ except when $s_0 = 1$ and $\rho = 0$ in Case I and $s_0 = \frac{3}{4}$, $\tau = 0$, and σ is as in Theorem 4.1 in Case II. If $\mathcal{E}[\rho, \Delta] = \mathcal{E}[s_0, \rho, \Delta]$ and $\mathrm{Re}(s_0) \geqslant \frac{1}{2}$, then we have automatically $\mathcal{E}[\bar{\rho}, \Delta] = \mathcal{E}[\bar{s}_0, \bar{\rho}, \Delta]$, since $\bar{s}_0 = s_0$ and $\bar{\rho} = \rho$ in those exceptional cases.

(2) The pair (σ, λ) corresponds to (s_0, ρ) and $(1 - s_0, \bar{\rho})$. Therefore, changing (s_0, ρ) for $(1 - s_0, \bar{\rho})$ if necessary, we can take s_0 such that $\mathrm{Re}(s_0) \geqslant \frac{1}{2}$ without changing λ.

Proposition 7.5. *The number of* ρ-regular cusp-classes of Δ *is equal to the number of $\bar{\rho}$-regular cusp-classes of* Δ.

PROOF. Given ρ and Δ, we can find s_0 so that $\mathcal{E}[\rho, \Delta] = \mathcal{E}[s_0, \rho, \Delta] = \mathcal{E}[1 - \bar{s}_0, \rho, \Delta]$, $\mathcal{E}[\bar{\rho}, \Delta] = \mathcal{E}[\bar{s}_0, \bar{\rho}, \Delta] = \mathcal{E}[1 - s_0, \bar{\rho}, \Delta]$, and λ of (7.1) is noncritical and simple (cf. Proposition 3.2). Then we have $\mathfrak{N}(\sigma, \lambda, \Delta) = \mathcal{E}(s_0, \rho, \Delta) = \mathcal{E}(1 - s_0, \bar{\rho}, \Delta)$, which proves our proposition.

Proposition 7.6 *Suppose* $\mathfrak{N}(\sigma, \lambda, \Delta) = \mathcal{E}(s_0, \rho, \Delta)$, *and* λ *is noncritical and simple. For* $f \in \mathfrak{A}(\sigma, \lambda, \Delta)$ *and* $\xi \in Y$, *put*

$$f \| \xi^{-1} = a_\xi y^p + a'_\xi y^q + \dots$$

with $p = s_0 u - \rho$ and $q = u - \sigma - p$. If $a_\xi = 0$ for all $\xi \in Y$, then f is a cusp form.

PROOF. Let $f = g(z, s_0) + h$ with $g \in \mathcal{E}[s_0, \rho, \Delta]$ and $h \in \mathcal{S}(\sigma, \lambda, \Delta)$. Writing g as in the proof of Proposition 7.2, we see that the assumption $a_\xi = 0$ implies that $g = 0$.

Theorem 7.7. *Suppose every ρ-*regular cusp-class of Δ *is also $\bar{\rho}$-regular. Define a $\mathbf{C}^Y$-valued meromorphic function $\mathbf{E}_\Delta(z, s, \rho)$ by*

$$\mathbf{E}_\Delta(z, s, \rho) = (E(z, s; \rho, \xi, \Delta))_{\xi \in Y}. \tag{7.6}$$

Then there exists an $\mathrm{End}(\mathbf{C}^Y)$*-valued meromorphic function $\Phi_\Delta(s, \rho)$ on $\mathbf{C}$ such that*

$$\mathbf{E}_\Delta(z, s, \rho) = \Phi_\Delta(s, \rho)\mathbf{E}_\Delta(z, 1 - s, \bar{\rho}), \tag{7.7a}$$

$$\Phi_\Delta(1 - s, \bar{\rho})\Phi_\Delta(s, \rho) = 1. \tag{7.7b}$$

Moreover, there is a diagonal matrix A, depending only on Δ and Y, whose diagonal entries are positive integers such that

$$\overline{\Phi_\Delta(s, \rho)}A \cdot {}^t\Phi_\Delta(1 - \bar{s}, \rho) = A. \tag{7.7c}$$

PROOF. Put $\lambda_v(s) = (s - \rho_v)(1 - s - \bar{\rho}_v)$, $p = su - \rho$, and $q = u - \sigma - p$. Suppressing the symbols z and Δ for simplicity, we have

$$E(s, \rho, \xi)\|\eta^{-1} = \delta_{\xi\eta}y^p + f_{\xi\eta}(s)y^q + \dots, \tag{7.8a}$$

$$E(1 - s, \bar{\rho}, \xi)\|\eta^{-1} = \delta_{\xi\eta}y^q + g_{\xi\eta}(1 - s)y^p + \dots \qquad (\xi, \eta \in Y) \tag{7.8b}$$

with meromorphic $f_{\xi\eta}$ and $g_{\xi\eta}$. Then, for every $\zeta \in Y$, we have

$$\left\{E(1 - s, \bar{\rho}, \xi) - \sum_{\eta \in Y} g_{\xi\eta}(1 - s)E(s, \rho, \eta)\right\} \Big\| \zeta^{-1} = \tag{7.9}$$

$$= 0y^p + \left\{\delta_{\xi\zeta} - \sum_{\eta \in Y} g_{\xi\eta}(1 - s)f_{\eta\zeta}(s)\right\}y^q + \dots .$$

Now we can find a nonempty open subset W of $\mathbf{C}$ such that $\mathcal{E}[\rho, \Delta] = \mathcal{E}[s, \rho, \Delta] = \mathcal{E}[1 - \bar{s}, \rho, \Delta]$, $\mathcal{E}[\bar{\rho}, \Delta] = \mathcal{E}[\bar{s}, \bar{\rho}, \Delta] = \mathcal{E}[1 - s, \bar{\rho}, \Delta]$, and that $\lambda(s)$ is noncritical and simple for every $s \in W$. (As to simple λ, see Proposition 3.2.) Now the left-hand side of (7.9) without $\|\zeta^{-1}$ belongs to $\mathfrak{N}(\sigma, \lambda(s), \Delta)$ for $s \in W$. By Proposition 7.6, we have

$$E(1 - s, \bar{\rho}, \xi) = \sum_{\eta \in Y} g_{\xi\eta}(1 - s)E(s, \rho, \eta),$$

$$\delta_{\xi\zeta} = \sum_{\eta} g_{\xi\eta}(1 - s)f_{\eta\zeta}(s).$$

Writing $\Phi(s, \rho)$ for the matrix $(f_{\xi\eta}(s))$, we obtain (7.7*a*, *b*). Now

$$E(1-\bar{s}, \rho, \xi) \parallel \eta^{-1} = \delta_{\xi\eta} y^{\bar{q}} + f_{\xi\eta}(1-\bar{s}) y^{\bar{p}} + \dots,$$

and $E(1-\bar{s}, \rho, \xi)$ belongs to $\mathfrak{A}(\sigma, \bar{\lambda}, \Delta)$ for $s \in W$. By (7.4), we have

$$\sum_{\eta \in Y} \nu_\eta [\delta_{\xi\eta}\delta_{\zeta\eta} - \overline{f_{\xi\eta}(s)} f_{\zeta\eta}(1-\bar{s})] = 0.$$

Denoting by D the diagonal matrix whose diagonal elements are ν_η, we obtain $D = \overline{\Phi(s, \rho)} D \cdot {}^t\Phi(1-\bar{s}, \rho)$, which proves (7.7*c*).

