Local Entropy Moduli and Eigenvalues of Operators in Banach Spaces

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Introduction

In the paper local entropy moduli of operators between Banach spaces are introduced. They constitute a generalization of entropy numbers and moduli, and localize these notions in an appropriate way. Many results regarding entropy numbers and moduli can be carried over to local entropy moduli.

We investigate relations between local entropy moduli and s-numbers, spectral properties, eigenvalues, absolutely summing operators. As applications local entropy moduli of identical and diagonal operators between l_p-spaces can be estimated. It is shown, that in general «local» and «global» degree of compactness considerably differ, but under certain type assumptions on the underlying Banach spaces they coincide. Finally, the results are applied to obtain (optimal) estimates for eigenvalues of certain integral operators.

0. Preliminaries

Throughout the paper all Banach spaces, $X, Y, Z, \ldots$, are complex. The dual and the closed unit ball of X are denoted by X' and B_X, respectively. For the class of all (bounded linear) operators from X into Y we shall write $\mathcal{L}(X, Y)$,

and for $\mathcal{L}(X, X)$ simply $\mathcal{L}(X)$. Concerning (quasi-normed) operator ideals we refer to the monograph [19]. We shall use mainly the ideals (Π_p, π_p) and $(\Pi_{p,2}, \pi_{p,2})$ of p- and $(p, 2)$-absolutely summing operators.

An important role will play the notion of type, see [18] for more informations. A Banach space X is of (*Rademacher*) *type* p, $1 \leqslant p \leqslant 2$, if there is a constant $c > 0$ such that for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$ the inequality

$$\mathbb{E}\left\| \sum_1^n x_i \epsilon_i \right\| \leqslant c \left(\sum_1^n \|x_i\|^p \right)^{1/p}$$

holds, where (ϵ_i) is a sequence of independent random variables, each taking the values $+1$ and -1 with probability $\frac{1}{2}$. The type p constant of X is then defined as $T_p(X) = \inf c$. Replacing the sequence (ϵ_i) by a sequence of independent standard Gaussian variables one can define Banach spaces of Gaussian type p, the Gaussian type p constant of X will be denoted by $\tilde{T}_p(X)$. A Banach space is of Gaussian type p iff it is of Rademacher type p, therefore we will not distinginsh between these two notions in the sequel, but only between the constants $T_p(X)$ and $\tilde{T}_p(X)$. As examples let us mention that the function spaces L_p (over arbitrary σ-finite measure spaces) are of type $\min(p, 2)$ if $1 \leqslant p < \infty$.

Moreover, we shall use the concept of s-numbers of operators, which also may be found in [19]. Here we only state the definitions of some s-numbers, for their properties see [19] and [20].

Given an operator $S \in \mathcal{L}(X, Y)$, the nth *approximation number* is defined by

$$a_n(S) = \inf \{ \|S\text{-}L\| : L \in \mathcal{L}(X, Y), \operatorname{rank} L < n \},$$

the nth *Gelfand number* by

$$c_n(S) = \inf \{ \|S J_M^X\| : M \subseteq X, \operatorname{codim} M < n \},$$

where J_m^X is the embedding from M into X,

the nth *Kolmogorov number* by

$$d_n(S) = \inf \{ \|Q_N^Y S\| : N \subseteq Y, \dim N < n \},$$

where Q_N^Y is the quotient map from Y onto Y/N, the nth *Hilbert number* by

$$h_n(S) = \sup \{ a_n(BSA) : \|A : l_2 \to X\| \leqslant 1, \|B : Y \to l_2\| \leqslant 1 \},$$

the nth *Weyl number* by

$$x_n(S) = \sup \{ a_n(SA) : \|A : l_2 \to X\| \leqslant 1 \}$$

and the nth *dual Weyl number* by

$$y_n(S) = \sup \{a_n(BS) \colon \|B \colon Y \to l_2\| \leqslant 1\}.$$

Given two sequences of positive real numbers (a_n) and (b_n) we shall write $a_n = 0(b_n)$ if $a_n \leqslant cb_n$ for some constant $c > 0$ and all $n \in \mathbb{N}$. The symbol $a_n \sim b_n$ means $a_n = 0(b_n)$ and $b_n = 0(a_n)$.

1. Entropy quantities

Let us start by defining the entropy quantities we are going to use in the sequel. For entropy numbers see e.g. [19], entropy moduli were introduced in [5], while local entropy moduli are considered here for the first time.

Given an operator $S \in \mathcal{L}(X, Y)$, the nth *entropy number* is

$$\epsilon_n(S) = \inf \left\{ \epsilon > 0 \colon \exists y_1, \ldots, y_n \in Y \quad \text{such that} \quad S(B_X) \subseteq \bigcup_1^n (y_i + \epsilon B_Y) \right\},$$

the nth *dyadic entropy number* is

$$e_n(S) = \epsilon_{2^{n-1}}(S),$$

the nth *entropy modulus* is

$$g_n(S) = \inf \{k^{1/2n} \epsilon_k(S) \colon k \in \mathbb{N}\}$$

and the nth *local entropy modulus* is

$$G_n(S) = \sup \{g_n(Q_N^Y S) \colon N \subseteq Y, \operatorname{codim} N \leqslant n\}.$$

By the definition of compactness of operators, $S \in \mathcal{L}(X, Y)$ is compact, iff $\epsilon_n(S) \to 0$ (or, equivalently, $e_n(S) \to 0$) as $n \to \infty$.

Thus entropy numbers quantify in a certain sense the notion of compactness. The «degree of compactness» of an operator can be characterized by the asymptotic behaviour of its (dyadic) entropy numbers. Another important point is the eigenvalue inequality [9]

$$\left(\prod_{i=1}^n |\lambda_i(S)| \right)^{1/n} \leqslant \inf_{k \in \mathbb{N}} k^{1/2n} \epsilon_k(S), \qquad S \in \mathcal{L}(X), \quad n \in \mathbb{N},$$

which motivated the introduction of entropy moduli. Finally, local entropy moduli are a local version of the concept of entropy moduli. The remarkable difference is that $G_n(S) \to 0$ as $n \to \infty$ not necessarily implies the compactness of S. But, on the other hand, the eigenvalue inequality for entropy moduli

$$\left(\prod_{i=1}^{n} |\lambda_i(S)| \right)^{1/n} \leqslant g_n(S), \qquad S \in \mathcal{L}(X), \qquad n \in \mathbb{N}$$

remains true for local entropy moduli, too. More detailed results will be stated later on in section 3.

Algebraic (and other) properties of entropy numbers and moduli can be found in [19] and [5], respectively. Therefore we want to list here only properties of local entropy moduli. Let $S \in \mathcal{L}(X, Y)$, $T \in \mathcal{L}(Y, Z)$ and $n \in \mathbb{N}$

(i) $\|S\| = G_1(S) \geqslant G_n(S)$.

