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Poincaré-Cartan Forms in Higher Order Variational Calculus on Fibred Manifolds

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Introduction

The aim of the present work is to present a geometric formulation of higher order variational problems on arbitrary fibred manifolds. The problems of Engineering and Mathematical Physics whose natural formulation requires the use of second order differential invariants are classic, but it has been the recent advances in the theory of «integrable» non-linear partial differential equations and the consideration in Geometry of invariants of increasingly higher orders that has highlighted the interest of being able to work with a general formalism for higher order variational problems (see for instance [5], [7], [8]).

As in the case of first order problems, the central point of the theory lies in the construction of the Poincaré-Cartan form associted to a Lagrangian density. The method followed here for such a construction has been to analyse the reiterated process of integration by parts classically employed in the local deduction of Euler-Lagrange equations. The conclusion reached is that if we wish to carry out this process for an arbitrary fibred manifold $p\colon Y \to X$, this depends essentially on the choice of a derivation law ∇ on the vertical bundle $V(Y)$ and on a linear connection ∇_0 of the manifold X. By means of the derivation laws ∇, ∇_0 it is possible to define an operator L, called the total Lie

derivative, which will allow us to globally perform this process and through which the diverse differential forms of the theory are expressed with intrinsic and explicit formulas. Furhtermore, the process itself, understood in this way, gives rise to the definition of the Poincaré-Cartan form Θ associated to an r-order Lagrangian density $\mathfrak{L}v$ (Proposition 5.1 and Theorem 5.2). In fact, from a methodological point of view, this work could be considered as an extension to higher order problems of the method used in [2] for first order problems. These are the main characteristics of our method for the construction of higher order Poincaré-Cartan forms. Other different procedures to obtain such forms have also been presented recently ([1], [9] and [12]).

In the classical cases (i.e. when either $r \leqslant 2$ and n is arbitrary, or $n = 1$ and r is arbitrary) the form Θ depends neither on ∇_0 nor on ∇ and its expression coincides with that obtained by different authors using different methods (see [2], [6], [7], [14], [16]). In the general case (i.e. when $r > 2$ and $n > 1$) Θ does not depend on ∇ but it does depend on ∇_0; that is, we have a family of Poincaré-Cartan forms $\Theta(\nabla_0, \mathfrak{L}v)$ which are parametrized by the linear connection ∇_0. Regarding this, it should be noted that the expression in local coordinates of the Poincaré-Cartan form for higher order variational problems which appears in certain works, even recent ones (and which in our construction corresponds to the form associated to the flat connection determined by a coordinate system) *is not covariant* in the general case; that is, when $r > 2$ and $n > 1$. Hence, the results obtained by using the aforementioned form are strictly local. In a more geometric sense, we could say that the only global results of higher order variational calculus are precisely those which remain convariant with respect to the linear connection ∇_0 which parametrizes the Poincaré-Cartan forms.

According to the procedure in [2] and by an adequate differential characterization of the notion of infinitesimal contact transformations, we formulate an r-order variational problem as a problem of invariance of the functional defined by an r-order Lagrangian density with respect to the Lie algebra of the infinitesimal contact transformations of the fibred manifold Y. This geometric presentation of variational problems (which is sufficient for the aims of this work) has the advantage of showing from the start the fundamental group of transformations which plays a part in the theory and which allows us to give a strictly differential treatment to it. In this context, the variational formula of Lagrangian density is expressed by an equation in the bundle $J^{2r-1}(Y)$ among the diverse differential forms of the theory. Naturally, when this equation is integrated along a holonomic section we obtain the expression in terms of the Euler-Lagrange operator well-known in the functional formulation of variational calculus (Proposition 9.1).

It is interesting to make the observation that, from this point of view, our construction of Poincaré-Cartan forms is invariant with respect to the group

of vertical automorphisms of the fibred manifold Y. Indeed, if ψ is a vertical automorphism of Y, we have $J^{2r-1}(\psi)^*\Theta(\nabla_0, \pounds v) = \Theta(\nabla_0, J^r(\psi)^*(\pounds v))$. In fact, this formula is a particular case of Theorem 10.1 in which the behaviour of $\Theta(\nabla_0, \pounds v)$ is analised with respect to an arbitrary automorphism of Y (not necessarily vertical). This theorem also has an important repercussion in the definition of the Poisson brackets on the space of Noether invariants (Proposition 10.5).

Both the variation formula of Lagrangian density and the principal results of §9 are based on an explicit formula for the exterior differential of the form Θ obtained in §7. This formula also contains good information about the geometric properties of the variational problem defined by $\pounds v$. For example, it allows us to decide when the form Θ (defined on $J^{2r-1}(Y)$) is projectable to $J^{2r-h}(Y)$ for $h = 2, \ldots, r$ (Corollary 7.7). This is an important question in that since Θ is defined in a jet bundle other than that of $\pounds v$, the notion of regularity for higher order problems in Field Theory has an aspect very different from its usual one. This problem will be dealt with in a later work.

A lot of the material contained in sections §1-§7 was part of the author's PH.D. thesis at the University of Salamanca and their main formulas were anticipated (without proof) in [3]. I should this like to reiterate my thanks to Professor P. L. García for his interest and encouragement. Nevertheless, other results are new (for example, the independence of the products $(\ ,\ )_{(k,l)}$ for $k + l = 2r + 1$ in Theorem 7.2 and Theorems 8.1 and 10.1). Such results complete the development of the theory.

1. Preliminaries and notations

Let $p\colon Y \to X$ be a fibred manifold (i.e., p is a surjective submersion). We shall use the notation $V(Y)$ for the sub-bundle of $T(Y)$ of vertical vectors over X. If $f\colon X' \to X$ is a differentiable mapping, we denote by $f^*(Y) \to X'$ the induced fibred manifold; we shall also write $Y_{X'} = f^*(Y)$ (specially when f is an open immersion). If $E \to X$ is a vector bundle, we denote by $\Gamma(E)$ the $C^\infty(X)$-module of differentiable sections of E over X; for any open set $U \subset X$ we write $\Gamma(U, E) = \Gamma(E_U)$. If $E_i \to X_i$ is a vector bundle and $f_i\colon X \to X_i$ is a differentiable mapping, with $i = 1, 2$, we shall denote by $E_1 \otimes_X E_2$, $\mathrm{Hom}_X(E_1, E_2)$ the vector bundles $f_1^*(E_1) \otimes f_2^*(E_2)$, $\mathrm{Hom}\,(f_1^*E_1, f_2^*E_2)$, respectively. Let E be a vector bundle over X and ω_q a E-valued q-form. We recall that by pulling ω_q back via a differentiable map $f\colon X' \to X$, we obtain a $f^*(E)$-valued q-form $f^*(\omega_q)$ over X'; we shall often denote this form by $\omega_{q|f}$ as well (specially when $f\colon X' \to X$ is a submanifold). All the definitions and results concerning valued differential calculus have been taken form [11]. The interior product and the Lie derivative (relative to a derivation law) of a valued q-form ω_q with respect

to a vector field D will be denoted by $i_D\omega_q$ and $L_D\omega_q$, respectively. The exterior product of valued forms with respect to a bilinear map of vector bundles $B: E_1 \times_X E_2 \to E_3$ will simply be denoted by $\omega_q \wedge \eta_r$, but we shall also write $\omega_q \wedge_{(B)} \eta_r$ when we need to specify the bilinear map under consideration. The interior product of a E_2-valued q-form with respect to an E_1-valued vector field relative to the bilinear map B can also be defined by imposing that $i_{D\otimes e_1}(\omega_q \otimes e_2) = (i_D\omega_q) \otimes B(e_1, e_2)$.

The k-jet bundle of local sections of $p: Y \to X$ is denoted by $p_k: J^k = J^k(Y/X) \to X$ and $p_k^h: J^h \to J^k$, $h \geqslant k$, stands for the projection $p_k^h(j_x^h x) = j_x^k s$; we set $n = \dim X$, $n + m = \dim Y$. Any fibred chart for Y with local coordinates (x_j, y_i), $1 \leqslant j \leqslant n$, $1 \leqslant i \leqslant m$, induces a fibred chart for J^k with local coordinates (x_j, y_α^i) defined by: $y_0^i = y_i$, $y_\alpha^i(j_x^k s) = (\partial^{|\alpha|}/\partial x^\alpha)(y_i \circ s)(x)$, $1 \leqslant |\alpha| \leqslant k$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. If $f: Y \to Y'$ is a morphism of fibred manifolds whose induced mapping on the base manifolds $\bar{f}: X \to X'$ is a local diffeomorphism, we can define a map $J^k(f): J^k(Y/X) \to \to J^k(Y'/X')$ by $J^k(f)(j_x^k s) = j_{x'}^k s'$, where $x' = \bar{f}(x)$ and $s' = f \circ s \circ \bar{f}^{-1}$.

As is well-known ([2], [3], [6], [13]), the k-jet bundle J^k is endowed with a canonical $V(J^{k-1})_{J^k}$-valued 1-form θ^k, called the *structure form of order k*, whose local expression is

$$\theta^k = \sum_i \sum_{|\alpha| < k} \theta_\alpha^i \otimes (\partial/\partial y_\alpha^i), \qquad \theta_\alpha^i = dy_\alpha^i - \sum_j y_{\alpha+(j)}^i \, dx_j, \tag{1.1}$$

where (j) stands for the multi-index $(j)_k = \delta_{jk}$ (δ_{jk} being the Kronecker index). For a geometric definition of the form θ^k by means of the notion of vertical differential of a section, one may consult [13]. We recall two basic properties of the structure forms:

(1.2) A section $\bar{s}$ of $p_k: J^k \to X$ is the k-jet prolongation of a section s of p (i.e., $\bar{s} = j^k s$) if and only if $\theta^k{}_{|\bar{s}} = 0$.

(1.3) The structure form θ^k is a section of the vector sub-bundle $\mathrm{Hom}_{J^k}(T(J^{k-1}), V(J^{k-1}))$ of $\mathrm{Hom}(T(J^k), V(J^{k-1})_{J^k}) = T^*(J^k) \otimes_{J^k} \otimes_{J^k} V(J^{k-1})$ and it determines a retract of the injection of $V(J^{k-1})_{J^k}$ into $T(\partial^{k-1})_{J^k}$. Thus, the exact sequence

$$0 \to V(J^{k-1})_{J^k} \to T(J^{k-1})_{J^k} \xrightarrow{(p_{k-1})_*} T(X)_{J^k} \to 0$$

splits canonically. We shall denote by $b_k: T(X)_{J^k} \to T(J^{k-1})_{J^k}$ the section of $(p_{k-1})_*$ associated to the retract determined by θ^k; that is, $b_k(z, D_x) = (j^{k-1}s)_*(D_x)$ with $z = j_x^k s$, $D_x \in T_x(X)$.

Let $p: Y \to X$ be a fibred manifold and $E \to Y$ be a vector bundle. We set: $T_h^i(X, E) = T_h^i(X) \otimes_Y E$, $T_h^i(X) = (\otimes^h T^*(X)) \otimes (\otimes^i T(X))$, and similarly for

the symmetric powers: $S^i_h(X, E) = S^i_h(X) \otimes_Y E$, $S^i_h(X) = S^h T^*(X) \otimes S^i T(X)$. Any bilinear mapping of vector bundles $B: E_1 \times_Y E_2 \to E_3$ induces a bilinear map on the symmetric powers

$$B^{i,i'}_{h,h'}: S^i_h(X, E_1) \times_Y S^{i'}_{h'}(X, E_2) \to S^{i+i'}_{h+h'}(X, E_3)$$

by the formula

$$(1.4) \qquad B^{i,i'}_{h,h'}(T_h \otimes T^i T_h \otimes T^i \otimes_{l_1} \otimes e_1, \bar{T}_{h'} \otimes \bar{T}^{i'} \otimes e_2) =$$

$$= \frac{h!h'!}{(h+h')!} (T_h \cdot \bar{T}_{h'}) \otimes (T^i \cdot \bar{T}^{i'}) \otimes B(e_1, e_2),$$

where the dot on the right hand-side stands for the symmetric product.

Let l be a now-negative integer such that $l \leqslant h$, $l \leqslant i$ and let $j: \{1, 2, \ldots, l\} \to \to \{1, 2, \ldots, h\}$, $k: \{1, 2, \ldots, k\} \to \{1, 2, \ldots, i\}$ be two injective mappings. We write $j = (j_1, \ldots, j_l)$, $k = (k_1, \ldots, k_l)$. We denote by $c^k_j: T^i_h(X, E) \to \to T^{i-l}_{h-l}(X, E)$ the contraction of the covariant indices j with the contravariant indices k; that is,

$$(1.5) \qquad c^k_j(w_1 \otimes \cdots \otimes w_h \otimes D_1 \otimes \cdots \otimes D_i \otimes e) =$$

$$= w_{j_1}(D_{k_1}) \cdots w_{j_l}(D_{k_l})(w_{j'_1} \otimes \cdots \otimes w_{j'_{h-l}} \otimes D_{k'_1} \otimes \cdots \otimes D_{k'_{i-l}} \otimes e),$$

where $j'_1 < \cdots < j'_{h-l}$, $k'_1 < \cdots < k'_{i-l}$ are the complementary sequences of $\{j_1, \ldots, j_l\}$, $\{k_1, \ldots, k_l\}$ in $\{1, \ldots, h\}$, $\{1, \ldots, i\}$, respectively and $w_1, \ldots, w_h \in T^*_x(X)$, $D_1, \ldots, D_i \in T_x(X)$, $e \in E_y$, with $p(y) = x$.

It is easily verified that c^k_j maps $S^i_h(X, E)$ onto $S^{i-l}_{h-l}(X, E)$ and also that the restriction of c^k_j to $S^i_h(X, E)$ does not depend on the indices j, k chosen. Thus, we can define a homomorphism $c^i_{h,l}: S^i_h(X, E) \to S^{i-l}_{h-l}(X, E)$ such that,

$$(1.6) \qquad c^i_{h,l}(w_1 \cdots w_h \otimes D_1 \cdots D_i \otimes e) =$$

$$= \sum_{j,k} (w_{j_1} \cdots w_{j_l})(D_{k_1} \cdots D_{k_l})(w_{j'_1} \cdots w_{j'_{h-l}} \otimes D_{k'_1} \cdots D_{k'_{i-l}} \otimes e),$$

where the indices j, k on the right hand-side run over all the sequences such that $1 \leqslant j_1 < \cdots < j_l \leqslant h$, $1 \leqslant k_1 < \cdots < k_l \leqslant i$, and $j'_1 < \cdots < j'_{h-l}$, $k'_1 < \cdots < k'_{i-l}$ are as above. In other words, $c^i_{h,l}$ is the contraction of l covariant indices with l contravariant ones in the vector sub-bundle of totally symmetric tensors of type (h, i).

The homomorphisms $c^i_{h,l}$ satisfy the following properties:

$$(1.7) \qquad c^{i-l}_{h-l,l'} \circ c^i_{h,l} = c^i_{h,l+l'}, \quad \text{if} \quad l + l' \leqslant h \quad \text{and} \quad l + l' \leqslant i.$$

(1.8) Let $B: E_1 \times_Y E_2 \to E_3$ be a bilinear map and ω_q, η'_r, ω'_q, η_r differential forms taking values in $S^i_h(X, E_1)$, $S^i_h(X, E_2)$, $S^0_{i-h}(X, E_1)$, $S^0_{i-h}(X, E_2)$, res-

pectively. Then

$$(a) \quad c^{i-l}_{i-l,\,i-l}((c^{i}_{h,\,l}\omega_q)\wedge\eta_r) = c^{i}_{i,\,i}(\omega_q\wedge\eta_r), \quad \text{for} \quad l\leqslant h\leqslant i,$$
$$(b) \quad c^{i-l}_{i-l,\,i-l}(\omega'_q\wedge(c^{i}_{h,\,l}\eta'_r)) = c^{i}_{i,\,i}(\omega'_q\wedge\eta'_r), \quad \text{for} \quad l\leqslant h\leqslant i,$$

where the exterior products are taken with respect to the bilinear mappings induced by B according to (1.4).

The proof follows from a simple computation and will thus be omitted.

2. Total lie derivative

Let $p\colon Y\to X$ be a submersion and E a vector bundle over Y. The *total contraction* is the homomorphism

$$c\colon \Lambda^q T^*(J^{k-1})\otimes_{J^k} S^i_h(X,E)\to \Lambda^{q-1}T^*(J^{k-1})\otimes_{J^k} S^i_{h+1}(X,E)$$

given by

$$(c\omega_q)(D_2,\ldots,D_q;D'_0,\ldots,D'_h,w_1,\ldots,w_i) =$$
$$= \frac{1}{h+1}\sum_{j=0}^{h}\omega_q(b_kD'_j,D_2,\ldots,D_q;D'_0,\ldots,\hat{D}'_j,\ldots,D'_h,w_1,\ldots,w_i),$$

where $b_k\colon T(X)_{J^k}\to T(J^{k-1})_{J^k}$ is the section defined in (1.3).

Proposition 2.1. *The total contraction satisfies the following conditions*:

(*a*) $c\circ c = 0$.

(*b*) $c(\eta_q\wedge\eta'_r) = (c\eta_q)\wedge\eta'_r + (-1)^q\eta_q\wedge(c\eta'_r)$, *where* η_q,η'_r *are differential forms with values in* $S^i_h(X,E_1)$, $S^{i'}_{h'}(X,E_2)$, *respectively, and the exterior products are taken with respect to the mappings induced by a bilinear map* $B\colon E_1\times_Y E_2\to E_3$.

(*c*) *On* $\Lambda^q T^*(J^{k-1})\otimes_{J^k} S^{i+1}_0(X,E)$, *we have* $c^{i}_{1,\,1}\circ c\circ c^{i+1}_{1,\,1}\circ c = 0$.

