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Oscillations of Anharmonic Fourier Series and the Wave Equation

A. Haraux V. Komornik

Introduction

Let Ω be a bounded domain in R^n, $n \geqslant 1$ and consider the usual wave equation with Dirichlet boundary conditions on $\Gamma = \partial\Omega$

$$(1) \qquad \begin{cases} u_{tt} - \Delta u = 0, & (t,x) \in R \times \Omega \\ u_{|\Gamma} = 0, & t \in R. \end{cases}$$

It is well-known that for any «initial data» $(u_0, v_0) \in H_0^1(\Omega) \times L^2(\Omega)$, there exists one and only one solution u of (1) in the functional class $C(R, H_0^1(\Omega)) \cap C^1(R, L^2(\Omega))$ such that $u(0,x) = u_0(x)$ and $u_t(0,x) = v_0(x)$. Moreover, for any such data, the vector $U(t) = (u(t,\cdot), u_t(t,\cdot))$ is almost periodic as a function: $R \to H_0^1(\Omega) \times L^2(\Omega)$ and the «anharmonic Fourier series» for u is given by the (generally formal) expansion formula

$$(2) \qquad u(t,x) = \sum_{n \geq 0} u_n \cos(t\sqrt{\lambda_n} + \alpha_n)\varphi_n(x)$$

where $(\lambda_n)_{n \geq 1}$ is the sequence of eigenvalues of $(-\Delta)$ in $H_0^1(\Omega)$, $\varphi_n(x)$ is an orthonormal (in $L^2(\Omega)$) associated sequence of eigenfunctions, $\{u_n\}$ and $\{\alpha_n\}$ are two sequences of real numbers which can be computed in terms of u_0, v_0 and n.

It is well-known (cf. for example [5]) that formula (2) does not define in general an absolutely convergent series for $x \in \Omega$ fixed. However, formula (2) makes sense pointwise if the initial data (u_0, v_0) lie in $H_0{}^m(\Omega) \times H_0^{m-1}(\Omega)$ with $m > \frac{n}{2}$, for example. In such a case, it becomes reasonable to ask about the behavior of the *sign* of $u(t, x_0)$ on a given interval $J \subset R$. Indeed, since $u(t, x_0)$ is then almost periodic with mean-value equal to 0, it is clear (cf. for example [3]) that $u(t, x_0)$ cannot keep a constant sign on an infinite interval unless $u(t, x_0) \equiv 0$ for $t \in R$.

In case $n = 1$, $\Omega =]0, l[$, $l > 0$, it is immediate that either $u(t, x_0) \equiv 0$, or $u(t, x_0)$ takes both positive and negative values on J as soon as $|J| \geqslant 2l$. This property has been generalized in [2] to a class of semi-linear wave equations.

In case $n > 1$, we know that $u(t, x)$ cannot remain $\geqslant 0$ in Ω for all $t \in J$ with $|J| > \pi/\sqrt{\lambda_1}$ (cf. [2]). However the *local* behavior of $u(t, x)$ is difficult to study already if $n = 2$ and Ω is a rectangle, for the usual wave equation (1).

In this paper, we have collected some partial results on the sign of $u(t, x)$ where u is a (sufficiently regular) solution of

$$\begin{cases} u_{tt} + (-1)^m \Delta^m u = 0 & (t, x) \in R \times \Omega \\ u_{|\Gamma} = \cdots = \Delta^{m-1} u_{|\Gamma} = 0 & t \in R. \end{cases} \tag{3}$$

These results rely on a study of the sign of almost periodic functions of a special form restricted to a bounded interval J.

1. Construction of positive functions orthogonal to some subspaces of $C([0, T])$

In this section, we consider a linear subspace X of the vector space AP, o of all (continuous) real-valued *almost periodic functions on R with mean-value* 0.

We try to answer the following question: find a function $p \in L^1(0, T)$ $(T > 0)$ such that

$$\begin{cases} p(t) > 0 & \text{a.e. on} \quad]0, T[\\ \forall f \in X, & \int_0^T p(t) f(t)\, dt = 0 \end{cases} \tag{1.2}$$

Our motivation for doing this is the following.

Proposition 1.1. *Let $T > 0$ be such that there exists $p \in L^1(0, T)$ satisfying* (1.1) *and* (1.2). *Then for any $f \in X$ we have the following alternative*

(*a*) either $f(t) = 0$, $\forall t \in [0, T]$;

(*b*) or there exists t_1, t_2 in $[0, T]$ with $f(t_1) > 0$ and $f(t_2) < 0$.

PROOF. Assume for example that $f(t) \geqslant 0$ on $[0, T]$. Then from (1.1) and (1.2) we deduce $p(t)f(t) = 0$ a.e. on $]0, T[$. Since $p(t) \neq 0$ a.e. on $]0, T[$, we conclude that $f(t) = 0$, $\forall t \in [0, T]$. □

The following simple result, although it will not be used in this paper, seems to be interesting in itself.

Proposition 1.2. *Assume that* $\dim(X) < +\infty$. *Then there exists* T_0 *such that for all* $T \geqslant T_0$, *there exists* $p \in C([0, T])$ *satisfying* (1.1) *and* (1.2).

PROOF. Let $\{f_j\}$ $1 \leqslant j \leqslant n$ be a basis of X. We can assume (as a consequence of Schmidt orthogonalization procedure) that

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T f_i(t) f_j(t)\, dt = \delta_{i,j} \tag{1.3}$$

Let E_T be the vector subspace of $L^2(0, T)$ generated by $\{f_{j|[0,T]}\}_{j \in \{1, \ldots, n\}}$. We denote by v_T the (orthogonal) projection of the constant function 1 on E_T in the Hilbert space $L^2(0, T)$. We have

$$v_T(t) = \sum_{j=1}^{n} v_j(T) f_j(t), \qquad \forall t \in [0, T]$$

and the property: $1 - v_T \in (E_T)^{\perp}$ yields

$$\int_0^T f_j(t)\, dt = \sum_{i \neq j} v_i(T) \int_0^T f_i(t) f_j(t)\, dt + v_j(T) \int_0^T |f_j(t)|^2\, dt$$

On dividing by $T > 0$:

$$\frac{1}{T} \int_0^T f_j(t)\, dt = \sum_{i \neq j} \left\{ \frac{1}{T} \int_0^T f_i(t) f_j(t)\, dt \right\} v_i(T) + \left\{ \frac{1}{T} \int_0^T |f_j(t)|^2\, dt \right\} v_j(T)$$

Since

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T f_j(t)\, dt = 0$$

and as a consequence of the orthonormality conditions (1.3) we deduce

$$\operatorname{Sup}_{1 \leq j \leq n} |v_j(T)| \to 0 \quad \text{as} \quad T \to +\infty.$$

As an immediate consequence, for all $T \geqslant T(\epsilon)$ we have $\|v_T\|_\infty \leqslant \epsilon$. Hence $p = 1 - v_T$ satisfies (1.2) and $p(t) \in C([0, T])$ with $1 - \epsilon \leqslant p(t) \leqslant 1 + \epsilon$ on $[0, T]$ for all $T \geqslant T(\epsilon)$.

Remarks 1.3. (*a*) It follows from the proof of Proposition 1.2 that $p(t)$ can be taken in the same regularity class as the vector function $F(t) = (f_1(t), \dots, f_n(t))$.