(ii) $G_n(TS) \leqslant G_n(T)G_n(S)$ (multiplicativity).

(iii) $G_n(S) = 0$ whenever rank $S < n$ and $G_n(I_X) = 1$ whenever dim $X \geqslant n$, where I_X is the identity on X.

(iv) $G_n(S) = g_n(S)$ if dim $Y = n$.

(v) $G_n(SQ) = G_n(S)$ for every metric surjection $Q \in \mathcal{L}(Z, X)$ (surjectivity).

(vi) Let (X_0, X_1) be an interpolation couple of Banach spaces and X be an intermediate space of K-type θ, $0 < \theta < 1$, and let $S \in \mathcal{L}(X_0 + X_1, Y)$. Then

$$G_n(S: X \to Y) \leqslant 2G_{[n/(1-\theta)]}(S: X_0 \to Y)^{1-\theta} \cdot G_{[n/\theta]}(S: X_1 \to Y)^{\theta}.$$

For interpolation theory see e.g. Bergh/Löfström [2].

Since all these properties can be easily derived from those for entropy numbers [19], proofs are omitted.

Moreover, for each of the quantities $s = \alpha, c, d, h, x, y, e, g, G$ we introduce the notion

$$s(S) = \inf \{s_n(S): n \in \mathbb{N}\} \quad \text{for} \quad S \in \mathcal{L}$$

and, for $0 < p \leqslant \infty$, the classes

$$\mathcal{L}_{p,t}^{(s)} = \left\{ S \in \mathcal{L}: \sum_{n=1}^{\infty} n^{(t/p)-1} s_n(S)^t < \infty \right\}, \qquad 0 < t < \infty,$$

$$\mathcal{L}_{p,\infty}^{(s)} = \left\{ S \in \mathcal{L}: \sup_{n \in \mathbb{N}} n^{1/p} s_n(S) < \infty \right\} \quad \text{and}$$

$$\mathcal{L}_{p,\omega}^{(s)} = \left\{ S \in \mathcal{L}: \lim_{n \to \infty} n^{1/p} s_n(S) = 0 \right\}.$$

2. *s*-Numbers

In this section we shall investigate realtions between entropy quantities, especially local entropy moduli, and *s*-numbers. We start with Hilbert numbers.

Proposition 1. *If $0 < p, t \leqslant \infty$, then*

$$\mathfrak{L}_{p,t}^{(G)} \subseteq \mathfrak{L}_{p,t}^{(h)}.$$

Moreover, $h_n(S) \leqslant G_n(S)$ for every $S \in \mathfrak{L}$.

PROOF. Given $S \in \mathfrak{L}(X, Y)$, $n \in \mathbb{N}$ and $\epsilon > 0$ one can find [19, 11.4.3.] operators $A \in \mathfrak{L}(l_2^n, X)$, $B \in \mathfrak{L}(Y, l_2^n)$ with $\|A\| \leqslant 1$, $\|B\| \leqslant 1$ and $BSA = = (1 - \epsilon)h_n(S)I_n$, where I_n is the identity in l_2^n. By the properties of local entropy moduli, $(1 - \epsilon)h_n(S) = G_n(BSA) \leqslant \|B\| G_n(S) \|A\|$, which implies $h_n(S) \leqslant G_n(S)$. The inclusion $\mathfrak{L}_{p,t}^{(G)} \subseteq \mathfrak{L}_{p,t}^{(h)}$ is an immediate consequence of this inequality. $\square$

Next let us state without proofs two lemmata from [6], that will be frequently used in what follows.

Lemma 2. *Let $s = c$ or d and $n \in \mathbb{N}$. Then for every $S \in \mathfrak{L}(X, Y)$*

$$\left(\prod_{i=1}^{n} s_i(S) \right)^{1/n} \leqslant n \sup \{ G_n(BSA) : \|A : l_1^n \to X\| \leqslant 1, \ \|B : Y \to l_\infty^n\| \leqslant 1 \}.$$

Lemma 3. *There are absolute constants $c_1, c_2 > 0$ such that for every $p, 1 \leqslant p \leqslant 2$, and $n \in \mathbb{N}$ the inequalities*

(i) $G_n(A) \leqslant c_1 n^{1/p - 1} T_p(X) \|A\|$ *for $A \in \mathfrak{L}(l_1^n, X)$*

and

(ii) $G_n(B) \leqslant c_2 n^{1/p - 1} T_p(Y') \|B\|$ *for $B \in \mathfrak{L}(Y, l_\infty^n)$*

hold.

A simple combination of these two lemmata yields the following relation between Gelfand, Kolmogorov and local entropy classes.

Proposition 4. *Let $s = c$ or d, let $1 \leqslant p, q \leqslant 2$ and $0 < r, t \leqslant \infty$ such that $\frac{1}{w} = 1 + \frac{1}{r} - \frac{1}{p} - \frac{1}{q} > 0$. If X and Y are Banach spaces such that X is of type p and Y' of type q, then*

$$\mathfrak{L}_{r,t}^{(G)}(X, Y) \subseteq \mathfrak{L}_{w,t}^{(s)}(X, Y).$$

Supposing that X and Y' are even of type 2, Gordon, König and Schütt [11] showed that

$$a_n(T) \sim c_n(T) \sim d_n(T)$$

for all operators $T \in \mathfrak{L}(X, Y)$.

Using this fact and Proposition 4 we derive

Proposition 5. *Let X and Y' be of type* 2, *and let* $0 < r, t \leqslant \infty$. *Then all classes* $\mathfrak{L}_{r,t}^{(s)}(X, Y)$ *with* $s \in \{a, c, d, e, g, G\}$ *coincide.*

Remark. In this special situation «local» and «global» degree of compactness are the same. As shown in Proposition 19, in general there is a big gap between them.

Finally let us consider Weyl and dual Weyl numbers.