PROOF. Condition (*a*) is an immediate consequence of the definition. First we shall prove (*b*) when η_q is an ordinary (non-valued) differential form and $B\colon (Y\times\mathbb{R})\times_Y E_2\to E_2$ is the natural bilinear mapping. We proceed by induction on q. If ω is an ordinary one-form, we have

$$c(\omega\wedge\eta'_r)(D_1,\ldots,D_r;D'_0,\ldots,D'_h,w_1,\ldots,w_i) =$$
$$= \frac{1}{h+1}\sum_{j=0}^{h} i_{(b_kD'_j)}(\omega\wedge\eta'_r)(D_1,\ldots,D_r;D'_0,\ldots,\hat{D}'_j,\ldots,D'_h,w_1,\ldots,w_i) =$$

$$= \frac{1}{h+1}\sum_{j=0}^{h}\omega(b_k D'_j)\eta'_r(D_1,\ldots,D_r;D'_0,\ldots,\hat{D}'_j,\ldots,D'_h,w_1,\ldots,w_i) -$$

$$- \frac{1}{h+1}\sum_{j=0}^{h}\sum_{l=1}^{r}(-1)^{l-1}\omega(D_l)\eta'_r(b_k D'_j,D_1,\ldots,\hat{D}_l,\ldots,D_r;$$

$$D'_0,\ldots,\hat{D}'_j,\ldots,D'_h,w_1,\ldots,w_i)$$

$$= (c\omega\wedge\eta'_r)(D_1,\ldots,D_r;D'_0,\ldots,D'_h,w_1,\ldots,w_i)$$

$$- \omega\wedge c\eta'_r)(D_1,\ldots,D_r;D'_0,\ldots,D'_h,w_1,\ldots,w_i).$$

proving (*b*) in this case. Now, if ω_q is an ordinary q-form, according to the induction hypothesis,

$$\begin{aligned} c((\omega\wedge\omega_q)\wedge\eta'_r) &= c(\omega\wedge(\omega_q\wedge\eta'_r)) = \\ &= c\omega\wedge(\omega_q\wedge\eta'_r) - \omega\wedge(c\omega_q\wedge\eta'_r + (-1)^q\omega_q\wedge c\eta'_r) = \\ &= c(\omega\wedge\omega_q)\wedge\eta'_r + (-1)^{q+1}(\omega\wedge\omega_q)\wedge c\eta'_r. \end{aligned}$$

In the general case,

$$\eta_q = \omega_q\otimes w_1\cdots w_h\otimes D'_1\cdots D'_i\otimes e,$$
$$\eta'_r = \omega'_r\otimes w'_1\cdots w'_{h'}\otimes D''_1\cdots D''_{i'}\otimes e'$$

and by setting $D_j = b_k(\partial/\partial x_j)$, from the above result we obtain

$$c(\eta_q\wedge\eta'_r) = \frac{h!h'!}{(h+h'+1)!}\sum_j i_{D_j}(\omega_q\wedge\omega'_r)\otimes dx_j\cdot w_1\cdots w_h\cdot w'_1\cdots w'_{h'}\otimes$$
$$\otimes D'_1\cdots D'_i\cdot D''_1\cdots D''_{i'}\otimes B(e,e')$$

$$(c\eta_q)\wedge\eta'_r = \frac{h!h'!}{(h+h'+1)}\sum_j((i_{D_j}\omega_q)\wedge\omega'_r)\otimes dx_j\cdot w_1\cdots w_h\cdot w'_1\cdots w'_{h'}\otimes$$
$$\otimes D_1\cdots D_i\cdot D''_1\cdots D''_{i'}\otimes B(e,e')$$

and similarly for $\eta_q\wedge(c\eta'_r)$. Hence $c(\eta_q\wedge\eta'_r) = (c\eta_q)\wedge\eta'_r + (-1)^q\eta_q\wedge(c\eta'_r)$, and thus (*b*) is proved. Finally,

$$(c^i_{1,1}\circ c\circ c^{i+1}_{1,1}\circ c)(\omega_q\otimes D'_0\cdots D'_i\otimes e) =$$

$$= \sum_{a\neq h}\sum_{j,l}(D'_h x_j)(D'_a x_l)(i_{D_l}i_{D_j}\omega_q)\otimes D'_0\cdots\hat{D}'_a\cdots\hat{D}'_h\cdots D'_i\otimes e = 0.$$

In what follows we shall consider a derivation law ∇_0 in $T(X)$ (i.e., a linear connection of X) and a derivation law ∇ in the vector bundle $E\to Y$. As is well known, ∇_0 induces a derivation law ∇^*_0 in the vector bundle $p^*T(X) = T(X)_Y$. We define an operator

$$L\colon\Gamma(\Lambda^q T^*(J^{k-1})\otimes_{J^{k-1}}S^i_h(X,E))\to\Gamma(\Lambda^q T^*(J^k)\otimes_{J^k}S^i_{h+1}(X,E)),$$

called the *total Lie derivative* associated to the pair of derivation laws (Δ_0, ∇), by the formula

$$L = c \circ d + d \circ c,$$

where d is the exterior differential with respect to the derivation law induced by ∇_0^* and ∇ in the symmetric powers $S_h^i(X, E)$.

It is clear that the operator L is $\mathbb{R}$-linear and commutes with c, that is,

$$L \circ c = c \circ L.$$

Furthermore, L satisfies the formal property of a derivation. Namely, we have

Proposition 2.2. *Let ∇_i be a derivation law in E_i, with $i = 1, 2, 3$, and $B: E_1 \times_Y E_2 \to E_3$ a bilinear map compatible with these derivation laws. Then*

$$L(\eta_q \wedge \eta'_r) = (L\eta_q) \wedge \eta'_r + \eta_q \wedge (L\eta'_r),$$

where η_q, η'_r are differential forms taking values in $S_h^i(X, E_1)$, $S_{h'}^{i'}(X, E_2)$, respectively, and the exterior products are taken with respect to the bilinear mappings $B_{h,h'}^{i,i'}$.

PROOF. This follows from (*b*) of Propostion 2.1 and the properties of the exterior differential for valued forms.

3. Structure forms associated to a pair of derivation laws

Given a linear connection ∇_0 of X and a derivation law ∇ in the vertical bundle $V(Y)$, we define a sequence of differential forms $(\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(k)}, \ldots)$ by the following recurrence relations:

(3.1) $\theta^{(1)} = \theta^1$, the structure form on J^1; $\theta^{(k)} = L\theta^{(k-1)}$, for $k > 1$.

Thus, $\theta^{(k)}$ is a $(S^{k-1}T^*(X) \otimes_{J^k} V(Y))$-valued one-form on J^k which is called the *structure form of order k associated to the pair* (∇_0, ∇).

We shall now determine the local expression of the forms $\theta^{(k)}$. Let (x_j, y_i) be a fibred coordinate system for Y; we set

$$\nabla_{0(\partial/\partial x_j)}(\partial/\partial x_k) = \sum_l \overset{\circ}{\Gamma}{}_{jk}^l (\partial/\partial x_l), \tag{3.2}$$

$$\nabla_{(\partial/\partial x_j)}(\partial/\partial y_i) = \sum_a \Gamma_{ji}^a (\partial/\partial y_a), \qquad \nabla_{(\partial/\partial y_h)}(\partial/\partial y_i) = \sum_a \bar{\Gamma}_{hi}^a (\partial/\partial y_a). \tag{3.3}$$

For the sake of simplicity, we shall henceforth denote by $\mathbb{D}_j$ the derivation of the ring $A = \varinjlim C^\infty(J^k)$ defined by the formula

$$\mathbb{D}_j = \partial/\partial x_j + \sum_i \sum_{|\alpha|=0}^{\infty} y^i_{\alpha+(j)}(\partial/\partial y^i_\alpha). \tag{3.4}$$

Note that $\mathbb{D}_j(f) = b_k(\partial/\partial x_j)(f)$ if $f \in C^\infty(J^{k-1})$, or, in other words, $(\mathbb{D}_j f) \circ j^k s = (\partial/\partial x_j)(f \circ j^{k-1} s)$ for every local section s of $p: Y \to X$. Moreover, $(dx)^\alpha$ stands for the symmetric product

$$(dx)^\alpha = dx_1^{\alpha_1} \cdots dx_n^{\alpha_n} = \overset{(\alpha_1}{dx_1} \cdots \overset{(\alpha_n}{dx_1} \cdots dx_n \cdots dx_n, \tag{3.5}$$

and similarly for $(\partial/\partial x)^\alpha$. Hence, with the notations of (1.1) we have

Proposition 3.1. *There exist unique functions* $\mathring{a}_{\beta\alpha} \in C^\infty(X)$, $|\beta| \leqslant |\alpha|$, $a^{hi}_\alpha \in C^\infty(J^{|\alpha|})$ *such that:*

$$L^u\theta^h_0 = \frac{1}{u!} \sum_{|\beta| \leq u} \sum_{|\alpha|=u} \mathring{a}_{\beta\alpha}\theta^h_\beta \otimes (dx)^\alpha, \tag{3.6}$$

$$L^v(\partial/\partial y_h) = \frac{1}{v!} \sum_i \sum_{|\alpha|=v} a^{hi}_\alpha (dx)^\alpha \otimes (\partial/\partial y_i). \tag{3.7}$$

These functions are completely determined by the following recurrence relations:

$$\mathring{a}_{00} = 1 \tag{3.8}$$

$$\mathring{a}_{\beta\alpha} = \sum_j [\partial \mathring{a}_{\beta,\alpha-(j)}/\partial x_j + \mathring{a}_{\beta-(j),\alpha-(j)}] - \sum_{j,k,l} (1 + \alpha_l - \delta_{jl} - \delta_{kl}) \mathring{\Gamma}^l_{jk} \mathring{a}_{\beta,\alpha+(l)-(j)-(k)} \tag{3.9}$$

$$a^{hi}_0 = \delta_{hi}, \qquad a^{hi}_{(j)} = \Gamma^i_{jh} + \sum_\nu y^\nu_{(j)} \bar{\Gamma}^i_{\nu h} \tag{3.10}$$

$$a^{hi}_\alpha = \sum_j \left[\mathbb{D}_j a^{hi}_{\alpha-(j)} + \sum_g a^{hg}_{\alpha-(j)} a^{gi}_{(j)} \right] - \sum_{j,k,l} (1 + \alpha_l - \delta_{jl} - \delta_{kl}) \mathring{\Gamma}^l_{jk} a^{hi}_{\alpha+(l)-(j)-(k)} \tag{3.11}$$

Finally, by setting

$$A^{hi}_{\beta\sigma} = \sum_{\alpha \leq \sigma, |\beta| \leq |\alpha|} \binom{|\sigma|}{|\alpha|} \mathring{a}_{\beta\alpha} a^{hi}_{\sigma-\alpha}, \tag{3.12}$$

we obtain

$$\theta^{(k)} = \frac{1}{(k-1)!} \sum_{h,i} \sum_{|\beta| \leq k-1} \sum_{|\sigma|=k-1} A^{hi}_{\beta\sigma} \theta^h_\beta \otimes (dx)^\sigma \otimes (\partial/\partial y_i). \tag{3.13}$$

PROOF. Formulas (3.6) and (3.7) can easily be proved by recurrence on u and v, respectively, using the formal properties of the operator L. In fact, assuming (3.6), we have

$$L^{u+1}\theta_0^h = L(L^u\theta_0^h) = \frac{1}{u!}\sum_{|\beta|\le u}\sum_{|\alpha|=u}\Big[\sum_j \frac{1}{u+1}(\mathbb{D}_j \mathring{a}_{\beta\alpha})\theta_\beta^h \otimes (dx)^{\alpha+(j)} +$$

$$+ \sum_j \frac{1}{u+1}\mathring{a}_{\beta\alpha}\theta^h_{\beta+(j)} \otimes (dx)^{\alpha+(j)} + \mathring{a}_{\beta\alpha}\theta_\beta^h \otimes L(dx)^\alpha\Big]$$

$$= \frac{1}{(u+1)}\sum_{|\beta|\le u}\sum_{|\alpha|=u}\Big[\sum_j((\mathbb{D}_j\mathring{a}_{\beta\alpha})\theta_\beta^h + \mathring{a}_{\beta\alpha}\theta^h_{\beta+(j)}) \otimes (dx)^{\alpha+(j)} -$$

$$- \sum_{j,k,l}\alpha_l \mathring{\Gamma}^l_{jk}\mathring{a}^h_{\beta\alpha}\theta_\beta^h \otimes (dx)^{\alpha-(l)+(j)+(k)}\Big],$$

and similarly for (3.7). From that, (3.9) and (3.11) also follow. On the other hand, we have

$$\theta^{(k)} = L^{k-1}\theta^1 = \sum_h L^{k-1}[\theta_0^h \otimes (\partial/\partial y_h)] =$$

$$= \sum_h \sum_{u+v=k-1}\binom{k-1}{u}(L^u\theta_0^h)\wedge(L^v(\partial/\partial y_h)) =$$

$$= \sum_{h,i}\sum_{u=0}^{k-1}\sum_{|\beta|\le u}\sum_{|\alpha|=u}\sum_{|\alpha'|=k-1-u}\frac{1}{u!(k-1-u)!}\mathring{a}_{\beta\alpha}a^{hi}_{\alpha'}\theta_\beta^h \otimes$$

$$\otimes (dx)^{\alpha+\alpha'} \otimes (\partial/\partial y_i) =$$

$$= \frac{1}{(k-1)!}\sum_{h,i}\sum_{|\beta|\le k-1}\sum_{|\sigma|=k-1}\Big[\sum_{\alpha\le\sigma,|\beta|\le|\alpha|}\binom{|\sigma|}{|\alpha|}\mathring{a}_{\beta\alpha}a^{hi}_{\sigma-\alpha}\Big]\theta_\beta^h \otimes (dx)^\sigma \otimes (\partial/\partial y_i),$$

thus proving formula (3.13).

Remark. It follows from (3.12) that

$$A^{hi}_{\beta\sigma} \in C^\infty(J^{|\sigma|-|\beta|}). \tag{3.14}$$

Moreover, from (3.9) we obtain by induction on $|\alpha|$,

$$\mathring{a}_{\beta\alpha} = \delta_{\beta\alpha}|\alpha|!/\alpha! \quad \text{for} \quad |\beta| = |\alpha|. \tag{3.15}$$

Then, from (3.10) and (3.12) we have

$$A^{hi}_{\beta\sigma} = \delta_{hi}\delta_{\beta\sigma}|\sigma|!/\sigma! \quad \text{for} \quad |\beta| = |\sigma|. \tag{3.16}$$

Corollary 3.2. *The homomorphism* P_r: $\oplus_{k=0}^{r} S^k T(X) \otimes_{J^r} V^*(Y) \to V^*(J^r)$ *mapping* $(f_0, \ldots, f_r)$ *into the restriction of* $\theta^{(1)} \circ f_0 + \theta^{(2)} \circ f_1 + \cdots + \theta^{(r+1)} \circ f_r$ *to* $V(J^r)$ *is an isomorphism of vector bundles.*

PROOF. First, note that the definition of P_r makes sense because locally $\theta^{(r+1)} \circ f_r$ belongs to the submodule generated over $C^\infty(J^r)$ by $(\theta^h_\beta)_{|\beta| \leq r}$, as follows from (3.13) and (3.14). Since the vector bundles $V^*(J^r)$ and $\oplus_{k=0}^{r} S^k T(X) \otimes_{J^r} V^*(Y)$ have the same rank, it will be sufficient to prove that P_r is injective. We proceed by induction on r. The case $r = 0$ is trivial. If $P_r(f_0, \ldots, f_r) = 0$ we have $\theta^{(1)}(D) \circ f_0 + \cdots + \theta^{(r+1)}(D) \circ f_r = 0$ for every vertical tangent vector D in J^r. In particular, by taking $D = \partial/\partial y^h_\beta$ with $|\beta| = r$, from (3.13) and (3.16) we obtain $[(dx)^\beta \otimes (\partial/\partial y_h)](f_r) = 0$; that is, $f_r = 0$. Hence, $P_{r-1}(f_0, \ldots, f_{r-1}) = 0$ and according to the induction hypothesis, $f_0 = 0, \ldots, f_{r-1} = 0$.

Corollary 3.3. *With the above notations, we have* $(\theta^{(1)}, \ldots, \theta^{(r)}) = P^*_{r-1} \circ \theta^r$. *Consequently a section* $\bar{s}$ *of* p_r: $J^r \to X$ *is the jet prolongation of a section* s *of* p: $Y \to X$ *if and only if* $\theta^{(k)}|_{\bar{s}}$ *vanishes for* $k = 1, \ldots, r$ (cf. [13, Proposition 2]).

PROOF. It follows from formula (3.13) that $\ker(\theta^{(1)}, \ldots, \theta^{(r)}) = \ker \theta^r$. Therefore it is sufficient to see that $(\theta^{(1)}, \ldots, \theta^{(r)})(\partial/\partial y^h_\beta) = (P^*_{r-1} \circ \theta^r)(\partial/\partial y^h_\beta)$, with $|\beta| \leqslant r - 1$. However, it is easily checked, by using (3.13) and the definition of P_{r-1}, that both sides of the preceding equation give the same result when applied to $(\partial/\partial x)^\sigma \otimes dy_i$ for $|\sigma| < r$.

4. Higher order variational problems

A vector field D in J^r is an *infinitestimal contact transformation of order* r if for any derivation law ∇ in $V(J^{r-1})$ there exists an endomorphism ϕ of the vector bundle $V(J^{r-1})_{J^r}$ such that $L_D \theta^r = \phi \circ \theta^r$, where the Lie derivative is taken with respect to the derivation law induced by ∇. Indeed, if the previous condition is fulfilled for a derivation law ∇, it is automatically verified for any other derivation law. We now recall some basic facts concerning higher order infinitesimal contact transformations. For the proof of these results and further information one may consult [13].