By Remark 7.4, (1), $E(z, s; \rho, \xi, \Delta)$ is finite at $s = \frac{1}{2}$ and hence $\Phi_\Delta(s, \rho)$ is finite at $s = \frac{1}{2}$. Moreover, we have $\Phi_\Delta\left(\frac{1}{2}, \rho\right)^2 = 1$ if $\tau = 0$.

Theorem 7.8. *Suppose λ is critical (and hence $\rho = \frac{\sigma}{2}$). Let $\mathcal{E}'\left(\frac{1}{2}, \rho, \Delta\right)$ denote the space spanned by $(\partial g/\partial s)\left(z, \frac{1}{2}\right)$ for all $g \in \mathcal{E}[\rho, \Delta]$, and $\mathcal{E}_0'\left(\frac{1}{2}, \rho, \Delta\right)$ the subspace of $\mathcal{E}'\left(\frac{1}{2}, \rho, \Delta\right)$ consisting of $(\partial g/\partial s)\left(z, \frac{1}{2}\right)$ for all such g satisfying $g\left(z, \frac{1}{2}\right) = 0$. Further let ν_+ (resp. ν_-) the multiplicity of 1 (resp. -1) in the eigenvalues of $\Phi\left(\frac{1}{2}, \rho\right)$. Then $\varkappa = \nu_+ + \nu_-$, $\dim \mathcal{E}\left(\frac{1}{2}, \rho, \Delta\right) = \nu_+$, $\dim \mathcal{E}_0'\left(\frac{1}{2}, \rho, \Delta\right) = \nu_-$, and*

$$\mathcal{E}'\left(\tfrac{1}{2}, \rho, \Delta\right) \subset \mathfrak{N}(\sigma, \lambda, \Delta) = \mathcal{E}\left(\tfrac{1}{2}, \rho, \Delta\right) \oplus \mathcal{E}_0'\left(\tfrac{1}{2}, \rho, \Delta\right).$$

Moreover, $\mathcal{E}\left(\frac{1}{2}, \rho, \Delta\right)$ consists of the elements of $\mathfrak{N}(\sigma, \lambda, \Delta)$ that do not involve $y^{(u-\sigma)/2} \log y^u$.

PROOF. For simplicity, let us suppress the symbols ρ and Δ occasionally. That an element of $\mathcal{E}'\left(\frac{1}{2}\right)$ satisfies (2.7*a*, *b*) can be verified immediately. That it satisfies (2.7*c*) is shown in the proof of Proposition 4.4 in Section 11, and hence $\mathcal{E}'\left(\frac{1}{2}\right) \subset \mathfrak{A}(\sigma, \lambda)$. The orthogonality with cusp forms can also be seen, because the integral expressing $\langle f, g(z, s)\rangle$ is uniformly convergent in a neighborhood of $s = \frac{1}{2}$ for every fixed cusp form f. Thus $\mathcal{E}'\left(\frac{1}{2}\right) \subset \mathfrak{N}(\sigma, \lambda)$. Put $p = (u - \sigma)/2$. From (7.8*a*) we obtain

$$E\left(\tfrac{1}{2}, \rho, \xi\right) \parallel \eta^{-1} = \left[\delta_{\xi\eta} + f_{\xi\eta}\left(\tfrac{1}{2}\right)\right] y^p + \dots,$$

$$(\partial E/\partial s)\left(\tfrac{1}{2}, \rho, \xi\right) \parallel \eta^{-1} = \left[\delta_{\xi\eta} - f_{\xi\eta}\left(\tfrac{1}{2}\right)\right] y^p \log y^u + (df_{\xi\eta}/ds)\left(\tfrac{1}{2}\right) y^p + \dots.$$

For $g(z, s) = \sum_\xi c_\xi E(s, \rho, \xi)$ with $c = (c_\xi)_{\xi \in Y} \in \mathbf{C}^Y$, we have $g\left(z, \frac{1}{2}\right) = 0$ if and only if ${}^t\Phi\left(\frac{1}{2}\right)c = -c$. Hence $\dim \mathcal{E}\left(\frac{1}{2}\right) = \nu_+$. If ${}^t\Phi\left(\frac{1}{2}\right)c = -c$, we have $(\partial g/\partial s)\left(z, \frac{1}{2}\right) \parallel \eta^{-1} = 2c_\eta y^p \log y^u + \dots$, which shows that $\dim \mathcal{E}_0'\left(\frac{1}{2}\right) = \nu_-$. Since no element of $\mathcal{E}\left(\frac{1}{2}\right)$ involves $y^p \log y^u$, we see that $\mathcal{E}\left(\frac{1}{2}\right)$ and $\mathcal{E}_0'\left(\frac{1}{2}\right)$ form a direct sum of dimension $\varkappa$. Now Theorem 6.1 shows that $\mathfrak{N}(\sigma, \lambda, \Delta)$ has dimension $\leqslant \varkappa$. Therefore we obtain all the remaining assertions.

Theorem 7.9. *With* λ, s_0, and ρ *as in* (7.1), *suppose that* λ is real, noncritical, and simple. Suppose $\mathcal{E}[\rho, \Delta] = \mathcal{E}^*[s_0, \rho, \Delta]$ *and a cusp-class of* Δ *is* ρ*-regular if and only if it is* $\bar{\rho}$*-regular. Then* $\mathfrak{N}(\sigma, \lambda, \Delta)$ *has dimension* $\varkappa$, *and is the direct sum of* $\mathcal{E}(s_0, \rho, \Delta)$ *and* $\mathcal{E}^*(s_0, \rho, \Delta)$.

PROOF. Define $R\colon \mathcal{E}[\rho, \Delta] \to \mathcal{E}^*(s_0, \rho, \Delta)$ by $R(g) = \mathrm{Res}_{s_0} g(z, s)$. Then $\mathcal{E}[s_0, \rho, \Delta] = \mathrm{Ker}(R)$, so that, by Proposition 7.2,

$$\dim \mathcal{E}(s_0, \rho, \Delta) + \dim \mathcal{E}^*(s_0, \rho, \Delta) = \varkappa.$$

Let $h \in \mathcal{E}(s_0, \rho, \Delta) \cap \mathcal{E}^*(s_0, \rho, \Delta)$. Then $h(z) = r(z, s_0) = R(g)$ with $r \in \mathcal{E}[s_0, \rho, \Delta]$ and $g \in \mathcal{E}[\rho, \Delta]$. Put

$$r = \sum_{\xi \in Y} a_\xi E(z, s; \rho, \xi, \Delta), \qquad g = \sum_{\xi \in Y} b_\xi E(z, s; \rho, \xi, \Delta) \tag{7.10}$$

with $a_\xi, b_\xi \in \mathbf{C}$. Then, for $\eta \in Y$, we have

$$\begin{aligned} h \parallel \eta^{-1} &= a_\eta y^p + \left(\textstyle\sum_\xi a_\xi f_{\xi\eta}\right)(s_0) y^q + \dots \\ &= 0 y^p + \left(\textstyle\sum_\xi b_\xi \mathrm{Res}_{s_0} f_{\xi\eta}\right) y^q + \dots, \end{aligned}$$

where $p = s_0 u - \rho$ and $q = u - \sigma - p$. Hence $a_\eta = 0$ for all η, so that $h = 0$. Thus $\mathcal{E}(s_0, \rho, \Delta)$ and $\mathcal{E}^*(s_0, \rho, \Delta)$ form a direct sum of dimension $\varkappa$. Consider again the map of (7.5) of $\mathfrak{A}(\sigma, \lambda, \Delta)/\mathcal{S}(\sigma, \lambda, \Delta)$ into $\mathbf{C}^{2\varkappa}$. Now the relation of Theorem 6.1 shows that the image of the map has dimension at most $\varkappa$. This completes the proof.