Proposition 6. *Let* $1 \leqslant p \leqslant 2$ *and* $0 < r, t \leqslant \infty$ *such that* $\frac{1}{s} = \frac{1}{r} + \frac{1}{2} - \frac{1}{p} > 0$. *Let X and Y be Banach spaces. If Y' has type p, then*

$$\mathfrak{L}_{r,t}^{(G)}(X, Y) \subseteq \mathfrak{L}_{s,t}^{(x)}(X, Y),$$

and if X has type p, then

$$\mathfrak{L}_{r,t}^{(G)}(X, Y) \subseteq \mathfrak{L}_{s,t}^{(y)}(X, Y).$$

Moreover, there is a constant $c > 0$ such that for all $S \in \mathfrak{L}(X, Y)$, all $1 \leqslant p \leqslant 2$ and $n \in \mathbb{N}$,

$$x_n(S) \leqslant cn^{(1/p) - (1/2)} T_p(Y')G_n(S)$$

and

$$y_n(S) \leqslant cn^{(1/p) - (1/2)} T_p(X)G_n(S).$$

PROOF. Given $A \in \mathfrak{L}(l_2, X)$, $\|A\| \leqslant 1$, and $B \in \mathfrak{L}(Y, l_2)$, $\|B\| \leqslant 1$ we conclude from Lemma 2 and Lemma 3

$$a_n(SA) = c_n(SA) \leqslant cn^{(1/p) - (1/2)} T_p(Y')G_n(S),$$

and

$$a_n(BS) = d_n(BS) \leqslant cn^{(1/p) - (1/2)} T_p(X)G_n(S),$$

where we used that

$$a_n(T) = c_n(T) \quad \text{for} \quad T \in \mathfrak{L}(H, X) \quad \text{and}$$
$$a_n(T) = d_n(T) \quad \text{for} \quad T \in \mathfrak{L}(Y, H),$$

H being a Hilbert space, see [19, 11.5.2. and 11.6.2.]. These inequalities imply the desired estimates for $x_n(S)$ and $y_n(S)$, from which the inclusions, stated in the first part of the proposition, easily follow. $\square$

3. Spectral properties and eigenvalues

In this section we want to describe spectral properties and the eigenvalue behaviour of operators in $\mathcal{L}(X)$ in terms of their local entropy moduli.

First of all let us briefly explain the notations we are going to use. Given an operator $S \in \mathcal{L}(X)$ consider the coset $S + \mathcal{K}(X)$ as an element of the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$, where $\mathcal{K}(X)$ denotes the ideal of compact operators in the algebra $\mathcal{L}(X)$. The spectral radius of this element is called the *essential spectral radius* of S, $r_{ess}(S)$. Let $\sigma(S)$ denote the usual spectrum of S, then for every $r > r_{ess}(S)$ the set

$$\{\lambda \in \mathbb{C} : \lambda \in \sigma(S), |\lambda| \geqslant r\}$$

consists only of finitely many points, each being an eigenvalue of S of finite algebraic multiplicity. Thus we can order all eigenvalues λ of S with $|\lambda| > r_{ess}(S)$ in such a way that

$$|\lambda_1(S)| \geqslant |\lambda_2(S)| \geqslant \cdots,$$

where each eigenvalue is counted according to its algebraic multiplicity. If there are only n eigenvalues λ with $|\lambda| > r_{ess}(S)$, then put $\lambda_{n+1}(S) = \lambda_{n+2}(S) = \cdots = r_{ess}(S)$. So we have assigned to every $S \in \mathcal{L}(X)$ the sequence $(\lambda_n(S))$. For more details we refer to Zemanek [24] and the references given therein.

An operator $S \in \mathcal{L}(X, Y)$ is called *strictly cosingular*, if $Q_N^Y S$ is never a surjection, whenever N is an infinite dimensional subspace of Y, see [19, 1.10.2.].

Proposition 7. *Every operator in $\mathcal{L}_{\infty,\omega}^{(G)}$ is strictly cosingular.*

PROOF. Let $S \in \mathcal{L}(X, Y)$ with $\lim_{n \to \infty} G_n(S) = 0$. Assuming S being not strictly consingular one could find an infinite codimensional subspace N of Y that $Q_N^Y S$ is a surjection. Hence the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\ S\ } & Y \\ {\scriptstyle Q_M^X}\Big\downarrow & & \Big\downarrow{\scriptstyle Q_N^Y} \\ X/M & \xrightarrow[\ S_0\ ]{} & Y/N \end{array}$$

where M is the kernel of $Q_N^Y S$ and S_0 is an isomorphism. Now, by the surjectivity of local entropy moduli, for every $n \in \mathbb{N}$,

$$1 = G_n(I_{X/M}) = G_n(S_0^{-1}S_0) \leqslant \|S_0^{-1}\| G_n(S_0 Q_M^X) =$$
$$= \|S_0^{-1}\| G_n(Q_N^Y S) \leqslant \|S_0^{-1}\| G_n(S).$$

Letting $n \to \infty$ this yields a contradiction, thus proving the proposition. $\square$

Next we need the notion of *Riesz operators*. These are operators $S \in \mathcal{L}(X)$ such that for every complex number λ the operator $I - \lambda S$ has finite dimensional kernel and finite codimensional closed range, see [19, 26.].

The essential spectral radius of Riesz operators is always equal to zero, hence all their non-zero eigenvalues have finite multiplicities and can be arranged in non-decreasing order. Since every strictly cosingular operator in $\mathcal{L}(X)$ is Riesz [19, 26.6.10.], we get

Corollary 8. *For every Banach space X the class $\mathcal{L}_{\infty,\omega}^{(G)}(X, X)$ consists of Riesz operators only.*

The essential spectral radius can be computed by local entropy moduli, as the following result shows.

Proposition 9. *If $S \in \mathcal{L}(X)$, then*

$$r_{ess}(S) = \lim_{N \to \infty} G(S^N)^{1/N}.$$

PROOF. It is known (see e.g. [24]), that

$$r_{ess}(S) = \lim_{N \to \infty} a(S^N)^{1/N} = \lim_{N \to \infty} e(S^N)^{1/N}.$$

By Lemma 2 $c_n(S) \leqslant nG_n(S)$ for arbitrary $n \in \mathbb{N}$.
  Combining this with

$$a_n(S) \leqslant 2n^{1/2}c_n(S) \qquad [19, 11.12.2.]$$

and the monotonicity of approximation numbers we obtain

$$a(S) \leqslant 2n^{3/2}G_n(S) \quad \text{for} \quad n \in \mathbb{N}.$$

Observing that $G_n(T) \leqslant \sqrt{2}\, e_n(T)$ for $T \in \mathcal{L}$ we get for all $n, k, N \in \mathbb{N}$

$$a(S^{Nk})^{1/Nk} \leqslant (2n^{3/2})^{1/Nk} G_n(S^{Nk})^{1/Nk} \leqslant$$

$$\leqslant (2n^{3/2})^{1/Nk} G_n(S^N)^{1/N} \leqslant (2n^{3/2})^{1/Nk} 2^{1/2N} e_n(S^N)^{1/N}.$$

Letting now $k \to \infty$ yields

$$r_{ess}(S) \leqslant G_n(S^N)^{1/N} \leqslant 2^{1/2N} e_n(S^N)^{1/N}.$$

Taking then the infimum over all $n \in \mathbb{N}$ and letting finally $N \to \infty$ the assertion follows. $\square$

Let us now turn to eigenvalue means.