(4.1) For *any* vector field D in Y (not necessarily p-projectable) there exists a unique infinitesimal contact transformation $D_{(r)}$ of order r projectable onto D.

(4.2) Moreover if $k > 0$ and $\bar{D}$ is an arbitrary infinitesimal contact transformation of order k, for every $r > k$ there exists a unique infinitesimal

contact transformation $\bar{D}_{(r)}$ of order r projectable onto $\bar{D}$. In particular, if D is a vector field in Y, it follows that $D_{(r)}$ is projectable onto $D_{(k)}$ for every $r > k$.

For any open set $U \subset X$ we denote by $T^r(U)$ the space of all the infinitesimal contact transformations of order r corresponding to the induced fibred manifold Y_U; we denote by $T^r_c(U)$ the set of vector fields in $T^r(U)$ whose support has compact image in U. Then

(4.3) $T^r(U)$ is a Lie algebra with respect to the Lie bracket of vector fields, and $T^r_c(U)$ is an ideal of $T^r(U)$. Furthermore, the map $D \to D_{(r)}$ is an injection of Lie algebras.

(4.4) Let τ_t be the local 1-parameter group of local transformations generated by a vector field D in Y. If D is p-projectable, each transformation τ_t defines an automorphism of the fibred manifold Y and $J^r(\tau_t)$ is the local 1-parameter group generated by the vector field $D_{(r)}$.

Proposition 4.1. *A vector field D in J^r is an infinitesimal contact transformation if an only if there exist homomorphisms*

$$\phi^k_l: S^{l-1}T^*(X) \otimes_{J^r} V(Y) \to S^{k-1}T^*(X) \otimes_{J^r} V(Y), \qquad 1 \leqslant k, \quad l \leqslant r,$$

such that

$$L_D\theta^{(k)} = \phi^k_1 \circ \theta^{(1)} + \cdots + \phi^k_r \circ \theta^{(r)} \quad \textit{for} \quad k = 1, \ldots, r.$$

PROOF. This is immediate from Corollary 3.3.

From now on we shall assume that the base manifold is orientable. Once a volume element v on X has been fixed, we can associate a functional $\mathbb{L}: S(U) \to \mathbb{R}$ to each function $\mathfrak{L} \in C^\infty(J^r)$ by the formula

$$\mathbb{L}(s) = \int_{j^r s} \mathfrak{L}v = \int_U (j^r s)^*(\mathfrak{L}v),$$

where $U \subset X$ is an open set and $S(U) \subset \Gamma(Y/U)$ is the space of those sections for which the above integral exists. For any section $s \in \Gamma(Y/U)$ we also define a linear form $\delta_s\mathbb{L}: T^r_c(U) \to \mathbb{R}$ by the formula

$$(\delta_s\mathbb{L})(D) = \int_{j^r s} L_D(\mathfrak{L}v).$$

According to the definition of infinitesimal contact transformations given above, the linear functional $\delta\mathbb{L}$ represents the infinitesimal variation of the functional $\mathbb{L}$ on the space of generalized infinitesimal transformations of J^r induced by the infinitesimal automorphisms of the fibred structure $p: Y \to X$. We shall say that a section s is *critical* for the Lagrangian density $\mathfrak{L}v$ when the linear functional $\delta\mathbb{L}$ has no variation at s; or in other words, when $\delta_s\mathbb{L} = 0$.

A basic problem in the Calculus in Variations is to characterize critical sections as solutions of a differential system defined on an appropiate jet bundle. In the following three sections we shall construct the Poincaré-Cartan forms associated to a higher order variational problem and examine their main properties. As will be shown below, these forms are the fundamental tools which will allow us to obtain not only the characterization of the critical sections but also the geometric properties of the manifold of solutions of a variational problem.

5. Poincaré-Cartan forms

Let $\mathfrak{L}$ be a differentiable function on J^r. We denote by $\overline{d\mathfrak{L}}$ the restriction of $d\mathfrak{L}$ to $V(J^r)$. According to Corollary 3.2 there exist unique sections f_k of $S^kT(X) \otimes_{J^r} V^*(Y)$, $0 \leqslant k \leqslant r$, such that: $P_r(f_0, \ldots, f_r) = \overline{d\mathfrak{L}}$; or in other words, $\theta^{(1)}(D) \circ f_0 + \cdots + \theta^{(r+1)}(D) \circ f_r = (d\mathfrak{L})(D)$ for every vertical vector field D in J^r. Let v be a volume element on X. We define

$$\omega_k = c^k_{1,1} c(v \otimes f_k) \quad \text{for} \quad k = 1, \ldots, r. \tag{5.1}$$

In what follows we shall consider ω_k as a $S^{k-1}T(X) \otimes_{J^r} V^*(Y)$-valued $(n-1)$-form on J^r.

Remark. The forms ω_k only depend on the Lagrangian density $\mathfrak{L}v$. In fact, in v' is another volume element on X and $\mathfrak{L}v = \mathfrak{L}'v'$, we have $v' = \rho v'$, $\mathfrak{L}' = \bar{\rho}^1 \mathfrak{L}$ for an invertible function $\rho \in C^\infty(X)$. Then $\overline{d\mathfrak{L}'} = \bar{\rho}^1 \overline{d\mathfrak{L}}$ and $f'_k = \bar{\rho}^1 f_k$, $0 \leqslant k \leqslant r$. Hence $\omega'_k = c^k_{1,1} c(v' \otimes f'_k) = c^k_{1,1} c(v \otimes f_k) = \omega_k$.

Locally, we set $f_k = \sum_i \sum_{|\alpha| = k} f^i_\alpha (\partial/\partial x)^\alpha \otimes dy_i$. From the definition of f_k and formula (3.13) we obtain

$$|\beta|! f^h_\beta + \sum_i \sum_{|\alpha| = |\beta|+1}^{r} \alpha! A^{hi}_{\beta\alpha} f^i_\alpha = \frac{\partial \mathfrak{L}}{\partial y^h_\beta}, \quad \text{for} \quad |\beta| < r, \tag{5.2}$$

$$r! f^h_\beta = \frac{\partial \mathfrak{L}}{\partial y^h_\beta}, \quad \text{for} \quad |\beta| = r. \tag{5.3}$$

Such equations determine the sections f_k by descending recurrence on k. By choosing the coordinates (x_j) so that $v = dx_1 \wedge \cdots \wedge dx_n$, we have

$$\omega_k = \sum_{i,j} \sum_{|\alpha| = k-1} (-1)^{j-1} (1 + \alpha_j) f^i_{\alpha + (j)} v_j \otimes (\partial/\partial x)^\alpha \otimes dy_i, \quad 1 \leqslant k \leqslant r, \tag{5.4}$$

as follows from (5.1), where $v_j = dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n$.

We shall now deal with the exterior product of a $S^kT^*(X) \otimes V(Y)$-valued form η and a $S^kT(X) \otimes V^*(Y)$-valued form η' with respect to the bilinear map

induced by duality. This product may be factorized as follows. With the same notations as in §1, let

$$B: S^k T^*(X) \otimes V(Y) \times_Y S^k T(X) \otimes V^*(Y) \to Yx_x S_k^k(X))$$

be the bilinear map canonically induced by duality on the vertical bundle. Then, we have

$$\eta \wedge \eta' = c_{k,k}^k \Big(\underset{(B)}{\eta \wedge \eta'} \Big).$$

Proposition 5.1. *Let* $\Theta_0 = \theta^{(1)} \wedge \omega_1 + \cdots + \theta^{(r)} \wedge \omega_r + \mathfrak{L}v$.

We have

$$\begin{aligned} d\Theta_0 = & -[\theta^{(1)} \wedge (d\omega_1 - v \otimes f_0) + \theta^{(2)} \wedge d\omega_2 + \cdots + \theta^{(r)} \wedge d\omega_r] + \\ & + \sum_{k=1}^{r} c_{k,k}^k c\Big(\underset{(B)}{d\theta^{(k)} \wedge (v \otimes f_k)} \Big), \end{aligned} \tag{5.5}$$

where the exterior differentials on the right hand-side are taken with respect to the derivation laws induced by ∇_0, ∇.

PROOF. Since $(\theta^{(1)} \circ f_0 + \cdots + \theta^{(r+1)} \circ f_r - d\mathfrak{L})(D) = 0$ for every vertical vector field D, it is clear that $\sum_{k=0}^{r} \theta^{(k+1)} \circ f_k - d\mathfrak{L}$ is a section of the vector sub-bundle $T^*(X)_{Jr}$. Hence,

$$\Big(\sum_{k=0}^{r} \theta^{(k+1)} \circ f_k - d\mathfrak{L} \Big) \wedge v = 0.$$

That is,

$$d\mathfrak{L} \wedge v = \sum_{k=0}^{r} (\theta^{(k+1)} \circ f_k) \wedge v = (\theta^{(1)} \circ f_0) \circ v + \sum_{k=1}^{r} (L\theta^{(k)} \circ f_k) \wedge v.$$

Then

$$\begin{aligned} d\Theta_0 = & -\Big[\theta^{(1)} \wedge (d\omega_1 - v \otimes f_0) + \sum_{k=2}^{r} \theta^{(k)} \wedge d\omega_k \Big] + \\ & + \sum_{k=1}^{r} [(L\theta^{(k)} \circ f_k) \wedge v + d\theta^{(k)} \wedge \omega_k]. \end{aligned}$$

Since the total contraction c vanishes on the structure forms $\theta^{(k)}$, we have

$$(L\theta^{(k)} \circ f_k) \wedge v = (cd\theta^{(k)} \circ f_k) \wedge v = c_{k,k}^k \Big[\underset{(B)}{cd\theta^{(k)} \wedge (v \otimes f_k)} \Big].$$

Moreover, from (*b*) of (1.8) and (5.1) we obtain:

$$d\theta^{(k)} \wedge \omega_k = c^{k-1}_{k-1,k-1}\Big[\underset{(B)}{d\theta^{(k)} \wedge c^k_{1,1} c(v \otimes f_k)}\Big]$$

$$= c^k_{k,k}\Big[\underset{(B)}{d\theta^{(k)} \wedge c(v \otimes f_k)}\Big]$$

The result follows from these three formulas.

Now, the central point of the theory is to reduce all the forms $\theta^{(k)}$ in formula (5.5) to the first, $\theta^{(1)}$, using the fundamental recurrence relationship (3.1). In this formalism the procedure constitutes the intrinsic version of the well-known classical method of reiterated integration by parts which is used in deducing higher order Euler-Lagrange equations. Furthermore, this procedure will lead to the definition of the Poincaré-Cartan form for Lagrangian densities of arbitrary order, starting from the form Θ_0 defined in the previous proposition.

Theorem 5.2 (fundamental). *With the above hypotheses and notations we have:*

$$d\Theta = \theta^{(1)} \wedge E + \Phi, \tag{5.6}$$

where we have set:

$$\Theta = \Theta_0 + \sum_{k=1}^{r-1} \theta^{(k)} \wedge \left(\sum_{i=1}^{r-k} (-1)^i c^k_{1,1} c \prod_{j=1}^{i-1} (c^{k+j}_{1,1} L)\, d\omega_{k+i} \right) \tag{5.7}$$

$$E = v \otimes f_0 - \sum_{h=0}^{r-1} (-1)^h \prod_{i=0}^{h-1} (c^{i+1}_{1,1} L)\, d\omega_{h+1} \tag{5.8}$$

$$\Phi = \sum_{k=1}^{r-1} c^k_{k,k} cd\left(\theta^{(k)} \underset{(B)}{\wedge} \left(\sum_{h=1}^{r-k} (-1)^h \prod_{i=1}^{h-1} (c^{k+i}_{1,1} L)\, d\omega_{k+h} \right) \right) + \tag{5.9}$$

$$+ \sum_{k=1}^{r} c^k_{k,k} c\Big(d\theta^{(k)} \underset{(B)}{\wedge} (v \otimes f_k) \Big).$$

We shall call the ordinary n-form Θ *the* Poincaré-Cartan form *associated to the Lagrangian density* $\mathfrak{L}v$ *relative to the derivation laws* ∇_0, ∇, *and we shall call the* $V^*(Y)$*-valued n-form* E *the* Euler-Lagrange form.

Remark. In §7 we shall see that the $(n+1)$-form Φ is a 2-contact form. We also note that the forms Θ, E and Φ are defined on J^{2r-1}, because operators c and L are applied $r-1$ times, at most, in the preceding formulas.

PROOF. First we shall prove by recurrence on $k = 0, \dots, r-1$ the following formula:

$$(*)\quad d\left[\Theta_0 + \sum_{h=1}^{k} \theta^{(r-h)} \wedge \left(\sum_{i=1}^{h} (-1)^i c_{1,1}^{r-h} c \prod_{j=1}^{i-1} (c_{1,1}^{r-h+j} L)\, d\omega_{r-h+i}\right)\right] =$$

$$= \theta^{(1)} \wedge (v \otimes f_0) - \sum_{h=1}^{r-k-1} \theta^{(h)} \wedge d\omega_h - \theta^{(r-k)} \wedge$$

$$\wedge \left(\sum_{h=0}^{k} (-1)^h \prod_{i=0}^{h-1} (c_{1,1}^{r-k+i} L)\, d\omega_{r-k+h}\right) +$$

$$+ \sum_{h=1}^{k} c_{r-h,r-h} c d\left[\theta^{(r-h)} \underset{(B)}{\wedge} \left(\sum_{i=1}^{h} (-1)^i \prod_{j=1}^{i-1} (c_{1,1}^{r-h+j} L)\, d\omega_{r-h+i}\right)\right] +$$

$$+ \sum_{h=1}^{r} c_{h,h}^{h} c\left(d\theta^{(h)} \underset{(B)}{\wedge} (v \otimes f_h)\right),$$

where we assume that the sums vanish and the products are the identity when the lower index is greater than the upper one.

Note that (*) for $k = 0$ reduces to formula (5.5). We set

$$\eta^{(k)} = \sum_{h=0}^{k} (-1)^h \prod_{i=0}^{h-1} (c_{1,1}^{r-k-i} L)\, d\omega_{r-k+h}.$$

By using part (*b*) of (1.8) and the fact that the exterior differential commutes with contractions, we have

$$\begin{aligned}
\theta^{(r-k)} \wedge \eta^{(k)} &= c_{r-k-1,r-k-1}^{r-k-1}\left(L\theta^{(r-k-1)} \underset{(B)}{\wedge} \eta^{(k)}\right) \\
&= c_{r-k-1,r-k-1}^{r-k-1}\left[L\left(\theta^{(r-k-1)} \underset{(B)}{\wedge} \eta^{(k)}\right) - \theta^{(r-k-1)} \underset{(B)}{\wedge} L\eta^{(k)}\right] \\
&= c_{r-k-1,r-k-1}^{r-k-1} c d\left(\theta^{(r-k-1)} \underset{(B)}{\wedge} \eta^{(k)}\right) + \\
&\quad + dc_{r-k-1,r-k-1}^{r-k-1} c\left(\theta^{(r-k-1)} \underset{(B)}{\wedge} \eta^{(k)}\right) - \\
&\quad - c_{r-k-2,r-k-2}^{r-k-2}\left(\theta^{(r-k-1)} \underset{(B)}{\wedge} c_{1,1}^{r-k-1} L\eta^{(k)}\right)
\end{aligned}$$

Moreover, since the total contraction is an anti-derivation which vanishes on the structure forms $\theta^{(k)}$, we have

$$\begin{aligned}
c_{r-k-1,r-k-1}^{r-k-1} c\left(\theta^{(r-k-1)} \underset{(B)}{\wedge} \eta^{(k)}\right) &= -c_{r-k-1,r-k-1}^{r-k-1}\left(\theta^{(r-k-1)} \underset{(B)}{\wedge} c\eta^{(k)}\right) \\
&= -\theta^{(r-k-1)} \wedge c_{1,1}^{r-k-1} c\eta^{(k)}
\end{aligned}$$

Thus, finally:

$$\theta^{(r-k)} \wedge \eta^{(k)} = c^{r-k-1}_{r-k-1,r-k-1} cd\Big(\theta^{(r-k-1)} \underset{(B)}{\wedge} \eta^{(k)}\Big) -$$
$$- d(\theta^{(r-k-1)} \wedge c^{r-k-1}_{1,1} c\eta^{(k)}) -$$
$$- \theta^{(r-k-1)} \wedge c^{r-k-1}_{1,1} L\eta^{(k)}$$

Then, sustituting this expression in (*) we find the corresponding formula for $k+1$. Thus, the proof of formula (*) is complete. In particular, for $k = r-1$ we obtain formula (5.6) of the statement.

We define valued $(n-1)$-forms $\Omega_1, \ldots, \Omega_r$ on J^{2r-1} by setting

$$\Omega_k = \omega_k + \sum_{i=1}^{r-k} (-1)^i c^k_{1,1} c \prod_{j=1}^{i-1} (c^{k+j}_{1,1} L)\, d\omega_{k+i} \qquad (1 \leqslant k \leqslant r). \tag{5.10}$$

We also define a $V^*(J^{r-1})$-valued $(n-1)$-form Ω on J^{2r-1}, which will be called the *Legendre form*, by the following formula:

$$\Omega = P_{r-1} \circ (\Omega_1, \ldots, \Omega_r), \tag{5.11}$$

with the same notations as in Corollary 3.2. Then, Poincaré-Cartan form can be rewritten as follows:

$$\Theta = \theta^{(1)} \wedge \Omega_1 + \cdots + \theta^{(r)} \wedge \Omega_r + \mathfrak{L}v = \theta^r \wedge \Omega + \mathfrak{L}v. \tag{5.12}$$

6. Recurrence relations for $\Omega_1, \ldots, \Omega_r$ and local expression of Poincaré-Cartan forms

Proposition 6.1. *The forms $\Omega_1, \ldots, \Omega_r$ satisfy the following conditions*:

(6.1) $\Omega_k = \omega_k - c^k_{1,1} cd\Omega_{k+1}$ *for* $k = 1, \ldots, r-1$, *and* $\Omega_r = \omega_r$.

