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Calderón-Zygmund Operators on Product Spaces

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1. Introduction

In their well-known theory of singular integral operators, Calderón and Zygmund [3] obtained the boundedness of certain convolution operators on $\mathbb{R}^d$ which generalize the Hilbert transform H in $\mathbb{R}^1$, defined for $f \in C_0^\infty(\mathbb{R}^1)$ by

$$(0.1) \qquad Hf(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} \frac{f(y)}{x-y}\, dy.$$

Typical examples of such operators are the Riesz transforms R_j, $j \in [1, d]$, defined for $f \in C_0^\infty(\mathbb{R}^d)$ by

$$(0.2) \qquad R_j f(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} \frac{x_j - y_j}{|x-y|^{d+1}} f(y)\, dy.$$

Their program can be decomposed into two steps. In the first one they prove L^2-boundedness using Plancherel's theorem. In the second step they use the smoothness and size properties of the kernel and the L^2-boundedness to prove L^p-boundedness for $p \in]1, +\infty[$ as well as the a.e. convergence of the r.h.s. of (0.2) for $f \in L^p$, $p \in]1, +\infty[$. Peetre [14] has shown that these operators are also bounded from $\mathrm{BMO}(\mathbb{R}^d)$ to $\mathrm{BMO}(\mathbb{R}^d)$.

The theory has been generalized in two ways.

In the first extension, one considers non-convolution operators associated to a kernel in the following sense. Let Δ be the diagonal set of $\mathbb{R}^d \times \mathbb{R}^d$ and let K be a locally bounded function defined from $\mathbb{R}^d \times \mathbb{R}^d \backslash \Delta$ to $\mathbb{C}$. Let T: $C_0^\infty(\mathbb{R}^d) \to [C_0^\infty(\mathbb{R}^d)]'$ be a linear operator defined in the weakest possible sense. Then K is the kernel of T if for $f, g \in C_0^\infty(\mathbb{R}^d)$ with disjoint supports, $\langle g, Tf \rangle$ is given by $\iint g(x)K(x, y)f(y)\,dx\,dy$. Suppose moreover that K satisfies some smoothness and size properties analogous to those enjoyed by the kernels of the Riesz transforms. Of course one cannot conclude that T is bounded on L^2 and if T is not a convolution operator one usually cannot use Plancherel's theorem. However it was observed that if the operator is known to be bounded on L^2 the second part of the program of Calderón and Zygmund can be carried out and one obtains a variety of results as in the convolution case. See [8] or [12]. In addition these operators are bounded from L^∞ to BMO, the obstruction for boundedness on BMO being purely algebraic; that is, they are bounded on BMO if and only if they are well defined on BMO, as for instance, in the convolution case. The most famous non-convolution operator of this kind is the Cauchy-operator on Lipschitz curves T_a defined for $a \in L_{\mathbb{C}}^\infty(\mathbb{R})$, $\|a\|_\infty < 1$, $f, g \in C_0^\infty(\mathbb{R})$ by

$$\langle g, T_a f \rangle = \iint \frac{g(x)f(y)}{(x-y) + \int_x^y a(u)\,du}\,dx\,dy. \tag{0.3}$$

This example also illustrates the weakness of the theory since it leaves open the question of the L^2-boundedness of such operators. See however [2] and [7] for the Cauchy kernel. This gap has been recently filled, up to a certain extent, by the so-called $T1$-theorem [9] which asserts that under a very weak regularity condition, T is bounded on L^2 if and only if $T1$ and T^*1, defined appropriately, both lie on BMO.

The second extension is due to R. Fefferman and E. Stein [11]. They study convolution operators which satisfy certain quantitative properties enjoyed by tensor products of operators of Calderón-Zygmund type, as for instance the double Hilbert transform defined for $f \in C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})$ by

$$[(H_1 \otimes H_2)f](x_1, x_2) = \lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_2 \to 0}} \iint_{\substack{|x_1 - y_1| > \epsilon_1 \\ |x_2 - y_2| > \epsilon_2}} \frac{f(y_1, y_2)}{(x_1 - y_1)(x_2 - y_2)}\,dy_1\,dy_2. \tag{0.4}$$

For such tensor products the L^p-boundedness for $p \in]1, +\infty[$ is a trivial consequence of Fubini's theorem but for the more general Fefferman-Stein operators a new machinery is built in [11] which unfortunately uses at each step that the operators under consideration are convolution operators. Moreover it ignores «the BMO aspect of things» in which we shall be mostly interested, while it gives sharp results on maximal operators, which we cannot handle.

Our purpose is to unify up to a certain extent these two generalizations and to define on a product of n Euclidean spaces a class of singular integraloperators which coincides with the extended Calderón-Zygmund class in the case $n = 1$ and coincides in the convolution case with the Fefferman-Stein class when $n = 2$. Actually we extend the non-convolution-Calderón-Zygmund class, and then proceed by induction for $n > 2$. The basic example of an operator considered in this setting is the «n^{th}-Cauchy operator» associated to the kernel K_a, defined for $a \in L^\infty_{\mathbb{C}}(\mathbb{R}^n)$ and $\|a\|_\infty < 1$ by

$$K_a(x, y) = \frac{1}{\prod_{i=1}^{n} (x_i - y_i) + \int_{x_1}^{y_1} \dots \int_{x_n}^{y_n} a(u_1, \dots, u_n)\, du}.$$

As in the case $n = 1$, this kernel K_a can be expanded in the sum $\sum_{j \in N} L_a^j$ of «commutators» where

$$L_a^j(x, y) = \left[\prod_{i=1}^{n} (x_i - y_i)\right]^{-j-1} \left[\int_{x_1}^{y_1} \dots \int_{x_n}^{y_n} a(u)\, du\right]^j.$$

Let $\bar{L}_a^j$ be the operator associated to L_a^j. Then we show $\|\bar{L}_a^j\|_{2,2} \leqslant C_n^j \|a\|_\infty^j$. Thus we can sum the series and obtain

$$\|\bar{K}_a\|_{2,2} \leqslant \frac{1}{1 - C_n \|a\|_\infty} \quad \text{for} \quad \|a\|_\infty < \frac{1}{C_n}.$$

The general case $\|a\|_\infty < 1$ remains open.

The connection between L^2 and BMO, emphasized by the $T1$-Theorem and its proof, turns out to be extremely useful in this setting too. The BMO-space to be considered is the space of Chang-Fefferman studied in [5] which takes into account the product structure of the underlying space. As in the classical situation one makes two kinds of size and smoothness assumptions (integral or pointwise) on the kernel according to whether the associated operator is known to be bounded on L^2 or not. In the first case we show under rather weak assumptions on the kernel that the operator is also bounded from L^∞ to BMO and therefore on all L^p's for $p \in]1, +\infty[$ and under somewhat stronger assumptions the boundedness on BMO, if there is no algebraic obstruction. In the second case we show a $T1$-theorem in the spirit of the classical one. In the case where T is given by a kernel K antisymmetric in each pair $(x_i, y_i)_{1 \le i \le n}$ as K_a or L_a^j for instance, the the $T1$-theorem reduces to: T is bounded on L^2 if and only if $T1 \in$ BMO.

In Sections 1 and 2 we recall some basic notations on singular integrals and Calderón-Zygmund operators in the classical situation and on BMO and Carleson measures on product spaces. The class of operators we wish to study is presented in Section 3, together with their more immediate properties.

In Section 4 we reduce the implication «L^2-boundedness → L^∞-BMO--boundedness» to a geometric lemma which we prove in Section 5. This lemma may be thought of as a substitute for the Whitney decomposition in the setting of product spaces. In Section 6 we state our «$T1$-theorem» and reduce its proof to two technical points which are studied in Section 7 and 8. Section 9 deals with a special property of antisymmetric kernels, which is new even when $n = 1$ and which is applied to the study of the kernel K_a for $\|a\| \leqslant \epsilon_n$. Finally we apply in Section 10 the geometric lemma of Section 5 to extend a result of J. L. Rubio de Francia on a Littlewood-Paley inequality of arbitrary intervals of $\mathbb{R}$ to the n-dimensional setting.

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1. Classical singular integral operators and Calderón-Zygmund operators on R^d

The definitions we shall adopt are standard. However, the terminology will be slightly different than usual ([8], [12]).

Let $\Omega = \mathbb{R}^d \times \mathbb{R}^d \backslash \Delta$, where $\Delta = \{(x, y), x = y\}$, and let $\delta \in]0, 1[$.

Definition 1. *Let K be a continuous function defined on Ω and taking its values in a Banach space B. This function K is a B-δ-standard kernel if the following are satisfied, for some constant $C > 0$.*

For all (x, y) in Ω,

$$|K(x, y)|_B \leqslant \frac{C}{|x-y|^d}. \tag{1.1}$$

For all (x, y) in Ω, and x' in $\mathbb{R}^d$ such that $|x - x'| < \frac{|x-y|}{2}$, (1.2)

$$|K(x, y) - K(x', y)|_B \leqslant C \frac{|x-x'|^\delta}{|x-y|^{d+\delta}} \quad \text{and}$$

$$|K(y, x) - K(y, x')|_B \leqslant C \frac{|x-x'|^\delta}{|x-y|^{d+\delta}}.$$

The smallest constant C for which (1.1) and (1.2) hold is denoted by $|K|_{\delta, B}$. We shall omit the subscript B when it creates no ambiguity.

Definition 2. *Let T: $C_0^\infty(\mathbb{R}^d) \to [C_0^\infty(\mathbb{R}^d)]'$ be a continuous linear mapping. T*

is a singular integral operator (SIO) *if, for some* $\delta \in]0,1[$, *there exists a* C-δ-*standard kernel* K *such that for all functions* f, g *in* $C_0^\infty(\mathbb{R}^d)$ *having disjoint supports,*

$$\langle g, Tf \rangle = \iint g(x) K(x,y) f(y)\, dx\, dx. \tag{1.3}$$

Here $\langle g, Tf \rangle$ denotes the action of the distribution Tf on the function g. We shall also say that T is a δ-SIO.

Definition 3. *Let* T *be a* δ-SIO. *It is a* δ-*Calderón-Zygmund operator* (δ-CZO) *if it extends boundedly from* L^2 *to itself.*

The following theorem gives necessary and sufficient conditions for a δ-SIO to be a δ-CZO. The statement of these conditions is explained afterwards.

Theorem 1 [9]. *Let* T *be a* δ-SIO. *It is a* δ-CZO *if and only if*

$$T1 \in \mathrm{BMO} \tag{1.4}$$

$$T^*1 \in \mathrm{BMO} \tag{1.5}$$

$$T \text{ has the weak-boundedness property} \tag{1.6}$$

In order to give a meaning to (1.4) we must show how T acts on bounded C^∞ functions. The meaning of (1.5) will then be clear since T^*, defined by $\langle g, T^*f \rangle = \langle f, Tg \rangle$ for all $f, g \in C_0^\infty(\mathbb{R}^d)$, is also a δ-SIO if T is.