Proposition 10. *Let $S \in \mathcal{L}(X)$ and $n \in \mathbb{N}$. Then*

$$\left(\prod_{i=1}^{n} |\lambda_i(S)| \right)^{1/n} = \lim_{N \to \infty} G_n(S^N)^{1/N}.$$

PROOF. There is an n-dimensional invariant subspace X_0 of X such that the restriction S_0 of S onto X_0 has exactly the eigenvalues $\lambda_1(S), \ldots, \lambda_n(S)$. If P is any projection from X onto X_0 and J is the canonical embedding from X_0 into X, then for all $N \in \mathbb{N}$, $S_0^N = PS^N J$. This implies

$$g_n(S_0^N) = G_n(PS^N J) \leqslant \|P\| G_n(S^N) \leqslant \|P\| g_n(S^N).$$

Now the result of Makai and Zemanek [17]

$$\left(\prod_{i=1}^{n} |\lambda_i(T)| \right)^{1/n} = \lim_{N \to \infty} g_n(T) \quad \text{for} \quad T \in \mathcal{L}(X)$$

and

$$\left(\prod_{i=1}^{n} |\lambda_i(S)| \right)^{1/n} = \left(\prod_{i=1}^{n} |\lambda_i(S_0)| \right)^{1/n}$$

yield the assertion. $\square$

4. Absolutely summing operators

The goal of this section is to establish some relationships between p- and $(p, 2)$-absolutely summing operators, Weyl numbers and local entropy moduli for operators acting in Banach spaces having certain (Rademacher) type.

Before doing that one more notion is required. Let (g_i) denote a sequence of independent standard Gaussian random variables. Then for any operator $T \in \mathcal{L}(l_2^n, X)$ one sets

$$l(T) = \left(\mathbb{E} \left\| \sum_{1}^{n} Te_i g_i \right\|^2 \right)^{1/2},$$

and for $T \in \mathcal{L}(H, X)$, H being an arbitrary Hilbert space, let

$$l(T) = \sup \{ l(TP) : P \in \mathcal{L}(l_2^n, H) \text{ unitary}, \quad n \in \mathbb{N} \}.$$

This so-called γ-summing norm was introduced by Linde and Pietsch [16]. We need the following relations to 2-absolutely summing operators (cf. [10]) and entropy numbers (due to Sudakov, see e.g. [14]), which we state as

Lemma 11. *There is a constant $c > 0$ such that for all $T \in \mathcal{L}(H, X)$, where H is any Hilbert space, X any Banach space,*

(i) $l(T) \leqslant \tilde{T}_2(X)\pi_2(T')$ *and*

(ii) $\sup_{n \geqslant 1}(\ln n)^{1/2}\epsilon_n(T') \leqslant cl(T)$.

Now we are prepared to prove the

Theorem 12. *Let $1 < p \leqslant 2 \leqslant q < \infty$ such that $\frac{1}{r} = \frac{1}{q} + \frac{1}{2} - \frac{1}{p} > 0$. If X is a Banach space whose dual is of type p, then for all Banach spaces Y,*

$$\Pi_{q,2}(X, Y) \subseteq \mathcal{L}_{r,\infty}^{(G)}(X, Y).$$

Moreover, there is some constant $c > 0$ (neither depending on p nor on q) such that for all $S \in \mathcal{L}(X, Y)$

$$G_n(S) \leqslant cn^{-1/r}\pi_{q,2}(S)\tilde{T}_p(X').$$

PROOF. Given any operator $S \in \mathcal{L}(X, Y)$ and any n-codimensional subspace N of Y one has the following factorization diagram

where $\pi_2(Q_N^Y S) = \pi_2(S_0)$ and $\|A\| = 1$.

Setting $Z = S_0'(l_2^n)$, which is an at most n-dimensional subspace of X', we obtain from the preceding lemma

$$n^{1/2}g_n(Q_N^Y S) \leqslant (2n)^{1/2}\epsilon_{2n}(Q_N^Y S) \leqslant (2n)^{1/2}\epsilon_{2n}(S_0)\|A\| \leqslant$$
$$\leqslant 2^{1/2}cl(S_0') \leqslant 2^{1/2}c\pi_2(S_0)\tilde{T}_2(Z).$$

By the results of Tomczak-Jaegermann [23],

$$\pi_2(S_0) = \pi_2(Q_N^Y S) \leqslant 2n^{(1/2) - (1/q)}\pi_{q,2}(Q_N^Y S) \leqslant 2n^{(1/2) - (1/q)}\pi_{q,2}(S)$$

and

$$\tilde{T}_2(Z) \leqslant 2n^{(1/p) - (1/2)}\tilde{T}_p(Z) \leqslant 2n^{(1/p) - (1/2)}\tilde{T}_p(X'),$$

yielding the estimate

$$G_n(S) \leqslant 2^{5/2}cn^{-1/r}\pi_{q,2}(S)\tilde{T}_p(X').$$

This inequality immediately implies the inclusion stated in the first part of the theorem. $\square$

We state now the most important special case as

Corollary 13. *Let* $2 \leqslant q < \infty$. *If the dual of the Banach space X is of type* 2, *then for all Banach spaces Y*

$$\Pi_{q,2}(X, Y) \subseteq \mathcal{L}_{q,\infty}^{(G)}(X, Y).$$

Next we want to investigate the inverse problem: Under which conditions on the underlying Banach spaces local entropy classes do consist of $(q, 2)$-summing operators only? To answer this question we need the following result by Pietsch [20] concerning Weyl classes:

$$\mathcal{L}_{p,p}^{(x)} \subseteq \Pi_{p,2} \quad \text{for} \quad p > 2 \quad \text{and} \quad \mathcal{L}_{2,1}^{(x)} \subseteq \Pi_2.$$

Combining this with Proposition 6 one can derive

Corollary 14. *Given* $1 \leqslant p \leqslant 2 < q < \infty$ *and* $0 < r < \infty$ *such that* $\frac{1}{q} = \frac{1}{r} - \frac{1}{p} + \frac{1}{2} > 0$, *the inclusions*

$$\mathcal{L}_{r,q}^{(G)}(X, Y) \subseteq \Pi_{q,2}(X, Y) \qquad and$$
$$\mathcal{L}_{p,1}^{(G)}(X, Y) \subseteq \Pi_2(X, Y)$$

hold for all Banach spaces X and Y, provided that Y' is of type p.

Corollaries 13 and 14 imply now a result similar to that for Weyl numbers [20].

Corollary 15. *Let X and Y be Banach spaces whose duals are of type* 2. *Then*

$$\mathcal{L}_{q,q}^{(G)}(X, Y) \subseteq \Pi_{q,2}(X, Y) \subseteq \mathcal{L}_{q,\infty}^{(G)} \quad for \quad 2 < q < \infty$$

and

$$\mathcal{L}_{2,1}^{(G)}(X, Y) \subseteq \Pi_2(X, Y) \subseteq \mathcal{L}_{2,\infty}^{(G)}(X, Y).$$

Remark. As shown in [7] the result for $q = 2$ is valid also for entropy numbers and moduli instead of local entropy moduli.

The next result is devoted to p-absolutely summing operators.