(6.2) $c^{k-1}_{1,1} c\Omega_k = c^{k-1}_{1,1} c\omega_k = 0$ *for* $k = 2, \ldots, r$.

(6.3) $\Omega_k = \omega_k - c^k_{1,1} L\Omega_{k+1}$ *for* $k = 1, \ldots, r-1$.

PROOF. It follows form (*c*) of Proposition 2.1 and (5.10) that

$$\Omega_k = \omega_k - c^k_{1,1} cd\omega_{k+1} + \sum_{i=2}^{r-k} (-1)^i c^k_{1,1} c \prod_{j=1}^{i-1} (c^{k+j}_{1,1} L)\, d\omega_{k+i}$$
$$= \omega_k - c^k_{1,1} cd\Big(\Omega_{k+1} - \sum_{i=1}^{r-k-1} (-1)^i c^{k+1}_{1,1} c \prod_{j=1}^{i-1} (c^{k+1+j}_{1,1} L)\, d\omega_{k+1+i}\Big) +$$
$$+ \sum_{i=2}^{r-k} (-1)^i c^k_{1,1} c \prod_{j=1}^{i-1} (c^{k+j}_{1,1} L)\, d\omega_{k+i} =$$

$$\Omega_k = \omega_k - c^k_{1,1} cd\Omega_{k+1} +$$
$$+ \sum_{i=1}^{r-k-1} (-1)^i c^k_{1,1} c c^{k+1}_{1,1} (dc + cd) \prod_{j=1}^{i-1} (c^{k+1+j}_{1,1} L)\, d\omega_{k+1+i} +$$
$$+ \sum_{i=2}^{r-k} (-1)^i c^k_{1,1} c \prod_{j=1}^{i-1} (c^{k+j}_{1,1} L)\, d\omega_{k+i}$$
$$= \omega_k - c^k_{1,1} cd\Omega_{k+1},$$

which proves (6.1).

On the other hand, by again using (*c*) of Proposition 2.1 and the definition of ω_k, we obtain $c^{k-1}_{1,1} c\omega_k = c^{k-1}_{1,1} c c^k_{1,1} c(v \otimes f_k) = 0$, thus proving the second part of (6.2). The first part is obtained from (6.1) by descending recurrence on k. Finally, (6.3) is a direct consequence of (6.1) and (6.2).

The forms $\Omega_1, \ldots, \Omega_r$ can be obtained in the same way as the forms $\omega_1, \ldots, \omega_r$ were derived. Namely, we have

Proposition 6.2. *There exist unique sections F_k, $1 \leqslant k \leqslant r$, of*

$$S^k T(X) \otimes_{J^{2r-k}} V^*(Y)$$

such that

$$\Omega_k = c^k_{1,1} c(v \otimes F_k).$$

Thus, Ω_k is a section of the vector sub-bundle

$$\Lambda^{n-1} T^*(X) \otimes_{J^{2r-k}} S^{k-1} T(X) \otimes_Y V^*(Y).$$

PROOF. The uniqueness part is easily verified. In order to prove the existence of such sections we proceed locally by setting $F_k = \sum_i \sum_{|\alpha|=k} F^i_\alpha (\partial/\partial x)^\alpha \otimes \otimes dy_i$.

Then, the proof is by descending recurrence on k. If $k = r$, it is sufficient to take $F_r = f_r$, because $\Omega_r = \omega_r$. Let us assume that the formula is also true for $k, k+1, \ldots, r$ with $k > 1$. That is, there exist sections F_l such that

$$\Omega_l = c^l_{1,1} c(v \otimes F_l) = \sum_{i,j} \sum_{|\alpha|=l-1} (-1)^{j-1} (1 + \alpha_j) F^i_{\alpha+(j)} v_j \otimes (\partial/\partial x)^\alpha \otimes dy_i, \qquad k \leqslant l \leqslant r,$$

with the same notations as in (5.4). Then it follows from a simple (but rather long) computation in local coordinates that

$$\Omega_{k-1} = \omega_{k-1} - c^{k-1}_{1,1} cd\Omega_k = \sum_{i,j} \sum_{|\alpha|=k-2} (-1)^{j-1} (1 + \alpha_j) F^i_{\alpha,j} v_j \otimes (\partial/\partial x)^\alpha \otimes dy_i,$$

where the coefficient $F^i_{\alpha,j}$ is given by

$$F^i_{\alpha,j} = f^i_{\alpha+(j)} - \sum_l (1+\alpha_l+\delta_{jl})\Big[\mathbb{D}_l F^i_{\alpha+(j)+(l)} - \sum_h a^{ih}_{(l)} F^h_{\alpha+(j)+(l)}\Big] -$$

$$- \sum_{l,q,u} (1+\alpha_l+\delta_{jl}-\delta_{lq})(1+\alpha_u+\delta_{lu}-\delta_{qu}+\delta_{ju})\mathring{\Gamma}^q_{ul} F^i_{\alpha+(j)+(l)+(u)-(q)}$$

Here $\mathbb{D}_l$ stands for the vector field defined in (3.4). The above formula shows that $F^i_{\alpha,j}$ only depends on $\alpha + (j)$, or in other words, that if $\alpha + (j) = \alpha' + (j')$, then $F^i_{\alpha,j} = F^i_{\alpha',j'}$. We can therefore define F^i_σ for $|\sigma| = k-1$ by setting $F^i_\sigma = F^i_{\alpha,j}$, where $\sigma = \alpha + (j)$ is an arbitrary decomposition of the multi-index σ. This proves the existence of F_{k-1} and completes the proof.

The previous formula can be rewritten as follows:

$$F^i_\sigma = f^i_\sigma - \sum_j (1+\sigma_j)\Big[\mathbb{D}_j F^i_{\sigma+(j)} - \sum_h a^{ih}_{(j)} F^h_{\sigma+(j)}\Big] - \tag{6.4}$$

$$- \sum_{j,k,l} (1+\sigma_k-\delta_{kl})(1+\sigma_j+\delta_{jk}-\delta_{jl})\mathring{\Gamma}^l_{jk} F^i_{\sigma+(j)+(k)-(l)} \quad (|\sigma| = 1,\dots,r-1).$$

Furthermore, since $F_r = f_r$, from (5.3) we obtain

$$F^i_\sigma = f^i_\sigma = \frac{1}{r!}(\partial \mathcal{L}/\partial y^i_\sigma), \qquad |\sigma| = r. \tag{6.5}$$

Formulas (6.4) and (6.5) together with (5.2) determine the sections $F_1,\dots,F_r$ by descending recurrence. They can also be used to obtain the local expression of Poincaré-Cartan forms. In fact, by (5.12) we have

$$\Theta = \sum_{h,j} \sum_{|\beta|=0}^{r-1} (-1)^{j-1} \lambda^h_{\beta j} \theta^h_\beta \wedge v_j + \mathcal{L} v \tag{6.6}$$

and

$$\lambda^h_{\beta j} = \sum_i \sum_{|\alpha|=|\beta|}^{r-1} (\alpha+(j))! A^{hi}_{\beta\alpha} F^i_{\alpha+(j)}, \qquad |\beta| = 0,\dots,r-1. \tag{6.7}$$

In particular

$$\lambda^h_{\beta j} = \frac{1+\beta_j}{r}(\partial \mathcal{L}/\partial y^h_{\beta+(j)}), \qquad |\beta| = r-1, \tag{6.8}$$

as follows from (6.7), (3.16) and (6.5). Note also that

$$\lambda^h_{\beta j} \in C^\infty(J^{2r-1-|\beta|}), \qquad |\beta| = 0,\dots,r-1. \tag{6.9}$$

Proposition 6.3. *With the above notations there exist functions* $\mu^h_{\beta j}$ *such that*:

$$\lambda^h_{\beta j} = \sum_{|\sigma|=0}^{r-1-|\beta|} C^\sigma_{\beta j} \mathbb{D}^\sigma (\partial \mathfrak{L} / \partial y^h_{\beta+(j)+\sigma}) + \mu^h_{\beta j}, \tag{6.10}$$

where the coefficient is given by $C^\sigma_{\beta j} = (-1)^{|\sigma|}(1+\beta_j)|\beta|!\,|\sigma|!\binom{\beta+(j)+\sigma}{\sigma}/$ $/|\beta+(j)+\sigma|!$ *and* $\mathbb{D}^\sigma = \mathbb{D}_1^{\sigma_1} = \mathbb{D}_n^{\sigma_n}$, $\mathbb{D}_j$ *being the vector field introduced in* (3.4). (Note that $[\mathbb{D}_j, \mathbb{D}_k] = 0$, which justifies the notation employed.)

$$\mu^h_{\beta j} \in C^\infty(J^{2r-2-|\beta|}) \tag{6.11}$$

and $\mu^h_{\beta j}$ *vanishes when* ∇_0 *and* ∇ *are the flat derivation laws associated to the coordinate system* (x_j, y_i).

PROOF. First note that all the functions $A^{hi}_{\beta\alpha}$ for $|\beta| < |\alpha|$ vanish when ∇_0 and ∇ are the flat derivation laws. Next, by descending recurrence on $|\sigma|$ and using (5.2), (5.3), (6.4), (6.5) it is not difficult to prove that there exist functions $G^i_\sigma \in C^\infty(J^{2r-|\sigma|-1})$, which vanish when ∇_0 and ∇ are the flat derivation laws, such that:

$$F^i_\sigma = \sum_{|\alpha|=0}^{r-|\sigma|} (-1)^{|\alpha|} \left[\frac{\binom{\sigma+\alpha}{\alpha}|\alpha|!}{|\sigma+\alpha|!} \right] \mathbb{D}^\alpha \left(\frac{\partial \mathfrak{L}}{\partial y^i_{\sigma+\alpha}} \right) + G^i_\sigma. \tag{6.12}$$

The result now follows from (6.7).

7. A more explicit formula for $d\Theta$

Proposition 7.1. *Let E be the Euler-Lagrange form associated to an r-order Lagrangian density relative to the derivation laws* ∇_0, ∇. *If D, D′ are vector fields in* J^{2r-1} *vertical over X, then* $i_D i_{D'} E = 0$.

PROOF. For $h = 0, \ldots, r-1$, let M^h_k be the module of $S^h T(X) \otimes V^*(Y)$--valued n-forms η on J^k such that $i_D i_{D'} \eta = 0$ for all vertical vector fields D, D'. Locally, M^h_k is spanned by the valued forms $v \otimes (\partial/\partial x)^\alpha \otimes dy_i$, $dy^i_\beta \wedge v_j \otimes$ $\otimes (\partial/\partial x)^\alpha \otimes dy_\alpha$ ($|\alpha| = h$, $|\beta| \leqslant k$). On the other hand, we note that, with the same notations as in (3.2) and (3.4), for any ordinary one-form w on J^k the following formula holds true:

$$L(w) = \sum_l \left(L_{\mathbb{D}_l} w - \sum_{j,\nu} \mathring{\Gamma}^\nu_{jl} w(\mathbb{D}_\nu)\, dx_j \right) \otimes dx_l. \tag{7.1}$$

Taking in particular $w = dy^i_\beta$ and $w = dx_\mu$ it is easily seen that

$$(c^{h+1}_{1,1} L) M^{h+1}_k \subset M^h_{k+1}. \tag{7.2}$$

Moreover, since

$$\prod_{i=0}^{h} (c_{1,1}^{i+1}L)(\eta) = \prod_{i=0}^{h-1} (c_{1,1}^{i+1}L)[(c_{1,1}^{h+1}L)(\eta)],$$

by induction on h we have that $\eta \in M_k^h$ implies

$$\prod_{i=0}^{h-1} (c_{1,1}^{i+1}L)(\eta) \in M_{k+h}^0.$$

The result now follows from (5.8) and (5.4), by setting $k = r$, $\eta = d\omega_{h+1}$.

Theorem 7.2. *Let E, Θ be the Euler-Lagrange and Poincaré-Cartan forms, respectively, associated to an r-order Lagrangian density $\mathcal{L}v$ with respect to the pair ∇_0, ∇. There exist unique bilinear mappings*

$$(\ ,\ )_{(k,l)}: (S^{k-1}T(X) \otimes V(Y)) \times_{J^{2r-1}} (S^{l-1}T(X) \otimes V(Y)) \to T(X)_{J^{2r-1}}$$

$$((k,l) \in I_r = \{(k,l) \in \mathbb{N} \times \mathbb{N};\quad 1 \leqslant k \leqslant l \leqslant 2r-1, k \leqslant r, k+l \leqslant 2r+1\})$$

which are alternating when $k = l$, such that:

$$d\Theta = \theta^{(1)} \wedge E + \sum_{(k,l) \in I_r} \left(\theta^{(k)} \underset{(k,l)}{\wedge} \theta^{(l)}\right) \cdot v, \tag{7.3}$$

where the form $(\theta^{(k)} \wedge_{(k,l)} \theta^{(l)})$. v is defined by the formula

$$\left(\left(\theta^{(k)} \underset{(k,l)}{\wedge} \theta^{(l)}\right) \cdot v\right)(D_0, \ldots, D_n) = \sum_{i<j} (-1)^{i+j-1} v\left(\left(\theta^{(k)} \underset{(k,l)}{\wedge} \theta^{(l)}\right)(D_i, D_j), D_0, \ldots, \hat{D}_i, \ldots, \hat{D}_j, \ldots, D_n\right).$$

Furthermore, the bilinear mappings $(\ ,\ )_{(k,l)}$ for $k + l = 2r + 1$ do not depend on the derivation laws chosen ∇_0 and ∇.

PROOF. First we shall prove that the form Φ of formula (5.6) locally belongs to the submodule M spanned by the forms: $\theta_\alpha^h \wedge \theta_\beta^i \wedge v_j$ ($|\alpha| < r$, $|\beta| < 2r - 1$, $|\alpha| + |\beta| < 2r$).

According to (5.9) it will be sufficient to prove that the forms

$$c_{k,k}^k c\left(d\theta^{(k)} \underset{(B)}{\wedge} (v \otimes f_k)\right) \qquad (k = 1, \ldots, r)$$

and

$$c_{k,k}^k cd\left(\theta^{(k)} \underset{(B)}{\wedge} \eta_k\right) \qquad (k = 1, \ldots, r-1)$$

belong to M, where

$$\eta_k = \sum_{h=1}^{r-k} (-1)^h \prod_{i=1}^{h-1} (c_{1,1}^{k+i} L)\, d\omega_{k+h} \qquad (k = 1, \ldots, r-1).$$

For the first group of these forms we proceed directly. Since dx_j, $\theta_\alpha^i (|\alpha| < k)$ is a local basis of $T^*(J^{k-1})_{J^k}$, it follows from (3.13) and (3.14) that there exist sections $S_{\alpha\beta}^{hi}$ of $S_{k-1}^k(X)_{J^k}$ such that

$$d\theta^{(k)} \underset{(B)}{\wedge} (v \otimes f_k) = \sum_{h,i} \sum_{|\alpha|<k} \sum_{|\beta|<k} \theta_\alpha^h \wedge \theta_\beta^i \wedge v \otimes S_{\alpha\beta}^{hi}.$$

This equation, when c and $c_{k,k}^k$ are applied, yields

$$c_{k,k}^k c\Big(d\theta^{(k)} \underset{(B)}{\wedge} (v \otimes f_k)\Big) = \frac{1}{k} \sum_{h,i} \sum_j \sum_{|\alpha|<k} \sum_{|\beta|<k} (-1)^{j-1} c_{k,k}^k (dx_j \cdot S_{\alpha\beta}^{hi}) \theta_\alpha^h \wedge \theta_\beta^i \wedge v_j.$$

thus proving our assertion in this case.