The action of an SIO, T on $C_b^\infty(\mathbb{R}^d)$, the set of bounded C^∞ functions, is described the following way ([8], [9]). For $f \in C_b^\infty(\mathbb{R}^d)$, Tf will be a distribution acting on $C_{00}^\infty(\mathbb{R}^d)$, the subspace of $C_0^\infty(\mathbb{R}^d)$ of functions g such that $\int g\, dx = 0$. Let g be such a function and let $h \in C_0^\infty(\mathbb{R}^d)$ be equal to f on a neighborhood of $\operatorname{supp} g$, so that g and $f - h$ have disjoint supports.

If f has compact support,

$$\langle g, Tf \rangle = \langle g, Th \rangle + \langle g, T(f-h) \rangle, \qquad \text{where, by (1.3),}$$

$$\langle g, T(f-h) \rangle = \iint g(x) K(x,y) [f(y) - h(y)]\, dx\, dy.$$

Since g has mean value 0, this is also equal to

$$\iint g(x) [K(x,y) - K(x_0,y)]\, [f(y) - h(y)]\, dx\, dy,$$

where x_0 is any point of $\operatorname{supp} g$. Notice that by (1.2), this integral is absolutely convergent even if $(f-h)$ has non-compact support, and is independent of x_0. This integral can therefore serve as a definition of $\langle g, T(f-h) \rangle$. Obviously $\langle g, Th \rangle + \langle g, T(f-h) \rangle$ does not depend on the choice of h. Hence we can set

$$\langle g, Tf \rangle = \langle g, T(f-h) \rangle + \langle g, Th \rangle,$$

and this defines the desired extension.

This description yields immediately an effective method for computing Tf when $f \in C_b^\infty(R^d)$.

Lemma 1. *Let θ be in $C_0^\infty(R^d)$ and equal to 1 on $\{x, |x| < 1\}$. Let θ_q be defined for $q \in \mathbb{N}$ by $\theta_q(x) = \theta\left(\frac{x}{q}\right)$, and for f on $C_b^\infty(R^d)$ let $f_q = f\theta_q$. Then for all g in $C_{00}^\infty(R^d)$,*

$$\langle g, Tf\rangle = \lim_{q \to +\infty} \langle g, Tf_q\rangle. \tag{1.7}$$

We shall now give the meaning of (1.6). See [9].

Definition 4. *Let T be a δ-SIO. It has the weak boundedness property if for any bounded subset B of $C_0^\infty(\mathbb{R}^d)$ there exists $C_B > 0$ such that for any pair (η, ξ) of elements of B and any (x, t) in $\mathbb{R}_{d+1}^+$,*

$$|\langle \eta_t^x, T\xi_t^x\rangle| \leqslant c_B t^{-d}, \tag{1.8}$$

where ξ_t^x is defined by $\xi_t^x(y) = \frac{1}{t^d}\xi\left(\frac{y-x}{t}\right)$ and η_t^x similarly.

We shall also write that T has the WBP.

Note that any operator T bounded on L^2 has the WBP since there exists a constant C'_B such that $\|\xi_t^x\|_2 \leqslant C'_B t^{-d/2}$ for all (x, t) in $\mathbb{R}_{d+1}^+$ and ξ in B.

It is easy to show that T has the WBP if there exists a constant C and an integer N such that for all cubes Q of length $\delta(Q)$ and all functions f and g supported in Q, $|\langle g, Tf\rangle| \leqslant C|Q|P(N, g, Q)P(N, f, Q)$, where

$$P(N, g, Q) = \sum_{|\alpha| \leq N} |\delta(Q)|^\alpha \left\| \frac{\partial^\alpha}{\partial x^\alpha} g \right\|_\infty. \tag{1.9}$$

It is well known that CZO's are bounded from L^∞ to BMO. However, there exist conditions much weaker than (1.2) that will ensure that an operator T, bounded on L^2, associated in the sense of (1.3) to a kernel K, is bounded from L^∞ to BMO. The weakest of the known conditions is

$$\int_{|x-y| > 2|x-x'|} |K(x, y) - K(x', y)|\, dy < C$$

and is due to Calderón and Zygmund. For our purposes it will be best to assume something slightly stronger

$$\int_{|x-y| > 2^k|x-x'|} |K(x, y) - K(x', y)|\, dy \leqslant C2^{-k\epsilon} \tag{1.10}$$

for some $\epsilon > 0$ and all $k \in \mathbb{N}$.

Definition 5. *A locally integrable function K satisfying to* (1.10) *is an ϵ-kernel. An operator T bounded on L^2 and associated to an ϵ-kernel is a*

Calderón-Zygmund operator of type ϵ (CZ_ϵ). ***If K takes its values in a normed space V, then it is a V-ϵ-kernel.***

We denote by $|K|_{\epsilon, V}$ the smallest C for which (1.10) holds.

This distinction between pointwise conditions like (1.2) and integral conditions like (1.10) becomes crucial when the operator T maps functions of $C_0^\infty(\mathbb{R})$ into Hilbert-space valued distributions, that is, distributions acting of functions taking their values in a Hilbert space H. In this case the kernel K takes its values in H and there are two possible ways to extend (1.10) in this setting, namely

$$\int_{|x-y|>2^k|x-x'|} \|K(x,y) - K(x',y)\|_H \, dy < C2^{-k\epsilon}$$

or, for all $\lambda \in H$ such that $\|\lambda\|_H = 1$,

$$\int_{|x-y|>2^k|x-x'|} |\langle \lambda, K(x,y) - K(x',y)\rangle_H| \, dy < C2^{-k\epsilon}. \tag{1.11}$$

Observe that an operator T bounded from L^2 to L^2_H associated to a kernel K satisfying (1.11) is bounded from L^∞ to BMO_H and therefore from L^p to L^p_H for all $p \in]2, +\infty[$, [15].

A slightly stronger version of (1.11) appears in the proof of the following theorem of J. L. Rubio de Francia.

Theorem 2 [15]. *Let $\{I_k\}_{k \in N}$ be a collection of disjoint intervals of $\mathbb{R}$ and let S_{I_k} be the Fourier multiplier with symbol χ_{I_k}. Finally let Δ be defined on L^2 by $\Delta f = [\Sigma (S_{I_k} f)^2]^{1/2}$. Then Δ is bounded on L^p for all $p \in [2, +\infty[$.*

We shall conclude this section with a lemma of Coifman and Meyer, some notations and a remark.

The letter φ will always denote a C_0^∞ radial function supported in the unit ball and such that $\int \varphi \, dx = 1$. Let us define φ_t by $\varphi_t(y) = \frac{1}{t^d} \varphi\left(\frac{y}{t}\right)$. Then P_t is the convolution with φ_t.

The letter ψ will denote a radial C_{00}^∞ function supported in the unit ball and **such that, for all $\xi \in R^d$, (1.12) $\int_0^{+\infty} |\hat{\psi}(t\xi)|^2 t^{-1} \, dt = 1$. We define ψ_t and Q_t like φ_t and P_t.**

Lemma 2. *Let T be a δ-*SIO *having the* WBP. *For all bounded subsets B of $C_0^\infty(R^d)$ and $\eta, \xi \in B$ such that $\int \eta \, dx = 0$ or $\int \xi \, dx = 0$,*

$$|\langle \eta_t^x, T\xi_t^y \rangle| \leqslant C_B \omega_{\delta, t}(x-y), \tag{1.13}$$

where $$\omega_{\delta, t}(x-y) = \frac{t^\delta}{t^{d+\delta} + |x-y|^{d+\delta}}.$$

Conversely every continuous operator T: $C_0^\infty(R^d) \to [C_0^\infty(R^d)]'$ having the WBP *and satisfying* (1.13) *is a δ'-*SIO *for all $\delta' < \delta$.*

We omit the proof of this lemma, which is elementary. For the converse part one uses the decomposition of T as $-\int_0^\infty \frac{\partial}{\partial t}(P_t T P_t)\,dt$.

This lemma suggests the following convention. In order to unify (1.1), (1.2) and (1.8) in an inequality analogous to (1.13) we shall remove the assumption «$\int \eta\,dx = 0$ or $\int \xi\,dx = 0$» when $x = y$. In the rest of the paper and without explicit mention we shall assume that if two functions η_t^x and ξ_t^y appear in an inequality of type (1.13) and $x \neq y$, then $\int \eta\,dx = 0$ or $\int \xi\,dx = 0$.

Finally let us observe that if a function f is, say, in L^2 and T is a $CZ\epsilon$, then Tf is a L^2-function and $Q_t Tf$ is a C^∞ function. Let $x \in \mathbb{R}^d$ and suppose $f(z) = 0$ when $|x - z| \leqslant 2t$. Then we can write

$$Q_t Tf(x) = \int_{|x-y|>2t} (Q_t T)_{xz} f(z)\,dz \tag{1.14}$$

where $(Q_t T)_{xz} = \int \psi_t(x-y)[K(y,z) - K(x,z)]\,dy$.

By (1.10), we have

$$\int_{|x-z|>2^k t} |(Q_t T)_{xz}|\,dz \leqslant C 2^{-k\epsilon}. \tag{1.15}$$

As a consequence, the following inequality holds for all $u \in \mathbb{R}_+$

$$\int_{\substack{r \leq u \\ 2t \vee u \leq |x-z|}} |(Q_t T)_{xz}|\,dz \frac{dt}{t} \leqslant C_{\epsilon, T}. \tag{1.16}$$

2. Carleson measures and BMO on product spaces

Let Ω be an open subset in $\mathbb{R} \times \mathbb{R}$. $S(\Omega)$ is the subset of $\mathbb{R}^2_+ \times \mathbb{R}^2_+$ of (x_1, t_1, x_2, t_2)'s such that $]x_1 - t_1, x_1 + t_1[\times]x_2 - t_2, x_2 + t_2[\subseteq \Omega$.

Definition 6 [4]. *A Carleson measure on* $\mathbb{R}^2_+ \times \mathbb{R}^2_+$ *is a measure* $d\mu(x_1, t_1, x_2, t_2) = d\mu(x, t)$ *such that for all* Ω

$$\int_{S(\Omega)} d\mu(x,t) \leqslant C_\mu |\Omega|.$$

Definition 7. *A function* b *is in* BMO($\mathbb{R} \times \mathbb{R}$) *if it can be written as* $a_0 + H_1 a_1 + H_2 a_2 + H_1 H_2 a_3$, *with* $\sum_{i=0}^3 \|a_i\|_\infty < +\infty$ *and where the* H_j's, $j \in \{1, 2\}$ *are the partial Hilbert transforms. Moreover,* $[\inf \sum_{i=0}^3 \|a_i\|_\infty]$, *where the* inf *is taken over all possible decompositions of* b, *is a norm that makes* BMO($\mathbb{R} \times \mathbb{R}$) *a Banach space.*

Let Q_{t_1} be defined on $C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})$ by $Q_{t_1}[f \otimes g] = [Q_{t_1} f] \otimes g$ and similarly for Q_{t_2}. Clearly Q_{t_1} and Q_{t_2} extend by linearity to $L^2_{\mathrm{loc}}(\mathbb{R}^2)$. A. Chang and R. Fefferman have proved the following.

Theorem A [5]. *A function* b *in* L^2_{loc} *is in* BMO *if and only if* $([Q_{t_1} Q_{t_2} b](x_1, x_2))^2 \, dx_1 \, dx_2 \, (t_1 t_2)^{-1} \, dt_1 \, dt_2$. *Is a Carleson measure on* $\mathbb{R}^2_+ \times \mathbb{R}^2_+$.