(*b*) For any $T \geqslant T_0$, and any vector $a \in R^n$, the real-valued function $f(t) = a \cdot F(t)$ satisfies the alternative described in Proposition 1.1. It is also possible to show this last result *directly* by working in $H = \mathrm{Vect}(F(R)) \neq \{0\}$. Indeed, if a_k is a sequence of vectors in H with $\|a_k\| = 1$ and $a_k \cdot F(t) \geqslant 0$ on $[0, k]$, any limiting point a of $\{a_k\}$ satisfies $\|a\| = 1$ and $a \cdot F(t) \geqslant 0$ on $R^+ \Rightarrow a \cdot F(t) \equiv 0$ on R and $a \cdot a = 0$, which is absurd.

(*c*) A variant of Hahn-Banach theorem shows that the *converse* of Proposition 1.1 is true if $\dim(X) < +\infty$. However

—If $\dim(X) = +\infty$, the converse is not true in general.

—If we used point (*b*) above to show Proposition 1.2, we would only have found $p \in L^2(0, T)$.

Now let $\tau > 0$ be arbitrary: we define

$$X_\tau = \left\{u \in C(R), u(t + \tau) = u(t) \quad \text{and} \quad \int_0^\tau u(t)\, dt = 0\right\}$$

We also set, *by definition* $X_0 = \{0\}$. The main result of this section is the following.

Theorem 1.4. *Let* $\{\tau_j\}_{1 \leq j < +\infty}$ *be a non-increasing sequence of* $\geqslant 0$ *numbers such that*

$$\tau_2 > 0 \quad \text{and} \quad \sum_{j=1}^{\infty} \tau_j = T < +\infty.$$

There exists a function $h: R \to R$ *such that*

$$\forall (x, y) \in R \times R, \quad |h(x) - h(y)| \leqslant |x - y| \tag{1.4}$$

$$\forall x \in]0, T[, \quad h(x) > 0 \tag{1.5}$$

$$\forall x \in R \backslash]0, T[, \quad h(x) = 0 \tag{1.6}$$

$$\forall j \in N \backslash \{0\}, \quad \forall \varphi \in X_{\tau_j}, \quad \int_R h(x)\varphi(x)\, dx = 0 \tag{1.7}$$

In addition, we have

$$\forall x \in R, \quad h(T - x) = h(x) \tag{1.8}$$

$$x \leqslant y \leqslant \tfrac{T}{2} \Rightarrow h(x) \leqslant h(y). \tag{1.9}$$

PROOF. We define inductively a sequence of functions $h_n: R \to R$ as follows

$$h_1(x) = \begin{cases} \tau_2 & \text{if} \quad x \in]0, \tau_1[\\ 0 & \text{if} \quad x \notin]0, \tau_1[\end{cases} \tag{1.10}$$

$$h_n(x) = \begin{cases} \dfrac{1}{\tau_n}\displaystyle\int_0^{\tau_n} h_{n-1}(x-t)\,dt & \text{if } n \geqslant 2, \quad \tau_n > 0 \\ h_{n-1}(x) & \text{if } n \geqslant 3, \quad \tau_n = 0 \end{cases} \tag{1.11}$$

Lemma 1.5. *For any $n \in N$, $n \geqslant 2$ the function $h_n(x)$ is such that*

$$\forall (x,y) \in R \times R, \quad |h_n(x) - h_n(y)| \leqslant |x-y| \tag{1.12}$$

$$\forall x \in \left]0, \sum_1^n \tau_j\right[, \quad h_n(x) > 0 \tag{1.13}$$

$$\forall x \in R \Big\backslash \left]0, \sum_1^n \tau_j\right[, \quad h_n(x) = 0 \tag{1.14}$$

$$\forall \varphi \in \bigcup_1^n X_{\tau_j}, \quad \int_R h_n(x)\varphi(x)\,dx = 0 \tag{1.15}$$

$$\forall x \in R, \quad h_n\left(\sum_1^n \tau_j - x\right) = h_n(x) \tag{1.16}$$

$$x \leqslant y \leqslant \frac{1}{2}\sum_1^n \tau_j \Rightarrow h_n(x) \leqslant h_n(y) \tag{1.17}$$

$$\int_R h_n(x)\,dx = \tau_1\tau_2 \tag{1.18}$$

$$0 < \epsilon \leqslant \frac{1}{2}\sum_1^{n-1} \tau_j \Rightarrow \int_0^{\epsilon} h_n(x)\,dx \geqslant \int_0^{\epsilon-\tau_n} h_{n-1}(x)\,dx \tag{1.19}$$

PROOF OF LEMMA 1.5. The proofs of (1.12)-(1.18) are by induction on n. The properties (1.13) $\rightarrow$ (1.18) are obviously satisfied for $n = 1$. Property (1.12) is true for $n = 2$, since

$$h_2(x) = \frac{1}{\tau_2}\int_0^{\tau_2} h_1(x-t)\,dt = \frac{1}{\tau_2}\int_{x-\tau_2}^{x} h_1(y)\,dy,$$

hence

$$h_2(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leqslant x \leqslant \tau_2 \\ \tau_2 & \text{if } \tau_2 \leqslant x \leqslant \tau_1 \\ \tau_1 + \tau_2 - x & \text{if } \tau_1 \leqslant x \leqslant \tau_1 + \tau_2 \\ 0 & \text{if } x > \tau_1 + \tau_2 \end{cases}$$

The inductive argument from $n-1$ to n is trivial if $\tau_n = 0$. If $\tau_n > 0$, we proceed as follows:

(1.12) $$|h_n(x) - h_n(y)| \leqslant \frac{1}{\tau_n}\int_0^{\tau_n} |h_{n-1}(x-t) - h_{n-1}(y-t)|\, dt \leqslant$$
$$\leqslant \frac{1}{\tau_n}\int_0^{\tau_n} |x-y|\, dt = |x-y|$$

(1.13) and (1.14): obvious from (1.11).

(1.15) $$\int_R h_n(x)\varphi(x)\, dx = \int_R \frac{1}{\tau_n}\int_0^{\tau_n} h_{n-1}(x-t)\, dt\varphi(x)\, dx =$$
$$= \frac{1}{\tau_n}\int_0^{\tau_n}\int_R h_{n-1}(x-t)\varphi(x)\, dx\, dt =$$
$$= \frac{1}{\tau_n}\int_0^{\tau_n}\int_R h_{n-1}(u)\varphi(u+t)\, du\, dt =$$
$$= \frac{1}{\tau_n}\int_R h_{n-1}(u)\left\{\int_0^{\tau_n} \varphi(u+t)\, dt\right\} du.$$

If $\varphi \in \bigcup_1^{n-1} X_{\tau_j}$, then $\int_R h_{n-1}(u)\varphi(u+t)\, du = 0$ for $t \in R$, and we deduce $\int_R h_n(x)\varphi(x)\, dx = 0$.

If $\varphi \in X_{\tau_n}$, then $\int_0^{\tau_n} \varphi(u+t)\, dt = 0$ for $u \in R$, and the result follows by integrating in u.