Remark 7.10. Given σ and λ, we can take s_0 and τ so that $\mathrm{Re}(s_0) \geqslant \frac{1}{2}$ and (7.1) is satisfied. By Theorem 4.2, we have $\mathcal{E}[\rho, \Delta] = \mathcal{E}^*[s_0, \rho, \Delta]$ if $\mathrm{Re}(s_0) \geqslant \frac{1}{2}$; moreover, the pole occurs only when $s_0 = 1$ or $= \frac{3}{4}$, and $\rho = \bar{\rho}$. Theorem 7.9 is applicable to such cases.

Remark 7.11. If $\rho = 0$ and $s_0 = 1$, we see that $\mathcal{E}^*(1, 0, \Delta)$ consists of the constants, as shown in Proposition 4.3. Therefore we obtain

$$\dim \mathcal{E}(s_0, \rho, \Delta) = \varkappa - 1 \quad \text{if} \quad \rho = 0 \quad \text{and} \quad s_0 = 1. \tag{7.11}$$

Combining Theorems 7.3, 7.8, 7.9 and Remarks 7.4, 7.10, we obtain

Theorem 7.12. *If* λ *is simple,* $\mathfrak{N}(\sigma, \lambda, \Delta)$ *has dimension* $\varkappa$.

In this section, we treated $\mathfrak{N}(\sigma, \lambda, \Delta)$ only for simple λ. If λ is multiple, $\mathfrak{N}(\sigma, \lambda, \Delta)$ is probably generated by Eisenstein series with several different (s_0, ρ), as Remark 5.5 suggests. The proof of this fact does not seem very difficult, though the author has no complete result.

8. Applications to holomorphic forms

Let $\mathcal{H}(\sigma, \Delta)$ denote the set of all holomorphic functions on $H^{\mathbf{a}}$ satisfying (2.7a, c), and $\mathcal{H}(\sigma)$ the union of $\mathcal{H}(\sigma, \Delta)$ for all congruence sugroups Δ of $\mathcal{G}_\sigma$. (It is well known that (2.7c) follows from (2.7a) and the holomorphy if $F \neq \mathbf{Q}$.) If $f \in \mathcal{H}(\sigma)$, it has an expansion

$$f(z) = b_0 + \sum_{0 \ll h \in \mathfrak{m}} b_h \mathbf{e}_{\mathbf{a}}(hz) \tag{8.1}$$

with a lattice $\mathfrak{m}$ in F and complex coefficients b_0 and b_h. Given a subfield K of $\mathbf{C}$, we denote by $\mathcal{H}(\sigma, K)$ and $\mathcal{H}(\sigma, \Delta, K)$ the subsets of $\mathcal{H}(\sigma)$ and $\mathcal{H}(\sigma, \Delta)$ consisting of all f such that the coefficients b_0 and b_h belong to K. We shall be especially interested in the case where K is the maximal abelian extension of $\mathbf{Q}$ which we denote by $\mathbf{Q}_{ab}$.

Proposition 8.1.
(i) $\mathcal{S}(\sigma, 0, \Delta) \subset \mathcal{H}(\sigma, \Delta) \subset \mathcal{A}(\sigma, 0, \Delta)$;
(ii) $\mathcal{S}(\sigma, 0, \Delta) = \mathcal{H}(\sigma, \Delta)$ *if* $\sigma \notin \mathbf{Q}u$.

PROOF. Assertion (i) is a restatement of Proposition 2.5. If $f \in \mathcal{H}(\sigma, \Delta)$, $\xi \in \mathcal{G}$, and c_0 is the constant term of $f \| \xi$, then (2.13b) shows that $c_0 = a^\sigma c_0$ for every a in a subgroup of $\mathfrak{g}^\times$ of finite index. Therefore $c_0 = 0$ if $\sigma \notin \mathbf{Q}u$, which proves (ii).

In order to study the holomorphic elements of $\mathcal{N}(\sigma, 0, \Delta)$, put

$$\mathcal{N}\mathcal{H}(\sigma, \Delta) = \mathcal{H}(\sigma, \Delta) \cap \mathcal{N}(\sigma, 0, \Delta). \tag{8.2}$$

From (2.23a) and the above (i), we obtain

$$\mathcal{H}(\sigma, \Delta) = \mathcal{S}(\sigma, 0, \Delta) \oplus \mathcal{N}\mathcal{H}(\sigma, \Delta). \tag{8.3}$$

The main purpose of this section is to show that $\mathcal{N}\mathcal{H}(\sigma, \Delta)$ can be obtained from Eisenstein series. By (ii) of the above proposition, the problem concerns only the case $\sigma \in \mathbf{Q}u$.

Proposition 8.2 *Let $E(z, s)$ denote any series of type* (4.1), (4.4), *or* (4.10) *with* $2\rho = \sigma = tu$, $0 < t \in \left(\frac{1}{2}\right)\mathbf{Z}$. *Suppose $k = 2t$ in Case II. Then the following assertions hold*:

(i) *E is finite at $s = t/2$.*
(ii) *If $t > 2$ or $t = 1$, $E(z, t/2)$ belongs to $\mathcal{H}(tu, \mathbf{Q}_{ab})$.*
(iii) *Suppose $t = 2$ or $t = 3/2$; suppose also $F \neq \mathbf{Q}$. Then $E(z, t/2)$ belongs to $\mathcal{H}(tu, \mathbf{Q}_{ab})$.*

(iv) *Suppose* $F = \mathbf{Q}$ *and* $t > 1$. *Then* $E_k(z, t/2; tu/2, \psi, \mathfrak{c})$ belongs to $\mathcal{H}(tu, \mathbf{Q}_{ab})$ *except in the following two cases*; (A) $t = 2$ *and* $\psi = 1$; (B) $t = 3/2$ *and* $\psi^2 = 1$.

(v) *Suppose* $t = 1/2$. *Then* $E(z, s)$ *has at most a simple pole at* $s = 3/4$ *and the residue is* $\pi^{-n}R_F$ *times an element of* $\mathcal{H}(tu, \mathbf{Q}_{ab})$, *where* R_F *is the regulator of* F.