Theorem B [6]. *A linear operator* T *bounded from* L^2 *to* L^2 *and from* $L^\infty(\mathbb{R}^2)$ *to* $\text{BMO}(\mathbb{R} \times \mathbb{R})$ *is bounded on all* L^p*'s for* $p \in]2, +\infty[$.

It is a routine exercise to rewrite these definitions and theorems when $\mathbb{R} \times \mathbb{R}$ is replaced by $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \times \ldots \times \mathbb{R}^{d_n}$ and $\mathbb{R}^2_+ \times \mathbb{R}^2_+$ is replaced by $\mathbb{R}^{d_1+1}_+ \times \times \mathbb{R}^{d_2+1}_+ \times \ldots \times \mathbb{R}^{d_n+1}_+$. Moreover Theorems A and B remain valid if the functions under consideration are Hilbert-space valued. This will be used without mention in Section 10. In order to avoid minor technical complication we shall suppose from now on that all the d_i's are equal to 1.

3. Extension of the definitions of Section 1 in the setting of product spaces

Let T_1 and T_2 be two classical δ-SIO's on $\mathbb{R}$ and let $T = T_1 \otimes T_2$. This operator T is a priori defined from $C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})$ to its algebraic dual by the formula

$$\langle g_1 \otimes g_2, Tf_1 \otimes f_2 \rangle = \langle g_1, T_1 f_1 \rangle \langle g_2, T_2 f_2 \rangle.$$

Let L_1 and L_2 be the kernels of T_1 and T_2. If g_1 and f_1 have disjoint supports, we can write

$$\langle g_1 \otimes g_2, Tf_1 \otimes f_2 \rangle = \int g_1(x) L_1(x, y) f_1(y) \langle g_2, T_2 f_2 \rangle \, dx \, dy.$$

Let us put on the set of δ-CZO's the norm $\| \ \|_{\delta CZ}$ defined by $\|S\|_{\delta CZ} = \|S\|_{2,2} + |K|_\delta$ where K is the kernel of S. This makes the set of δ-CZO's a Banach space which we denote by δCZ. Let $K_1(x, y) = L_1(x, y) T_2$. Then K_1 is a δCZ-valued function and is actually a δCZ-δ-standard kernel and one has

$$\langle g_1 \otimes g_2, Tf_1 \otimes f_2 \rangle = \iint g_1(x) \langle g_2, K_1(x, y) f_2 \rangle f_1(y) \, dx \, dy.$$

We can define $K_2(x, y)$ in a similar fashion. Now we forget that T is a tensor product and set the following definition.

Definition 8. *Let* $T\colon C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R}) \to [C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})]'$ *be a continuous linear mapping. It is a* δ-SIO *on* $\mathbb{R} \times \mathbb{R}$ *if there exists a pair* (K_1, K_2) *of* δCZ-δ*-standard kernels so that, for all* $f, g, h, k \in C_0^\infty(\mathbb{R})$, *with* $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$,

$$\langle g \otimes k, Tf \otimes h \rangle = \iint g(x) \langle k, K_1(x, y) h \rangle f(y) \, dx \, dy, \tag{3.1}$$

$$\langle k \otimes g, Th \otimes f \rangle = \iint g(x) \langle k, K_2(x, y) h \rangle f(y) \, dx \, dy \tag{3.2}$$

Let $\tilde{T}$ be defined by

$$\langle g \otimes k, \tilde{T} f \otimes h \rangle = \langle f \otimes k, Tg \otimes h \rangle.$$

It is readily seen that $\tilde{T}$ is a δ-SIO if T is. Its kernels $\tilde{K}_1$ and $\tilde{K}_2$ will be given by $\tilde{K}_1(x,y) = K_1(y,x)$ and $\tilde{K}_2(x,y) = [K_2(x,y)]^*$.

Definition 9. *A δ-SIO T on $\mathbb{R} \times \mathbb{R}$ is a δ-CZO if T and $\tilde{T}$ are bounded on L^2.*

The role of $\tilde{T}$ becomes clear in Section 6.

We can again put a norm on the set of δ-CZO's on $\mathbb{R} \times \mathbb{R}$ by setting

$$\| T \|_{\delta CZ(\mathbb{R} \times \mathbb{R})} = \| T \|_{2,2} + \| \tilde{T} \|_{2,2} + \sum_{i=1}^{2} |K_i|_{\delta, \delta CZ(\mathbb{R})}.$$

Using this remark one can easily define δ-CZO's on a product space with an arbitrary number of factors, by induction on this number.

We can repeat the same procedure to define $CZ\epsilon$'s on product spaces. However for $CZ\epsilon$'s there is no need to consider the partial adjoints as for δ-CZO's.

Let T be a $CZ\epsilon$ on $\mathbb{R}$ and K its kernel. We define $\| T \|_{CZ\epsilon}$ as $\| T \|_{2,2} + |K|_\epsilon$. A $CZ\epsilon$ T on $\mathbb{R} \times \mathbb{R}$ will be a bounded operator on L^2 associated in the sense of Definition 8 to a pair fo $CZ\epsilon$-ϵ-kernels and we shall put $\| T \|_{CZ\epsilon} = \| T \|_{2,2} + \sum_{i=1}^{2} |K_i|_{\epsilon, CZ\epsilon}$.

In order to state an analogue of Theorem 1 in the product setting we need to observe that a δ-SIO on $\mathbb{R} \times \mathbb{R}$ has a natural extension from $C_b^\infty \otimes C_b^\infty$ to $[C_{00}^\infty \otimes C_{00}^\infty]'$. This can be shown by an iteration of the argument sketched in Section 1. It also follows that Lemma 1 can be extended, using the same notations.

Lemma 3. *For all* $g_1, g_2 \in C_{00}^\infty(\mathbb{R})$ and $f_1, f_2 \in C_b^\infty(\mathbb{R})$,

$$\begin{aligned} \lim_{q \to +\infty\, q' \to +\infty} \langle g_1 \otimes g_2, T[(f_1)_q \otimes (f_2)_{q'}] \rangle &= \lim_{q \to +\infty} \langle g_1 \otimes g_2, T[(f_1)_q \otimes f_2] \rangle \\ &= \langle g_1 \otimes g_2, Tf_1 \otimes f_2 \rangle. \end{aligned}$$

In order to extend the definition of the WBP in the product setting it is convenient to introduce the following notations.

Let T be a δ-SIO on $\mathbb{R} \times \mathbb{R}$ and $f, g \in C_0^\infty(\mathbb{R})$. The operator $\langle g, T^1 f \rangle : C_0^\infty(\mathbb{R}) \to$ $\to [C_0^\infty(\mathbb{R})]'$ is defined by

$$\langle h, \langle g, T^1 f \rangle k \rangle = \langle g \otimes h, Tf \otimes k \rangle.$$

It is easy to see that $\langle g, T^1 f \rangle$ is a δ-SIO on $\mathbb{R}$ with kernel $\langle g, T^1 f \rangle (x,y) = \langle g, K_2(x,y) f \rangle$. One defines $\langle g, T^2 f \rangle$ similarly. The notation $T^1 f = 0$ simply means $\langle g, T^1 f \rangle = 0$ for all g. Notice that all this makes sense if $f \in C_b^\infty(\mathbb{R})$ and

$g \in C_{00}^{\infty}(\mathbb{R})$. In particular $T^1 1 = 0$ is equivalent to $\langle k, T^2 h\rangle 1 = 0$ for all $k, h \in C_0^{\infty}(\mathbb{R})$. Similarly $T^1{}^*1 = 0$ means $\langle k, T^2 h\rangle^* 1 = 0$ in the same conditions. Exchanging the role of indices we obtain the meaning of $T^2 1 = 0$ or $T^2{}^*1 = 0$.

In the following, the notations are those of Definition 4.

Definition 10. *Let T be a δ*-SIO *on $\mathbb{R} \times \mathbb{R}$. T has the* WBP *if for $i \in \{1, 2\}$*

$$\| \langle \eta_t^x, T^i \xi_t^x \rangle \|_{CZ\delta} \leqslant C_B t^{-1}. \tag{3.3}$$

It is easy to see that a δ-CZO on $\mathbb{R} \times \mathbb{R}$ has the WBP.

Next we indicate the extension of Lemma 2 in the product setting.

Lemma 4. *Let T be a δ*-SIO *with the* WBP. *Then for all B, $(\eta, \xi) \in B \times B$, $(x, y) \in \mathbb{R} \times \mathbb{R}$, $t > 0$ and $i \in \{1, 2\}$,*

$$\| \langle \eta_t^x, T^i \xi_t^y \rangle \|_{CZ\delta} \leqslant C_B \omega_{\delta, t}(x - y). \tag{3.4}$$

Conversely every bounded operator T defined from $C_0^{\infty} \otimes C_0^{\infty}$ to its dual satisfying to (3.4) *is a δ'-SIO having the* WBP *for all $\delta' < \delta$.*

In this statement we made use of the convention of Section 1. The proof of this lemma is routine and we omit the details. Of course lemmas 3 and 4 extend in the setting of an arbitrary product of copies of $\mathbb{R}$.

To conclude this section, we shall give the analogue of (1.14), (1.15) and (1.16) in the product setting.

Suppose first that there are only two factors in the product. Let T be a $CZ\epsilon$ on $\mathbb{R} \times \mathbb{R}$ and $f \in L^2(\mathbb{R}^2)$. Then $(Q_{t_1} Q_{t_2} Tf)(x_1, x_2)$ is a C^{∞} function of (x_1, x_2). If x_1, t_1 are fixed and $f(z_1, z_2) = 0$ for $|x_1 - z_1| \leqslant 2t_1$, then we can write

$$[Q_{t_1} Q_{t_2} Tf](x_1, x_2) = \int Q_{t_2}[(Q_{t_1} T)_{x_1 \bar{z}_1} f(\bar{z}_1, z_2)](x_2)\, d\bar{z}_1, \tag{3.5}$$

where $(Q_{t_1} T)_{x_1 \bar{z}_1}$ is a $CZ\epsilon$ acting on functions of z_2, and given by

$$(Q_{t_1} T)_{x_1 \bar{z}_1} = \int \psi_{t_1}(x_1 - y_1)[K_1(y_1, \bar{z}_1) - K_1(x_1, \bar{z}_1)]\, dy_1.$$

Here K_1 is the first kernel of T and the symbol « $^-$ » over z_1 simply means that z_1 has become a parameter in (3.5). It is not clear that the integral in (3.5) converges absolutely. However by (3.6) below, that will be the case if $f(\bar{z}_1, z_2)$ is uniformly in $L^2(dz_2)$, in particular if f is bounded with compact support.

The definition of a $CZ\epsilon$ on $\mathbb{R} \times \mathbb{R}$ immediately yields the following generalization of (1.15),

$$\int_{|x_1 - z_1| \geq 2^k t_1} \| (Q_{t_1} T)_{x_1 z_1} \|_{CZ\epsilon}\, dz_1 \leqslant C 2^{-k\epsilon}. \tag{3.6}$$

The case of a product of three spaces or more is very similar.