From now on, we use the notation:

$$T_n = \sum_{j=1}^{n} \tau_j, \qquad \forall n \geqslant 1.$$

(1.16) $$h_n(T_n - x) = \frac{1}{\tau_n}\int_0^{\tau_n} h_{n-1}(T_n - x - t)\, dt =$$
$$= \frac{1}{\tau_n}\int_0^{\tau_n} h_{n-1}(x + t - \tau_n)\, dt =$$
$$= \frac{1}{\tau_n}\int_0^{\tau_n} h_{n-1}(x-u)\, du = h_n(x)$$

for all $n \geqslant 2$.

(1.17) This property is obviously true for $n = 2$. If > 2, we remark that

$$\tau_n h_n(x) = \int_{x-\tau_n}^{x} h_{n-1}(y)\, dy, \qquad \forall x \in R \Rightarrow h_n \in C^1(R)$$

and

$$\tau_n h_n'(x) = h_{n-1}(x) - h_{n-1}(x - \tau_n)$$

Hence $h_n' \geqslant 0$ on $]-\infty, \frac{1}{2}T_{n-1}]$. Moreover, if $x \in [\frac{1}{2}T_{n-1}, \frac{1}{2}T_n]$, we have $T_{n-1} - x \leqslant \frac{1}{2}T_{n-1}$ and $T_{n-1} - x \geqslant x - \tau_n$, hence $\tau_n h_n'(x) = h_{n-1}(T_{n-1} - x) - h_{n-1}(x - \tau_n) \geqslant 0$. Finally, h_n is non-decreasing on $]-\infty, \frac{1}{2}T_n]$.

(1.18)
$$\int_R h_n(x)\,dx = \int_R \frac{1}{\tau_n}\int_{x-\tau_n}^{x} h_{n-1}(u)\,du\,dx =$$
$$= \int_R h_{n-1}(u)\left[\int_u^{u+\tau_n} \frac{1}{\tau_n}dx\right]du = \int_R h_{n-1}(u)\,du$$

(1.19) As a consequence of (1.14) and (1.17) we have

$$\int_0^{\epsilon} h_n(x)\,dx = \int_0^{\epsilon}\frac{1}{\tau_n}\int_0^{\tau_n} h_{n-1}(x-t)\,dt\,dx \geqslant \int_0^{\epsilon}\frac{1}{\tau_n}\int_0^{\tau_n} h_{n-1}(x-\tau_n)\,dt\,dx =$$
$$= \int_0^{\epsilon} h_{n-1}(x-\tau_n)\,dx = \int_{-\tau_n}^{\epsilon-\tau_n} h_{n-1}(u)\,du = \int_0^{\epsilon-\tau_n} h_{n-1}(u)\,du. \quad \square$$

END OF PROOF OF THEOREM 1.4. If $\tau_{n_0} = 0$ for some $n_0 \geqslant 2$, there is nothing left to prove. If $\tau_n > 0$ for all $n \geqslant 1$, we remark that

$\forall n \geqslant 2, \quad \forall x \in R,$
$$|h_n(x) - h_{n-1}(x)| \leqslant \frac{1}{\tau_n}\int_0^{\tau_n} |h_{n-1}(x-t) - h_{n-1}(x)|\,dt \leqslant$$
$$\leqslant \frac{1}{\tau_n}\int_0^{\tau_n} t\,dt = \frac{1}{2}\tau_n.$$

Since $\sum_{n=1}^{\infty}\tau_n < +\infty$, $\{h_n(x)\}$ is a Cauchy sequence in $C_B(R)$.
Let

$$h(x) = \lim_{n\to+\infty} h_n(x), \qquad \forall x \in R.$$

We claim that h satisfies (1.4) → (1.9). Since the sequence h_n satisfies (1.12), from (1.13) → (1.18) we deduce easily all the properties required on h *except* (1.5). Now if (1.5) is not satisfied, then for some $\epsilon > 0$ we have $h(x) = 0$ on $[0, \epsilon]$. We pick $m \in N$ large enough so that

$$\epsilon \leqslant \frac{1}{2}\sum_1^m \tau_k \quad \textit{and} \quad \sum_{m+1}^{\infty}\tau_k < \epsilon.$$

For all $n \in N$, $n > m$ we find as a consequence of (1.19):

$$\int_0^\epsilon h_n(x)\,dx \geqslant \int_0^{\epsilon - \tau_n} h_{n-1}(x)\,dx \geqslant \cdots \geqslant \int_0^{\epsilon - \sum_{m+1}^{n} \tau_j} h_m(x)\,dx.$$

Hence:

$$\int_0^\epsilon h_n(x)\,dx \geqslant \int_0^{\epsilon - \sum_{m+1}^{\infty} \tau_j} h_m(x)\,dx > 0.$$

By letting $n \to +\infty$ we find $\int_0^\epsilon h(x)\,dx > 0$.

This contradiction with $h \equiv 0$ on $[0, \epsilon]$ shows that in fact $h > 0$ on $]0, T[$.

Corollary 1.6. *Let* $\{\tau_j\}_{1 \le j < +\infty}$ *be as in the statement of Theorem* 1.4 *and let* $f_j \in X\tau_j$, $\forall j \in \{1, 2, \ldots, \}$ *be such that* $\sum_{j=1}^{+\infty} \|f_j\|_\infty < +\infty$. *We set*

$$f(t) = \sum_{j=1}^{+\infty} f_j(t), \qquad \forall t \in R.$$

Then for any interval $J \subset R$ such that $|J| \geqslant T$, *we have either* $f(t) \equiv 0$ *on* J, *or* $\exists (t_1, t_2)$ *in* J *with* $f(t_1) > 0$ *and* $f(t_2) < 0$.

PROOF. Let X be the closure in $C_B(R)$ of the *algebraic sum* $\sum_{j=1}^{\infty} X\tau_j$. Clearly, $X \subset AP, o$ and X is translation-invariant, i.e. $f(t + \alpha) \in X$ for all $f \in X$, $\alpha \in R$. Assume that $f \in X$ and $f \geqslant 0$ on J with $|J| \geqslant T$. Let $a \in R$ be such that $[a, a + T] \subset J$. Then $g(t) = f(t + a) \in X$.

As a consequence of Proposition 1.1 and Theorem 1.4, we obtain $g \equiv 0$ on $[0, T]$, hence $f \equiv 0$ on $[a, a + T]$. Since a is arbitrary in $[\operatorname{Inf} J, \operatorname{Sup} J - T]$ we conclude that $f \equiv 0$ on J.

2. Oscillation length and pseudo-analyticity measure

Let X be as in section 1. We define three nonnegative numbers, possibly infinite, which play an important role in the study of oscillation properties.

Definition 2.1. *The oscillation* legth *of* X *is the number* $l_1(X) = \inf\{l > 0, \forall a \in R, \forall f \in X, f \geqslant 0 \text{ on } [a, a + l] \Rightarrow f \equiv 0 \text{ on } [a, a + l]\}$.

The pseudo-analyticity measure of X *is* $l_2(X) = \inf\{l > 0, \forall a \in R, \forall f \in X, f \equiv 0 \text{ on } [a, a + l] \Rightarrow f \equiv 0 \text{ on } R\}$.

We also define $l_3(X) = \inf\{l > 0, \forall a \in R, \forall f \in X, f \geqslant 0 \text{ on } [a, a + l] \Rightarrow \Rightarrow f \equiv 0 \text{ on } R\}$.