Theorem 16. *If H is a Hilbert space and X a Banach space, then*

$$\Pi_p(H, X) \subseteq \mathcal{L}_{2,\infty}^{(G)}(H, X) \quad for \quad 0 < p < \infty.$$

Moreover, there is a constant $c > 0$ such that for all $2 \leqslant p < \infty$, $n \in \mathbb{N}$ and $S \in \mathcal{L}(H, X)$

$$G_n(S) \leqslant cp^{1/2}n^{-1/2}\pi_p(S).$$

PROOF.   Let $S \in \mathcal{L}(H, X)$, $2 \leqslant p < \infty$, $n \in \mathbb{N}$ and $N \subseteq X$ with $\operatorname{codim} N \leqslant n$. Denote by Q the quotient map from X onto X/N, let H_0 be the orthogonal complement of the kernel of QS, and let J be the embedding from H_0 into H and P the orthogonal projection from H onto H_0. Then clearly $QS = QSJP$. By the Pietsch factorization theorem [19] we have the following commutative diagram

$$
\begin{array}{ccc}
H \xrightarrow{\;QS\;} X/N \xrightarrow{\;J_0\;} Y \\
A \downarrow \qquad\qquad \qquad \downarrow B \\
L_\infty(K, \gamma) \xrightarrow{\qquad\qquad I \qquad\qquad} L_p(K, \gamma)
\end{array}
$$

where J_0 is a metric injection into an appropriate Banach space Y (possessing the metric extension property), γ is a probability measure on a compact Hausdorff space K, I is the identity and

$$\pi_p(QS) = \|A\| \, \|B\|.$$

Let $m := \dim H_0$. Then by a result of Kashin (see Szarek and Tomczak-Jaegermann [22]), there is an m-dimensional subspace E of l_1^{2m} with $d(E, l_2^m) < 4e$. (Here $d(X, Y)$ denotes the Banach-Mazur distance of two isomorphic Banach spaces X and Y.)

Hence also $d(E, H_0) < 4e$ and one can find an isomorphism $T \in \mathcal{L}(E, H_0)$ with $\|T\| \, \|T^{-1}\| \leqslant 4e$. Now we have the commutative diagram

$$
\begin{array}{ccccc}
H \xrightarrow{\;T^{-1}P\;} E \xrightarrow{\;JT\;} H \xrightarrow{\;J_0QS\;} Y \\
J_1 \downarrow \qquad A_0 \quad A \downarrow \qquad I \quad \uparrow B \\
l_1^{2m} \dashrightarrow L_\infty(K, \gamma) \xrightarrow{\;I\;} L_p(K, \gamma),
\end{array}
$$

where J_1 is the canonical injection and A_0 is an extension of AJT with $\|A_0\| = \|AJT\|$. Such A_0 exists, since $L_\infty(K, \gamma)$ has the metric extension property. By Lemma 3 we have (see [6])

$$g_m(IA_0) \leqslant aT_2(L_p(K, \gamma))m^{-1/2}\|IA_0\|$$

with some absolute constant $a > 0$. Observing that $T_2(L_p(K, \gamma)) \leqslant p^{1/2}$ we get from the monotonicity and injectivity of entropy moduli [5]

$$g_n(QS) \leqslant g_m(QS) \leqslant 2g_m(J_0 QSJP) = 2g_m(BIA_0 J_1 T^{-1}P) \leqslant$$
$$\leqslant 2g_m(IA_0)\|B\|\ \|J_1 T^{-1}P\|$$
$$\leqslant 2ap^{1/2}\|A\|\ \|T\|\ \|B\|\ \|T^{-1}\|m^{-1/2} \leqslant$$
$$\leqslant 8eap^{1/2}\pi_p(QS)n^{-1/2}.$$

Since N was arbitrarily chosen this finally yields the desired estimate

$$G_n(S) \leqslant cp^{1/2}\pi_p(S)n^{-1/2} \quad \text{with} \quad c = 8ea.$$

Hence, also

$$\Pi_p(H, X) \subseteq \mathcal{L}_{2,\infty}^{(G)}(H, X) \quad \text{for} \quad 0 < p < \infty. \qquad \square$$

5. Identical and diagonal operators

As concrete examples we consider in this section identical and diagonal operators between l_p-spaces, and determine the exact asymptotic behaviour of their local entropy moduli. As an application we shall see that there are non-compact operators whose local entropy moduli tend to zero, in contrast to the situation for entropy numbers or moduli. We start with identity operators.

Proposition 17. *Let* $1 < p < q < \infty$ *and* $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. *Then* $G_n(I: l_p \to l_q) \sim n^{-1/r}$, *hence* $I \in \mathcal{L}_{r,\infty}^{(G)}(l_p, l_q)$.

PROOF. The *estimate from below* is quite simple. Given $n \in \mathbb{N}$ one has the relation

$$1 = G_n(I: l_q^n \to l_q^n) \leqslant \|I: l_q^n \to l_p^n\|\, G_n(I: l_p^n \to l_q^n) \leqslant n^{(1/p) - (1/q)}G_n(I: l_p \to l_q),$$

hence $G_n(I: l_p \to l_q) \geqslant n^{-1/r}$.

Now let us turn to the *estimate from above*.

First we consider the *case* $1 < p < q \leqslant 2$.

By Bennett [1] or [3] it holds $I \in \Pi_{u,1}(l_p, l_q)$ with $\frac{1}{u} = \frac{1}{p} - \frac{1}{q} + \frac{1}{2}$. Moreover, there is a constant $c_0 > 0$ such that for all p, q with $1 < p < q \leqslant 2$ the estimate $\pi_{u,1}(I: l_p \to l_q) \leqslant c_0$ holds. This implies

$$\pi_{r,2}(I: l_p \to l_q) \leqslant c_0$$

and by Theorem 12

$$G_n(I: l_p \to l_q) \leqslant c_1 n^{-1/r}\tilde{T}_2(l_{p'})\pi_{r,2}(I) \leqslant c_2(p')^{1/2}n^{-1/r}.$$

Note, that c_2 is independent of p and q. Next let us treat the *case* $2 = p <$ $< q < \infty$.