For all $I \subseteq [1, n]$, $(x_i, i \in I) \in \mathbb{R}^I$ and $(t_i, i \in I) \in (\mathbb{R}_+)^I$ and $(z_i, i \in I) \in \mathbb{R}^I$ such that for all $i \in I$, $|z_i - x_i| \geqslant 2t_i$ (we write also $|z_I - x_I| \geqslant 2t_I$) the symbol $[Q_{t_I}T]_{x_I z_I}$ denotes a $CZ\epsilon$ acting on $L^2(\mathbb{R}^J)$, where $J = [1, n] \backslash I$. This $CZ\epsilon$ is defined by induction on $|I|$. If $I = \{i\}$ and K_i is the kernel of T in the variable i, then

$$[Q_{t_I}T]_{x_I z_I} = \int \psi_{t_i}(x_i - y_i)[K_i(y_i, z_i) - K_i(x_i, z_i)]\, dy_i.$$

Now if $[Q_{t_I}T]_{x_I z_I}$ is defined and $I' = I \cup \{i\}$ we define $[Q_{t_{I'} z_{I'}} = [Q_{t_i}[Q_{t_I}T]_{x_I \mp z_I}]_{x_i z_i}$. This makes sense since $[Q_{t_I}T]_{x_I z_I}$ is itself a $CZ\epsilon$ and has a kernel in the i-variable. On the other hand it is readily seen that $[Q_{t_{I'}}T]_{x_{I'} z_{I'}}$ depends only on $t_{I'}, x_{I'}$ and $z_{I'}$ and not on the decomposition of I' as $I \cup \{i\}$. So the notation is consistent.

Let $I \subseteq [1, n]$ and $J = [1, n] \backslash I$ and let $f \in L^\infty(\mathbb{R}^n)$ have compact support and suppose $f(z) = 0$ if $|x_i - z_i| \leqslant 2t_i$ for some $i \in I$. Then with obvious notations we write

$$[Q_t Tf](x) = \int Q_{t_J}[[Q_{t_I}T]_{x_I \bar{z}_I}]f(\bar{z}_I, z_J)](x_J)\, d\bar{z}_I. \tag{3.7}$$

From (3.10) below it follows that this integral is absolutely convergent. Indeed (1.15) and (1.16) extend easily to the following, where $i \notin I$ and $I' = I \cup \{i\}$:

$$\int_{|x_i - z_i| \geq 2^{k_i} t_i} \| [Q_{t_{I'}}T]_{x_{I'} z_{I'}} \|_{CZ\epsilon}\, dz_i \leqslant C 2^{-k_i \epsilon} \| [Q_{t_I}T]_{x_I z_I} \|_{CZ\epsilon}, \tag{3.8}$$

and for $u \in \mathbb{R}_+$

$$\int_{\substack{t_i \leq u \\ 2t_i \vee u \leq |x_i - z_i|}} \| [Q_{t_{I'}}T]_{x_{I'} z_{I'}} \|_{CZ\epsilon}\, dz_i \frac{dt_i}{t_i} \leqslant C_\epsilon \| [Q_{t_I}T]_{x_I z_I} \|_{CZ\epsilon}. \tag{3.9}$$

Moreover it follows from (3.8),

$$\int_{|x_I - z_I| \geq 2^{k_I} t_I} \| (Q_{t_I}T)_{x_I z_I} \|_{CZ\epsilon}\, dz_I \leqslant C_\epsilon \| T \|_{CZ\epsilon} \times 2^{-\epsilon \Sigma_{i \in I} k_i}. \tag{3.10}$$

4. L^∞-BMO boundedness of $CZ\epsilon$'s on $\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R} = \mathbb{R}^n$

We wish to show the following.

Theorem 3. *Let T be a $CZ\epsilon$ on $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}$. Then T admits a bounded extension from $L^\infty(\mathbb{R}^n)$ to* BMO$(\mathbb{R} \times \ldots \times \mathbb{R})$.

By interpolation it follows that T is bounded on all L^p's for $p \in]2, +\infty[$ and if T^* is also a $CZ\epsilon$, then T is bounded on all L^p's for $p \in]1, +\infty[$. This situation occurs automatically in the convolution case where we can conclude the following.

Corollary. *Let T be a $CZ\epsilon$ on $\mathbb{R}^n = \mathbb{R} \times \ldots \times \mathbb{R}$ and a convolution operator. Then T admits a bounded extension from* BMO($\mathbb{R}^n$) *to itself.*

To prove the corollary we use the H^1-BMO duality [5] and an argument of [10], p. 150. Since L^2 is dense in H^1 (this is a trivial consequence of the atomic decomposition for H^1 [6]) it is enough to show that for all $f \in L^2 \cap H^1$, $\|T^*f\|_{H^1} \leqslant C\|f\|_{H^1}$, or equivalently that T^*f, H_1T^*f, H_2T^*f and $H_1H_2T^*f$ are all in L^1 with a norm less that $C\|f\|_{H^1}$. But as functions of L^2 these four functions are equal to T^*f, T^*H_1f, T^*H_2f and $T^*H_1H_2f$ which are in L^1 since by Theorem 3 T^* maps H^1 in L^1 and f, H_1f, H_2f, and H_1H_2f are all in H^1. The corollary is proved.

There are other $CZ\epsilon$'s which are candidates for being bounded on BMO, namely those defined on BMO. In the case $n = 2$ to be defined on BMO is equivalent to the conditions $T^1 1 = 0 = T^2 1$. It turns out that one can still prove that T is then bounded on BMO but the assumptions on the kernels of the $CZ\epsilon$'s have to be strengthened in order to know that TH_1, TH_2 and TH_1H_2 are also $CZ\epsilon$'s if T is and satisfies $T^1 1 = 0 = T^2 1$. We omit the details.

We now turn to the proof of Theorem 3. In order to use the induction hypothesis it is convenient to have the following formulation of Theorem 3, which is clearly equivalent by Theorem A.

Theorem 3′. *There exists $C_{n,\epsilon} > 0$ such that for all bounded open subsets Ω of $\mathbb{R}^n$, all $b \in L^\infty(\mathbb{R}^n)$ with compact support and all $T \in CZ\epsilon(\mathbb{R} \times \ldots \times \mathbb{R})$,*

$$\int_{S(\Omega)} |Q_t Tb(x)|^2 \, dx \frac{dt}{t} \leqslant C_{n,\epsilon} \|T\|^2_{CZ\epsilon} \|b\|^2_\infty |\Omega|. \tag{4.1}$$

We shall need the following lemma.

Lemma 5. *There exists a constant $C_{n,\epsilon}$ such that for all bounded open subsets Ω of $\mathbb{R}^n$ there exists n functions $T_1, \ldots, T_n$ defined from $S(\Omega)$ to $\mathbb{R}_+$ such that $T_i(x,t) \geqslant 2t_i$ and with the following properties:*

$$\textit{If} \quad \Omega_n = \bigcup_{(x,t)\in S(\Omega)} \prod_{1 \le i \le n}]x_i - T_i, x_i + T_i[, \tag{4.2}$$

then $|\Omega_n| \leqslant C_n |\Omega|$.

For all $T \in CZ\epsilon(\mathbb{R} \times \ldots \times \mathbb{R})$, all $I \subseteq [1,n]$, $I \neq \phi$, let

$$E_{x_I t_I z_I} = \bigcup_{T_I(x,t) < |x_I - z_I|} \prod_{j \notin I}]x_j - t_j, x_j + t_j[.$$

Then

$$\int |E_{x_I t_I z_I}| \, \|[Q_{t_I} T]_{x_I z_I}\|_{CZ\epsilon} \frac{dt_I}{t_I} \, dz_I \, dx_I \leqslant C_{n,\epsilon} |\Omega| \, \|T\|_{CZ\epsilon}. \tag{4.3}$$

Of course when $I=[1,n]$ (4.3) has to be interpreted the following way: $[Q_{t_I}T]_{x_Iz_I}$ is a real number and $|E_{x_It_Iz_I}|=1$ if $T_i(x,t)\leqslant|x_i-z_i|$ for all $i\in[1,n]$ and $|E_{x_It_Iz_I}|=0$ otherwise.

We postpone the proof of Lemma 5 to the next section.

Let Ω and Ω_n be as in Lemma 5, $b\in L^\infty(\mathbb{R}^n)$ with compact support and $\|b\|_\infty\leqslant 1$ and let $T\in CZ\epsilon(\mathbb{R}^n)$ with $\|T\|_{CZ\epsilon}\leqslant 1$. We want to prove

$$\int_{S(\Omega)}|Q_tTb(x)|^2\frac{dx\,dt}{t}\leqslant C_{n,\epsilon}|\Omega|. \tag{4.4}$$

Using (4.2) we immediately reduce to the case where b is supported out of Ω_n. Just write $b=b\chi_{\Omega_n}+b\chi_{\Omega_n^c}$ and observe that

$$\int_{S(\Omega)}|(Q_tTb\chi_{\Omega_n})(x)|^2\frac{dx\,dt}{t}\leqslant C_n\|b\chi_{\Omega_n}\|_2^2\leqslant C_n|\Omega_n|\leqslant C_n|\Omega|.$$

Suppose from now on that b is supported out of Ω_n. Then, for each $(x,t)\in S(\Omega)$ and $z\in\operatorname{supp}b$, $|z_i-x_i|\geqslant T_i$ for at least one index i. This yields the following decomposition for b:

$$b=\sum_{\substack{I\subseteq[1,n]\\ I\neq\phi}}(-1)^{|I|-1}b_{x,t,I},$$

where

$$b_{x,t,I}(z)=b(z)\prod_{i\in I}\chi_{\{z_i,\,|x_i-z_i|\geq T_i(x,t)\}}.$$

Thus

$$(Q_tTb)(x)=\sum_{\substack{I\subseteq[1,n]\\ I\neq\phi}}(-1)^{|I|-1}(Q_tTb_{x,t,I})(x).$$

Therefore, to prove (4.4) it is enough to prove for all $I\subseteq[1,n]$, $I\neq\phi$,

$$\int_{S(\Omega)}|Q_tTb_{x,t,I}(x)|^2\frac{dx\,dt}{t}\leqslant C_{n,\epsilon}|\Omega|. \tag{4.5}$$

Since $T_i(x,t)\geqslant 2t_i$, we can use (3.7), which reads

$$(Q_tTb_{x,t,I})(x)=\int_{\bar z_I}[Q_{t_J}[Q_{t_I}T]_{x_I\bar z_I}b(\bar z_I,z_J)](x_J)\chi_{|x_I-\bar z_I|\geq T_I}\,d\bar z_I. \tag{4.6}$$

For x_I, t_I fixed, let $E_{x_It_I}=\bigcup\prod_{j\notin I}]x_j-t_j,x_j+t_j[$, the union being over the t_J's such that $(x_I,x_J,t_I,t_J)\in S(\Omega)$. Minkowski's inequality and (4.6) yield

$$\int_{S(E_{x_I t_I})} |Q_t T b_{x,t,I}(x)|^2 \, dx_J \frac{dt_J}{t_J} \leqslant$$

$$\leqslant \left[\int_{|z_I - x_I| \geq 2t_I} \left(\int_{S(E_{x_I t_I})} [Q_{t_J}[Q_{t_I} T]_{x_I \bar{z}_I} b(\bar{z}_I, z_J)](x_J)|^2 \times \right. \right.$$

$$\left. \left. \times \, \chi_{|x_I - z_I| \geq T_I} \frac{dx_J \, dt_J}{t_J} \right)^{1/2} d\bar{z}_I \right]^2 .$$