PROOF. The assertions in Case II are included in [13, Theorem 2.3]. In Case I, the results are essentially due to Hecke [1] when $F = \mathbf{Q}$, and to Kloosterman [2] and Klingen in the case $[F: \mathbf{Q}] > 1$, though our formulation is different from theirs. In the present formulation, the assertions in Case I are included in [12, Theorem 7.1] as special cases.

Theorem 8.3. *Let* $2\rho = \sigma = tu$ *with* $0 < t \in \left(\frac{1}{2}\right)\mathbf{Z}$. *If* $t > \frac{1}{2}$, *one has*

$$\mathfrak{N}\mathcal{H}(\sigma, \Delta) = \mathcal{E}\left(\tfrac{t}{2}, \rho, \Delta\right) \cap \mathcal{H}(\sigma, \Delta). \tag{8.4}$$

Moreover

$$\mathfrak{N}\mathcal{H}(\sigma, \Delta) = \mathcal{E}\left(\tfrac{t}{2}, \rho, \Delta\right) \tag{8.5}$$

except in the following three cases: (i) $t = \frac{1}{2}$; (ii) $t = \frac{3}{2}$ *and* $F = \mathbf{Q}$; (iii) $t = 2$ *and* $F = \mathbf{Q}$.

PROOF. The last assertion follows from (8.4) and Proposition 8.2. Now Proposition 3.2 shows that λ is simple if $\lambda = 0$. Moreover, λ is critical if and only if $t = 1$. Therefore, putting $s_0 = \frac{t}{2}$ with $t > 1$ in Theorem 7.3, we obtain

$$\mathfrak{N}(\sigma, 0, \Delta) = \mathcal{E}\left(\tfrac{t}{2}, \rho, \Delta\right) \quad \text{if} \quad t > 1, \tag{8.6}$$

which proves (8.4). If $t = 1$, the last part of Theorem 7.8 proves (8.4).

As for the case $t = \frac{1}{2}$, we have

Theorem 8.4. $\mathfrak{N}\mathcal{H}\left(\frac{u}{2}, \Delta\right) = \mathcal{E}^*\left(\frac{3}{4}, \frac{u}{4}, \Delta\right)$.

PROOF. By Proposition 8.2, (v), $\mathcal{E}^*\left(\frac{3}{4}, \frac{u}{4}, \Delta\right) \subset \mathcal{H}\left(\frac{u}{2}, \Delta\right)$. In view of Theorem 7.9, it is sufficient to prove that 0 is the only holomorphic element of $\mathcal{E}\left(\frac{3}{4}, \frac{u}{4}, \Delta\right)$. To see this, let $r \in \mathcal{E}\left[\frac{3}{4}, \frac{u}{4}, \Delta\right]$ and express r as in (7.10). Then we see that

$$r\left(z, \tfrac{3}{4}\right) \,\|\, \eta^{-1} = a_\eta y^{u/2} + c_\eta + \dots$$

with $c_\eta \in \mathbf{C}$ for every $\eta \in Y$. If $r\left(z, \frac{3}{4}\right)$ is holomorphic, we have $a_\eta = 0$ for every η, so that $r = 0$, which proves the desired fact.

Remark 8.5. The result of [13, Proposition 6.4] tog·ther with (4.3), (4.12), and (4.24) shows that the elements of $\mathcal{E}^*\left(\frac{3}{4}, \frac{u}{4}, \Delta\right)$ are theta series.

As to the previous investigations on $\mathfrak{N}\mathfrak{H}(\sigma, \Delta)$, the reader is referred to the papers mentioned in the introduction.

9. Cyclopean forms

We call an element f of $\mathfrak{A}(\sigma, \lambda)$ a *cyclopean form* (or simply a *cyclops*) *of exponent q*, if the following conditions (9.1*a*, *b*, *c*) are satisfied:

(9.1*a*) $f \in \mathfrak{N}(\sigma, \lambda)$;

(9.1*b*) *for every* $\xi \in \mathcal{G}$, *the constant term of* $f\|\xi$ *is of the form* $c_\xi y^q$ *with* $c_\xi \in \mathbf{C}$; *that is, it has no term of the form* by^p *with p other than q*;

(9.1*c*) $$(1-\sigma_v)/2 < \mathrm{Re}(q_v) < \begin{cases}(2-\sigma_v)/2 & \\ (3-2\sigma_v)/4 & \end{cases} \textit{for every}\quad v \in \mathbf{a} \qquad \begin{matrix}(\textit{Case I}),\\ (\textit{Case II}).\end{matrix}$$

By Proposition 3.2, (9.1*c*) implies that λ is noncritical and simple. By (3.6*b*), we can put $q = (1-s_0)u - \bar{\rho}$ and $\rho = (\sigma - i\tau)/2$ with $s_0 \in \mathbf{C}$ and an admissible τ. Then (9.1*c*) is equivalent to

(9.2) $$\tfrac{1}{2} > \mathrm{Re}(s_0) > \begin{cases}0 & \text{(Case I)},\\ \tfrac{1}{4} & \text{(Case II)}.\end{cases}$$

We also note that

(9.3) $$\lambda_v = q_v(1-\sigma_v-q_v) = (s_0-\rho_v)(1-s_0-\bar{\rho}_v).$$

Put $p = s_0 u - \rho$. If $f \in \mathfrak{A}(\sigma, \lambda)$, we have, for $\xi \in \mathcal{G}$,

$$f\|\xi = b_\xi y^p + c_\xi y^q + \dots .$$

Thus (9.1*b*) means that $b_\xi = 0$ for *every* $\xi \in \mathcal{G}$.

Theorem 9.1. *Let* $\rho = (\sigma - i\tau)/2$ *and* $q = (1-s_0)u - \bar{\rho}$ *with* $s_0 \in \mathbf{C}$ *and an admissible* τ. *In Case II, let k be an arbitrarily fixed odd integer. If there exists a nonzero cyclopean form of* $\mathfrak{A}(\sigma, \lambda)$ *of exponent q, then there exists a Hecke character* ψ *of F such that*

(9.4*a*) $L(2s_0, \psi) = 0$ (*Case I*),

(9.4*b*) $L(4s_0 - 1, \psi^2) = 0$ (*Case II*),

(9.5) $\psi(x) = |x|^{\pm i\tau}(x/|x|)^{\sigma'}$ *for* $x \in F_{\mathbf{a}}^\times$, *where* $\sigma' = \sigma$ *in Case I and* $\sigma' = \sigma - ku/2$ *in Case II.*

Conversely, suppose there exist a Hecke character ψ *of* F *and a complex number* s_0 *satisfing* (9.2), (9.4*a* or *b*), *and* (9.5). *Then there exists a nonzero cyclops of* $\mathfrak{A}(\sigma, \lambda)$ *of exponent* q. *More explicitly,*

$$[L(2s, \psi)E(z, s; \rho, \psi, \mathfrak{c})]_{s=s_0} \qquad (\textit{Case I}),$$

$$[L(4s-1, \psi^2)E_k(z, s; \rho, \psi, \mathfrak{c})]_{s=s_0} \qquad (\textit{Case II})$$

are cyclopes, for every multiple $\mathfrak{c}$ *of the conductor of* ψ *that is divisible by* 4 *in Case II.*