Let N be an arbitrary n-codimensional subspace of l_q, and let $Q: l_q \to l_q/N$ be the canonical quotient map. Our aim is to estimate $\pi_q(QI)$ from above. Denoting the orthogonal projection from l_2 onto the orthogonal complement of the kernel of QI by P, and the injection of this space into l_2 by J, one has obviously

$$QI = QIJP.$$

Therefore, using the inequality

$$\pi_q(S) \leqslant \pi_q(S') \quad \text{for all} \quad S \in \mathcal{L}(l_2, l_q)$$

(see Pietsch [19, 19.5.2.]) we obtain

$$\pi_q(QI) \leqslant \pi_q(IJ) \leqslant \pi_q(J'I') \leqslant \pi_2(J'I').$$

Since $J'I'$ has rank at most n, by Tomczak-Jaegermann [23],

$$\pi_2(J'I') \leqslant 2n^{(1/2)-(1/r)}\pi_{r,2}(I': l_{q'} \to l_2) \leqslant 2c_0 n^{(1/2)-(1/r)},$$

where c_0 is the constant from the previous case. As shown in the proof of Theorem 16, there is a constant $c_3 > 0$ such that

$$g_n(QI) \leqslant c_3 q^{1/2} n^{-1/2} \pi_q(QI) \leqslant c_4 q^{1/2} n^{-1/2},$$

where again c_4 does not depend on q. This finally gives the desired estimate

$$G_n(I) \leqslant c_4 q^{1/2} n^{-1/r}.$$

Combining these two cases we get the result in the *case* $1 < p < 2 < q < \infty$. It holds with some absolute constant $c_5 > 0$

$$G_n(I: l_p \to l_q) \leqslant G_n(I: l_p \to l_2)G_n(I: l_2 \to l_q) \leqslant$$
$$= c_5(p')^{1/2} n^{(1/2)-(1/p)} q^{1/2} n^{(1/q)-(1/2)} =$$
$$= c_5(p'q)^{1/2} n^{-1/r}.$$

Finally, the remaining *case* $2 < p < q < \infty$ can be treated by interpolation. Let $0 < \theta < 1$ such that $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{q}$. Since l_p is then of K-type θ with respect to the interpolation couple (l_q, l_2), we get the estimate

$$G_n(I: l_p \to l_q) \leqslant 2G_{[n/(1-\theta)]}(I: l_q \to l_q)^{1-\theta} G_{[n/\theta]}(I: l_2 \to l_q)^{\theta} \leqslant$$
$$\leqslant 2c_4^{\theta}\left[\frac{n}{\theta}\right]^{\theta[(1/q)-(1/2)]} \leqslant c_6 n^{(1/q)-(1/p)} = c_6 n^{-1/r},$$

where c_6 only depends on θ but not on n. $\square$

As a useful consequence let us state

Corollary 18. *There is a constant $c > 0$ such that for all $n \geqslant 8$ $n^{-1} \leqslant$*
$\leqslant G_n(I: l_1 \to l_\infty) \leqslant c\,(\ln n)n^{-1}$, *therefore $I \in \mathfrak{L}_{p,t}^{(G)}(l_1, l_\infty)$ for all $p > 1$, $0 <$*
$< t \leqslant \infty$.

PROOF. The estimate from below can be proved in the same way as in Proposition 14. The estimate from above follows from the factorization

$$G_n(I: l_1 \to l_\infty) \leqslant \|I: l_1 \to l_p\| \, G_n(I: l_p \to l_q) \, \|I: l_q \to l_\infty\| \leqslant c_5(p'q)^{1/2}n^{(1/q)-(1/p)},$$

where $1 < p < 2 < q < \infty$ are arbitrary, and c_5 is the constant from the previous proposition. Specifying now p and q as $p' = q = \ln n$ we get

$$G_n(I: l_1 \to l_\infty) \leqslant c_5 q n^{2/q - 1} = c_5\,(\ln n)n^{-1}n^{2/\ln n} = c\,(\ln n)n^{-1} \quad \text{with} \quad c = c_5 e^2.$$

The inclusion $I \in \mathfrak{L}_{p,t}^{(G)}$ for $p > 1$, $0 < t \leqslant \infty$, is a consequence of the estimate from above. $\square$

Now let us return to the question, how «local» and «global» degree of compactness are related to each other. The following result shows that there is in general a big difference between them. This supplements Proposition 5. Let $\mathcal{K}$ denote the ideal of compact operators.

Proposition 19. *Let $0 < p, t < \infty$. Then*

$$\mathfrak{L}_{p,t}^{(G)} \subseteq \mathcal{K} \quad iff \quad p \leqslant 1.$$

PROOF. Since the (clearly non-compact) identity $I \in \mathfrak{L}(l_1, l_\infty)$ belongs to all classes $\mathfrak{L}_{p,t}^{(G)}$ with $p > 1$, it remains to prove the «if» part. Let $S \in \mathfrak{L}_{1,t}^{(G)}$ for some $0 < t < \infty$. Assuming S being non-compact one had $\inf_{n \in \mathbb{N}} c_n(S) > 0$, and the inequality (Lemma 2)

$$c_n(S) \leqslant nG_n(S)$$

would imply the contradiction

$$\infty = \sum_{n=1}^{\infty} \frac{c_n(S)^t}{n} \leqslant \sum_{n=1}^{\infty} (n^{1-(1/t)}G_n(S))^t < \infty.$$

hence $S \in \mathcal{K}$, and the proof is finished. $\square$

Remark. The exact asymptotic behaviour of $G_n(I: l_1 \to l_\infty)$ is not known, but the conjecture $G_n(I: l_1 \to l_\infty) \approx n^{-1}$ seems reasonable. The validity of this con-

jecture would imply, that even the class $\mathfrak{L}^{(G)}_{1,\infty}$ contains non-compact operators. Obviously, $\mathfrak{L}^{(G)}_{1,\omega} \subseteq \mathfrak{K}$.

Similar estimates as for the identity $I \in \mathfrak{L}(l_1, l_\infty)$, involving certain logarithmic terms, can be derived also for the identities $I \in \mathfrak{L}(l_p, l_q)$, where either $p = 1$ or $q = \infty$.

Now let us turn to diagonal operators. The diagonal operator D_σ, generated by a given sequence $\sigma = (\sigma_n) \in l_\infty$, and acting between appropriate Banach sequence spaces, is defined by $D_\sigma(\zeta_n) = (\sigma_n \zeta_n)$.

Proposition 20. *Let* $1 < p, q < \infty, 0 < r < \infty, 0 < t \leqslant \infty$ *and* $\frac{1}{s} = \frac{1}{p} + \frac{1}{r} - \frac{1}{q} > 0$. *Then* $D_\sigma \in \mathfrak{L}^{(G)}_{s,t}(l_p, l_q)$ *iff* $\sigma \in l_{r,t}$.

PROOF. If $\sigma \in l_{r,t}$, then by [4], $D_\sigma \in \mathfrak{L}^{(e)}_{s,t}(l_p, l_q) \subseteq \mathfrak{L}^{(G)}_{s,t}(l_p, l_q)$. It remains to prove the «only if» part. Without loss of generality let us assume that $\sigma_1 \geqslant \sigma_2 \geqslant \cdots > 0$. Let D_n be the operator D_σ restricted to the first n coordinates.