Now let x_I, t_I, z_I be fixed and $E_{x_I t_I z_I}$ as defined in Lemma 5. If $(x_J, t_J) \in S(E_{x_I t_I})$ and $T_I(x, t) \leqslant |x_I - z_I|$, then $(x_J, t_J) \in S(E_{x_I t_I z_I})$. Therefore we need only to dominate

$$\left[\int_{|z_I - x_I| \geq 2t_I} \left[\int_{S(E_{x_I t_I z_I})} |[Q_{t_J}[Q_{t_I} T]_{x_I \bar{z}_I} b(\bar{z}_I, z_J)](x_J)|^2 \frac{dx_J \, dt_J}{t_J} \right]^{1/2} d\bar{z}_I \right]^2 .$$

The induction hypothesis under the form (4.1) yields the following majorant

$$\left[\int_{|z_I - x_I| \geq 2t_I} |E_{x_I t_I z_I}|^{1/2} \| [Q_{t_I} T]_{x_I z_I} \|_{CZ_\epsilon} \, dz_I \right]^2 .$$

By (3.10) and Cauchy-Schwarz, this is less than

$$C_{n,\epsilon} \left[\int_{|z_I - x_I| \geq 2t_I} |E_{x_I t_I z_I}| \, \| [Q_{t_I} T]_{x_I z_I} \|_{CZ_\epsilon} \, dz_I \right] .$$

It remains to integrate against $dx_I \, dt_I / t_I$ and use (4.3). In the case where $I = [1, n]$ some minor modifications in notations are needed. They are left to the reader. The proof is therefore reduced to showing Lemma 5.

5. Proof of Lemma 5

When $n = 1$ this lemma is trivial. Let Ω be a bounded open subset of $\mathbb{R}$ and for $x \in \Omega$, let $I(x)$ be the connected component of x in Ω. Then simply set $T(x, t) = |I(x)|$ for $(x, t) \in S(\Omega)$. Clearly $T(x, t) \geqslant 2t$ since $]x - t, x + t[\subseteq I(x)$. Moreover $]x - I(x), x + I(x)[\subseteq 3I(x)$ which implies (4.2) with $C_1 = 3$. Finally (4.3) reduces to

$$\int_{S(\Omega)} \int_z |(Q_t T)_{x,z}| \chi_{|x - z| > |I(x)|} \, dz \, dx \frac{dt}{t} \leqslant C_\epsilon |\Omega| \, \| T \|_{CZ_\epsilon}, \tag{5.1}$$

which follows trivially from (3.9) with $u = |I(x)|/2$. This observation will permit us to illustrate in a simple case one point of the strategy of the proof.

Lemma 6. *Suppose we have built* $T_1 \dots T_n$ *such that* (4.3) *holds for* $I = [2, n]$. *Then if* $T_1 \geqslant |I_{x_I t_I}(x_1)|$, (4.3) *holds for* $I' = [1, n]$.

Here $I_{x_I t_I}(x_1)$ denotes the connected component of x_1 in $E_{x_I t_I}$ as defined in Section 4.

Let x_I, t_I, z_I be fixed. To deduce (4.3) for I' form (4.3) for I, it is enough to show that

$$\int |E_{x_{I'} t_{I'} z_{I'}}| \, |[Q_{t_{I'}} T]_{x_{I'} z_{I'}}| \frac{dt_1}{t_1} dz_1 \, dx_1 \leqslant C |E_{x_I t_I z_I}| \, \| [Q_{t_I} T]_{x_I z_I} \|_{CZ\epsilon},$$

and then integrate against $dx_I \dfrac{dt_I}{t_I} dz_I$.

This inequality actually means

$$\int_{(x_1, z_1, t_1)/|x_{I'} - z_{I'}| \geq T_{I'}} |[Q_{t_{I'}} T]_{x_{I'} z_{I'}}| \frac{dt_1}{t_1} dz_1 \, dx_1 \leqslant C |E_{x_I t_I z_I}| \, \| [Q_{t_I} T]_{x_I z_I} \|_{CZ\epsilon}.$$

Thanks to the formula

$$[Q_{t_{I'}} T]_{x_{I'} z_{I'}} = [Q_{t_1} [Q_{t_I} T]_{x_I z_I}]_{x_1 z_1},$$

we are almost in position to use (5.1). We only need that the conditions on (x_1, z_1, t_1) imply

i) $(x_1, t_1) \in S(E_{x_I t_I z_I})$

ii) $|x_1 - z_1| \geqslant |I_{x_I t_I z_I}(x_1)|$, where $I_{x_I t_I z_I}(x_1)$ is the component of x_1 in $E_{x_I t_I z_I}$.

i) follows from the definition of $E_{x_I x_I z_I}$ and from the condition $|x_I - z_I| \geqslant$ $\geqslant T_I$. ii) follows from the fact that $E_{x_I t_I z_I} \subseteq E_{x_I t_I}$ for all z_I. Therefore $|x_1 - z_1| \geqslant T_1 \geqslant |I_{x_I t_I}(x_1)| \geqslant |I_{x_I t_I z_I}(x_1)|$. This implies ii) and lemma 6 is proved.

In general one point in the strategy will be to define the T_i's by induction on i in such a way that if I is a set of indices and $i_0 < \inf I$, and if the T_i's are such that (4.3) holds for I, then it holds for $\{i_0\} \cup I$, almost independently of the choice of the T_i's for $i > i_0$.

Lemma 7. *Let* $\Omega \subseteq \mathbb{R}^n$, $(x_1, t_1) \in \mathbb{R}^2_+$ *and for* $(x_2, \dots, x_n) \in E_{x_1 t_1}$, *let* $\tau(x_1, t_1, x_2, \dots, x_n) = 2t_1 \vee \inf \left\{ \alpha, (\chi_{E_{x_1 t_1} \setminus E_{x_1 \alpha}})^* (x_2, \dots, x_n) \geqslant \frac{1}{2} \right\}$. *For* $(x, t) \in S(\Omega)$, *let* $\tau(x, t) = \sup \tau(x_1, t_1, y_2, \dots, y_n)$, *the* sup *being over those* $(y_i)_{2 \leq i \leq n}$ *such that* $|x_i - y_i| \leqslant t_i$ *for all* $i \in [2, n]$. *For* $z_1 \in \mathbb{R}$ *such that* $|z_1 - x_1| \geqslant 2t_1$ *let*

$$\tilde{E}_{x_1 t_1 z_1} = \bigcup_{\tau(x,t) < |x_1 - z_1|} \prod_{i \leq 2}]x_i - t_i, x_i + t_i[.$$

Then, for $T \in CZ\epsilon(\mathbb{R} \times \ldots \times \mathbb{R})$ with $\|T\|_{CZ_\epsilon} \leqslant 1$

$$(5.2) \qquad \int_{|x_1 - z_1| \geq 2t_1} |\tilde{E}_{x_1 t_1 z_1}| \, \|(Q_{t_1} T)_{x_1 z_1}\|_{CZ_\epsilon} \, dx_1 \frac{dt_1}{t_1} dz_1 \leqslant C_n |\Omega|.$$

Moreover, if

$$\Omega' = \bigcup_{(x,t) \in S(\Omega)}]x_1 - \tau, x_1 + \tau[\times \prod_{i \geq 2}]x_i - t_i, x_i + t_i[,$$

then

$$(5.3) \qquad \chi_{\Omega'} \leqslant \tfrac{1}{2} (\chi_\Omega)^*,$$

where * *is the strong Hardy-Littlewood maximal operator.*

In order to prove (5.3), it is enough to prove that, for all $(x, t) \in S(\Omega)$,

$$\frac{|]x_1 - \tau, x_1 + \tau[\times \prod_{i \geq 2}]x_i - t_i, x_i + t_i[\cap \Omega|}{2^n \tau_1 \times \prod_{i \geq 2} t_i} \geqslant \frac{1}{2}.$$

If $\tau = 2t_1$ this is obvious since $(x, t) \in S(\Omega)$. If $\tau > 2t_1$, we can choose β such that $\tau > \beta > 2t_1$ and $(y_i)_{2 \leq i \leq n}$ such that $|x_i - y_i| \leqslant t_i$ for $i \in [2, n]$, and $\tau_1(x_1, t_1, y_2, \ldots, y_n) > \beta$. Therefore $(\chi_{E_{x_1 t_1} \setminus E_{x_1 \beta}})^*(y_2, \ldots, y_n) < \frac{1}{2}$ and in particular

$$\frac{\left| \prod_{i \geq 2}]x_i - t_i, x_i + t_i[\cap E_{x_1 t_1} \setminus E_{x_1 \beta} \right|}{2^{n-1} \prod_{i \geq 2} t_i} < \frac{1}{2}.$$

Since $\prod_{i \geq 2}]x_i - t_i, x_i + t_i[\subseteq E_{x_1 t_1}$, this is equivalent to

$$\frac{\left| \prod_{i \geq 2}]x_i - t_i, x_i + t_i[\cap E_{x_1 \beta} \right|}{2^{n-1} \prod_{i \geq 2} t_i} > \frac{1}{2}.$$

Since $]x_1 - \beta, x_1 + \beta[\times E_{x_1 \beta} \subseteq \Omega$, this implies

$$\frac{\left|]x_1 - \beta, x_1 + \beta[\times \prod_{i \geq 2}]x_i - t_i, x_i + t_i[\cap \Omega \right|}{2^n \left[\beta \times \prod_{i \geq 2} t_i \right]} > \frac{1}{2}.$$

Letting β tend to τ, we obtain the desired inequality and (5.3).

To prove (5.2) observe that

$$\tilde{E}_{x_1 t_1 z_1} \subseteq \{(y_2, \ldots, y_n) \in \mathbb{R}^{[2,n]}, \ \tau_1(x_1, t_1, y_2, \ldots, y_n) < |x_1 - z_1|\} \subseteq$$
$$\subseteq \left\{ (y_2, \ldots, y_n), (\chi_{E_{x_1 t_1} \setminus E_{x_1 |x_1 - z_1|}})^*(y_2 \ldots y_n) \geqslant \tfrac{1}{2} \right\}.$$

This latter inclusion follows from the trivial fact that $(\chi_{E_{x_1t_1}\setminus E_{x_1\alpha}})^*(y_2,\ldots,y_n)$ is an increasing function of α. These inclusions imply $|\tilde{E}_{x_1t_1z_1}| < C_n|E_{x_1t_1}\setminus E_{x_1|x_1-z_1|}|$. At this point we need the following.

Lemma 8. *Let $x_1 \in \mathbb{R}$, $T \in CZ\epsilon(\mathbb{R}\times\ldots\times\mathbb{R})$ with $\|T\|_{CZ\epsilon} \leqslant 1$, and let F: $\mathbb{R}_+ \to \mathbb{R}_+$ be a decreasing function vanishing for t large. Then*

$$(5.4)\quad \int_{|x_1-z_1|\geq 2t_1} [F(t_1) - F(|x_1-z_1|)]\,\|(Q_{t_1}T)_{x_1z_1}\|_{CZ\epsilon}\frac{dt_1}{t_1}\,dz_1 \leqslant C_\epsilon F(0^+).$$

It is easy to reduce to the case where F is C^1. In this case write $F(t_1) - F(|x_1-z_1|) = -\int_{t_1}^{|x_1-z_1|}F'(u)\,du$. Using (3.9) with $I=\phi$ and $I'=\{1\}$, we obtain, since $-F'>0$, a domination of the L.H.S. of (5.4) by $-\|T\|_{CZ\epsilon}\times \int_0^{+\infty}F'(u)\,du$, which proves Lemma 8.