PROOF. We prove this only in Case II; Case I can be treated in a similar way. Suppose $L(4s_0 - 1, \psi^2) \neq 0$ for every ψ of type (9.5). Then $\bar{s}_0$ has the same property. Let f be a cyclops of exponent q belonging to $\mathfrak{M}(\sigma, \lambda, \Delta)$. Theorem 4.1 together with (4.3) and (4.24) shows that $\mathcal{E}[\rho, \Delta] = \mathcal{E}[s_0, \rho, \Delta]$ and $\mathcal{E}[\bar{\rho}, \Delta] = \mathcal{E}[\bar{s}_0, \bar{\rho}, \Delta]$. By Theorem 7.3, we have $f(z) = h(z, s_0)$, $h = \sum_{\xi \in Y} a_\xi E_k(z, s; \rho, \xi, \Delta)$ with $a_\xi \in \mathbf{C}$. Putting $p = s_0 u - \rho$ and employing the notation of Proposition 5.2, we have

$$f \| \eta^{-1} = a_\eta y^p + \Big(\sum_{\xi \in Y} a_\xi f_{\xi\eta}\Big)(s_0) y^q + \ldots$$

for $\eta \in Y$. Hence $a_\eta = 0$ for all $\eta \in Y$, so that $f = 0$, a contradiction.

Conversely, suppose $L(4s_0 - 1, \psi^2) = 0$ for s_0 and ψ satisfying (9.2) and (9.5). Take any common multiple $\mathfrak{c}$ of 4 and the conductor of ψ, and put

$$g(z, s) = L_\mathfrak{c}(4s - 1, \psi^2)E_k(z, s; \rho, \psi, \mathfrak{c}).$$

By Theorem 4.1, g is finite at s_0. Hence $g(z, s_0)$ belongs to $\mathfrak{M}(\sigma, \lambda)$ by Propositions 7.1 and 5.3. Now, for every $\zeta \in \mathfrak{G}$, we have, by Proposition 5.4.

$$g(z, s) \| \zeta = ac^s L_\mathfrak{c}(4s - 1, \psi^2) y^{su - \rho} + \ldots$$

with $a \in \mathbf{C}$ and $0 < c \in \mathbf{R}$. Therefore $g(z, s_0)$ satisfies (9.1*b*). To show that $g(z, s_0) \neq 0$, we consider an element η_0 of G as in [13, (4.10)]. Then the Fourier coefficients of $g \| \Lambda_\sigma^k(\eta_0)$ has been determined in [13, §6]. In particular, its constant term at s_0 is a nonzero constant times $L_\mathfrak{c}(4s_0 - 2, \psi^2)y^q$. Since $-1 < 4s_0 - 2 < 0$, this term is nonvanishing. This completes the proof, since $L_\mathfrak{c}/L$ is nonvanishing for this value.

Proposition 9.2. *Let* s_0 *be a complex number satisfying* (9.2). *Define* Φ_Δ *as in Theorem* 7.7 *for each* Δ *such that a cusp-class of* Δ *is* ρ*-regular if and only if it is* $\bar{\rho}$*-regular. Then a Hecke character* ψ *of* F *satisfying* (9.4*a* or *b*) *and* (9.5) *exists if and only if* $\det \Phi_\Delta(s, \rho)$ *has a pole at* s_0 *for some* Δ. *Moreover, the maximum number of linearly independent cyclopes in* $\mathfrak{M}(\sigma, \lambda, \Delta)$ *with* λ *of* (9.3) *is* $\varkappa - \operatorname{rank} \Phi_\Delta(1 - s_0, \bar{\rho})$.

Proof. By Theorem 7.3 and Remark 7.4, (1), we have $\mathfrak{N}(\sigma, \lambda, \Delta) = \mathcal{E}(1 - s_0, \bar{\rho}, \Delta)$. Given (a row vector) $c \in \mathbf{C}^Y$, we have

$$\sum_{\xi} c_\xi E(1 - s_0, \bar{\rho}, \xi) \Big\| \eta^{-1} = c_\eta y^q + \sum_{\xi} c_\xi g_{\xi\eta}(1 - s_0) y^p + \dots$$

with the same notation as in (7.8*b*). This gives a nontrivial cyclops of exponent q if and only if $c \neq 0$ and $c\Phi_\Delta(1 - s_0, \bar{\rho}) = 0$, which proves the last assertion. The first assertion follows from this fact, Theorem 9.1, (7.7*b*), and Proposition 3.4.

10. Appendix I: Whittaker functions

For $y > 0$ and $(\alpha, \beta) \in \mathbf{C}^2$, we put

$$\tau(y, \alpha, \beta) = \int_0^\infty e^{-yt}(1 + t)^{\alpha - 1} t^{\beta - 1}\, dt. \tag{10.1}$$

This is convergent if $\mathrm{Re}(\beta) > 0$. We have obviously

$$\left(\frac{\partial}{\partial y}\right)\tau(y, \alpha, \beta) = -\tau(y, \alpha, \beta + 1). \tag{10.2}$$

Since $(1 + t)^\alpha = (1 + t)^{\alpha - 1}(1 + t)$, we obtain

$$\tau(y, \alpha + 1, \beta) = \tau(y, \alpha, \beta) + \tau(y, \alpha, \beta + 1). \tag{10.3}$$

Integration by parts shows

$$\beta\tau(y, \alpha + 1, \beta) = y\tau(y, \alpha + 1, \beta + 1) - \alpha\tau(y, \alpha, \beta + 1). \tag{10.4}$$

From these formulas, we obtain easily

$$\left\{y\left(\frac{\partial}{\partial y}\right)^2 + (\alpha + \beta - y)\cdot\frac{\partial}{\partial y} - \beta\right\}\tau(y, \alpha, \beta) = 0. \tag{10.5}$$

Let us now put

$$V(y, \alpha, \beta) = e^{-y/2} y^\beta \Gamma(\beta)^{-1} \tau(y, \alpha, \beta). \tag{10.6}$$

From (10.3), we obtain

$$V(y, \alpha + 1, \beta) = V(y, \alpha + 1, \beta + 1) - \alpha y^{-1} V(y, \alpha, \beta + 1). \tag{10.7}$$

This shows that V can be continued as a holomorphic function in (α, β) to the whole $\mathbf{C}^2$. Now we have

$$y^\beta \tau(y, \alpha, \beta) = \int_0^\infty e^{-t}(1 + y^{-1}t)^{\alpha - 1} t^{\beta - 1}\, dt.$$

Therefore we see, at least for $\mathrm{Re}(\beta) > 0$, that

$$\lim_{y\to\infty} e^{y/2} V(y, \alpha, \beta) = 1. \tag{10.8}$$

Since this is consistent with (10.7), we can easily verify that (10.8) holds uniformly for (α, β) in any compact subset of $\mathbf{C}^2$.

We now consider a differential equation

$$y^2 f''(y) + \sigma y f'(y) + (\lambda + A\sigma y - A^2 y^2) f(y) = 0 \tag{10.9}$$

with $A \in \mathbf{R}^\times$, $(\sigma, \lambda) \in \mathbf{C}^2$, and $0 < y \in \mathbf{R}$.