In the case $p \leqslant q$ Proposition 17 implies

$$n^{(1/q) - (1/p)} \leqslant G_n(I: l_p^n \to l_q^n) \leqslant G_n(D_n: l_p^n \to l_q^n) \| D_n^{-1}: l_q^n \to l_q^n \| \leqslant$$
$$\leqslant G_n(D_\sigma: l_p \to l_q) \sigma_n^{-1}.$$

if $p > q$, we proceed as follows, again using Proposition 17,

$$1 = G_n(I: l_p^n \to l_p^n) \leqslant$$
$$\leqslant G_n(D_n: l_p^n \to l_q^n) \| D_n^{-1}: l_q^n \to l_q^n \| G_n(I: l_q^n \to l_p^n) \leqslant$$
$$\leqslant c G_n(D_\sigma) \sigma_n^{-1} n^{(1/q) - (1/p)}.$$

In both cases, $G_n(D_\sigma) \geqslant c_1 \sigma_n n^{1/q - 1/p}$, hence $D_\sigma \in \mathfrak{L}^{(G)}_{s,t}$ implies the desired assertion $\sigma \in l_{r,t}$. $\square$

6. Eigenvalues of integral operators

In this section we apply the results obtained till now to certain integral operators, namely to Hille-Tamarkin and weakly singular integral operators. For the latter ones we only consider the critical case where the order of the singularity is half of the dimension of the domain on which the operator acts.

Throughout the section let (Ω, Σ, μ) be any σ-finite measure space. By a kernel we mean a $\mu \times \mu$-measurable function $K: \Omega \times \Omega \to \mathbb{C}$. To every kernel K we assign the integral operator T_K, defined as

$$T_K f(s) = \int_\Omega K(s, t) f(t) \, d\mu(t), \qquad s \in \Omega$$

for measurable functions f, provided the integral exists. We shall pose such assumptions on K, that T_K acts as a bounded operator between appropriately chosen function spaces. Thus let us introduce the classes $(L_q)L_p = (L_q(\Omega, \Sigma, \mu))L_p(\Omega, \Sigma, \mu)$, $1 \leqslant p, q < \infty$, consisting of all kernels K with

$$\|K\|_{(L_q)L_p} := \left(\int_\Omega \left(\int_\Omega |K(s, t)|^q \, d\mu(s) \right)^{p/q} d\mu(t) \right)^{1/p} < \infty.$$

These kernels are called *Hille-Tamarkin kernels*.

Theorem 21. *Let* $1 \leqslant p < \infty$, $1 < q \leqslant 2$ *and* $K \in (L_q)L_p$. *Then*

$$T_K \in \mathcal{L}_{q', \infty}^{(G)}(L_{p'}, L_q).$$

Moreover, for all $n \in \mathbb{N}$

$$G_n(T_K) \leqslant c n^{(1/q) - 1} \|K\|_{(L_q)L_p}$$

with some constant c, *depending on* p *and* q *but not on the underlying measure space.*

PROOF. Put

$$g(t) = \left(\int_\Omega |K(s, t)|^q \, d\mu(s) \right)^{1/q} \quad \text{for} \quad t \in \Omega,$$

and for $s, t \in \Omega$ set

$$\tilde{K}(s, t) := \begin{cases} K(s, t)/g(t) & \text{if} \quad g(t) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then $g \in L_p$ and $\int_\Omega |\tilde{K}(s, t)|^q \, d\mu(s) \leqslant 1$ for every $t \in \Omega$. This implies $M_g \in$ $\in \mathcal{L}(L_{p'}, L_1)$, where $M_g f = gf$, and for $f \in L_1$

$$\|T_{\tilde{K}} f\|_{L_q} = \left(\int_\Omega \left| \int_\Omega \tilde{K}(s, t) f(t) \, d\mu(t) \right|^q d\mu(s) \right)^{1/q} \leqslant$$

$$\leqslant \int_\Omega \left(\int_\Omega |\tilde{K}(s, t)|^q \, d\mu(s) \right)^{1/q} |f(t)| \, d\mu(t) \leqslant \|f\|_{L_1}.$$

Hence $T_{\tilde{K}} \in \mathcal{L}(L_1, L_q)$, and $T_K = T_{\tilde{K}} M_g$.

If $2 \leqslant p < \infty$, then L_p has type 2. By Kwapién [15] one has

$$\mathcal{L}(L_1, L_q) = \pi_{q', 2}(L_1, L_q)$$

and with some $c_1 > 0$

$$\pi_{q', 2}(S) \leqslant c_1 \|S\| \quad \text{for all} \quad S \in \mathcal{L}(L_1, L_q).$$

Now Corollary 13 implies for $n \in \mathbb{N}$

$$G_n(T_K) \leqslant c_2 n^{-1/q'} \|T_K\| \leqslant c_2 n^{-1/q'} \|K\|_{(L_q)L_p},$$

where the constant c_2 depends only on p, q, but not on the measure space.

In the case $1 < p < 2$ one can determine $2 < r < \infty$ such that $\frac{1}{p} = \frac{1}{2} + \frac{1}{r}$. Then g can be splitted as $g = g_1 g_2$, where $g_1 \in L_2$ and $g_2 \in L_r$. Therefore $M_{g_1} \in \mathcal{L}(L_2, L_1)$, $M_{g_2} \in \mathcal{L}(L_{p'}, L_2)$ and $M_g = M_{g_2} M_{g_1}$. Hence, similar as in the first case one can conclude

$$G_n(T_k) \leqslant \|M_{g_2}\| G_n(T_{\bar{K}} M_{g_1}) \leqslant \|g_2\|_{L_r} c_3 n^{-1/q} \|\bar{K}\| \|g_1\|_{L_2} = c_3 n^{-1/q'} \|K\|_{(L_q)L_p}.$$

Thus, in both cases,

$$T_K \in \mathcal{L}^{(G)}_{q', \infty}(L_{p'}, L_q). \qquad \square$$

Next we apply this result in order to get estimates for eigenvalues of Hille-Tamarkin operators with kernels from the space $(L_{p'})L_p, 2 \leqslant p < \infty$.

These results are already known, see [20], but the approach via local entropy moduli is new. In [8] entropy numbers were used to obtain similar results. But there some additional restrictions (e.g. finiteness of the underlying measure space) had to be posed, which can be omitted now.