Next, we shall consider the second group of forms. We have

$$(*) \qquad c_{k,k}^k cd\Big(\theta^{(k)} \underset{(B)}{\wedge} \eta_k\Big) = c_{k,k}^k L\Big(\theta^{(k)} \underset{(B)}{\wedge} \eta_k\Big) - c_{k,k}^k dc\Big(\theta^{(k)} \underset{(B)}{\wedge} \eta_k\Big).$$

But from (5.10), (5.1) and Proposition 6.2, we conclude that

$$c_{1,1}^k c\eta_k = \Omega_k - \omega_k = c_{1,1}^k c(v \otimes (F_k - f_k)).$$

Thus, by using (b) of (1.8) and the fact that d commutes with contractions, the last term in (*) can be transformed as follows:

$$-c_{k,k}^k dc\Big(\theta^{(k)} \underset{(B)}{\wedge} \eta_k\Big) = dc_{k,k}^k \Big(\theta^{(k)} \underset{(B)}{\wedge} c\eta_k\Big) = dc_{k-1,k-1}^{k-1}\Big(\theta^{(k)} \underset{(B)}{\wedge} (c_{1,1}^k c\eta_k)\Big) =$$

$$= dc_{k-1,k-1}^{k-1}\Big(\theta^{(k)} \underset{(B)}{\wedge} c_{1,1}^k c(v \otimes (F_k - f_k))\Big) =$$

$$= dc_{k,k}^k \Big(\theta^{(k)} \underset{(B)}{\wedge} c(v \otimes (F_k - f_k))\Big) = -c_{k,k}^k dc\Big(\theta^{(k)} \underset{(B)}{\wedge} v \otimes (F_k - f_k)\Big) =$$

$$= -c_{k,k}^k L\Big(\theta^{(k)} \underset{(B)}{\wedge} v \otimes (F_k - f_k)\Big) + c_{k,k}^k cd\Big(\theta^{(k)} \underset{(B)}{\wedge} v \otimes (F_k - f_k)\Big) =$$

$$= -c_{k,k}^k L\Big(\theta^{(k)} \underset{(B)}{\wedge} v \otimes (F_k - f_k)\Big) + c_{k,k}^k c\Big(d\theta^{(k)} \underset{(B)}{\wedge} v \doteq (F_k - f_k)\Big) -$$

$$- (-1)^n c_{k,k}^k c\Big(\theta^{(k)} \underset{(B)}{\wedge} v \wedge d(F_k - f_k)\Big).$$

Upon substituting this expression into Eq. (*), we obtain

$$c^k_{k,k}cd\Big(\underset{(B)}{\theta^{(k)}\wedge}\eta_k\Big) = c^k_{k,k}L\Big(\underset{(B)}{\theta^{(k)}\wedge}(\eta_k - v\otimes(F_k - f_k))\Big) +$$

$$+ c^k_{k,k}c\Big(\underset{(B)}{d\theta^{(k)}\wedge}v\otimes(F_k - f_k)\Big) - (-1)^n c^k_{k,k}c\Big(\underset{(B)}{\theta^{(k)}\wedge}v\wedge d(F_k - f_k)\Big).$$

We shall now show that η_k belongs to the submodule M^k_{2r-k-1} introduced in the proof of Proposition 7.1. In fact, this in an immediate consequence of (7.2) and the following recurrence relations

$$\eta_{r-1} = -d\omega_r, \qquad \eta_k = -d\omega_{k+1} - c^{k+1}_{1,1}L\eta_{k+1} \qquad (k = 1, \ldots, r-2),$$

which follow directly from the definition of η_k. Therefore, η_k can be expressed as $\eta_k = v\otimes S^k + \sum_{i,j}\sum_{|\beta|<2r-k} dy^i_\beta\wedge v_j\otimes S^{ki}_{\beta j}$, for certain local sections S^k, $S^{ki}_{\beta j}$ of $S^kT(X)\otimes_{J^{2r-k-1}}V^*(Y)$. Or equivalently,

$$\eta_k = v\otimes\Big(S^k + \sum_{i,j}\sum_{|\beta|<2r-k}(-1)^{j-1}y^i_{\beta+(j)}S^{ki}_{\beta j}\Big) + \sum_{i,j}\sum_{|\beta|<2r-k}\theta^i_\beta\wedge v_j\otimes S^{ki}_{\beta j}.$$

Hence, the relation $c^k_{1,1}c(\eta_k - v\otimes(F_k - f_k)) = 0$ obtained before, implies:

(i) $$F_k - f_k = S^k + \sum_{i,j}\sum_{|\beta|<2r-k}(-1)^{j-1}y^i_{\beta+(j)}S^k_{\beta j}$$

and

(ii) $$(-1)^{j+l-1}c^k_{1,1}(dx_j\otimes S^{ki}_{\beta l}) + c^k_{1,1}(dx_l\otimes S^{ki}_{\beta j}) = 0.$$

From (i) we deduce:

$$\eta_k - v\otimes(F_k - f_k) = \sum_{i,j}\sum_{|\beta|<2r-k}\theta^i_\beta\wedge v_j\otimes S^{ki}_{\beta j}.$$

Hence,

$$c^k_{k,k}L\Big(\underset{(B)}{\theta^{(k)}\wedge}(\eta_k - v\otimes(F_k - f_k))\Big) = \sum_{i,j}\sum_{|\beta|<2r-k}\theta^{(k+1)}\wedge(\theta^i_\beta\wedge v_j\otimes S^k_{\beta j}) +$$

$$+\sum_i\sum_{j,l}\sum_{|\beta|<2r-k}\theta^{(k)}\wedge(\theta^i_{\beta+(l)}\wedge v_j\otimes(dx_l\otimes S^{ki}_{\beta j})) + \sum_{i,j}\sum_{|\beta|<2r-k}\theta^{(k)}\wedge$$

$$\wedge\theta^i_\beta\wedge L(v_j\otimes S^{ki}_{\beta j}),$$

where we have used the equality $L(\theta^i_\beta) = \sum_j\theta^i_{\beta+(j)}\otimes dx_j$.

Using (7.1) with $w = dx_\mu$, it is easily checked that $c^k_{k,k}L(\theta^{(k)}\wedge_{(B)}(\eta_k - v\otimes\otimes(F_k - f_k)))$ belong to M when $k > 1$ (i.e. when $k = 2, \ldots, r-1$). In addition, $v\wedge d(F_k - f_k)$ can be written as a linear combination of the forms $\theta^i_\sigma\wedge v\otimes(\partial/\partial x)^\alpha\otimes dy_a$ $(|\sigma|\leqslant 2r-k, |\alpha| = k)$. Thus, the term $c^k_{k,k}c(\theta^{(k)}\wedge_{(B)}v\wedge$
$\wedge d(F_k - f_k))$ also belongs to M when $k > 1$. On the other hand, by means of the same argument considered in the first part of the proof, it is easily seen

that the term $c^k_{k,k}c(d\theta^{(k)} \wedge_{(B)} v \otimes (F_k - f_k))$ belongs to M even for $k = 1$. Therefore, it only remains to prove that

$$c^1_{1,1}L\Big(\theta^{(1)} \underset{(B)}{\wedge} (\eta_1 - v \otimes (F_1 - f_1))\Big) - (-1)^n c^1_{1,1}c\Big(\theta^{(1)} \underset{(B)}{\wedge} v \wedge d(F_1 - f_1)\Big)$$

also belongs to M. To this end, we first separate the highest order terms in the above form, obtaining

$$\begin{aligned}
&c^1_{1,1}L\Big(\theta^{(1)} \underset{(B)}{\wedge} (\eta_1 - v \otimes (F_1 - f_1))\Big) - (-1)^n c^1_{1,1}c\Big(\theta^{(1)} \underset{(B)}{\wedge} v \wedge d(F_1 - f_1)\Big) = \\
&\quad = \sum_{h,i}\sum_{j,l}\sum_{|\beta|=2r-2}[(-1)^{j+l-1}B(\partial/\partial y_h \otimes c^1_{1,1}(dx_j \otimes S^{1i}_{\beta l})) + \\
&\qquad + B(\partial/\partial y_h \otimes c^1_{1,1}(dx_l \otimes S^{1i}_{\beta j}))]\theta^h_0 \wedge \theta^i_{\beta+(l)} \wedge v_j + \\
&\qquad + \sum_{i,j}\sum_{|\beta|<2r-2}(\theta^{(2)} \wedge (\theta^i_\beta \wedge v_j \otimes S^{1i}_{\beta j}) + \\
&\qquad + \sum_l \theta^{(1)} \wedge (\theta^i_{\beta+(l)} \wedge v_j \otimes dx_l \otimes S^{1i}_{\beta j})) + \\
&\qquad + \sum_{i,j}\sum_{|\beta|\le 2r-2}\theta^{(1)} \wedge (\theta^i_\beta \wedge L(v_j \otimes S^{1i}_{\beta j})) + (-1)^n\theta^{(1)} \wedge c(v \wedge dS^1) + \\
&\qquad + (-1)^n\theta^{(1)} \wedge c\Big(\sum_{i,j}\sum_{|\beta|<2r-2}(-1)^{j-1}v \wedge \theta^i_{\beta+(j)} \otimes S^{1i}_{\beta j}\Big) + \\
&\qquad + (-1)^n\theta^{(1)} \wedge c\Big(\sum_{i,j}\sum_{|\beta|\le 2r-2}(-1)^{j-1}y^i_{\beta+(j)}v \wedge dS^{1i}_{\beta j}\Big).
\end{aligned}$$

The first term on the right-hand side vanishes by virtue of (ii), while all the other terms lie in the submodule M. We have thus completed the proof of our first statement.

We shall now consider the uniqueness of the bilinear maps $(\,,\,)_{(k,l)}$. Locally, each one of these mappings determines n bilinear forms $(\,,\,)^j_{(k,l)}$ given by

$$(\,,\,)_{(k,l)} = \sum_j (\,,\,)^j_{(k,l)}\partial/\partial x_j.$$

Then, as $(\,,\,)_{(k,k)}$ is alternating, applying both sides of formula (7.3) to $(\partial/\partial y^h_\alpha, \partial/\partial y^i_\beta, \mathbb{D}_1, \ldots, \hat{\mathbb{D}}_j, \ldots, \mathbb{D}_n)$ ($|\alpha| < r$, $|\beta| < 2r-1$, $|\alpha| \leqslant |\beta|$, $|\alpha| + |\beta| < 2r$), we obtain:

$$\begin{aligned}
(7.4)\quad &\Phi(\partial/\partial y^h_\alpha, \partial/\partial y^i_\beta, \mathbb{D}_1, \ldots, \hat{\mathbb{D}}_j, \ldots, \mathbb{D}_n) = \\
&\quad = (-1)^{j-1}\epsilon_{\alpha\beta}((dx)^\alpha \otimes \partial/\partial y_h, (dx)^\beta \otimes \partial/\partial y_i)^j_{(|\alpha|+1,\,|\beta|+1)} + \\
&\qquad + \sum_{(k,l)\in I_{\alpha\beta}}\Big(\Big(\theta^{(k)} \underset{(k,l)}{\wedge} \theta^{(l)}\Big)\cdot v\Big)(\partial/\partial y^h_\alpha, \partial/\partial y^i_\beta, \mathbb{D}_1, \ldots, \hat{\mathbb{D}}_j, \ldots, \mathbb{D}_n),
\end{aligned}$$

where we have set,

$$\epsilon_{\alpha\beta} = 1/\alpha!\beta! \quad \text{for} \quad |\alpha| < |\beta| \quad \text{and} \quad \epsilon_{\alpha\beta} = 2/\alpha!\beta! \quad \text{for} \quad |\alpha| = |\beta|,$$

and

$$I_{\alpha\beta} = \{(k, l)\epsilon I_r; |\alpha| < k, |\beta| < l, (k, l) \neq (\widehat{|\alpha| + 1}, |\beta| + 1)\}.$$

In particular,

$$(7.5) \quad \Phi(\partial/\partial y^h_\alpha, \partial/\partial y^i_\beta, \mathbb{D}_1, \ldots, \widehat{\mathbb{D}_j}, \ldots, \mathbb{D}_n) =$$
$$= (-1)^{j-1}\epsilon_{\alpha\beta}((dx)^\alpha \otimes \partial/\partial y_h, (dx)^\beta \otimes \partial/\partial y_i)^j_{(|\alpha|+1, |\beta|+1)} \quad (|\alpha| + |\beta| = 2r - 1).$$

It is now clear that formulas (7.4) and (7.5) completely determine, by descending recurrence on $|\alpha| + |\beta|$, the bilinear mapping in question.

Because of the uniqueness of such mappings, in order to prove their existence it will be sufficient to give a local definition of them so that Eq. (7.3) will be fulfilled locally. First we use the above formulas to define $(\ ,\ )_{(k,l)}$ by descending recurrence on $k + l$. Next, we note that $(\theta^{(k)} \wedge_{(k,l)} \theta^{(l)}) \cdot v$ belongs to the submodule M when $(k, l)\epsilon I_r$. Thus, we only need to check that forms Φ and $\sum_{(k,l)\epsilon I_r}(\theta^{(k)} \wedge_{(k,l)} \theta^{(l)}) \cdot v$ coincide when applied to

$$(\partial/\partial y^h_\alpha, \partial/\partial y^i_\beta, \mathbb{D}_1, \ldots, \widehat{\mathbb{D}_j}, \ldots, \mathbb{D}_n),$$
$$(|\alpha| < r, |\beta| < 2r - 1, |\alpha| \leqslant |\beta|, |\alpha| + |\beta| < 2r).$$

However, this condition leads us to Eq. (7.4), which is fulfilled by the very definition of the bilinear mapping $(\ ,\ )_{(k,l)}$.

Finally, we shall prove the independence of the products $(\ ,\ )_{(k,l)}$, $k + l = 2r + 1$, by a method which will provide further information.

Let ψ: $V(J^{r-1}) \otimes_{J^{2r-1}} T(J^{2r-1}/J^{r-1}) \to T(X)_{J^{2r-1}}$ be the bilinear mapping given by the formula

$$\psi(D, D') = \tilde{v}^{-1}(i_D, i_D d\Theta),$$

where $\tilde{v}$: $T(X) \to \Lambda^{n-1}T^*(X)$ is the isomorphism induced by the volume element: $\tilde{v}(D) = i_D v$. Note that the definition makes sense, because formulas (6.6) and (6.7) imply that $i_{D'} d\Theta = \sum_{h,j}\sum_{|\beta|=0}^{r-1}(-1)^{j-1}D'(\lambda^h_{\beta j})\theta^h_\beta \wedge v_j$ for every vector field D' in J^{2r-1} vertical over J^{r-1}. Hence, $i_{D'} d\Theta$ is a section of the vector sub-bundle $\Lambda^n T^*(J^{r-1})_{J^{2r-1}}$. Moreover, from this we also obtain the local expression of ψ:

$$\psi(D, D') = -\sum_{h,j}\sum_{|\beta|=0}^{r-1}(D'\lambda^h_{\beta j})(Dy^h_\beta)(\partial/\partial x_j).$$

Then, since $\lambda^h_{\beta j}$ is a function on $J^{2r-1-|\beta|}$ (formula (6.9)), we have:

(*a*) The bilinear mapping ψ vanishes on the vector sub-bundle

$$T(J^{r-1}/J^{2r-2-k}) \otimes_{J^{2r-1}} T(J^{2r-1}/J^k) \quad \text{of} \quad V(J^{r-1}) \otimes_{J^{2r-1}} T(J^{2r-1}/J^{r-1})$$
$$\text{for} \quad r-1 \leqslant k \leqslant 2r-2.$$

Let us fix an index k such that $r-1 \leqslant k \leqslant 2r-2$, and let ψ_k be the restriction of ψ to $V(J^{r-1}) \otimes_{J^{2r-1}} T(J^{2r-1}/J^k)$. According to (*a*), ψ_k induces a bilinear mapping on the quotient $\bar{\psi}_k$: $V(J^{2r-2-k}) \otimes_{J^{2r-1}} T(J^{2r-1}/J^k) \to T(X)_{J^{2r-1}}$. Let $\bar{\psi}'_k$ be the restriction of $\bar{\psi}_k$ to the vector sub-bundle $S^{2r-2-k}T^*(X) \otimes \otimes V(Y) \otimes_{J^{2r-1}} T(J^{2r-1}/J^k)$. (Recall that $T(J^k/J^{k-1})$ is canonically isomorphic to $S^kT^*(X) \otimes_{J^k} V(Y)$.) Then, as above, we have:

(*b*) The mapping $\bar{\psi}'_k$ vanishes on the vector sub-bundle

$$S^{2r-2-k}T^*(X) \otimes V(Y) \otimes_{J^{2r-1}} T(J^{2r-1}/J^{k+1}).$$

Thus, $\bar{\psi}'_k$ finally induces a bilinear map

$$B_k\colon (S^{2r-2-k}T^*(X) \otimes_{J^{2r-1}} V(Y)) \otimes (S^{k+1}T^*(X) \otimes_{J^{2r-1}} V(Y)) \to T(X)_{J^{2r-1}}$$
$$(r-1 \leqslant k \leqslant 2r-2)$$

given locally by

$$(7.6)\quad B_k((dx)^\alpha \otimes (\partial/\partial y_h), (dx)^\beta \otimes (\partial/\partial y_i)) = -\alpha!\beta! \sum_j (\partial\lambda^h_{\alpha j}/\partial y^i_\beta)(\partial/\partial x_j) =$$
$$= \alpha!\beta!\bar{v}^{-1}(i_{(\partial/\partial y^i_\beta)} i_{(\partial/\partial y^h_\alpha)} d\Theta) \quad (|\alpha| = 2r-2-k, |\beta| = k+1)$$

We shall now compare (7.6) with (7.5). Let us fix two multi-indices α, β such that $|\alpha| < r$, $|\beta| < 2r-1$, $|\alpha| \leqslant |\beta|$, $|\alpha| + |\beta| = 2r-1$. We set $k = |\beta| - 1$, so that $|\alpha| = 2r-2-k$ and $r-1 \leqslant k \leqslant 2r-3$. Since $|\alpha| > 0$, $|\beta| > 0$ in this case, from Proposition 7.1 and formula (5.6) we obtain $i_{(\partial/\partial y^i_\beta)} i_{(\partial/\partial y^h_\alpha)}\Phi = i_{(\partial/\partial y^i_\beta)} i_{(\partial/\partial y^h_\alpha)}(d\Theta)$, and (7.5) becomes

$$((dx)^\alpha \otimes (\partial/\partial y_h), (dx)^\beta \otimes (\partial/\partial y_i))^j_{(2r-1-k,k+2)} = -\alpha!\beta!(\partial\lambda^h_j/\partial y^i_\beta).$$

Hence,

$$(7.7)\qquad B_k = (\,,\,)_{(2r-1-k,k+2)} \qquad (r-1 \leqslant k \leqslant 2r-3)$$

Lemma 7.3. *The bilinear mappings B_k, $r-1 \leqslant k \leqslant 2r-2$, do not depend on the derivation laws chosen. Actually, they only depend on the Hessian metric of the Lagrangian $\mathcal{L}$.*

PROOF OF THE LEMMA. According to (6.10) and (6.11) we have

$$\partial\lambda^h_{\alpha j}/\partial y^i_\beta = \sum_{|\sigma|=0}^{r-1-|\alpha|} C^\sigma_{\alpha j}(\partial/\partial y^i_\beta)\mathbb{D}^\sigma(\partial\mathfrak{L}/\partial y^h_{\alpha+(j)+\sigma})$$

$$(|\alpha| = 2r-2-k, |\beta| = k+1, r-1 \leqslant k \leqslant 2r-2).$$

On the other hand, we note that for every $f \in C^\infty(J^k)$ the following formulas hold true:

$$\partial/\partial y^i_\beta(\mathbb{D}^\sigma f) = 0, \quad \text{if} \quad |\sigma| + k < |\beta|$$
$$\partial/\partial y^i_\beta(\mathbb{D}^\sigma f) = \partial f/\partial y^i_{\beta-\sigma}, \quad \text{if} \quad |\sigma| + k = |\beta|.$$

(This is proved by induction on $|\sigma|$ using the identity $[\partial/\partial y^i_\beta, \mathbb{D}_j] = \partial/\partial y^i_{\beta-(j)}$.) Hence,

$$\partial\lambda^h_{\alpha j}/\partial y^i_\beta = \sum_{|\sigma|=r-1-|\alpha|} C^\sigma_{\alpha j}(\partial^2\mathfrak{L}/\partial y^i_{\beta-\sigma}\partial y^h_{\alpha+(j)+\sigma}). \tag{7.8}$$

As $|\beta - \sigma| = |\alpha + (j) + \sigma| = r$, the lemma follows and the proof of the theorem is complete.