Proposition 10.1. *Let α and β be complex numbers such that $\alpha - \beta = \sigma$ and $\beta(1 - \alpha) = \lambda$. For fixed α, β, and A, define a function f_A by*

$$f_A(y) = \begin{cases} V(2Ay, \alpha, \beta) & \text{if } A > 0, \\ |2Ay|^{-\sigma} V(-2Ay, \beta, \alpha) & \text{if } A < 0. \end{cases}$$

Then f_A is a solution of (10.9). *Moreover, if f is a solution of* (10.9) *and $f(y) = O(y^B)$ with $B \in \mathbf{R}$ when $y \to \infty$, then f is a constant multiple of f_A.*

PROOF. That f_A is a solution of (10.9) follows from (10.5) in a straightforward way. Let f be a solution of (10.9) such that $f(y) = O(y^B)$. Then

$$(y^\sigma f')' = y^\sigma(f'' + \sigma y^{-1} f') = y^\sigma(A^2 - A\sigma y^{-1} - \lambda y^{-2}) f = O(y^C)$$

with $C \in \mathbf{R}$ when $y \to \infty$. It follows that $y^\sigma f'$, as well as f', is $O(y^D)$ with $D \in \mathbf{R}$. Now put $h = f_A f' - f_A' f$. Then $h' = f_A f'' - f_A'' f = -\sigma y^{-1} h$, and hence $h = a y^{-\sigma}$ with a constant a. Since both f_A and f_A' are $O(e^{-|A|y/2})$ as can easily be seen from (10.8) and (10.2), we see that $a = 0$. Therefore f is a constant multiple of f_A.

In Proposition 10.1, we can change (α, β) for $(1 - \beta, 1 - \alpha)$ without changing σ and λ. Therefore $V(2Ay, 1 - \beta, 1 - \alpha)$ for $A > 0$ is a solution of (10.9), and hence must be a constant multiple of f_A. In view of (10.8), we thus obtain

$$V(y, 1 - \beta, 1 - \alpha) = V(y, \alpha, \beta). \tag{10.10}$$

We note also that, given a compact subset K of $\mathbf{C}^2$, there exist two positive constants B and C depending only on K such that

$$|V(y, \alpha, \beta)| \leqslant C e^{-y/2}(1 + y^{-B}) \qquad \text{for } y > 0 \text{ and } \qquad (\alpha, \beta) \in K. \tag{10.11}$$

This can be proved in an elementary way by means of (10.1) and (10.8); for details, see [11, pp. 282-283].

With σ, λ, A, and f_A as in Proposition 10.1, define a function φ_A on H by

$$(10.12) \qquad \varphi_A(x+iy,\sigma,\lambda) = e^{iAx} f_A(y).$$

Further define operators ϵ and δ^σ on H by $\epsilon f = -y^2 \partial f/\partial\bar{z}$ and $\delta^\sigma f = y^{-\sigma}\partial(y^\sigma f)/\partial z$. Then we can easily verify, employing (10.2), (10.3), and (10.4), that

$$(10.13a) \qquad \epsilon\varphi_A(z,\sigma,\lambda) = \begin{cases} \lambda(4Ai)^{-1}\varphi_A(z,\sigma-2,\lambda+2-\sigma) & \text{if } A>0, \\ (4Ai)^{-1}\varphi_A(z,\sigma-2,\lambda+2-\sigma) & \text{if } A<0, \end{cases}$$

$$(10.13b) \qquad \delta^\sigma\varphi_A(z,\sigma,\lambda) = \begin{cases} iA\varphi_A(z,\sigma+2,\lambda+\sigma) & \text{if } A>0, \\ (\lambda+\sigma)iA\varphi_A(z,\sigma+2,\lambda+\sigma) & \text{if } A<0. \end{cases}$$

11. Appendix II: Proofs of Propositions 2.1, 2.2, 2.3, and 4.4

Throughout this section, we put $U_F = \{a \in \mathfrak{g}^\times \mid a \gg 0\}$, $\mu(y) = \mathrm{Min}\{y_v \mid v \in \mathbf{a}\}$ for $y \in \mathbf{R}^\mathbf{a}$, $|z| = (|z_v|)_{v\in\mathbf{a}}$ and $\{z\} = \sum_{v\in\mathbf{a}} z_v$ for $z \in \mathbf{C}^\mathbf{a}$. For example, we have $\mathbf{e}_\mathbf{a}(i|h|y) = \exp(-2\pi\{|h|y\})$ for $h \in F$ and $0 \ll y \in \mathbf{R}^\mathbf{a}$.

Lemma 11.1 *Let $\mathfrak{a}$ be a fractional ideal of F, and β an element of $\mathbf{R}^\mathbf{a}$. Then there exist positive constants A, B, and C such that*

$$\sum_{0\neq h\in\mathfrak{a}} |h|^\beta \mathbf{e}_\mathbf{a}(i|h|y) \leqslant A(1+\mu(y)^{-B})\exp(-Cy^{u/n}).$$

for $0 \ll y \in \mathbf{R}^\mathbf{a}$.

Proof. Let $\|h\| = \{h^2\}^{1/2}$. If $c \geqslant 0$, then $|h_v|^c \leqslant \|h\|^c$, and

$$|h_v|^{-c} = \left|h^{-u}\prod_{w\neq v} h_w\right|^c \leqslant N(\mathfrak{a})^{-c}\|h\|^{(n-1)c} \quad \text{for} \quad 0 \neq h \in \mathfrak{a}.$$

Therefore $|h|^\beta \leqslant A\|h\|^b$ for $0 \neq h \in \mathfrak{a}$ with positive constants A and b. Now $\{|h|y\} \geqslant n|hy|^{u/n} \geqslant nN(\mathfrak{a})^{1/n}y^{u/n}$ for such h. Put $C = \pi nN(\mathfrak{a})^{1/n}$. Then $2\pi\{|h|y\} \geqslant \pi\{|h|y\} + Cy^{u/n} \geqslant \pi\mu(y)\|h\| + Cy^{u/n}$. Therefore we have $\sum_h |h|^\beta \exp(-2\pi\{|h|y\}) \leqslant A\cdot\exp(-Cy^{u/n})\sum_h \|h\|^b\exp(-\pi\mu(y)\|h\|)$. Since there are only finitely many h's in $\mathfrak{a}$ such that $\|h\| < 1$, we may assume, changing A for a larger constant, that b is a positive integer. For $0 < m \in \mathbf{Z}$, let p_m be the number of elements h of $\mathfrak{a}$ such that $m-1 < \|h\| \leqslant m$. Then $p_m \leqslant Dm^{n-1}$ with a constant D, and the last sum $\sum_h$ is majorized by $D\sum_{m=1}^\infty m^{b+n-1}e^{t-mt}$ with $t = \pi\mu(y)$. This is $\leqslant E(1+t^{-b-n})$ with a constant E, which completes the proof.