To prove (5.3) we apply (5.4) with $F(t) = E_{x_1t}$. The restriction $|x_1-z_1| \geqslant 2t_1$ is irrelevant since otherwise $\tilde{E}_{x_1t_1z_1} = \phi$. An application of (5.4) and the inequality $|\tilde{E}_{x_1t_1z_1}| \leqslant C(|E_{x_1t_1}| - |E_{x_1|x_1-z_1|}|)$ yield

$$\int |\tilde{E}_{x_1t_1z_1}|\,\|(Q_{t_1}T)_{x_1z_1}\|_{CZ\epsilon}\frac{dt_1}{t_1}\,dz_1 \leqslant C_\epsilon|E_{x_10^+}|, \quad\text{where}\quad E_{x_10^+} = \bigcup_{t_1>0}E_{x_1t_1}.$$

An integration in x_1 yields $C_\epsilon\int|E_{x_10^+}|\,dx_1$ as a majorant of the l.h.s. of (5.3). But this is exactly $C_\epsilon|\Omega|$, and Lemma 7 is proved.

We shall use Lemma 7 with many indices playing the role of index 1 and with many sets instead of Ω; we shall specify which index and which set are considered, e.g. $\tau_1(x_1,t_1,x_2,t_2,\ldots,x_n,t_n,\Omega)$.

A direct consequence of Lemma 7 is the following. If $T_1(x,t) \geqslant \tau_1(x,t,\Omega)$, then $E_{x_1t_1z_1} \subseteq \tilde{E}_{x_1t_1z_1}$ and (5.2) implies (4.3) for $I=\{1\}$. Now we define the *T_i's* by induction on i. The letter ω will denote an open subset of $\mathbb{R}^k$ for some $k\in[1,n]$ which will be specified by the context. We shall use the notation $E_{x_It_I}$ as in Section 4 but we shall specify the set under consideration, e.g. $E_{x_It_I}(\Omega)$. Finally $I_{xt} = I_{x_It_I}(x_1)$ with the notations of Lemma 6.

We set

$$T_1 = |I_{xt}| \vee \sup_{\substack{I\subseteq[2,n]\\ J=[2,n]\setminus I\\ \omega\subseteq E_{x_It_I}(\Omega)\\ (x_1,t_1,x_J,t_J)\in S(\Omega)}} \tau_1(x_1,t_1,x_J,t_J,\omega),$$

$$\Omega_1 = \bigcup\,]x_1-T_1,x_1+T_1[\,\times\prod_{i\geq 2}\,]x_i-t_i,x_i+t_i[,$$

the union being taken over $(x, t) \in S(\Omega)$,

$$T_2 = \sup_{\substack{I \subseteq [3, n] \\ J = [3, n] \setminus I \\ \omega \subseteq E_{x_I t_I}(\Omega_1) \\ (x_1, T_1, x_2, t_2, x_J, t_J) \in S(\omega)}} \tau_2(x_1, T_1, x_2, t_2, x_J, t_J, \omega),$$

and $\Omega_2 = \cup]x_1 - T_1, x_1 + T_1[x]x_2 - T_2, x_2 + T_2[\times \prod_{i \geq 3}]x_i - t_i, x_i + t_i[$.

Suppose $T_1 \ldots T_{i-1} T_i$ are already defined and let

$$\Omega_i = \cup \prod_{j \geq i}]x_j - T_j, x_j + T_j[\times \prod_{k \geq i}]x_k - t_k, x_k + t_k[.$$

We define T_{i+1} as follows

$$T_{i+1} = \sup_{\substack{I \subseteq [i+2, n] \\ J = [i+2, n] \setminus I \\ \omega \subseteq E_{x_I t_I}(\Omega_i) \\ (x_1, T_1, \ldots, x_i, T_i, x_{i+1}, t_{i+1}, x_J, t_J) \in S(\omega)}} \tau_{i+1}(x_1, T_1, \ldots, x_i, T_i, x_{i+1}, t_{i+1}, x_J, t_J, \omega).$$

Finally let

$$\Omega_{n-1} = \cup \Big(\prod_{i \leq n-1}]x_i - T_i, x_i + T_i[\Big) \times]x_n - t_n, x_n + t_n[,$$

and let $T_n = \tau_n(x_1, T_1, \ldots, x_{n-1}, T_{n-1}, x_n, t_n, \Omega_{n-1})$.

The property (4.2) will be a trivial consequence of the following.

Lemma 9. *For all* $i \in [1, n-1]$, $(\chi_{\Omega_i})^* \geqslant \frac{1}{2} \chi_{\Omega_{i+1}}$.

If $i = n - 1$, this is an immediate consequence of (5.3) applied with index n and set Ω_{n-1}.

If $i < n - 1$, let $(x, t) \in S(\Omega)$, $\alpha > 0$ be such that $t_{i+1} < \alpha < T_{i+1}$. There exists $I \subseteq [i+2, n]$ and $\omega \subseteq E_{x_I t_I}(\Omega_i)$ such that $(x_1, T_1, \ldots, x_i, T_i, x_{i+1}, t_{i+1}, x_J, t_J) \in S(\omega)$ and $\tau_{i+1}(x_1, T_1, \ldots, x_i, T_i, x_{i+1}, t_{i+1}, x_J, t_J, \omega) > \alpha$. The proof of Lemma 7 shows that

$$\prod_{j \leq i}]x_j - T_j, x_j + T_j[\times]x_{i+1} - \tau_{i+1}, x_{i+1} + \tau_{i+1}[\times]x_J - t_J, x_J + t_J[$$

has at least half of its volume in ω. Since $\omega \subseteq E_{x_I t_I}(\Omega_i)$, $\omega \times]x_I - t_I, x_I + t_I[\subseteq \subseteq \Omega_i$. Hence

$$\prod_{j\leq i}]x_j - T_j, x_j + T_j[\times]x_{i+1} - \tau_{i+1}, x_{i+1} + \tau_{i+1}[\times \prod_{j>i+1}]x_j - t_j, x_j + t_j[$$

has half of its volume in Ω_i. Let α tend to T_{i+1} and the same is proved for T_{i+1} instead of τ_{i+1}. Finally we have proved that Ω_{i+1} is the union of rectangles that have at least half of their volume in Ω_i. This implies the lemma. Actually we have skipped the case where $i = 1$ and $T_1 = I_{xt}$, but then the argument is trivial.

We are left with proving (4.3). To do so we replace $E_{x_I t_I z_I}$ by a larger set $F_{x_I t_I z_I}$ defined as follows. Let $i_0 = \inf I$. Then

$$F_{x_I t_I z_I} = \bigcup_{T_I(x,t) < |x_I - z_I|} \prod_{j<i_0}]x_j - T_j, x_j + T_j[\times \prod_{\substack{j>i_0 \\ j\notin I}}]x_j - t_j, x_j + t_j[.$$

Now we shall prove by induction on $|I|$ that

$$\int |F_{x_I z_I t_I}| \, \| [Q_{t_I} T]_{x_I z_I} \|_{CZ_\epsilon} \frac{dt_I}{t_I} \, dz_I \, dx_I \leqslant C_{n\epsilon} |\Omega| \, \| T \|_{CZ_\epsilon}. \tag{5.5}$$

This will be sufficient since $t_j < T_j$ for all j and $E_{x_I t_I z_I} \subseteq F_{x_I t_I z_I}$. Also, by Lemma 6 it is enough to consider the case $|I| < n$.

If I has a single element i, then (5.5) is a direct consequence of (5.2) applied with the set Ω_i and the index i, since $T_i(x, t) \geqslant \tau_i(x_1, T_1, \ldots, x_{i-1}, T_{i-1}, x_i, t_i, \ldots \ldots, x_n, t_n, \Omega_i)$ and $|\Omega_i| \leqslant C_n |\Omega|$ by Lemma 9.

If I has more than a single element let $K = I \backslash \{ i_0 \}$, and let $G^{i_0}_{x_K t_K z_K}$ be defined as

$$\bigcup_{T_K \leq |x_K - z_K|} \prod_{j<i_0}]x_j - T_j, x_j + T_j[\times]x_{i_0} - t_{i_0}, x_{i_0} + t_{i_0}[\times \prod_{\substack{j>i_0 \\ j\notin I}}]x_j - t_j, x_j + t_j[.$$

Clearly $G^{i_0}_{x_K t_K z_K} \subseteq F_{x_K t_K z_K}$. Moreover we have the following.

Lemma 10.

$$\int |F_{x_I t_I z_I}| \, \| [Q_{t_I} T]_{x_I z_I} \|_{CZ_\epsilon} \frac{dt_{i_0}}{t_{i_0}} \, dz_{i_0} \leqslant C_{n,\epsilon} |(Q_{t_K} T)_{x_K z_K} \|_{CZ_\epsilon} |G^{i_0}_{x_K t_K z_K}|.$$

With Lemma 10, one deduces immediately (5.5) for I from (5.5) for K. Therefore the induction, and the proof of Lemma 5, reduce to Lemma 10 which we now prove. To do so we shall apply (5.2) with the set $G^{i_0}_{x_K t_K z_K}$, the index i_0 and the operator $(Q_{t_K} T)_{x_K z_K}$. Let $\tilde{F}_{x_I t_I z_I} = \bigcup \prod_{j\notin I}]y_j - s_j, y_j + s_j[$, where the union is taken over those $(y_j, s_j)_{j\notin I}$ such that

$$\tau_{i_0}(x_{i_0}, t_{i_0}, y, s, G^{i_0}_{x_K t_K z_K}) \leqslant |x_{i_0} - z_{i_0}| \quad \text{and} \quad (x_{i_0}, t_{i_0}, y, s) \in S(G^{i_0}_{x_K t_K z_K}).$$

Then (5.2) reads as follows:

$$\int |\tilde{F}_{x_I t_I z_I}| \, \|(Q_{t_I}T)_{x_I z_I}\|_{CZ\epsilon} \, dx_{i_0} \frac{dt_{i_0}}{t_{i_0}} \, dz_{i_0} \leqslant C_n \|(Q_{t_K}T)_{x_K z_K}\|_{CZ\epsilon} \|G^{i_0}_{x_K t_K z_K}\| .$$

Therefore we need only to prove $F_{x_I t_I z_I} \subseteq \tilde{F}_{x_I t_I z_I}$. In other words we must show that if $T_I(x, t) < |x_I - z_I|$, that is $T_K(x, t) < |x_K - z_K|$ and $T_{i_0}(x, t) < |x_{i_0} - z_{i_0}|$, then $((x_j, T_j)_{j<i_0}, (x_{i_0}, t_{i_0}), (x_j, t_j)_{j>i_0, j\notin I}) \in S(G^{i_0}_{x_K t_K z_K})$ and the associated $\tau_{i_0}(\cdot, G^{i_0}_{x_K t_K z_K}) \leqslant |x_{i_0} - z_{i_0}|$. The first assertion follows from the definition of T_{i_0}. Indeed $G^{i_0}_{x_K t_K z_K} \subseteq E_{x_K t_K}(\Omega_{i_0 - 1})$ (with $\Omega_0 = \Omega$), and therefore $T_{i_0} \geqslant \tau_{i_0}(\cdot, G^{i_0}_{x_K t_K z_K})$ where « · » means $((x_j, T_j)_{j<i_0}, (x_{i_0}, t_{i_0}), (x_j, t_j)_{j>i_0, j\notin I})$. Now $\tau_{i_0}(\cdot, G^{i_0}_{x_K t_K z_K}) \leqslant T_{i_0}(x, t) \leqslant |x_{i_0} - z_{i_0}|$ and the lemma is proved.