Proposition 2.2. *We have*

$$l_3(X) = \mathrm{Sup}\{l_1(X), l_2(X)\}$$

PROOF. This is an obvious consequence of the definitions of the numbers $l_i(X)$.

Remark 2.3. If $X = AP, o$ we have $l_1(X) = l_2(X) = +\infty$.

We have $l_1(X) > 0$ as soon as $X \neq \{0\}$.

In contrast with this property of l_1, it is clear that if $X \subset$ {real analytic functions}, then $l_2(X) = 0$.

It is impossible to compare *in general* the values of $l_1(X)$ and $l_2(X)$. Indeed, if $\{0\} \neq X \subset$ {real analytic functions}, we have $0 = l_2(X) < l_1(X)$. On the other hand, it is not difficult to find $f \in AP, o$ such that f is 1-periodic, with $f = 0$ on $[0, 1-\epsilon]$, $f \neq 0$ (hence $l_2(Rf) \geqslant 1-\epsilon$) and $f(t)$ takes positive and negative values *in any neighbourhood of* $1-\epsilon$ *and* 1. Hence if $f(t)$ has a constant sign on some interval J, we must have either $J \subset [m-1, m-\epsilon]$ or $J \subset [m-\epsilon, m]$ for some $m \in Z$. In particular, if $|J| > \epsilon$ we deduce $f \equiv 0$ on J. This obviously implies that $l_1(RF) \leqslant \epsilon$.

A major result of this section is the following.

Theorem 2.4. *Let* $\{\tau_j\}_{1 \leq j \leq n}$ *be a finite sequence of positive numbers. Then*

$$l_3\left(\sum_{j=1}^{n} X\tau_j\right) \leqslant \sum_{j=1}^{n} \tau_j.$$

PROOF. Let $X = \sum_{j=1}^{n} X\tau_j$. It follows from Corollary 1.6 that $l_1(X) \leqslant \sum_{j=1}^{n} \tau_j$. Hence Theorem 2.4 will be proved as soon as we establish the following lemma.

Lemma 2.5. *Let* $a \in R$ *be arbitrary and* $f \in X$ *be such that* $f \equiv 0$ *on* $J = \left[a, a + \sum_{j=1}^{n} \tau_j\right]$. *Then* $f \equiv 0$ *on* R.

PROOF. By induction on n. The result is obviously true if $n = 1$. Assume that we have the result for $n-1$ with $n \geqslant 2$. Let $f = \sum_{j=1}^{n} f_j$ with $f_j \in X\tau_j$ and $f \equiv 0$ on J. Then

$$g(t) = f(t+\tau_n) - f(t) = \sum_{j=1}^{n-1} \{f_j(t+\tau_n) - f_j(t)\} \in \sum_{j=1}^{n-1} X_{\tau_j}$$

and $g \equiv 0$ on $J^* = \left[a, a + \sum_{j=1}^{n-1} \tau_j\right]$. By the induction hypothesis, $g \equiv 0 \Rightarrow f$ is τ_n-periodic. The result follows immediately. □

Remark 2.6. In our applications to hyperbolic equations of the second order in t, Lemma 2.5 will not be very useful since the results that we shall obtain will follow by taking each «harmonic oscillation» in a different $X\tau_j$, so that for an infinite number of harmonics we get nothing, while when the harmonics are in finite number we have analycity in t!. Therefore, the following extension of Theorem 2.4 will in fact reveal essential for our purpose.

Theorem 2.7. *Let* $\{\tau_j\}_{1\leq j<+\infty}$ *be an infinite sequence of positive numbers. We set* $Y=\{f\in AP,\ o,\ \exists\{f_j\}_{j\geq 1}$ *such that* $\sum_{j=1}^{\infty}\|f_j\|_\infty<+\infty$ *and* $f(t)=\sum_{j=1}^{\infty}f_j(t)$ *on* $R\}$. *Then:*

$$l_3(Y)\leqslant\sum_{j=1}^{\infty}\tau_j=T.$$

PROOF. If $T=+\infty$, there is nothing to prove. If $T<+\infty$, we know already that $l_1(Y)\leqslant T$ as a consequence of Corollary 1.6. Therefore to have the result it is sufficient to prove the following lemma.

Lemma 2.8. *Let* $a\in R$ *be arbitrary and* $f\in Y$ *be such that* $f\equiv 0$ *on* $[a,a+T]=J$. *Then* $f\equiv 0$ *on* R.

PROOF. Since Y is translation-invariant, it suffices to consider the case $a=0$. let $f(t)=\sum_{j=1}^{\infty}f_j(t)$, $f_j\in X\tau_j$, $\sum_1^\infty\|f_j\|<+\infty$. We assume $T<+\infty$ and we set $\epsilon_k=\sum_k^\infty\tau_j$. Let $p_k=R\to R$ be a continuous function such that

$$\begin{aligned}&\operatorname{Supp}(p_k)\subset[0,\epsilon_k]\\&p_k>0\quad\text{on}\quad]0,\epsilon_k[\\&p_k(\epsilon_k-t)=p_k(t)\\&\int_0^{\epsilon_k}p_k(s)\varphi(s)\,ds=0,\qquad\forall\varphi\in\bigcup_k^\infty X\tau_j.\end{aligned}$$

We introduce

$$g_k(t)=\int_0^{\epsilon_k}f(t+s)p_k(s)\,ds\quad\text{for all}\quad t\in R.$$

We have

$$\begin{aligned}g_k(t)&=\int_0^{\epsilon_k}\sum_{j=1}^{\infty}f_j(t+s)p_k(s)\,ds=\sum_{j=1}^{\infty}\int_0^{\epsilon_k}f_j(t+s)p_k(s)\,ds=\\&=\sum_{j=1}^{k-1}\int_0^{\epsilon_k}f_j(t+s)p_k(s)\,ds,\end{aligned}$$

therefore $g_k \in \sum_1^{k-1} X\tau_j$ for all $k \in N$, $k \geqslant 2$. From $f \equiv 0$ on $[0, T]$ we deduce $g_k = 0$ on $[0, T - \epsilon_k]$, hence as a consequence of Lemma 2.5 (note that $T - \epsilon_k = \sum_1^{k-1} \tau_j$) we have $g_k \equiv 0$ on R. Now let

$$\lambda_k = \int_0^{\epsilon_k} p_k(s)\, ds > 0 \quad \text{and} \quad \mu_k(t) = \frac{1}{\lambda_k} p_k(t), \quad t \in R.$$

Because of the properties of p_k, it is immediate to check that $\mu_k \to \delta_0$, the Dirac mass at 0 for the weak-star topology of $M_B(-1, 1)$ (say) as $k \to +\infty$. We deduce immediatly:

$$\forall t \in R, \qquad \lim_{k \to +\infty} \frac{1}{\lambda_k} g_k(t) = f(t).$$

Since $g_k \equiv 0$, this convergence clearly implies that in fact $f \equiv 0$ on R. Hence the proof of Lemma 2.8 is completed. □

3. Optimality of the results in sections 1 and 2

In this section, X_τ is defined as previously. We also use the following subspaces of X_τ:

$$\hat{X}_{\tau, k} = \left\{ u \in X_\tau, \quad \exists \{u_j\} \in R^k, \quad \exists \{\alpha_j\} \in R^k, \quad u(t) = \sum_{j=1}^{k} u_j \cos\left(2j \frac{\pi t}{\tau} + \alpha_j\right) \right\}$$

$$\hat{X}_\tau = \bigcup_{k=1}^{+\infty} \hat{X}_{\tau, k}.$$

a) On the optimality of Theorem 1.4

It will appear as an easy consequence of the following density result.