Lemma 11.2. *Let Δ be a congruence subgroup of $\mathcal{G}$, and f a continuous function on $H^{\mathbf{a}}$ satisfying* (2.7*a*, *c*). *Then there exist two positive constants A and B such that*

$$(11.1)\qquad |y^{\sigma/2}f(x+iy)| \leqslant A(y^{Bu}+y^{-Bu}) \quad \textit{for all} \quad x+iy \in H^{\mathbf{a}}.$$

PROOF. With a compact fundamental domain M of $\mathbf{R}^{\mathbf{a}}/\mathfrak{g}$ and $0 < c \in \mathbf{R}$, put

$$(11.2)\qquad T_c = \{x+iy \mid x \in M, \mu(y) > c\}.$$

Then we can take a finite subset X of $\mathcal{G}$ so that $H^{\mathbf{a}} = \bigcup_{\beta \in \Delta, \xi \in X} \beta\xi(T_c)$. By (2.7*c*), we can find two positive constants A and B such that $|y^{\sigma/2}(f \| \xi)(x+iy)| \leqslant Ay^{Bu}$ if $\mu(y) > c$ and $\xi \in X$. Given $z = x+iy \in H^{\mathbf{a}}$, take $\beta \in \Delta$ and $\xi \in X$ so that $z = \beta\xi(z')$ with $z' = x' + iy' \in T_c$. Let $\mathrm{pr}(\xi^{-1}) = \begin{pmatrix} * & * \\ p & q \end{pmatrix}$ and $\mathrm{pr}(\beta\xi)^{-1} = \begin{pmatrix} * & * \\ r & s \end{pmatrix}$. To prove our lemma, we may assume that $\mathrm{pr}(\Delta) \subset SL_2(\mathfrak{g})$. Then $r, s \in p\mathfrak{g} + q\mathfrak{g}$. Let D be the smallest of $N(p\mathfrak{g}+q\mathfrak{g})$ for all $\xi \in X$. If $r \neq 0$, we have $|r^u| \geqslant D$. Now $y'^u = y^u|rz+s|^{-2u} \leqslant D^{-2}y^{-u}$, and hence $|y^{\sigma/2}f(z)| = |y'^{\sigma/2}(f \| \beta\xi)(z')| \leqslant Ay'^{Bu} \leqslant A(D^2y^u)^{-B}$ if $r \neq 0$. When $r = 0$, we have $y'^u = s^{-2u}y^u \leqslant D^{-2}y^u$, so that $|y^{\sigma/2}f(z)| \leqslant A(D^{-2}y^u)^B$. This proves our lemma.

PROOF OF PROPOSITION 2.1. Given $f \in \mathfrak{A}(\sigma, \lambda, \Delta)$, define b_h as in (2.21). By Lemma 11.2, we see easily that

$$(11.2')\qquad |y^{\sigma/2}b_h W(hy; \sigma, \lambda)| \leqslant A'(y^{Bu}+y^{-Bu})$$

with positive constants A' and B independent of y and h. let U be a subgroup of U_F of finite index such that $\Lambda_\sigma^k(\mathrm{diag}[a, a^{-1}]) \subset \Delta$ for every $a \in U$. Now we can find two positive constants c_1 and c_2 with the following property: given $0 \ll y \in \mathbf{R}^{\mathbf{a}}$, there exists an element a of U such that $c_1 y^{u/n} \leqslant a_v^2 y_v \leqslant c_2 y^{u/n}$ for every $v \in \mathbf{a}$. Hereafter c_m for $m = 3, 4, \ldots$ will denote constants independent of h and y. Given $0 \neq h \in F$, take $a \in U$ so that $c_1|h|^{u/n} \leqslant |a_v h_v| \leqslant c_2|h|^{u/n}$. By (10.8), we can find a constant $d > 1$ so that $|V(g; \alpha_v, \beta_v)| \geqslant 2^{-1}e^{-g/2}$ and $|V(g; \beta_v, \alpha_v)| \geqslant 2^{-1}e^{-g/2}$ if $g \geqslant d$. Put $t = c_1^{-1}d|h|^{-u/n}$. Then $ta_v|h_v| \geqslant d$, so that

$$(11.3)\qquad |(tah)^{\sigma/2}W(tah; \sigma, \lambda)| \geqslant c_3|tah|^{\sigma'/2}e^{-2\pi t\{|ah|\}},$$

where $\sigma'_v = \mathrm{sgn}(h_v)\sigma_v$. Taking ta to be y in (11.2′), we find that

$$|(ta)^{\sigma/2}b_h W(tah; \sigma, \lambda)| \leqslant A'(t^{nB}+t^{-nB}),$$

which together with (11.3) shows that

$$|h^{-\sigma/2}b_h| \leqslant c_4(t^{nB}+t^{-nB})|tah|^{-\sigma'/2}e^{2\pi t\{|ah|\}}.$$

Since $t = c_1^{-1}d|h|^{-u/n}$ and $|a_v h_v| \leqslant c_2|h|^{u/n}$, we have $|(tah)_v| \leqslant dc_2/c_1$, and hence $|h^{-\sigma/2}b_h| \leqslant c_5|h|^{Bu}$. This proves (1) of Proposition 2.1. Next, we see from (10.11) that

$$(11.4) \qquad |W(hy; \sigma, \lambda)| \leqslant c_6 \sum_{s \in S} |hy|^s \mathbf{e_a}(ihy)$$

with a finite subset S of $\mathbf{R^a}$. Hence, by Lemma 11.1, we obtain

$$(11.5) \qquad y^{\sigma/2} \sum_{h \neq 0} |b_h W(hy; \sigma, \lambda)| \leqslant c_7 \sum_{s \in S} y^s(1 + \mu(y)^{-B}) \exp(-Cy^{u/n})$$

with constants B and C independen of s. Now the left-hand side is invariant under $y \to a^2y$ with $a \in U$. Given y, take $a \in U$ so that $c_1y^{u/n} \leqslant (a^2y)_v \leqslant c_2y^{u/n}$ for every $v \in \mathbf{a}$. Then $\mu(a^2y) \geqslant c_1y^{u/n}$ and $(a^2y)^s \leqslant c_8(y^{Du} + y^{-Du})$ with D independent of s. Hence, substituting a^2y for y in (11.5), we obtain

$$(11.6) \qquad y^{\sigma/2} \sum_{h \neq 0} |b_h W(hy; \sigma, \lambda)| \leqslant c_9(y^{Eu} + y^{-Eu}) \exp(-Cy^{u/n})$$

with a constant E, which proves (2) of Proposition 2.1. Assertion (3) is now an easy consequence of (2) and (2.7c). To prove (4), take $f \in \mathcal{S}(\sigma, \lambda, \Delta)$ and take X as in the proof of Lemma 11.2. Applying (2) to $f \| \xi$ for each $\xi \in X$, we see that $y^{\sigma/2}f$ is bounded on the whole $H^{\mathbf{a}}$. Therefore we can take $B = 0$ in (11.1) and also in (11.2′). Repeating the proof of (1) with $B = 0$, we can conclude that $h^{-\sigma/2}b_h$ is bounded. This completes the proof.