Corollary 7.4. *For every vector field D in J^{2r-1} vertical over J^{2r-2} the valued $(n-1)$-form $i_D E$ does not depend on the derivation laws chosen.*

PROOF. Since D lies in $T(J^{2r-1}/J^{2r-2})$, from (7.3) we obtain $i_D d\Theta = -\theta^{(1)} \wedge (i_D E)$. Locally, there exist ordinary n-forms such that $E = \sum_h w_h \otimes dy_h$, and by applying $i_{(\partial/\partial y_h)}$ to the first equation, by virtue of Proposition 7.1, we have $i_D w_h = -i_{(\partial/\partial y_h)} i_D d\Theta$. Thus, $i_D w_h$ is a section of $\Lambda^{n-1}T^*(X)_{J^{2r-1}}$. Therefore, it is completely determined by $\tilde{v}^{-1}(i_D w_h) = B_{2r-2}(\partial/\partial y_h, D)$. The result now follows from the previous lemma.

Corollary 7.5. *Let E, E' be the Euler-Lagrange forms associated to an r-order Lagrangian density with respect to the derivation laws (∇_0, ∇), (∇'_0, ∇'), respectively. Then, there exists a unique $Hom_{J^{2r-1}}(V(J^{2r-2}), V^*(Y))$-valued $(n-1)$-form η on J^{2r-1} horizontal over X such that*

$$E' - E = \theta^{2r-1} \wedge \eta.$$

In particular, for every local section s of Y, the valued form $E_{|j^{2r-1}s}$ does not depend on the derivation laws chosen.

PROOF. We set $E = \sum_i w_i \otimes dy_i$, $E' = \sum_i w'_i \otimes dy_i$ and $G_i = w'_i(\mathbb{D}_1, \ldots, \mathbb{D}_n) - w_i(\mathbb{D}_1, \ldots, \mathbb{D}_n)$. From (7.3) we obtain

$$i_{\mathbb{D}_n}\cdots i_{\mathbb{D}_1}(d\Theta) = \sum_{j=1}^{n}(-1)^{j+n}L_{\mathbb{D}_j}\Big(\sum_{k=1}^{r}\theta^{(k)}\circ\Omega_k(\mathbb{D}_1,\ldots,\hat{\mathbb{D}}_j,\ldots,\mathbb{D}_n)\Big) +$$
$$+ i_{\mathbb{D}_n}\cdots i_{\mathbb{D}_1}(d\mathfrak{L}\wedge v) = (-1)^n\theta^{(1)}\circ(i_{\mathbb{D}_n}\cdots i_{\mathbb{D}_1}E).$$

Writing down the corresponding equation for E' and subtracting, we have

$$\sum_i G_i\theta_0^i = \sum_j L_{\mathbb{D}_j}\Big(\sum_i\sum_{|\alpha|<r} g_{\alpha j}^i\theta_\alpha^i\Big) = \sum_{i,j}\sum_{|\alpha|<r}(\mathbb{D}_j g_{\alpha j}^i)\theta_\alpha^i + \sum_{i,j}\sum_{|\alpha|<r} g_{\alpha j}^i\theta_{\alpha+(j)}^i,$$

for certain differentiable functions $g_{\alpha j}^i$ on J^{2r-1}. Therefore,

(*a*) $G_i = \sum_j(\mathbb{D}_j g_{0j}^i)$.

(*b*) $\sum_j(\mathbb{D}_j g_{\alpha j}^i) + \sum_{\beta+(j)=\alpha} g_{\beta j}^i = 0 \quad (0<|\alpha|<r)$.

(*c*) $\sum_{\beta+(j)=\alpha} g_{\beta j}^i = 0 \quad (|\alpha| = r)$.

From (*b*) it is verified that the function $G_i^k = (-1)^k\sum_j\sum_{|\beta|=k}\mathbb{D}^{\beta+(j)}g_{\beta j}^i$ does not depend on the index $k = 0,\ldots,r-1$. But $G_i^0 = G_i$ and $G_i^{r-1} = 0$, as follows from (*a*) and (*c*), respectively. Hence, $G_i = 0$. Therefore, by Proposition 7.1, we can write

$$E' - E = \sum_{h,i}\sum_j\sum_{|\alpha|\le 2r-1}\eta_{\alpha j}^{hi}\theta_\alpha^h\wedge v_j\otimes dy_i \qquad (\eta_{\alpha j}^{hi}\in C^\infty(J^{2r-1}))$$

and, by virtue of the preceding corollary, the coefficients $\eta_{\alpha j}^{hi}$ for $|\alpha| = 2r-1$ must vanish. Thus, $\eta = \sum_{h,i}\sum_j\sum_{|\alpha|<2r-1}\eta_{\alpha j}^{hi}v_j\otimes dy_\alpha^h\otimes dy_i$ is the unique form fulfilling the conditions of the statement.

Corollary 7.6. *Let* Θ, Θ' *be the Poincaré-Cartan forms associated to an r-order Lagrangian density with respect to the derivation laws* (∇_0,∇), (∇_0',∇'), *respectively. There exists a* $\mathrm{Hom}_{J^{2r-1}}(V(J^{2r-2}), V^*(J^{r-1}))$*-valued* $(n-1)$*-form* $\bar\eta$ *on* J^{2r-1} (*not necessarily unique*) *such that*

$$d\Theta' - d\Theta = \theta^r\wedge(\theta^{2r-1}\wedge\bar\eta).$$

PROOF. This is an immediate consequence of formula (7.3) and the previous corollary.

Corollary 7.7. *The Poincaré-Cartan form* Θ *is projectable to* J^{2r-h} *for* $h = 2,\ldots,r$ *if and only if* B_k *vanishes for* $k = 2r-h,\ldots,2r-2$. *Thus, since the bilinear mapping* B_k *do not depend on the derivation laws chosen, if the form* Θ *corresponding to the pair* ∇_0, ∇ *is projectable to* J^{2r-h}, *then it is also true for the form* Θ' *corresponding to any other pair of derivation laws.*

PROOF. This follows from the first equality in formula (7.6).

Remark. According to formula (7.8) a sufficient condition for the form Θ to be projectable to J^r is that $\mathfrak{L}: J^r \to \mathbb{R}$ must be an affine function over J^{r-1}. This condition is also necessary if $\dim X = n = 1$. Note that in this case formula (7.8) reads $\partial \lambda^h_{\alpha 1}/\partial y^i_\beta = (-1)^{r-1-\alpha}(\partial^2 \mathfrak{L}/\partial y^h_r y^i_r)$.

8. Analysis of how Poincaré-Cartan forms depend on the derivation laws chosen

Theorem 8.1. *The Poincaré-Cartan form Θ associated to an r-order Lagrangian density $\mathfrak{L}v$ with respect to a pair of derivation laws ∇_0, ∇ does not depend on the vertical derivation law ∇. In fact, the value taken by Θ at a point $j^{2r-1}_x(s)$ only depends on $j^{r-2}_x(\mathrm{sym}\,\nabla_0)$, where sym ∇_0 means the symmetric connection associated to ∇_0.*

PROOF. Let Θ' be the Poincaré-Cartan form constructed with the same linear connection ∇_0 of the manifold X and another derivation law ∇' in the vertical bundle. According to formula (6.6), locally we have

$$\Theta = \sum_{h,j} \sum_{|\beta|=0}^{r-1} (-1)^{j-1} \lambda^h_{\beta j} \theta^h_\beta \wedge v_j + \mathfrak{L}v \quad \text{and} \quad \Theta' = \sum_{h,j} \sum_{|\beta|=0}^{r-1} (-1)^{j-1} \lambda'^h_{\beta j} \theta^h_\beta \wedge v_j + \mathfrak{L}v.$$

We set:

$$G^h_{\alpha j} = \sum_i \sum_{\alpha \le \sigma,\, |\sigma| \le r-1} (\sigma + (j))! \binom{|\sigma|}{|\alpha|} a^{hi}_{\sigma-\alpha} F^i_{\sigma+(j)} \qquad (0 \leqslant |\alpha| \leqslant r-1),$$

and similarly for the form Θ'. Hence, from formulas (6.7) and (3.12) we obtain

$$\lambda^h_{\beta j} = \sum_{|\alpha|=|\beta|}^{r-1} \mathring{a}_{\beta\alpha} G^h_{\alpha j} \qquad (0 \leqslant |\beta| \leqslant r-1).$$

Functions $G^h_{\alpha j}$ satisfy the following property: If $\alpha + (j) = \alpha' + (j')$, then $G^h_{\alpha j} = G^h_{\alpha' j'}$. Actually, if $\alpha + (j) = \alpha' + (j')$, we have $\alpha = \tau + (j')$, $\alpha' = \tau + (j)$ for a certain multi-index τ, and, thus, from the definition of $G^h_{\alpha j}$ we obtain

$$G^h_{\alpha j} = G^h_{\tau+(j'),j} = \sum_i \sum_{\substack{|\sigma| \le r-1 \\ \tau \le \sigma - (j')}} (\sigma + (j))! \binom{|\sigma|}{|\tau|+1} a^{hi}_{\sigma-(j')-\tau} F^i_{\sigma+(j)} =$$

(by setting $\sigma' = \sigma - (j')$)

$$G^h_{\alpha j} = \sum_i \sum_{\substack{|\sigma'| \leq r-2 \\ \tau \leq \sigma'}} (\sigma' + (j) + (j'))! \binom{|\sigma'| + 1}{|\tau| + 1} a^{hi}_{\sigma' - \tau} F^i_{\sigma' + (j) + (j')} =$$

(by setting $\sigma'' = \sigma' + (j)$)

$$= \sum_i \sum_{\substack{|\sigma''| \leq r-1 \\ \tau \leq \sigma'' - (j)}} (\sigma'' + (j'))! \binom{|\sigma''|}{|\tau| + 1} a^{hi}_{\sigma'' - (j) - \tau} F^i_{\sigma'' + (j')} = G^h_{\tau + (j), j'} =$$

$$= G^h_{\alpha' j'}.$$

We can therefore define functions $\bar{G}^h_\alpha$ $(1 \leqslant |\alpha| \leqslant r)$ such that $G^h_{\alpha j} = \bar{G}^h_{\alpha + (j)}$, for $|\alpha| = 0, \ldots, r-1$.

On the other hand, from formula of Corollary 7.6 we deduce that

$$i_{\mathbb{D}_1} \cdots i_{\mathbb{D}_n} d(\Theta' - \Theta) = 0.$$

Hence,

(a) $\sum_j \mathbb{D}_j(\lambda'^h_{0j} - \lambda^h_{0j}) = 0$

(b) $\sum_i \mathbb{D}_j(\lambda'^h_{\beta j} - \lambda^h_{\beta j}) + \sum_i (\lambda'^h_{\beta - (j), j} - \lambda^h_{\beta - (j), j}) = 0 \qquad (0 < |\beta| < r)$

(c) $\sum_j (\lambda'^h_{\beta - (j), j} - \lambda^h_{\beta - (j), j}) = 0 \qquad (|\beta| = r).$

We shall now prove by descending recurrence on $k = 0, \ldots, r-1$ that

(*) $$\lambda^h_{\beta j} = \lambda'^h_{\beta j} \quad \text{and} \quad G^h_{\alpha j} = G'^h_{\alpha j} \qquad (|\alpha| = |\beta| = k).$$

For $k = r-1$, it follows from (6.8) that $\lambda^h_{\beta j} = \lambda'^h_{\beta j}$ $(|\beta| = r-1)$, and from the definition of $G^h_{\alpha j}$ we obtain $G^h_{\alpha j} = (\alpha!/|\alpha|!)\lambda^h_{\alpha j}$ when $|\alpha| = r-1$. Hence, in this case $G^h_{\alpha j} = G'^h_{\alpha j}$. Let us assume that conditions (*) are fulfilled for $r-1, r-2, \ldots, k > 0$. Thus, equation (b) for $|\beta| = k$ becomes,

$$0 = \sum_j (\lambda'^h_{\beta - (j), j} - \lambda^h_{\beta - (j), j}) = \sum_j \sum_{|\alpha| = k-1} \mathring{a}_{\beta - (j), \alpha} (\bar{G}'^h_{\alpha + (j)} - \bar{G}^h_{\alpha + (j)}) =$$

$$= \left(\sum_j \frac{(k-1)!}{(\beta - (j))!} \right) (\bar{G}'^h_\beta - \bar{G}^h_\beta) = (k!/\beta!)(\bar{G}'^h_\beta - \bar{G}^h_\beta).$$

Therefore, $\bar{G}'^h_\beta = \bar{G}^h_\beta$; or in other words, $G^h_{\alpha j} = G'^h_{\alpha j}$ for $|\alpha| = k-1$. Furthermore, for $|\beta| = k-1$ we have

$$\lambda'^h_{\beta j} = \sum_{|\alpha| = |\beta|}^{r-1} \mathring{a}_{\beta\alpha} G'^h_{\beta j} = \frac{(k-1)!}{\beta!} G'^h_{\beta j} + \sum_{|\alpha| = k}^{r-1} \mathring{a}_{\beta\alpha} G^h_{\alpha j} =$$

$$= \frac{(k-1)!}{\beta!} G^h_{\beta j} + \sum_{|\alpha| = k}^{r-1} \mathring{a}_{\beta\alpha} G^h_{\alpha j} = \sum_{|\alpha| = |\beta|}^{r-1} \mathring{a}_{\beta\alpha} G^h_{\alpha j} = \lambda^h_{\beta j}.$$

Thus, (*) is proved for $|\alpha| = |\beta| = k - 1$. In particular, we conclude that the corresponding coefficients of forms Θ and Θ' coincide. Hence, $\Theta = \Theta'$.

Since Θ does not depend on ∇, this form can be calculated using the flat vertical derivation law associated to the local basis $\partial/\partial x_j$, $\partial/\partial y_i$ (i.e., by taking $\Gamma^a_{ji} = \bar{\Gamma}^a_{hi} = 0$). In this case, we have $a^{hi}_\alpha = 0$ when $\alpha > 0$ and $A^{hi}_{\beta\alpha} = \delta_{hi}\mathring{a}_{\beta\alpha}$. Consequently, equations (5.2), (6.4) and (6.7) can now be written respectively as follows:

$$(8.1)\qquad |\beta|!f^h_\beta + \sum_{|\alpha| = |\beta|+1}^{r} \alpha!\mathring{a}_{\beta\alpha}f^h_\alpha = \sum_{|\alpha|=|\beta|}^{r} \alpha!\mathring{a}_{\beta\alpha}f^h_\alpha = \partial\mathcal{L}/\partial y^h_\beta \qquad (0 \leqslant |\beta| \leqslant r-1).$$

$$(8.2)\qquad F^i_\sigma = f^i_\sigma - \sum_j (1+\sigma_j)(\mathbb{D}_j F^i_{\sigma+(j)}) - \sum_{j,k,l} (1+\sigma_k-\delta_{kl})(1+\sigma_j+\delta_{jk}-\delta_{jl})\mathring{\Gamma}^l_{jk}F^i_{\sigma+(j)+(k)-(l)}, \qquad (|\sigma| = 1,\ldots,r-1).$$

$$(8.3)\qquad \lambda^h_{\beta j} = \sum_{|\alpha|=|\beta|}^{r-1} (\alpha+(j))!\mathring{a}_{\beta\alpha}F^h_{\alpha+(j)} = \sum_{|\sigma|=|\beta|+1}^{r} \sigma!\mathring{a}_{\beta,\sigma-(j)}F^h_\sigma \qquad (0 \leqslant |\beta| \leqslant r-1).$$

Moreover, a direct computation shows that formulas (8.2) and (3.9) remain true when the components $\mathring{\Gamma}^l_{jk}$ are substituted by the functions $\mathring{\Gamma}^{*l}_{jk} = \frac{1}{2}(\mathring{\Gamma}^l_{jk} + \mathring{\Gamma}^l_{kj})$, which obviously proves tht Θ only depends on sym (∇_0) (see [10; Proposition 7.9 of Chapter III]).