Theorem 3.1. *Let $n \geqslant 1$ be an integer, $\tau_1, \ldots, \tau_n$ some positive numbers such that $\tau_j/\tau_i \notin Q$ if $i \neq j$, and T such that $0 < T < \sum_{j=1}^{n} \tau_j$. Then the restrictions to $[0, T]$ of functions in $\sum_{j=1}^{n} \hat{X}_{\tau_j}$ are dense in $C([0, T])$.*

PROOF. It has been pointed out to us by Y. Meyer that Theorem 3.1 can be derived as an easy consequence of general results from the theory of meanperiodic functions (cf. [6]). For completeness we will give below a more direct proof based on a density result for the «limiting case» $T = \sum_{j=1}^{n} \tau_j$.

Theorem 3.2. *Let n and* $\{\tau_j\}_{1 \le j \le n}$ *be as in the statement of Theorem* 3.1. *We denote by* $\mathcal{P}_n$ *the set of polynomial functions of degree* $\leqslant n$. *The restrictions to* $\left[0, \sum_{j=1}^{n} \tau_j\right]$ *of functions in* $\sum_{j=1}^{n} \hat{X}_{\tau_j} + \mathcal{P}_n$ *are dense in* $C\left(\left[0, \sum_{j=1}^{n} \tau_j\right]\right)$.

PROOF. We rely on the following two simple lemmas concerning the map $\mathcal{C}: C(R) \to C(R)$ defined by $(\mathcal{C}f)(t) = f(t+\tau) - f(t)$. [$\tau > 0$ is given].

Lemma 3.3. $\forall k \geqslant 1$, $\mathcal{C}(\mathcal{P}_k) = \mathcal{P}_{k-1}$.

Lemma 3.4. $\forall \sigma > 0$ *with* $\tau/\sigma \notin Q$, *we have* $\mathcal{C}(\hat{X}_\sigma) = \hat{X}_\sigma$.

The proof of lemma 3.3 is obvious. To prove lemma 3.4, it is sufficient to check that $\forall k \in N$, $k \geqslant 1$, $\mathcal{C}(\hat{X}_{\sigma,k}) = \hat{X}_{\sigma,k}$. But obviously $\mathcal{C}(\hat{X}_{\sigma,k}) \subset \hat{X}_{\sigma,k}$ and if we denote by $\mathcal{C}_k$ the restriction of $\mathcal{C}$ to $\hat{X}_{\sigma,k}$, we have $\mathcal{C}_k^{-1}(0) = \{0\}$ because $\tau/\sigma \notin Q$. Since $\hat{X}_{\sigma,k}$ is finite dimensional, the result of lemma 3.4 is now obvious.

PROOF OF THEOREM 3.2 CONTINUED. We proceed by induction on n.

—For $n = 1$ the result is obviously true since

$$C([0, \tau_1]) = X_{\tau_1} + \mathcal{P}_1$$

and $\hat{X}_{\tau_1}$ is dense in X_{τ_1} for the topology of $C([0, \tau_1])$.

—For $n \geqslant 1$, we consider an arbitrary function $f \in C([0, T])$ with

$$T = \sum_{j=1}^{n} \tau_j.$$

We define

$$\tilde{f}(t) = f(t + \tau_n) - f(t), \qquad \forall t \in [0, T - \tau_n]. \tag{3.1}$$

By the induction hypothesis, for any $\delta > 0$ there exists $\tilde{f}_j \in \hat{X}_{\tau_j}$ for $1 \leqslant j \leqslant n-1$ and $\tilde{p} \in \mathcal{P}_{n-1}$ such that

$$\left\| \tilde{f} - \tilde{p} - \sum_{j=1}^{n-1} \tilde{f}_j \right\|_{C([0, T-\tau_n])} \leqslant \delta. \tag{3.2}$$

As a consequence of lemmas 3.3 and 3.4, we may assume for all j as above

$$\tilde{f}_j(t) = f_j(t + \tau_n) - f_j(t), \qquad \forall t \in R; \quad f_j \in \hat{X}_{\tau_j} \tag{3.3}$$

$$\tilde{p}(t) = p(t + \tau_n) - p(t), \qquad \forall t \in R; \quad p \in \mathcal{P}_n. \tag{3.4}$$

Also by the case $n = 1$ we can find $f_n \in \hat{X}_{\tau_n}$ and $q \in \mathcal{P}_1$ such that

$$\left\| f - (p + q) - \sum_{j=1}^{n} f_j \right\|_{C([0, \tau_n])} \leqslant \delta. \tag{3.5}$$

Clearly, $\tilde{q}$ is a constant and we have

$$|\tilde{q}(0)| \leqslant 2\delta + \left| \left(\tilde{f} - \tilde{p} - \sum_{j=1}^{n-1} f_j \right)(0) \right| \leqslant 3\delta.$$

Finally, let

$$h: = f - (p + q) - \sum_{j=1}^{n} f_j \quad \text{on} \quad [0, T].$$

Then we have

$$\| h \|_{C([0, \tau_n])} \leqslant \delta \tag{3.6}$$
$$|h(t + \tau_n) - h(t)| \leqslant 4\delta, \qquad \forall t \in [0, T - \tau_n]. \tag{3.7}$$

From (3.6) and (3.7) it is immediate to deduce

$$\left\| f - \sum_{j=1}^{n} f_j - (p + q) \right|_{C([0, T])} \leqslant \left(1 + \frac{4T}{\tau_n} \right) \delta. \tag{3.8}$$

Since δ can be taken arbitrarily small and $f_j \in X_{\tau_j}$, $p + q \in \mathcal{P}_n$, the induction step is achieved, and the proof of Theorem 3.2 is completed. □

PROOF OF THEOREM 3.1. Let $\mathfrak{M}$ be the space of bounded measures on R which are supported by $[0, \sum_{j=1}^{n} \tau_j]$ and consider

$$Z = \left\{ \mu \in \mathfrak{M}, \quad \forall f \in \sum_{j=1}^{n} \hat{X}_{\tau_j}, \quad \mu(f) = 0 \right\}.$$

As a consequence of Theorem 3.2, we have

$$\dim(Z) \leqslant n + 1 < +\infty.$$

Let now $0 < T < \sum_{j=1}^{n} \tau_j$ and consider a bounded measure ν on R with $\operatorname{supp}(\nu) \subset [0, T]$, such that $\nu \in Z$.

For $a \in [0, \sum_{j=1}^{n} \tau_j - T]$, the translated measure $\nu(\cdot + a) = \nu_a$ is also in Z. On the other hand, if $\nu \neq 0$, it is obvious to show, by looking at the supports, that all the measures ν_a are linearly independent. Since Z is finite-dimensional, we *must* have $\nu = 0$. Hence for $0 < T < \sum_{j=1}^{n} \hat{X}_{\tau_j}$, we obtain that $\sum_{j=1}^{n} \hat{X}_{\tau_j}$ is dense in $C([0, T])$.