6. A «$T1$-theorem» in the product setting

If T is a δ-SIO on $\mathbb{R} \times \mathbb{R}$ and has the WBP, the conditions $T1 = 0$ and $T^*1 = 0$ do not imply that T is bounded on L^2. This is why we introduced in Section 3 the partial adjoint $\tilde{T}$. Now if $T1 = T^*1 = \tilde{T}1 = \tilde{T}^*1 = 0$, then T is bounded on L^2. Moreover the following is true.

Theorem 4. *Let T be a δ-SIO on $\mathbb{R} \times \mathbb{R}$ having the* WBP *and such that* $T1$, T^*1, $\tilde{T}1$ and $\tilde{T}^*1$ *lie in* BMO($\mathbb{R} \times \mathbb{R}$). *Then T extends boundedly from L^2 to L^2.*

Let us consider an example. Let $(a_{k_1,k_2})_{\mathbb{Z}\times\mathbb{Z}}$ be a bounded real-valued sequence on $\mathbb{Z} \times \mathbb{Z}$ and let $\tilde{a} = \sum\sum a_{k_1 k_2} e^{i2^{k_1}x} e^{i2^{k_2}y}$ be the tempered distribution such that $\langle \tilde{a} | \psi \rangle = \sum\sum a_{k_1 k_2} \hat{\psi}(-2^{k_1}, -2^{k_2})$ for all $\psi \in S(\mathbb{R}^2)$. Let $\psi_0 \in S(\mathbb{R}^1)$ be such that $\hat{\psi}_0(0) = 0$ and $\sum_{k\in\mathbb{Z}} \hat{\psi}_0(2^{-k}\xi) = 1$ for $\xi = 0$ and let E_k be the Fourier multiplier of symbol $\hat{\psi}_0(2^{-k}\xi)$. Let $T_a : C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R}) \to C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})$ be defined by $\langle g_1 \otimes g_2, T_a f_1 \otimes f_2 \rangle = \sum_{k_1} \sum_{k_2} a_{k_1 k_2} \langle \Delta^*_{k_1} g_1, e^{i2^{k_1}} f_1 \rangle \langle g_2, e^{i2^{k_2}} \Delta_{k_2} f_2 \rangle$. It is easy to show that this series is absolutely convergent. Moreover, if the rows and columns of the matrix $((a_{k_1 k_2}))$ are uniformly bounded, then T_a is a 1-SIO, satisfies $T_a 1 = T^*_{\tilde{a}} 1 = \tilde{T}^*_{\tilde{a}} 1 = 0$ and $\tilde{T}_a 1 = \tilde{a}$, and T_a has the WBP. Finally T_a is bounded if and only if $((a_{k_1 k_2}))$ is bounded on $l^2(\mathbb{Z})$ and $\tilde{T}_a 1$ is in BMO only if $((a_{k_1 k_2}))$ is Hilbert-Schmidt.

This example shows that $\tilde{T}1$ and $\tilde{T}^*1$ have to be taken into account in order to obtain L^2-boundedness but the conditions $\tilde{T}1$ and $\tilde{T}^*1 \in$ BMO are not necessary.

From Theorems 3 and 4 applied to T and $\tilde{T}$ we obtain the following.

Corollary. *Let T be a δ-SIO on $\mathbb{R}\times\mathbb{R}$. It is a δ-CZO if and only if $T1$, T^*1, $\tilde{T}1$ and $\tilde{T}^*1$ lie in BMO and it has the WBP.*

To avert the suspicion of vain aesthetism, we shall now explain why we require $\tilde{T}$ to be bounded on L^2 in the definition of a CZO. This is not merely to have a nice characterization of CZO's but to have statements which extend in the setting of an arbitrary finite product of copies of $\mathbb{R}$. It is a very good exercise to extend the proof of Theorem 4 that we shall give below and an opportunity to see why one needs to take into account $\tilde{T}$ in the definition of $\| \ \|_{CZ\delta(\mathbb{R}\times\mathbb{R})}$. We shall leave it to the reader and stick from now on to the case $n = 2$ (except in Section 10).

The proof of Theorem 4 can be decomposed in three steps.

In the first step, one simply observes that if T satisfies $T^11 = T^{1*}1 = 0$ and has the WBP, then if can be viewed as a classical vector valued SIO, $\bar{T}$ acting on $C_0^\infty(\mathbb{R})\otimes H$, where $H = L^2(\mathbb{R}, dx_2)$, and for which $\bar{T}1 = \bar{T}^*1 = 0$. The proof of the L^2-boundedness of such an operator is the hilbertian version of the proof of [9] based on the Cotlàr-Knapp-Stein lemma.

The second step is the decomposition of an operator T having the WBP, such that $T1 = T^*1 = \tilde{T}1 = \tilde{T}^*1 = 0$ as the sum of two operators S and $T - S$ having the WBP and such that $S^21 = S^{*2}1 = 0$ and $(T-S)^11 = (T-S)^{*1}1 = 0$. The L^2-boundedness of T is then a consequence of the first step. The construction of S is given in Section 7.

The last step is, as in the classical situation, to construct for all functions $b\in$ BMO, a CZO V_b such that $V_b1 = b$ and $V_b^*1 = \tilde{V}_b1 = \tilde{V}_b^*1 = 0$. Now if T satisfies the assumptions of the theorem and b_1, b_2, b_3 and b_4 are $T1$, T^*1, $\tilde{T}1$ and $\tilde{T}^*1$ respectively, the operator $T - V_{b_1} - V_{b_2}^* - \tilde{V}_{b_3} - \tilde{V}_{b_4}^*$ is of the type studied in the second step, so that T is bounded on L^2. The operator V_b is described in Section 8.

7. Decomposition of T when $T1 = T^*1 = \tilde{T}1 = \tilde{T}^*1 = 0$

Let $\beta\in\mathrm{BMO}(\mathbb{R})$ and let $U_\beta: C_0^\infty(\mathbb{R})\to[C_0^\infty(\mathbb{R})]'$ be defined by $\langle g, U_\beta f\rangle = \int_0^{+\infty}\langle (Q_tg), (Q_t\beta)(P_tf)\rangle\, dt/t$. It is classical that this integral is absolutely convergent and that U_β is a 1-CZO with $\|U_\beta\|_{1CZ}\leqslant C\|\beta\|_{\mathrm{BMO}}$. Moreover $U_\beta1 = \beta$ and $U_\beta^*1 = 0$ ([9]).

Now let T be a δ-SIO on $\mathbb{R}\times\mathbb{R}$ such that $T1 = T^*1 = \tilde{T}1 = \tilde{T}^*1 = 0$. We define the operator N as follows.

For all $f_1, f_2, g_1, g_2 \in C_0^\infty(\mathbb{R})$

$$\langle g_1\otimes g_2, Nf_1\otimes f_2\rangle = \langle g_1, U_{\{\langle g_2, T^2f_2\rangle 1\}}f_1\rangle. \tag{7.1}$$

Lemma 11. *The operator N is a δ'-SIO having the* WBP *for all $\delta' < \delta$. Moreover $N^2 1 = (N^2)^* 1 = (N^1)^* 1 = 0$ and $(T - N)^1 1 = 0$.*

In order to prove Lemma 11 we shall need the following.

Lemma 12. *Let T: $C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R}) \to [C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})]'$ be a continuous linear mapping. Suppose that for every bounded subset B of $C_0^\infty(\mathbb{R})$, there exists a constant C_B such that:*

i) *for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$, all $t_1, t_2 \in \mathbb{R}_+$, and all $\eta_1, \xi_1, \eta_2, \xi_2 \in B$*

$$|\langle \eta_{1t_1}^{x_1} \otimes \eta_{2t_2}^{x_2}, T\xi_{1t_1}^{y_1} \otimes \xi_{2t_2}^{y_2} \rangle| \leqslant C_B \omega_{\delta, t_1}(x_1 - y_1) \omega_{\delta, t_2}(x_2 - y_2);$$

ii) *for all $(x, y) \in \mathbb{R}$, $t > 0$, $i \in \{1, 2\}$ and $\eta, \xi \in B$,*

$$\| \langle \eta_t^x, T^i \xi_t^y \rangle 1 \|_{\mathrm{BMO}} \leqslant C_B \omega_{\delta, t}(x - y) \text{ and } \| \langle \eta_t^x, T^i \xi_t^y \rangle^* 1 \|_{\mathrm{BMO}} \leqslant C_B \omega_{\delta, t}(x - y).$$

Then T is a δ'-SIO on $\mathbb{R} \times \mathbb{R}$ and has the WBP *for all $\delta' < \delta$. Conversely any δ-SIO having the* WBP *satisfies* i) *and* ii).

This lemma is an immediate consequence of Lemma 2 and of Theorem 1.

Let us apply it to N. Applying the converse part of Lemma 12 to T, using (7.1) and the properties of the operators U_β for $\beta \in \mathrm{BMO}(\mathbb{R})$, one obtains easily the property i) for N as well as the property ii) for $i = 2$. From the formula $\langle f, N^2 g \rangle = U_{\{\langle f, T^2 g \rangle 1\}}$ we also conclude $(T - N)^1 1 = 0$ and $(N^1)^* 1 = 0$. We are left with showing that N satisfies ii) with $i = 1$. In fact we shall prove $N^2 1 = (N^2)^* 1 = 0$, or, in other words $\langle g, N^1 f \rangle 1 = \langle g, N^1 f \rangle^* 1 = 0$ for all $f, g \in C_0^\infty(\mathbb{R})$. For this we shall use the assumptions $T1 = \tilde{T}1 = 0$.

To show $\langle g, N^1 f \rangle 1 = 0$, it is enough by Lemma 1 to show that for all $h \in C_{00}^\infty(\mathbb{R})$, $\lim_{q \to +\infty} \langle h, \langle g, N^1 f \rangle \theta_q \rangle = 0$, which means $\lim_{q \to +\infty} \langle g, U_{\{\langle h, T^2 \theta_q \rangle 1\}} f \rangle = 0$, where θ_q is defined in Lemma 1. This is immediate from the two following lemmas.