Finally, we shall prove our last assertion of the statement. First we note that $\mathring{a}_{\beta\alpha}(x)$ for $|\beta| < |\alpha|$ only depends on $j^{|\alpha|-|\beta|-1}_x(\nabla_0)$, as is easily checked by induction on $|\alpha|$ using the recurrence relations for the functions $\mathring{a}_{\beta\alpha}$ (formula (3.9)). Formula (8.1) thus implies that f^h_β only depends on $j^{r-1-|\beta|}(\nabla_0)$ for $|\beta| = 0,\ldots,r-1$. Similarly, from (8.2) we derive that F^i_σ only depends on $j^{r-1-|\sigma|}(\nabla_0)$ for $|\sigma| = 1,\ldots,r-1$. Thus, from (8.3) we conclude that $\lambda^h_{\beta j}$ only depends on $j^{r-2-|\beta|}(\nabla_0)$ for $|\beta| = 0,\ldots,r-2$. As the coefficients $\lambda^h_{\beta j}$ $(|\beta| = r-1)$ do not depend on ∇_0, the proof of the theorem is complete.

Corollary 8.2. *The Legrende form Ω associated to an r-order Lagrangian density with respect to a pair of derivation laws ∇_0, ∇ does not depend on the vertical derivation law ∇.*

Remark. The forms $\Omega_1,\ldots,\Omega_{r-1}$, defined in (5.10) do depend on ∇.

As an example we shall now compute the first group of coefficients of Poincaré-Cartan form Θ depending on the linear connection ∇_0, for an arbitrary

r-order variational problem; that is, $\lambda^h_{\beta j}$ ($|\beta| = r - 2$). According to the above three formulas, we have

$$\lambda^h_{\beta u} = \frac{1+\beta_u}{r-1}(\partial\mathfrak{L}/\partial y^h_{\beta+(u)}) - \frac{1+\beta_u}{r(r-1)}\sum_j(1+\beta_j+\delta_{ju})\mathbb{D}_j(\partial\mathfrak{L}/\partial y^h_{\beta+(j)+(u)}) -$$

$$- \frac{1+\beta_u}{r!(r-1)}\sum_{|\alpha|=r}\alpha!\mathring{a}_{\beta+(u),\alpha}(\partial\mathfrak{L}/\partial y^h_\alpha) + \sum_{|\alpha|=r}(\alpha!/r!)\mathring{a}_{\beta,\alpha-(u)}(\partial\mathfrak{L}/\partial y^h_\alpha) +$$

$$- \frac{1+\beta_u}{r(r-1)}\sum_{j,k,l}(1+\beta_k+\delta_{ku}-\delta_{kl})(1+\beta_j+\delta_{ju}+$$

$$+\delta_{jk}-\delta_{jl})\mathring{\Gamma}^l_{jk}(\partial\mathfrak{L}/\partial y^h_{\beta+(u)+(j)+(k)+(l)})$$

Moreover, using the recurrence formulas for the functions $\mathring{a}_{\beta\alpha}$ we find

$$\mathring{a}_{\beta+(u),\alpha} = \sum_j \mathring{a}_{\beta+(u)-(j),\alpha-(j)} -$$

$$- \sum_{j,k,l}(1+\alpha_l-\delta_{jl}-\delta_{kl})\mathring{\Gamma}^l_{jk}\mathring{a}_{\beta+(u),\alpha+(l)-(j)-(k)},$$

and finally,

$$\lambda^h_{\beta u} = \frac{1+\beta_u}{r-1}(\partial\mathfrak{L}/\partial y^h_{\beta+(u)}) - \tag{8.4}$$

$$- \frac{1+\beta_u}{r(r-1)}\sum_j(1+\beta_j+\delta_{ju})\mathbb{D}_j(\partial\mathfrak{L}/\partial y^h_{\beta+(j)+(u)}) -$$

$$- \sum_{|\alpha|=r}(\alpha!/r!)\left[\frac{1+\beta_u}{r-1}\sum_j\mathring{a}_{\beta+(u)-(j),\alpha-(j)} - \mathring{a}_{\beta,\alpha-(u)}\right](\partial\mathfrak{L}/\partial y^h_\alpha)$$

$$(|\beta| = r-2).$$

Proposition 8.3. *For variational problems of order $r \leqslant 2$ on an arbitrary fibred manifold and for variational problems of arbitrary order on a fibred manifold with a 1-dimensional base manifold the Poincaré-Cartan form does not depend on the linear connection ∇_0.*

PROOF. For first order variational problems the result is well-known and, in fact, follows from formula (6.8). For second order variational problems the result follows from the above formula, since in this case the last summand on the right hand-side vanishes. Let us now consider the case $\dim X = n = 1$. Dropping the corresponding index to the base manifold in the lo-

cal expressions, formulas (3.9), (8.1), (8.2) and (8.3) read, respectively, as follows:

$$\mathring{a}_{\beta\alpha} = \partial \mathring{a}_{\beta,\alpha-1}/\partial x + \mathring{a}_{\beta-1,\alpha-1} - (\alpha-1)\mathring{\Gamma}\mathring{a}_{\beta,\alpha-1},$$

$$\beta! f^h_\beta + \sum_{\alpha=\beta+1}^{r} \alpha! \mathring{a}_{\beta\alpha} f^h_\alpha = \partial \mathfrak{L}/\partial y^h_\beta$$

$$F^i_\sigma = f^i_\sigma - (\sigma+1)(\mathbb{D}F^i_{\sigma+1}) - \sigma(\sigma+1)\mathring{\Gamma}F^i_{\sigma+1},$$

$$\lambda^h_\beta = \sum_{\sigma=\beta+1}^{r} \sigma! \mathring{a}_{\beta,\sigma-1} F^h_\sigma$$

It is then easily checked that the following recurrence relation holds true,

$$\lambda^i_{\beta-1} = \partial \mathfrak{L}/\partial y^i_\beta - \mathbb{D}\lambda^i_\beta,$$

thus proving the independence of form Θ in this case.

Remark. The above proposition can also be proved without calculation using corollary 7.6 and the system (*a*) – (*b*) – (*c*) in the proof of the previous theorem (see [4], [16]).

We shall now calculate the coefficients of the Poincaré-Cartan form for a third order variational problem. From formula (6.8) we derive in particular,

$$\lambda^h_{(kl)j} = \frac{1}{3}(1 + \delta_{jk} + \delta_{jl})\left(\frac{\partial \mathfrak{L}}{\partial y^h_{(jkl)}}\right).$$

The values of the intermediate coefficients are directly deduced from the general formula (8.4). In fact, we have

$$\lambda^h_{(u)u} = \partial \mathfrak{L}/\partial y^h_{(uu)} - \frac{1}{6}\sum_j (juu)! \mathbb{D}_j(\partial \mathfrak{L}/\partial y^h_{(juu)})$$

$$\lambda^h_{(v)u} = \frac{1}{2}(\partial \mathfrak{L}/\partial y^h_{(uv)}) - \frac{1}{6}\sum_j (juv)! \mathbb{D}_j(\partial \mathfrak{L}/\partial y^h_{(juv)}) +$$

$$+ \frac{1}{12}\sum_{j,k} (jkv)! \mathring{\Gamma}^u_{jk}(\partial \mathfrak{L}/\partial y^h_{(jkv)}) - \frac{1}{12}\sum_{j,k} (jku)! \mathring{\Gamma}^v_{jk}(\partial \mathfrak{L}/\partial y^h_{(jku)}) \qquad (u \neq v).$$

In all these formulas we have used the following notations: $(jk) = (j) + (k)$, $(jkl) = (j) + (k) + (l)$, etc. Finally, the first group of coefficients may be calculated by reiterating the above method. We obtain

$$\lambda^h_{0u} = \partial\mathfrak{L}/\partial y^h_{(u)} - \frac{1}{2}\sum_j (ju)!\,\mathbb{D}_j(\partial\mathfrak{L}/\partial y^h_{(ju)}) +$$

$$+ \frac{1}{6}\sum_{j,k}(1+\delta_{ju})(1+\delta_{jk}+\delta_{ku})\mathbb{D}_j\mathbb{D}_k(\partial\mathfrak{L}/\partial y^h_{(jku)}) +$$

$$+ \frac{1}{12}\sum_{j,k,l}(jkl)!\,\mathbb{D}_l\{\overset{\circ}{\Gamma}{}^u_{jk}(\partial\mathfrak{L}/\partial y^h_{(jkl)})\} -$$

$$- \frac{1}{12}\sum_{j,k,l}(jku)!\,\mathbb{D}_l\{\overset{\circ}{\Gamma}{}^l_{jk}(\partial\mathfrak{L}/\partial y^h_{(jku)}\}.$$

9. Variation formula of Lagrangian density: characterization of critical sections

Proposition 9.1 (Variation formula of Lagrangian density.) *Let Θ be a Poincaré-Cartan form associated to an r-orden Lagrangian density $\mathfrak{L}v$. For every infinitesimal contact transformation $\bar{D}$ in J^{2r-1} there exists a $V^*(J^{2r-2})$-valued $(n-1)$-form ξ on J^{2r-1} such that,*

$$L_{\bar{D}}(\mathfrak{L}v) = \theta^{(1)}(\bar{D})\circ E + d(i_{\bar{D}}\Theta) + \theta^{2r-1}\wedge\xi. \tag{9.1}$$

The linear functional $\delta_s\mathbb{L}$ defined in §4 is thus given by the following formula:

$$(\delta_s\mathbb{L})(D) = \int_{j^r s} L_D(\mathfrak{L}v) = \int_{j^{2r-1}s}\theta^{(1)}(D_{(2r-1)})\circ E \qquad (D\in T^r_c(U)). \tag{9.2}$$

PROOF. According to formula (5.12) and the definition of infinitesimal contact transformations, there exists an endomorphism ϕ of the vertical vector bundle so that,

$$\begin{aligned} L_{\bar{D}}(\mathfrak{L}v) &= L_{\bar{D}}\Theta - (L_{\bar{D}}\theta^r)\wedge\Omega - \theta^r\wedge(L_{\bar{D}}\Omega) = \\ &= i_{\bar{D}}\,d\Theta + d(i_{\bar{D}}\Theta) - \theta^r\wedge(\phi^*\circ\Omega + L_{\bar{D}}\Omega). \end{aligned}$$

On the other hand, decomposition (7.3) implies in particular that there exists $\mathrm{Hom}_{J^{2r-1}}(V(J^{2r-2}), V^*(J^{r-1}))$-valued $(n-1)$-form $\tilde{\eta}$ on J^{2r-1} such that

$$d\Theta = \theta^{(1)}\wedge E + \theta^r\wedge(\theta^{2r-1}\wedge\tilde{\eta}).$$

Hence,

$$\begin{aligned} i_{\bar{D}}\,d\Theta = \theta^{(1)}(\bar{D})\circ E - \theta^{(1)}\wedge(i_{\bar{D}}E) + \theta^r(\bar{D})\circ(\theta^{2r-1}\wedge\tilde{\eta}) - \\ - \theta^r\wedge(\theta^{2r-1}(\bar{D})\circ\tilde{\eta}) + \theta^r\wedge(\theta^{2r-1}\wedge(i_{\bar{D}}\tilde{\eta})). \end{aligned}$$

Thus, in order to obtain formula (9.1) it is sufficient to take

$$\xi = -\phi^*\circ\Omega - L_{\bar{D}}\Omega - i_{\bar{D}}E + \theta^r(\bar{D})\circ\tilde{\eta} - \theta^{2r-1}(\bar{D})\circ\tilde{\eta} - \theta^r\wedge(i_{\bar{D}}\tilde{\eta}).$$

Since the support of D has compact image in U, formula (9.2) follows directly from Stokes' theorem.

Theorem 9.2. (First Characterization.) *Let E be the Euler-Lagrange form associated to an r-order Lagragian density $\mathfrak{L}v$ on the fibred manifold Y with respect to a pair of derivation laws ∇_0, ∇. A section s of Y is critical for the variational problem defined by $\mathfrak{L}v$ if and only if:*

$$E_{|j^{2r-1}s} = 0$$

Furthermore, this condition does not depend on the pair of derivation laws chosen. Thus, the valued differential system on J^{2r-1} given by

$$(\theta^{(k)}, E) \qquad (k = 1, \ldots, 2r-1)$$

constitutes a global and intrinsic version of the Euler-Lagrange equations for higher order variational problems.

PROOF. A section s is critical if and only if $(\delta_s \mathbb{L})(D) = 0$ for every $D \in T_c^r(D)$, where U is the domain of s. By virtue of formula (9.2), this is equivalent to the following condition

$$\int_{j^{2r-1}s} \theta^{(1)}(D_{(2r-1)}) \circ E = 0 \qquad (D \in T_c^r(U)).$$

Since this equation must hold for all vector fields of $T_c^r(U)$, we conclude that $E_{|j^{2r-1}s} = 0$, and conversely. The independence of the derivation laws ∇_0, ∇ follows immediately from Corollary 7.5. Moreover, according to the same corollary, in order to compute $E_{|j^{2r-1}s}$ we can locally use the flat derivation laws. Then, with the notations of (5.2) and (5.4), we have $f_0 = \sum_i (\partial \mathfrak{L}/\partial y_i)\, dy_i$, and

$$\omega_k = \frac{1}{k!} \sum_{i,j} \sum_{|\alpha| = k-1} (-1)^{j-1}(1 + \alpha_j) \left(\frac{\partial \mathfrak{L}}{\partial y^i_{\alpha + (j)}} \right) v_j \otimes \left(\frac{\partial}{\partial x} \right)^{\alpha} \otimes dy_i \, (1 \leqslant k \leqslant r).$$

Thus, from formula (5.8) we obtain

$$(9.3)\, E = \sum_i \left\{ \frac{\partial \mathfrak{L}}{\partial y_i} v + \sum_{h=0}^{r-1} (-1)^{h+1} \sum_j \sum_{|\alpha| = h} (-1)^{j-1} \frac{1 + \alpha_j}{1 + h} d\left(\mathbb{D}^{\alpha} \frac{\partial \mathfrak{L}}{\partial y^i_{\alpha + (j)}} \right) \wedge \right.$$

$$\left. \wedge v_j \right\} \otimes dy_i.$$

Hence,

$$E_{|j^{2r-1}s} = \sum_i \left\{ \sum_{k=0}^{r} \sum_{|\beta| = k} (-1)^k \left(\frac{\partial^{|\beta|}}{\partial x^{\beta}} \right) \left(\frac{\partial \mathfrak{L}}{\partial y^i_{\beta}} \circ j^r s \right) \right\} v \otimes dy_i.$$

Theorem 9.3. (Second characterization.) *Let Θ be a Poincaré-Cartan form associated to an r-order Lagrangian density $\mathfrak{L}v$ on the fibred manifold Y. A section s of Y is critical for the variational problem defined by $\mathfrak{L}v$ if an only if:*

$$(i_D d\Theta)_{|j^{2r-1}s} = 0 \quad \textit{for all vector fields } D \textit{ in } \quad J^{2r-1}.$$

Furthermore, this condition does not depend on the particular Poincaré-Cartan form chosen.

PROOF. From formula (7.3) we deduce that for every vector field D in J^{2r-1}, we have $(i_D d\Theta)_{|j^{2r-1}s} = (\theta^{(1)}(D) \circ E)_{|j^{2r-1}s}$. The result now follows from Theorem 9.2 and Corollary 7.6.

For every open set $U \subset X$, we denote by $\Gamma(U, \mathcal{V})$ the set of critical sections of the variational problem determined by $\mathfrak{L}v$ which are defined on the domain U. Since $\Gamma(U, \mathcal{V})$ is the «set of solutions» of a globally defined differential operator, it is clear that $\mathcal{V}$ is a sheaf of sets over the manifold X.

A section $\bar{s}$ (not necesarily holonomic) of the canonical projection $p_{2r-1}: J^{2r-1} \to X$ is said to be a *Hamilton extremal* of the variational problem defined by $\mathfrak{L}v$ with respect to the linear connection ∇_0 if $(i_D d\Theta)|_{\bar{s}}$ vanishes for all vector fields D in J^{2r-1}, where Θ is the Poincaré-Cartan form associated with ∇_0. We shall denote by $\bar{\mathcal{V}}(\nabla_0)$ the sheaf of Hamiltonian extremals with respect to the linear connection ∇_0.

According to the second characterization of the critical sections, the jet prolongation $s \to j^{2r-1}s$ induces an injection of $\mathcal{V}$ into each $\bar{\mathcal{V}}(\nabla_0)$.

Remark. Note that condition $(i_D d\Theta)|_{\bar{s}} = 0$ can also be viewed as a differential equation on the linear connection ∇_0.

EXAMPLE. Let us consider the third order variational problem defined on the canonical projection $p: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ by the Lagrangian density $\mathfrak{L}v = \frac{1}{2} y^2_{(3,0)}\, dx_1 \wedge dx_2$.