This density result immediately implies the following. □

Corollary 3.3. *If all the τ_j such that $\tau_j \neq 0$ are pairwise incommensurable, the result of Theorem* 1.4 *and Corollary* 1.6 *are optimal in the sense that the number $T = \sum_{j=1}^{\infty} \tau_j$ cannot be replaced by any number $T' < T$.*

b) On the optimal character of Theorem 2.4 *when* $n = 2$

The result of Theorem 2.4 (just like Theorem 1.4) is not optimal *in general* for $n = 2$. Indeed, if $\tau_1 \in N\tau_2$, the conclusions of both Theorem 1.4 and 2.4 are still valid with $T = \tau_1 + \tau_2$ replaced by $T' = \tau_1 < T$. On the other hand, the following result shows that Theorem 2.4 is optimal when $n = 2$ and $\tau_1/\tau_2 \notin Q$.

Theorem 3.4. *Let $0 < \tau_2 < \tau_1$ with $\tau_2/\tau_1 \notin Q$, and $0 < T < \tau_1 + \tau_2$. Then there exists $u_1 \in X_{\tau_1}$ and $u_2 \in X_{\tau_2}$ such that $u_1 \neq 0$ and $u_1 + u_2 \equiv 0$ on $[0, T]$.*

The proof of Theorem 3.4 relies on the following.

Lemma 3.5. *Let J be a closed interval such that $|J| < \tau_2$. For any $p \in N$, there exists a* finite *set $F \subset Z$, such that $[-p, p] \cap Z \subset F$ and having the following property: setting $X = F\tau_1 + Z\tau_2$, for all $t \in J$ we have*

$$t \in X \Leftrightarrow t + \tau_1 \in X. \tag{3.9}$$

PROOF. Since $\tau_2/\tau_1 \notin Q$, $N\tau_1 - N\tau_2$ and $N\tau_2 - N\tau_1$ are everywhere dense in R. We set $J = [a, b]$.

1. There exists l, s in N with $l > p$ and such that $b - \tau_2 < l\tau_1 - s\tau_2 < a$. As a consequence, for any $m \in Z$ we have either $m > -s$ and $l\tau_1 + m\tau_2 > b$, or $m \leqslant -s$ and $l\tau_1 + m\tau_2 < a$.
   This implies in particular $(l\tau_1 + Z\tau_2) \cap J = \emptyset$.
2. There exists k, r in N with $k > p$ and such that

$$b - \tau_2 + \tau_1 < -k\tau_1 + r\tau_2 < a + \tau_1.$$

   This implies $(-k\tau_1 + Z\tau_2) \cap (J + \tau_1) = \emptyset$.
3. We consider $F = \{-k, -k + 1, \ldots, l\}$.
   —If $t \in J$ and $t = n\tau_1 + m\tau_2$ with $-k \leqslant n \leqslant l$, then we have in fact $n \leqslant l - 1$, hence $t + \tau_1 \in X$.
   —If $t \in J$ and $t + \tau_1 = n\tau_1 + m\tau_2$ with $-k \leqslant n \leqslant l$, then in fact $n \geqslant -k + 1$, hence $t \in X$. □

PROOF OF THEOREM 3.4. The result is obvious if $T \leqslant \tau_1$. If $T > \tau_1$, we fix $\delta > 0$ small enough so that $|J| < \tau_2$ with $J = [-\delta, T - \tau_1 + \delta]$.

Let X be as in Lemma 3.6: then

$$\alpha = \text{Inf}\{|x - y|, x \in X, y \in X, x \neq y\} > 0$$

We choose ρ such that $0 < \rho < \frac{1}{2}\inf\{\alpha, \delta\}$ and a function $\varphi \not\equiv 0$, $\varphi \in \mathfrak{D}\,(]0, +\infty[)$ such that $\text{Supp}\,(\varphi) \subset [0, \rho]$.

Let $w(t) = \varphi(\text{dist}\,(t, X))$, $\forall t \in R$. Clearly, $w \in C^\infty(R) \cap Y_{\tau_2}$, and $w \not\equiv 0$, since the function $\text{dist}\,(t, X)$ takes at least all values of $[0, \rho]$ as t ranges over R. We now show that $w(t + \tau_1) = w(t)$ for all $t \in [0, T - \tau_1]$. Indeed:

(*a*) If $\text{dist}\,(t, X) \geqslant \rho$, we cannot have $\text{dist}\,(t + \tau_1, X) < \rho$: assuming this inequality, since $\rho < \frac{1}{2}\alpha$ there would exist a unique point $x \in X$ such that $|x - (t + \tau_1)| < \rho$. Because $\rho < \delta$, we deduce $x - \tau_1 \in J \cap (X - \tau_1) \Rightarrow x - \tau_1 \in X$, a contradiction since $|t - (t - \tau_1)| < \rho$. Hence we *must* have $\text{dist}\,(t + \tau_1, X) \geqslant \rho$. In this case we have $w(t) = w(t + \tau_1) = 0$.

(*b*) If $\text{dist}\,(t, X) < \rho$, let $x \in X$ be such that $|x - t| = \text{dist}\,(t, X)$. Because $t \in [0, T - \tau_1]$ we have $x \in J$, hence $x \in J \cap X \Rightarrow x + \tau_1 \in X$. Now $\text{dist}\,(t + \tau_1, X) = |x - t|$, because there is at most one point $y \in X$ such that $|t + \tau_1 - y| \leqslant \rho$, and $y = x + \tau_1$ precisely fulfills this condition with $|t + \tau_1 - y| = |x - t|$.

We conclude that $\text{dist}\,(t + \tau_1, X) = \text{dist}\,(t, X) \Rightarrow w(t + \tau_1, X) = w(t, X)$. Finally, let $u(t) = w'(t)$: we have $u \in X_{\tau_2}$ and $u \not\equiv 0$ since w is not constant. We finally have $u + v \equiv 0$ on $[0, T]$ where $v(t)$ is the unique τ_1-periodic function such that $v(t) = -w'(t)$ on $[0, T]$. Clearly $v \in X_{\tau_1}$, hence the proof of Theorem 3.4 is completed. □

4. Applications to some hyperbolic problems of the second order with respect to t

a) An abstract oscillation theorem

Let H be a real Hilbert space and $A: D(A) \subset H \to H$ be a (possibly unbounded) linear operator such that $A = A^* \geqslant 0$ and A is strongly positive with A^{-1} *compact*.

If we set $V = D(A^{1/2})$, it is well-known that for any $(u_0, v_0) \in V \times H = \mathcal{H}$, the abstract second-order equation

$$u'' + Au(t) = 0, \qquad t \in R \tag{4.1}$$

has a unique solution $u \in C(R, V) \cap C^1(R, H)$ such that $u(0) = u_0$ and $u'(0) = v_0$.

Moreover, the equation (4.1) generates a group of isometries $T(t)$ on $\mathcal{H}$ endowed with the norm $\|(u, v)\|_{\mathcal{H}} = \{|A^{1/2}u|_H^2 + |v|_H^2\}^{1/2}$.

Therefore, for all $(u_0, v_0) \in \mathcal{H}$, the function $t \to (u(t), u'(t)) \in \mathcal{H}$ is almost periodic. (Cf. for example [4], lecture 24, Proposition 9.) On the other hand u' is bounded: $R \to V'$ and it follows that $u(t) = -A^{-1}u''(t)$ has mean-value 0 in V.