Lemma 11.3. *Let f be a C^∞-function of form* (2.21) *satisfying* (2.7a, b). *Suppose $|b_h| \leqslant p|h|^{qu+\sigma/2}$ for $0 \neq h \in \mathfrak{m}$ with positive constants p and q. Then f satisfies* (2.7c).

PROOF. Applying the above proof of (2) to $f - b_0(y)$, we obtain, from (11.6) that

$$y^{\sigma/2}|f - b_0(y)| \leqslant A(y^{Eu} + y^{-Eu}) \exp(-Cy^{u/n}).$$

Since $b_0(y)$ is a linear combination of the functions of Proposition 3.1, we have $y^{\sigma/2}|f(z)| \leqslant A'(y^{Ju} + y^{-Ju})$ on $H^{\mathbf{a}}$ with constants A' and J. Then (2.7c) can easily be verified.

PROOF OF PROPOSITION 2.2. Let $g \in \mathcal{A}(\sigma, \lambda, \Delta)$. It is straightforward to see that $\epsilon_v g$ and $\delta_v^\sigma g$ satisfy (2.7a, b) with modified σ, λ, and Δ as stated in the proposition. To verify (2.7c), take $\xi \in \mathcal{G}_\sigma$. Then $(\epsilon_v g) \| \xi_* = \epsilon_v(g \| \xi)$ by (2.5b). Since $g \| \xi \in \mathcal{A}(\sigma, \lambda)$, it has an expansion of type (2.21) with b_h as in (1) of Proposition 2.1. By (10.13a), we see that

$$(\epsilon_v g) \| \xi_* = \epsilon_v b_0 + \sum_{0 \neq h} c_h W(hy; \sigma - 2v, \lambda + (2 - \sigma_v)v)$$

with c_h satisfying (1) of Proposition 2.1 with $\sigma - 2v$ instead of σ. Therefore, by Lemma 11.3, $\epsilon_v g$ satisfies (2.7c). The assertion for $\delta^{\sigma}{}_v g$ can be proved in a similar way.

PROOF OF PROPOSITION 2.3. We first note that given two positive integers a and p, and a positive real number $r < 1$, one has

$$\sum_{m=p}^{\infty} m^a x^m \leqq C(a, r) p^a x^p \quad \text{for} \quad 0 \leqslant x \leqslant r \tag{11.7}$$

with a constant $C(a, r)$ independent of p and x. In fact,

$$\sum_{m=p}^{\infty} m^a x^{m-p} = \sum_{n=0}^{\infty} (n+p)^a x^n \leqslant \sum_{i=0}^{\infty} \binom{a}{i} p^{a-i} \sum_{n=0}^{\infty} n^i r^n.$$

Now take X as in the proof of Lemma 11.2. For $f \in \mathcal{S}(\sigma, \lambda, \Delta)$ and $\xi \in X$, put $M_f = \mathrm{Max}|y^{\sigma/2} f|$ and

$$f \| \xi = \sum_h b_{h,\xi} W(hy; \sigma, \lambda) \mathbf{e}_{\mathbf{a}}(hx).$$

Since $|y^{\sigma/2}(f\|\xi)| \leqslant M_f$, we have $|b_{h,\xi}| \leqslant AM_f|h|^{\sigma/2}$ with a constant A independent of f, as can be seen from the proof of (1) and (4) of Proposition 2.1. Fix an integer $p > 1$ and suppose $b_{h,\xi} = 0$ for all $\xi \in X$ and all h such that $\|h\| < p$. Then, by (11.4), we have

$$|y^{\sigma/2} f \| \xi| < BM_f \sum_{s \in S} |hy|^{\sigma/2 + s} \mathbf{e}_{\mathbf{a}}(i|h|y)$$

with a constant B independent of f. The same reasoning as in the proof of Lemma 11.1 shows that, for any fixed $q > 0$, we have

$$|y^{\sigma/2} f \| \xi| \leqslant CM_f \sum_{m=p}^{\infty} m^a e^{-\pi m \mu(y)} \quad \text{if} \quad \mu(y) > q$$

with a constant C and a positive integer a independent of f. By (11.7), we have $|y^{\sigma/2} f\|\xi| \leqslant D_q M_f p^a e^{-\pi p \mu(y)}$ for $\mu(y) > q$ with a constant D_q independent of f and p. Take q to be c of (11.2). For every $z \in H^{\mathbf{a}}$, take $\beta \in \Delta$ and $\xi \in X$ so that $z = \beta\xi(z')$ with $z' = x' + iy' \in T_c$. Then

$$|y^{\sigma/2} f(z)| = |y^{\sigma/2} f(\beta\xi(z'))| = |y'^{\sigma/2}(f\|\beta\xi)(z')|,$$

and hence $M_f \leqslant D_q M_f p^a e^{-\pi p q}$. If p is sufficiently large, we obtain $M_f = 0$. This shows that $f = 0$ if $b_{h,\xi} = 0$ for $\|h\| < p$ and for all $\xi \in X$. Thus $\mathcal{S}(\sigma, \lambda, \Delta)$ is finite-dimensional. Now the constant term of an element of $\mathcal{A}(\sigma, \lambda)$ belongs to a 2^n-dimensional space as shown in Section 3, and hence $\mathcal{A}(\sigma, \lambda, \Delta)/\mathcal{S}(\sigma, \lambda, \Delta)$ is finite-dimensional. This completes the proof.

PROOF OF PROPOSITION 4.4. By (4.3) and (4.24), we may restrict E to the functions of type (4.10). Then our first assertion follows immediately from Theorem 4.1. Suppose g is finite at s_0. By (4.4) and analytic continuation, we see that $L_v^\sigma g(z, s_0) = \lambda_v g(z, s_0)$. In order to verify (2.7c) for $g(z, s_0)$, we consider a function D' which is obtained from D of Theorem 4.1 by replacing E by E', where E' is defined by (4.16) in Case I and by [13, (4.10)] in Case II. Then we take a Fourier expansion

$$l(s)D'(z,s) = a_0(s,y) + \sum_{h \neq 0} a_h(s) W(hy; \sigma, \lambda) \mathbf{e_a}(hx), \tag{11.8}$$

where $l(s)$ is the polynomial 1, $s(s-1)$, or $s - \frac{3}{4}$ that cancels the pole(s) of D'. Then for every compact subset K of $\mathbf{C}$, we have $|a_h(s)| \leqslant A|h|^{\sigma/2 + Bu}$ for $s \in K$ with positive constants A and B depending only on D' and K. This follows from the explicit form of $a_h(s)$ given by (4.20) in Case I and by [13, Theorem 6.1] in Case II. Then Lemma 11.3 shows that $l(s)D'(z,s)$ satisfies (2.7c). Put $q(z,s) = l(s)D'(z,s)\|\xi$ with any $\xi \in \mathcal{G}$. Then $q(z,s)$ belongs to $\mathfrak{A}(\sigma, \lambda)$ and satisfies (2.7c) uniformly on K. By a well-known principle, the same type of estimate holds for $\partial^m q/\partial s^m$ for every m. Now we consider a finite linear combination $\sum f_q(s) q(z,s)$ with meromorphic functions f_q on $\mathbf{C}$. We observe that if it is finite at s_0, it satisfies (2.7c). This completes the proof.

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