The Poincaré-Cartan form corresponding to ∇_0 is given by the formula

$$\Theta = y_{(3,0)}\, dy_{(2,0)} \wedge dx_2 - y_{(4,0)}\, dy_{(1,0)} \wedge dx_2 - \frac{1}{2} \overset{\circ}{\Gamma}{}^2_{11} y_{(3,0)}\, dy_{(1,0)} \wedge dx_1 -$$

$$- \frac{1}{2} \overset{\circ}{\Gamma}{}^2_{11} y_{(3,0)}\, dy_{(0,1)} \wedge dx_2 + \left[y_{(5,0)} - \frac{1}{2} \frac{\partial \overset{\circ}{\Gamma}{}^2_{11}}{\partial x_2} y_{(3,0)} - \right.$$

$$\left. - \frac{1}{2} \overset{\circ}{\Gamma}{}^2_{11} y_{(3,1)} \right] dy \wedge dx_2 - \frac{1}{2} \left[\frac{\partial \overset{\circ}{\Gamma}{}^2_{11}}{\partial x_1} y_{(3,0)} + \overset{\circ}{\Gamma}{}^2_{11} y_{(4,0)} \right] dy \wedge dx_1 +$$

$$+\left[y_{(2,0)}y_{(4,0)} - y_{(1,0)}y_{(5,0)} - \frac{1}{2}y_{(3,0)}^2 - \frac{1}{2}\frac{\partial\overset{\circ}{\Gamma}{}^2_{11}}{\partial x_1}y_{(0,1)}y_{(3,0)} + \right.$$

$$\left. + \frac{1}{2}\frac{\partial\overset{\circ}{\Gamma}{}^2_{11}}{\partial x_2}y_{(1,0)}y_{(3,0)} - \frac{1}{2}\overset{\circ}{\Gamma}{}^2_{11}y_{(0,1)}y_{(4,0)} + \frac{1}{2}\overset{\circ}{\Gamma}{}^2_{11}y_{(1,0)}y_{(3,1)}\right] dx_1 \wedge dx_2,$$

and the differential system which determines Hamiltonian extremals $\bar{s} = (s_\alpha)_{|\alpha| \leq 5}$, is the following:

$$s_{(1,0)} = \frac{\partial s_0}{\partial x_1}, \qquad s_{(2,0)} = \frac{\partial s_{(1,0)}}{\partial x_1}, \qquad s_{(4,0)} = \frac{\partial s_{(3,0)}}{\partial x_1},$$

$$s_{(5,0)} = \frac{\partial s_{(4,0)}}{\partial x_1} + \frac{1}{2}\overset{\circ}{\Gamma}{}^2_{11}\left(s_{(3,1)} - \frac{\partial s_{(3,0)}}{\partial x_2}\right)$$

$$s_{(3,0)} = \frac{\partial s_{(2,0)}}{\partial x_1} + \frac{1}{3}\overset{\circ}{\Gamma}{}^2_{11}\left(\frac{\partial s_{(1,0)}}{\partial x_2} - \frac{\partial s_{(0,1)}}{\partial x_1}\right) + \frac{1}{2}\frac{\partial\overset{\circ}{\Gamma}{}^2_{11}}{\partial x_1}\left(\frac{\partial s_0}{\partial x_2} - s_{(0,1)}\right)$$

$$\frac{\partial s_{(5,0)}}{\partial x_1} = \frac{1}{2}\overset{\circ}{\Gamma}{}^2_{11}\left(\frac{\partial s_{(3,1)}}{\partial x_1} - \frac{\partial s_{(4,0)}}{\partial x_2}\right) + \frac{1}{2}\frac{\partial\overset{\circ}{\Gamma}{}^2_{11}}{\partial x_1}\left(s_{(3,1)} - \frac{\partial s_{(3,0)}}{\partial x_2}\right) +$$

$$+ \frac{1}{2}\frac{\partial\overset{\circ}{\Gamma}{}^2_{11}}{\partial x_2}\left(\frac{\partial s_{(3,0)}}{\partial x_1} - s_{(4,0)}\right)$$

Note that even for sections which are holonomic up to third order the above system depends on the connection chosen; only for 4-holonomic sections does the system become independent of ∇_0.

10. Functoriality of Poincaré-Cartan forms, infinitesimal symmetries and Noether invariants

In order to emphasize the dependence on the linear connection ∇_0, in this section we shall denote by $\Theta(\nabla_0, \mathcal{L}v)$ the Poincaré-Cartan form associated to the Lagrangian density $\mathcal{L}v$ constructed with the connection ∇_0. As we have seen in Theorem 8.1, $\Theta(\nabla_0, \mathcal{L}v)$ only depends on $j^{r-2}(\nabla_0)$. Let us denote by $K \to X$ the affine bundle of linear connections of the manifold X. We can thus define an ordinary n-form $\Theta(\mathcal{L}v)$ on the manifold $Z = J^{r-2}(K) \times_X J^{2r-1}(Y)$ by the formula

$$\Theta(\mathcal{L}v)_{(j_x^{r-2}\nabla_0, j_x^{2r-1}s)} = \Theta(\nabla_0, \mathcal{L}v)_{(j_x^{2r-1}s)}.$$

We shall call the form $\Theta(\mathcal{L}v)$ the *universal Poincaré-Cartan form* associated to the Lagrangian density $\mathcal{L}v$.

Remark. It follows from the previous definition that $j^{r-2}(\nabla_0)^* \Theta(\mathcal{L}v) = \Theta(\nabla_0, \mathcal{L}v)$. Note also that $\Theta(\mathcal{L}v)$ is horizontal over $J^{2r-1}(Y)$. The local expression of the form $\Theta(\mathcal{L}v)$ is also given by formulas (8.1), (8.2) and (8.3); but functions $\lambda^h_{\beta j}$ must now be considered as differentiable functions on $J^{r-2}(K)$.

Theorem 10.1. (Functoriality of Poincaré-Cartan forms.) *Let ψ be an automorphism of the fibred manifold Y; that is,*

$$\begin{array}{ccc} Y & \xrightarrow{\psi} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[\bar{\psi}]{} & X \end{array}$$

The automorphism $\tilde{\psi}: L(X) \to L(X)$ of the bundle of linear frames induced by $\bar{\psi}$ maps ∇_0 into a connection $\bar{\nabla}_0 = \tilde{\psi}(\nabla_0)$ (see [10, pp. 79 and 226]). *Then*

$$J^{2r-1}(\psi)^* (\Theta(\bar{\nabla}_0, \mathcal{L}v)) = \Theta(\nabla_0, J^r(\psi)^* (\mathcal{L}v)).$$

In particular, if ψ is a vertical automorphism of Y,

$$J^{2r-1}(\psi)^* (\Theta(\nabla_0, \mathcal{L}v)) = \Theta(\nabla_0, J^r(\psi)^* (\mathcal{L}v)).$$

PROOF. Let (x_j, y_i) be a fibred coordinate system for Y. We define a new fibred coordinate system $(\bar{x}_j, \bar{y}_i)$ by setting $\bar{x}_j = x_j \circ \bar{\psi}^{-1}$, $\bar{y}_i = y_i \circ \psi^{-1}$, and denote by $(\bar{y}^i_\alpha)$ the corresponding coordinate system induced on the jet bundles. We also denote by $\bar{\theta}^i_\alpha$ the components of the structure forms in the coordinate system $(\bar{y}^i_\alpha)$. Let $\bar{\nabla}$ be the flat derivation law associated to $(\bar{x}_j, \bar{y}_i)$. According to (3.6), there exist unique functions $\overset{\circ}{a}_{\beta\alpha} \in C^\infty(X)$ such that

$$\bar{L}^u \bar{\theta}^h_0 = \frac{1}{u!} \sum_{|\beta| \leq u} \sum_{|\alpha| = u} \overset{\circ}{a}_{\beta\alpha} \bar{\theta}^h_\beta \otimes (d\bar{x})^\alpha,$$

where the total Lie derivative $\bar{L}$ is taken with respect to $(\bar{\nabla}_0, \bar{\nabla})$. Functions $\overset{\circ}{a}_{\beta\alpha}$ fulfil the following conditions $\overset{\circ}{a}_{\beta\alpha} \circ \bar{\psi} = \mathring{a}_{\beta\alpha}$, where $\mathring{a}_{\beta\alpha}$ stand for the functions associated to ∇_0 and the falt derivation law ∇ determined by (x_j, y_i). This is easily verified by induction on $|\alpha|$ using the recurrence relations for these functions and the fact that the components of the linear connection $\bar{\nabla}_0$ with respect to $(\bar{x}_j)$ are $\overset{\circ}{\bar{\Gamma}}{}^k_{ij} = \overset{\circ}{\Gamma}{}^k_{ij} \circ \bar{\psi}^{-1}$. On the other hand, let $\bar{\lambda}^h_{\beta j}$ be the coefficients of the Poincaré-Cartan form $\Theta(\bar{\nabla}_0, \mathcal{L}v)$ in the coordinate system $(\bar{x}_j, \bar{y}^i_\alpha)$. We shall also use the obvious notations for the sections $\bar{f}_k$, $\bar{F}_k$ associated to this form with respect to the derivation laws $\bar{\nabla}_0$ and $\bar{\nabla}$.

We can check by induction on $|\beta|$ that the following formula holds true,

$$\bar{y}^h_\beta \circ J^k(\psi) = y^h_\beta \qquad (|\beta| \leq k).$$

Hence, $dy^h_\beta = J^k(\psi)^*(d\bar{y}^h_\beta)$, and consequently,

$$J^r(\psi)^*\bar{\theta}^h_\beta = \theta^h_\beta \quad \text{and} \quad \frac{\partial \mathfrak{L}}{\partial \bar{y}^h_\beta} \circ J^r(\psi) = \frac{\partial}{\partial y^h_\beta}(\mathfrak{L} \circ J^r(\psi)).$$

Let $\mathfrak{L}' \in C^\infty(J^r)$ be the unique function such that $J^r(\psi)^*(\mathfrak{L}v) = \mathfrak{L}'v$; or equivalently, $\mathfrak{L}' = \rho(\mathfrak{L} \circ J^r(\psi))$, where ρ is defined by the condition $\bar{\psi}^* v = \rho v$. Let us denote by $\lambda^h_{\beta j}$ the coefficients of the Poincaré-Cartan form $\Theta(\nabla_0, \mathfrak{L}'v)$ in the coordinate system (x_j, y^i_α). The components (in the same coordinate system) of the sections f_k, F_k associated to this form with respect to the derivation laws ∇_0, ∇ will be denoted by f^h_β, F^i_σ, respectively. Then, from the last formula above and Eq. (8.1) for $\bar{f}_k$ we derive $f^h_\beta = \bar{f}^h_\beta \circ J^r(\psi)$. Now, by recurrence on $|\sigma|$, it follows from formula (8.2) that $F^i_\sigma = \bar{F}^i_\sigma \circ J^{2r-1}(\psi)$. Thus, we obtain $\lambda^h_{\beta j} = \bar{\lambda}^h_{\beta j} \circ J^{2r-1}(\psi)$, and therefore

$$J^{2r-1}(\psi)^*(\Theta(\bar{\nabla}_0, \mathfrak{L}v)) = \sum_{h,j} \sum_{|\beta|=0}^{r-1} (-1)^{j-1} \lambda^h_{\beta j} \theta^h_\beta \wedge (\bar{\psi}^* \bar{v}_j) + J^r(\psi)^*(\mathfrak{L}v) =$$
$$= \Theta(\nabla_0, \mathfrak{L}'v),$$

since $\bar{\psi}^* \bar{v}_j = v_j$. This completes the proof of the theorem.

Corollary 10.2. (Infinitesimal functoriality of the universal Poincaré-Cartan form.) *Let D be a p-projectable vector field in the fibred manifold Y. We denote by $\bar{D}$ its projection on X. Let $\tilde{D}$ be the vector field induced by $\bar{D}$ in the affine bundle of linear connections of X. Then*

$$L_{(\tilde{D}_{(r-2)}, D_{(2r-1)})}(\Theta(\mathfrak{L}v)) = \Theta(L_{D_{(r)}}(\mathfrak{L}v)). \tag{10.1}$$

In particular, if D is p-vertical,

$$L_{D_{(2r-1)}}(\Theta(\mathfrak{L}v)) = \theta(L_{D_{(r)}}(\mathfrak{L}v)).$$

PROOF. Let τ_t, $\bar{\tau}_t$, $\tilde{\tau}_t$ be the local 1-parameter groups generated by D, $\bar{D}$, $\tilde{D}$, respectively. The 1-parameter group generated by $(\tilde{D}_{(r-2)}, D_{(2r-1)})$ is $(J^{r-2}(\tilde{\tau}_t), J^{2r-1}(\tau_t))$, as was pointed out in (4.4). Thus, formula (10.1) is equivalent to the following

$$(J^{r-2}(\tilde{\tau}_t), J^{2r-1}(\tau_t))^* \Theta(\mathfrak{L}v) = \Theta(J^r(\tau_t)^*(\mathfrak{L}v)). \tag{a}$$

Moreover, since $\Theta(\mathfrak{L}v)$ is horizontal over $J^{2r-1}(Y)$, in order to verify (b) it will be sufficient to prove that for every linear connection ∇_0 of X, we have:

$$j^{r-2}(\nabla_0)^*(J^{r-2}(\tilde{\tau}_t), J^{2r-1}(\tau_t))^* \Theta(\mathfrak{L}v) = J^{2r-1}(\tau_t)^* j^{r-2}(\tilde{\tau}_t(\nabla_0))^* \Theta(\mathfrak{L}v) =$$
$$= j^{r-2}(\nabla_0)^* \Theta(J^r(\tau_t)^*(\mathfrak{L}v)).$$

But according to the previous remark this is equivalent to

$$J^{2r-1}(\tau_t)^*(\Theta(\tilde{\tau}_t(\nabla_0), \mathcal{L}v)) = \Theta(\nabla_0, J^r(\tau_t)^*(\mathcal{L}v)).$$

The result follows immeditely from the preceding theorem.

A p-projectable vector field D in the manifold Y is said to be an *infinitesimal symmetry* of an r-order Lagrangian density $\mathcal{L}v$ if $L_{D_{(r)}}(\mathcal{L}v) = 0$.

For every open set $V \subset Y$ we denote by $\Gamma(V, \mathfrak{D})$ the set of infinitesimal symmetries of the Lagrangian density $\mathcal{L}v$. It follows directly from the above definition that $\mathfrak{D}$ is a sheaf of Lie algebras over Y. Moreover, we denote by $\mathfrak{D}^v$ the ideal of $\mathfrak{D}$ determined by the p-vertical infinitesimal symmetries. We have thus an exact sequence of sheaves over Y,

$$0 \to \mathfrak{D}^v \to \mathfrak{D} \to p^{-1}(\mathrm{Der}_X) \to 0.$$

Corollary 10.3. *A p-projectable vector field D in Y is an infinitesimal symmetry of $\mathcal{L}v$ if and only if,*

$$L_{(\tilde{D}_{(r-2)}, D_{(2r-1)})}(\Theta(\mathcal{L}v)) = 0.$$

In particular, p-vertical infinitesimal symmetries are characterized by the condition

$$L_{D_{(2r-1)}}(\Theta(\mathcal{L}v)) = 0.$$

If D is an infinitesimal symmetry of $\mathcal{L}v$, then the ordinary $(n-1)$-form $i_{D_{(2r-1)}}\Theta(\mathcal{L}v)$ will be called the *Noether invariant* corresponding to D. Note that $i_{D_{(2r-1)}}\Theta(\mathcal{L}v)$ is a differential form on the manifold $Z = J^{r-2}(K) \times_X J^{2r-1}(Y)$. The Noether invariant corresponding to D with respect to the connection ∇_0 is, by definition, the ordinary $(n-1)$-form on the manifold $J^{2r-1}(Y)$,

$$j^{r-2}(\nabla_0)^*(i_{D_{(2r-1)}}\Theta(\mathcal{L}v)) = i_{D_{(2r-1)}}\Theta(\nabla_0, \mathcal{L}v).$$

Note also that the Noether invariant corresponding to D really only depends on $D_{(r)}$.

For a different approach to the theory of Noether invariants one may consult [15].

Proposition 10.4. *If D is an infinitesimal symmetry of $\mathcal{L}v$, for every critical section s we have*:

$$d[(i_{D_{(2r-1)}}\Theta(\nabla_0, \mathcal{L}v))_{|j^{2r-1}s}] = 0.$$

Thus, once a linear connection ∇_0 has been fixed, each Noether invariant defines a function on v with values in the space of closed $(n-1)$-forms of X by the formula $f_D(s) = (i_{D_{(2r-1)}}\Theta(\nabla_0, \mathcal{L}v))_{|j^{2r-1}s}$.

PROOF. Since D is an infinitesimal symmetry of $\pounds v$, we have

$$L_{D_{(2r-1)}}\Theta(\nabla_0, \pounds v) = L_{D_{(2r-1)}}(\theta^r \wedge \Omega) = (L_{D_{(r)}}\theta^r) \wedge \Omega + \theta^r \wedge (L_{D_{(2r-1)}}\Omega)$$
$$= \theta^r \wedge (\phi^* \circ \Omega + L_{D_{(2r-1)}}\Omega).$$

Hence,

$$(L_{D_{(2r-1)}}\Theta(\nabla_0, \pounds v))_{|j^{2r-1}s} = d[(i_{D_{(2r-1)}}\Theta(\nabla_0, \pounds v))_{|j^{2r-1}s}] +$$
$$+ (i_{D_{(2r-1)}} d\Theta(\nabla_0, \pounds v))_{|j^{2r-1}s} = 0,$$

and the result follows from the second characterization of critical sections.

Proposition 10.5. *The mapping τ which associates to each infinitesimal symmetry $D \in \mathfrak{D}$ its Noether invariant is $\mathbb{R}$-linear and* Ker τ *is an ideal of $\mathfrak{D}$. According to this, we can translate by τ the Lie algebra structure of $\mathfrak{D}$ to the set J of Noether invariants. J will be called the* Poisson algebra *associated to the variational problem under consideration. Furthermore, $J^v = \tau(\mathfrak{D}^v)$ is an ideal of the Poisson algebra.*

PROOF. Let D, D' be two infinitesimal symmetries of $\pounds v$. Assume that $D \in \text{Ker } \tau$ (i.e., $i_{D_{(2r-1)}}\Theta(\pounds v) = 0$). Then, from Corollary 10.3 we obtain,

$$i_{[D,D']_{(2r-1)}}\Theta(\pounds v) = i_{D_{(2r-1)}}L_{D'_{(2r-1)}}\Theta(\pounds v) = -i_{D_{(2r-1)}}L_{\bar{D}'_{(r-2)}}\Theta(\pounds v) +$$
$$= -L_{\bar{D}'_{(r-2)}}(i_{D_{(2r-1)}}\Theta(\pounds v)) = 0.$$

Hence, $[D, D'] \in \text{Ker } \tau$.

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