As a consequence, for any $\zeta \in V'$, the function $t \to \langle \zeta, u(t) \rangle$ cannot remain nonnegative on an infinite interval except if $\langle \zeta, u(t) \rangle \equiv 0$ on R. The results of section 1 now allow us to state a more precise property.

Theorem 4.1. *In addition to the above hypotheses on A, assume that the eigenvalues of A on H, denoted by* $\{\lambda_n\}_{n \in N \setminus \{0\}}$ *and repeated according to their multiplicity order, are such that*

$$\sum_{n=1}^{+\infty} \frac{2\pi}{\sqrt{\lambda_n}} = T < +\infty. \tag{4.2}$$

Then, for any $\zeta \in [D(A^{1/4})]'$ *and any solution u of* (4.1), *we have the following alternative: either,* $\langle \zeta, u(t) \rangle \equiv 0$ *on R, or for any interval J of R with* $|J| \geqslant T$, *there exists* τ_1 *and* τ_2 *in J such that*

$$\langle \zeta, u(\tau_1) \rangle > 0 \quad \text{and} \quad \langle \zeta, u(t_2) \rangle < 0.$$

PROOF. Let $\{\varphi_n\}_{n \geq 1}$ be an orthonormal (in H) sequence of eigenfunctions relative to $\{\lambda_n\}_{n \geq 1}$.

We set $\tau_n = 2\pi/\sqrt{\lambda_n}$ for $n \geqslant 1$, and we consider first the case where

$$u_0 = \sum_{j=1}^{k} u_j \varphi_j, \qquad v_0 = \sum_{j=1}^{k} v_j \varphi_j$$

In this case, $u(t)$ is given by

$$u(t) = \sum_{j=1}^{k} \left\{ u_j \cos(\sqrt{\lambda_j}\, t) + \frac{v_j}{\sqrt{\lambda_j}} \sin(\sqrt{\lambda_j}\, t) \right\} \varphi_j$$

Hence for any $\zeta \in V'$, $\langle \zeta, u(t) \rangle \in \sum_{j=1}^{k} X_{\tau_j}$.

We claim that *in general*, the series defining $u(t)$ is in fact absolutely convergent in $D(A^{1/4})$: as a consequence we shall have, for any $\zeta \in [D(A^{1/4})]'$.

$\langle \zeta, u(t) \rangle = f(t) \in Y = \{f \in AP,\ \exists f_n \in X_{\tau_n}$ such that $\sum_{n=1}^{\infty} \|f_n\|_\infty < +\infty$ and $f = \sum_{n=1}^{\infty} f_n\}$. Indeed, let $u_0 = \sum_{j=1}^{\infty} u_j \varphi_j$, $v_0 = \sum_{j=1}^{n} v_j \varphi_j$ be the Fourier expansions of the initial data (u_0, v_0) and

$$u_0^n = \sum_{j=1}^{n} u_j \varphi_j, \qquad v_0^n = \sum_{j=1}^{n} v_j \varphi_j.$$

It is clear that

$$u(t) = \lim_{n \to +\infty} \sum_{j=1}^{n} \left\{ u_j \cos(\sqrt{\lambda_j}\, t) + \frac{v_j}{\sqrt{\lambda_j}} \sin(\sqrt{\lambda_j}\, t) \right\} \varphi_j$$

in $C_B(R, V)$.

Also we have

$$\sum_{j=1}^{\infty} \left\| \left\{ u_j \cos(\sqrt{\lambda_j}\, t) + \frac{v_j}{\sqrt{\lambda_j}} \sin(\sqrt{\lambda_j}\, t) \right\} \varphi_j \right\|_{D(A^{1/4})} \leqslant \sum_{j=1}^{\infty} \left\{ |u_j| + \frac{|v_j|}{\sqrt{\lambda_j}} \right\} \|\varphi_j\|_{D(A^{1/4})} =$$

$$= \sum_{j=1}^{\infty} \{ \lambda_j^{1/4} |u_j| + \lambda_j^{-1/4} |v_j| \} \leqslant$$

$$\leqslant \left(\sum_{j=1}^{\infty} \frac{1}{\sqrt{\lambda_j}} \right)^{1/2} \left(\sum_{j=1}^{\infty} \lambda_j |u_j|^2 \right)^{1/2} + \left(\sum_{j=1}^{\infty} \frac{1}{\sqrt{\lambda_j}} \right)^{1/2} \left(\sum_{j=1}^{\infty} |v_j|^2 \right)^{1/2} =$$

$$= \left(\sum_{j=1}^{\infty} \frac{1}{\sqrt{\lambda_j}} \right)^{1/2} (\|u_0\|_V + \|v_0\|_H) < +\infty.$$

Hence the claim is proved and for any $\zeta \in [D(A^{1/4})]'$ we have $<\zeta, u(t)> \in Y$ as explained above. The conclusion of Theorem 4.1 is now an immediate consequence of Theorem 2.7.

b) Examples of application

Let us start with a one-dimensional case.

Example 4.2. Let $\Omega =]0, l[$, $l > 0$ and $h(x) \in L^\infty(\Omega)$. We consider the equation

$$(4.3) \qquad \begin{cases} u_{tt} + u_{xxxx} + h(x)u = 0, & t \in R, x \in \Omega \\ u(t,0) = u(t,l) = u_{xx}(t,0) = u_{xx}(t,l) = 0, & t \in R \end{cases}$$

We set $H = L^2(\Omega)$ and

$$D(A) = \{ u \in H_0^1(\Omega) \cap H^4(\Omega), \quad u_{xx} \in H_0^1(\Omega) \}$$
$$Au = u_{xxxx} + hu \quad \text{for all} \quad u \in D(A).$$

Let $A_0 u = u_{xxxx}$ for $u \in D(A)$. The eigenvalues of A_0 are given by

$$\lambda_n^0 = \left(\frac{n\pi}{l} \right)^4, \qquad \forall n \in N \backslash \{0\}.$$

As a consequence, the eigenvalues of A are such that $\lambda_n \geqslant \left(\frac{n\pi}{l}\right)^4 - \|h^-\|_{L^\infty(\Omega)}$. Hence, if we assume $\|h^-\|_{L^\infty(\Omega)} < \left(\frac{\pi}{l}\right)^4$, we have A strongly positive with compact inverse in H. Also in this case, $D(A^{1/4}) = H_0^1(\Omega)$. From Theorem 4.1, we obtain that for any $\zeta \in H^{-1}(\Omega)$ and any solution $u \in C(R, H^2 \cap H_0^1(\Omega)) \cap$ $\cap\, C^1(R, L^2(\Omega))$ of (4.3), we have the alternative

—either $\langle \zeta, u(t)\rangle \equiv 0$ on R

—or for any interval J of R such that

$$|J| \geqslant \sum_{n=1}^{+\infty} \frac{2\pi}{\sqrt{\left(\dfrac{n\pi}{l}\right)^4 - \|h^-\|_\infty}} = T,$$

there exists t_1 and $t_2 \in J$ such that $\langle \zeta, u(t_1)\rangle > 0$ and $\langle \zeta, u(t_2)\rangle < 0$. As a particular case, for any $x_0 \in]0, l[$, we have $\delta_{x_0} \in H^{-1}(\Omega)$: hence the function $u(t, x_0)$ is either $\equiv 0$ on R, or must take >0 and <0 values on each interval J such that $|J| \geqslant T$, for any weak solution u of (4.3).