Theorem 22. *Let $2 \leqslant p < \infty$ and $K \in (L_{p'})L_p$. Then $(\lambda_n(T_K)) \in l_p$.*

PROOF. Without loss of generality we may and do assume that $\|K\|_{(L_{p'})L_p} \leqslant 1$. Then, by the proof of the preceding theorem, there is a constant $c > 0$ only depending on p, but not on the underlying measure space, such that

$$G_n(T_K) \leqslant cn^{-1/p} \quad \text{for all} \quad n \in \mathbb{N}.$$

Hence, by Corollary 8, T_K is a Riesz operator, and by Proposition 10,

$$|\lambda_n(T_K)| \leqslant G_n(T_K) \leqslant cn^{-1/p} \quad \text{for all} \quad n \in \mathbb{N}.$$

Hence for arbitrary $r > p$

$$\sum_{n=1}^{\infty} |\lambda_n(T_K)|^r \leqslant \frac{cr}{r-p}. \qquad (*)$$

Now we proceed in an analogous way as it was done in Pietsch [21]. Define the kernel $\bar{K}$ on $\Omega^2 \times \Omega^2$ by $\bar{K}(s_1, s_2, t_1, t_2) = K(s_1, t_1)K(s_2, t_2)$. Then $\bar{K}$ belongs to the space $(L_{p'})L_p$ over the product measure space $(\Omega, \Sigma, \mu) \times (\Omega, \Sigma, \mu)$, and again $\|\bar{K}\|_{(L_{p'})L_p} \leqslant 1$. As in [21] it can be shown, that if λ, μ are eigenvalues of T_K with algebraic multiplicities l and m, then $\lambda\mu$ is an eigenvalue of $T_{\bar{K}}$ hav-

ing multiplicity at least $l \cdot m$. Applying now inequality (*) to $T_{\tilde{K}}$ instead of T_K we get

$$\left(\sum_{n=1}^{\infty} |\lambda_n(T_K)|^r\right)^2 = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} |\lambda_{n_1}(T_K)\lambda_{n_2}(T_K)|^r \leqslant \sum_{n=1}^{\infty} |\lambda_n(T_{\tilde{K}})|^r \leqslant \frac{cr}{r-p}.$$

Thus (*) holds even with constant $\left(\frac{cr}{r-p}\right)^{1/2}$. Iterating this argument it follows

$$\sum_{n=1}^{\infty} |\lambda_n(T_K)|^r \leqslant 1 \quad \text{for every} \quad r > p,$$

which implies

$$\sum_{n=1}^{\infty} |\lambda_n(T_k)|^p \leqslant 1,$$

hence the proof is finished. $\quad\square$

Finally let us consider *weakly singular integral operators*. Here we want to restrict our attention to the border case where the order of singularity is half of the dimension of the domain on which the operator acts. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain, N any positive integer, and let $\Delta = \{(x, x): x \in \Omega\}$. Suppose the measurable kernel $K: \Omega^2\backslash\Delta \to \mathbb{C}$ is of the form

$$K(x, y) = \frac{L(x, y)}{|x - y|^{N-\alpha}}, \quad 0 > \alpha \geqslant N,$$

with $L \in L_\infty(\Omega^2)$. (Here $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^N$.) Then the operator T_K is compact in $L_\infty(\Omega)$. As shown by König, Retherford and Tomczak-Jaegermann [13, Proposition 12] its eigenvalues are square summable whenever $\alpha > N/2$ and satisfy

$$\lambda_n(T_K) = 0(n^{-\alpha/N}) \quad \text{if} \quad 0 < \alpha < N/2.$$

In both cases the result is optimal. The conjecture

$$\lambda_n(T_K) = 0(n^{-1/2}) \quad \text{in the border case} \quad \alpha = N/2$$

fails, as was proved by König [12]. The optimal asymptotic behaviour is

$$\lambda_n(T_K) = 0\left(\left(\frac{\ln n}{n}\right)^{1/2}\right).$$

In order to illustrate the usefulness of the concept of local entropy moduli we want to give a proof of the last mentioned result via local entropy moduli. We replace the condition $L \in L_\infty(\Omega^2)$ of König [12] by a weaker one.

Theorem 23. *Let* Ω *be a bounded domain of* $\mathbb{R}^N$, $N \geqslant 2$, *and let* $\Delta = \{(x,x): x \in \Omega\}$. *Given a measurable kernel* $K: \Omega^2 \backslash \Delta \to \mathbb{C}$ *of the form*

$$K(x,y) = \frac{L(x,y)}{|x-y|^{N/2}}$$

with

$$l(y) := \sup_{x \in \Omega} |L(x,y)| \in L_p(\Omega), \qquad 2 < p < \infty,$$

one has

$$\lambda_n(T_K) = 0\left(\left(\frac{\ln n}{n}\right)^{1/2}\right).$$

PROOF. Put

$$\tilde{K}(x,y) = \begin{cases} \dfrac{K(x,y)}{l(y)} & \text{if} \quad l(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then T_K as operator in $L_{p'}(\Omega)$ can be factorized as follows

$$
\begin{array}{ccc}
L_{p'} & \xrightarrow{\quad T_K \quad} & L_{p'} \\
\scriptstyle M_1 \downarrow & & \uparrow \scriptstyle I_1 \\
L_1 & \xrightarrow[\;T_{\tilde{K}}\;]{} L_{2,\infty} \xrightarrow[\;I_0\;]{} & L_s \quad,
\end{array}
$$

where s is any real with $p' \lessdot s < 2$, and I_0, I_1 are the respective identities. Since L_p (the dual of $L_{p'}$) is of type 2 and since every $S \in \mathcal{L}(L_1, L_s)$ is $(s', 2)$-absolutely summing with $\pi_{s',2}(S) \leqslant \rho \|S\|$, for some constant $\rho > 0$ not depending on S and s, Corollary 13 implies with some constans $c_1, c_2 > 0$ independent of s,

$$G_n(T_K) \leqslant G_n(I_0 T_{\tilde{K}} M_1) \|I_1\| \leqslant$$

$$\leqslant c_1 T_2(L_p) n^{-1/s'} \pi_{s',2}(I_0 T_{\tilde{K}} M_1) \|I_1\| \leqslant$$

$$\leqslant c_2 n^{-1/s'} \|I_0 T_{\tilde{K}} M_1\| \; \|I_1\|.$$

Observing that

$$\|I_0\| \leqslant \left(\frac{2}{2-s}\right)^{1/s} \mu(\Omega)^{(1/s)-(1/2)},$$

$$\|I_1\| \leqslant \mu(\Omega)^{(1/p')-(1/s)}, \qquad \|M_1\| \leqslant \|l\|_{L_p(\Omega)} \quad \text{and}$$

$$\|T_{\tilde{K}}\| \leqslant \|k\|_{L_{2,\infty}(\Omega-\Omega)} < \infty, \quad \text{where} \quad k(x) = |x|^{-N/2},$$

we obtain the estimate

$$G_n(T_K) \leqslant c_3 n^{-1/s'}\left(\frac{2}{2-s}\right)^{1/s},$$

with $c_3 = c_3(\Omega, p, l)$ not depending on s. For n large enough one can specify now s such that $\frac{2}{2-s} = \ln n$, $p' < s < 2$. Then $n^{-1/s'}\left(\frac{2}{2-s}\right)^{1/s} \leqslant c_4\left(\frac{\ln n}{n}\right)^{1/2}$, and we finally get $G_n(T_K) \leqslant c_5\left(\frac{\ln n}{n}\right)^{1/2}$ for n large enough. Via Proposition 10 this implies the desired assertion. $\square$

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