EXAMPLE 4.3. Let Ω be any bounded domain of R^n, $n \geqslant 1$ with sufficiently regular boundary $\Gamma = \partial\Omega$. We consider the equation

$$(4.4) \quad \begin{cases} u_{tt} + (-1)^m \Delta^m u = 0 & t \in R, \quad x \in \Omega \\ \Delta^s u(t, x) = 0 & \text{for} \quad s \in \{0, 1, \dots, m-1\}, \quad t \in R, \quad x \in \Gamma \end{cases}$$

where $m \in N$ is such that $m > n$.

In the case where $\Omega = (]0, \pi[)^n$, the eigenvalues of $A = (-1)^m \Delta^m u$ with $D(A) = \{u \in H^{2m}(\Omega), \Delta^s u = 0 \text{ on } \Gamma \text{ for all } s \in \{0, 1, \dots, m-1\}\}$ are given by the formula

$$\lambda_{j_1, j_2, \dots, j_n} = (j_1^2 + j_2^2 + \cdots + j_n^2)^m$$

By using the variational characterisation of the eigenvalues of $(-\Delta)$ in $H_0^1(\Omega)$, it is easy to show for any Ω the existence of two constants $c(\Omega)$, $C(\Omega)$ with $0 < c(\Omega) < C(\Omega) < +\infty$, such that

$$\frac{c(\Omega)}{(j_1^2 + \cdots + j_n^2)^{m/2}} \leqslant \frac{2\pi}{\sqrt{\lambda_{j_1, \dots, j_n}}} \leqslant \frac{C(\Omega)}{(j_1^2 + \cdots + j_n^2)^{m/2}}$$

where $\lambda_{j_1, \dots, j_n}$ are the eigenvalues of A associated with Ω.

As a consequence:

$$\sum_{j_1, \dots, j_n} \frac{2\pi}{\sqrt{\lambda_{j_1, \dots, j_n}}} < +\infty \Leftrightarrow \sum_{j_1, \dots, j_n} \frac{1}{(j_1^2 + \cdots + j_n^2)^{m/2}} < +\infty \Leftrightarrow m > n.$$

Under the *same condition*, we have $D(A^{1/4}) \hookrightarrow C(\bar{\Omega})$.

Hence for any $x_0 \in \Omega$, the map $u \in D(A^{1/4}) \to u(x_0)$ is well-defined and can be considered as an element of $[D(A^{1/4})]'$. As a consequence of Theorem 4.1, we obtain that for some $T < +\infty$ (increasing, in fact, with the diameter of Ω), the following property holds: for any u solution of (4.4) and any $x_0 \in \Omega$, we have either $u(t, x_0) \equiv 0$, or for any interval J with length $> T$, there exists t_1, t_2 in J with $u(t_1, x_0)\, u(t_2 x_0) < 0$.

c) *A counterexample*

The example 4.3 does not include the wave equation (case $m = 1$) in *any* dimension and even for $n = 1$. This clearly means that our method cannot give always the best possible result, since for $n = 1$ the solutions *do* oscillate, for a seemingly quite special reason (namely the periodicity of solutions in t). According to this remark, it becomes essential to decide whether in fact the oscillation property is (or is not) always true for the wave equation, at least for C^∞ solutions, say.

The following construction shows that *it is not the case*, therefore one should be careful while attempting to generalize our example 4.3 under weaker conditions on m.

Theorem 4.4. *Let a, b be positive and such that $b^2/a^2 \notin Q$. Let $\Omega =]0, a[\times$ $\times\,]0, b[\subset R^2$ and $(x_0, y_0) \in \Omega$ be any point such that $x_0/a \notin Q$, $y_0/b \notin Q$.*

Then for any $T > 0$, there exists a solution $u \in C^\infty(R \times \bar{\Omega})$ of

$$\begin{cases} u_{tt} - u_{xx} - u_{yy} = 0 & (t; x, y) \in R \times \Omega \\ u(t, x, y) = 0 & (t; x, y) \in R \times \partial\Omega \\ u(t, x_0, y_0) \geqslant 1 & \text{on} \quad [0, T] \end{cases}$$

PROOF. Let $u_{m,n}$ be a double sequence of real numbers, $m \geqslant 1$, $n \geqslant 1$ and such that $u_{m,n} = 0$ for $m > m_0$ or $n > n_0$.

We set

$$u(t, x, y) = \sum_{m,n} u_{m,n} \frac{\cos\left(\dfrac{m\pi t}{a}\sqrt{n^2 + \dfrac{a^2}{b^2} + \alpha_{m,n}}\right)}{\sin\left(\dfrac{mn\pi x_0}{a}\right)\sin\left(\dfrac{m\pi y_0}{b}\right)} \sin\left(\frac{mn\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)$$

for $(t, x, y) \in R \times \Omega$.

It is easy to check that $u \in C(R \times \bar{\Omega})$ and $u_{tt} - u_{xx} - u_{yy} = 0$ in $R \times \Omega$, with $u(t, x, y) = 0$ on $R \times \partial\Omega$.

Moreover, we have

$$u(t, x_0, y_0) = \sum_{m,n} u_{m,n} \cos\left(\frac{m\pi t}{a}\sqrt{n^2 + \frac{a^2}{b^2}} + \alpha_{m,n}\right) =$$

$$= \sum_n \left\{\sum_m u_{m,n} \cos\left(\frac{m\pi t}{a}\sqrt{n^2 + \frac{a^2}{b^2}} + \alpha_{m,n}\right)\right\}$$

For $n \in N$ fixed and m_0 ranging over N, the function

$$\varphi_n(t) = \sum_{m=1}^{m_0} u_{m,n} \cos\left(\frac{m\pi t}{a}\sqrt{n^2 + \frac{a^2}{b^2}} + \alpha_{m,n}\right)$$

can be taken equal to *any* element of the space $\hat{X}_{\tau_n}$ with $\tau_n = 2ab(a^2 + n^2b^2)^{-1/2}$.

Also for $n_1 \neq n_2$, the numbers τ_{n_1} and τ_{n_2} are incommensurable since $b^2/a^2 \notin Q$.

Finally, we have

$$\sum_{n \in N} \tau_n = +\infty.$$

Now we pick n_0 such that

$$\sum_1^{n_0} \tau_n > T.$$

As a consequence of Theorem 3.1, there exists

$$f(t) \in \sum_{n=1}^{n_0} \hat{X}_{\tau_n}$$

such that $f \geqslant 1$ on $[0, T]$.

As a consequence of the remarks above, we can choose first m_0 large enough, and then the coefficients $u_{m,n}$ such that $u(t, x_0, y_0) = f(t)$.

This concludes the proof of Theorem 4.4.

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A. Haraux
Université Pierre et Marie Curie
Laboratoire d'Analyse Numérique
(U. A. 189)
Tour 55-65-5ème étage
4, place Jussieu
75230 Paris Cedex 05 (France)

V. Komornic
Université Eötvös Loránd
Département de Mathématiques
Chaire d'Analyse
Budapest, 8
Pf: 323—1445 (Hungrie)