Lemma 13. *Let $(\beta_q)_{q \in \mathbb{N}}$ be a bounded sequence taking its values in* $\mathrm{BMO}(\mathbb{R})$. *If $\lim_{q \to +\infty} \beta_q = 0$ for $\sigma^*(H^1, \mathrm{BMO})$, then for all $f, g \in C_0^\infty(\mathbb{R})$, $\lim_{q \to +\infty} \langle g, U_{\beta_q} f \rangle = 0$.*

Lemma 14. *Let T be a δ-SIO on $\mathbb{R} \times \mathbb{R}$ such that $T1 = 0$. Then for all $h \in C_{00}^\infty(\mathbb{R})$ and for $i \in \{1, 2\}$, the sequence $(\langle h, T^i \theta_q \rangle 1)_{q > q_0}$ satisfies the hypothesis of Lemma 13 for q_0 big enough.*

To prove Lemma 13, observe that the integrals $\iint (Q_t g)(x)(Q_t \beta_q)(x) \cdot (P_t f)(x) t^{-1} \, dx \, dt$ are uniformly absolutely convergent since $\| Q_t \beta_q \|_\infty \leqslant C$, $\int_0^{+\infty} \| Q_t g \| t^{-1} \, dt < +\infty$ and $\sup_{t > 0} \| P_t f \|_2 < +\infty$. Therefore we can take the limit under the integral sign. Since by assumption $\lim_{q \to +\infty} (Q_t \beta_q)(x) = 0$ for all $(x, t) \in \mathbb{R}_+^2$, Lemma 13 is proved.

To prove Lemma 14 we pick a function $k \in C_0^\infty(\mathbb{R})$ and we want to prove that $|\langle k, \langle h, T^i\theta_q\rangle 1\rangle|$ is less than $C\|k\|_{H^1}$ for $q > q_0$, q_0 and C being independent of k, and that $\lim_{q\to+\infty} |\langle k, \langle h, T^i\theta_q\rangle 1\rangle| = 0$. This latter fact follows from $T1 = 0$ and from Lemma 3 in Section 3. To prove the first fact it is enough to prove that for $q > q_0$ and $q' > q_0$

$$|\langle k, \langle h, T^i(\theta_q - \theta_{q'})\rangle 1\rangle| \leqslant C\|k\|_{H^1}, \tag{7.2}$$

and then take the limit when $q' \to +\infty$. Notice now that if $\operatorname{supp} h \cap \cap \operatorname{supp}(\theta_q - \theta_{q'}) = \phi$, then $\langle h, T^i(\theta_q - \theta_{q'})\rangle = \iint h(x)K_i(x,y)(\theta_q - \theta_{q'})(y)\,dx\,dy$. This will be true if q_0 is chosen large enough and in this case a straightforward computation (using $\int h\,dx = 0$) yields $\|\langle h, T^i(\theta_q - \theta_{q'})\rangle\|_{\delta CZ} \leqslant C$, which implies (7.2).

We have proved $N^2 1 = 0$. The proof of $N^{2*}1 = 0$ is identical. One just has to use $\tilde{T}1 = 0$ instead of $T1 = 0$. This proves Lemma 11.

We also need another operator M, similar to N, defined by

$$\langle g_1 \otimes g_2, Mf_1 \otimes f_2\rangle = \langle g_1, U^*_{\{\langle g_2, T^2 f_2\rangle^* 1\}} f_1\rangle.$$

This operator M is also an SIO and has the WBP. Moreover, $M^2 1 = M^{2*}1 = M^1 1 = 0$ and $(T-M)^{1*}1 = 0$. This can be shown using the same arguments as for N.

We now set $S = M + N$ so that S has the WBP, $S^2 1 = S^{2*}1 = 0$ and $(T-S)^1 1 = (T-S)^{1*}1 = 0$.

8. Construction of the operator V_b

The construction of V_b is inspired by the construction of the operators U_β, $\beta \in \in \mathrm{BMO}$ of Section 7; see [9].

The family of operators $(P_t)_{t>0}$ is defined as in Section 2, but now Q_t denotes $-t\frac{\partial}{\partial t}P_t$, so that $\int Q_t \frac{dt}{t} = I$ and $\int Q_t^2 \frac{dt}{t} = C_0 I$, where C_0 is not necessarily 1.

Let $b \in \mathrm{BMO}(\mathbb{R}\times\mathbb{R})$ and let W_b: $C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R}) \to [C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})]'$ be defined by

$$\langle f_1 \otimes f_2, W_b g_1 \otimes g_2\rangle = $$
$$= \iint \langle Q_{t_1} f_1 \otimes Q_{t_2} f_2, (Q_{t_1}Q_{t_2}b)P_{t_1}g_1 \otimes P_{t_2}g_2\rangle \frac{dt_1}{t_1}\frac{dt_2}{t_2}.$$

The L^2-boundedness of W_b is, as in the classical situation [9], a consequence of the fact that $(Q_{t_1}Q_{t_2}b)^2\,dx_1 dx_2\,(t_1 \cdot t_2)^{-1}\,dt_1\,dt_2$ is a Carleson measure [5] and of the properties of such measures on product spaces [4]. On the other hand one sees easily that W_b is a 1-SIO whose kernels take their values in

$\{U_\beta, \beta \in \mathrm{BMO}(\mathbb{R})\}$. Moreover, an application of Lemma 3 shows that $W_b 1 = C_0 b$, $W_b^* 1 = 0$, $\tilde{W}_b 1 = 0$ and $\tilde{W}_b^* 1 = 0$. It remains to show that $\tilde{W}_b$ is bounded on L^2. In order to do this, it is enough to show that $\tilde{W}_b$ maps L^∞ into BMO. Similarly $\tilde{W}_b^*$ will map L^∞ into BMO so that the L^2-boundedness of W_b will follow by interpolation between $H^1 \to L^1$ and $L^\infty \to \mathrm{BMO}$ [13].

We want the estimate $\| \tilde{W}_b f \|_{\mathrm{BMO}} \leqslant C \| b \|_{\mathrm{BMO}} \| f \|_\infty$. Consider the operator T_f: $b \to \tilde{W}_b f$. We need to show that it maps BMO to itself. Observe that T_f, which is given by

$$\langle h_1 \otimes h_2, T_f b_1 \otimes b_2 \rangle = \iint \langle P_{t_1} h_1 \otimes Q_{t_2} h_2, (Q_{t_1} P_{t_2} f) Q_{t_1} b_1 \otimes Q_{t_2} b_2 \rangle \frac{dt_1}{t_1} \frac{dt_2}{t_2},$$

is itself a 1-SIO, and satisfies $T_f^2 1 = T_f^{2*} 1 = T_f^1 1 = 0$. Therefore we already know that T_f maps L^2 to L^2. From Theorem 3 it follows that T_f maps L^∞ to BMO. To show that T_f maps BMO to itself, we observe that $T_f H_1$, $T_f H_2$ and $T_f H_1 H_2$ are SIO's, because the kernel of $Q_t H$ satisfies the same estimates as the kernel of P_t. Since these operators are bounded on L^2 as well as T_f, they also map L^∞ to BMO. Hence T_f is bounded on BMO.

The proof of Theorem 4 is complete.

9. Bicommutators of Calderón-Coifman type

In the classical situation, a standard kernel K is antisymmetric if $K(x, y) = -K(y, x)$ for all $(x, y) \in \Omega$. Such a kernel induces automatically and SIO T defined for all $f, g \in C_0^\infty(\mathbb{R})$ by

$$\langle g, Tf \rangle = \lim_{\epsilon \to 0} \iint_{|x-y| > \epsilon} g(x) K(x, y) f(y)\, dx\, dy. \tag{9.1}$$

The existence of the limit is a consequence of the antisymmetry of the kernel K and of the smoothness of f and g. Actually,

$$\langle g, Tf \rangle = \tfrac{1}{2} \iint K(x, y) [g(x) f(y) - f(x) g(y)]\, dx\, dy, \tag{9.2}$$

so that $|\langle g, Tf \rangle| \leqslant C\, (\mathrm{diam}\, [\mathrm{supp}\, g \cup \mathrm{supp} f])^3 \| g' \|_\infty \| f' \|_\infty$.

This clearly implies that T has the WBP. Since $T = -T^*$, T is bounded on L^2 if and only if $T1 \in \mathrm{BMO}$, by Theorem 1.

The best known examples of CZO's generated by antisymmetric kernels in the manner just described are the Calderón commutators associated to the kernels $[(A(x) - A(y))/(x - y)]^k \cdot (x - y)^{-1}$ where $k \geqslant 0$ and A: $\mathbb{R} \to \mathbb{C}$ satisfies $A' = a \in L^\infty$. Calderón proved in [2] that $\| T_k \|_{\mathrm{CZ1}} \leqslant C^k$. This estimate which has been improved in [7], can be easily obtained from Theorem 1 ([9]).

Actually, this can also be derived from a more general result on antisymmetric kernels.

Let K be an antisymmetric standard kernel and A: $\mathbb{R} \to \mathbb{C}$ be such that $A' = a \in L^\infty$, and let K_a be defined by $K_a(x,y) = K(x,y)\,[A(x) - A(y)] \cdot \cdot (x-y)^{-1}$ for all $(x,y) \in \Omega$. Clearly K_a is also an antisymmetric standard kernel and defines an SIO T_a having the WBP.

Proposition 1. *If T is a* CZO, *then T_a is a* CZO, *and for all $\delta \in]0,1]$ there exists $C_\delta > 0$ such that*

$$\|T_a\|_{\mathrm{CZ}\delta} \leqslant C_\delta \|a\|_\infty \|T\|_{\mathrm{CZ}\delta}. \tag{9.3}$$

This propostion can be generalized to the product setting.

Let L: $\Omega \times \Omega \to \mathbb{C}$ be a function such that for all (x_1, y_1) and $(x_2, y_2) \in \Omega$

$$|L(x_1, y_1, x_2, y_2)| \leqslant \frac{C}{|x_1 - y_1|\,|x_2 - y_2|}. \tag{9.4}$$

If L is antisymmetric in each couple it defines a continuous operator T: $C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R}) \to [C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})]'$ by

$$\langle g_1 \otimes g_2, Tf_1 \otimes f_2 \rangle =$$
$$= \lim_{\epsilon_1 \to 0\, \epsilon_2 \to 0} \iiiint\limits_{|x_1 - y_1| > \epsilon_1,\, |x_2 - y_2| > \epsilon_2} g_1(x_1) g_2(x_2) L(x_1, y_1, x_2, y_2) f_1(y_1) f_2(y_2)\, dx_1\, dx_2\, dy_1\, dy_2$$

for all $f_1, f_2, g_1, g_2 \in C_0(\mathbb{R})$.

As in the classical situation, the existence of this limit is a consequence of the antisymmetry of L, of (9.4) and of the smoothness of f_1, f_2, g_1 and g_2. It is easy to see that T has two kernels K_1 and K_2 in the sense of Definition 8, in Section 3. These are given by

$$\langle g, K_1(x) f \rangle = \lim_{\epsilon \to 0} \iint\limits_{|u-v|} g(u) K(x, y, u, v) f(v)\, du\, dv$$

and

$$\langle g, K_2(x) f \rangle = \lim_{\epsilon \to 0} \iint\limits_{|u-v|} g(u) K(u, v, x, y) f(v)\, du\, dv$$

for all $f, g \in C_0^\infty(\mathbb{R})$.

Let A: $\mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be a function such that

$$\frac{\partial^2 A}{\partial_{x_1} \partial_{x_2}} = a \in L^\infty$$

(in the distributional sense) and let $\tilde{A}$: $\Omega \times \mathrm{A} \to \mathbb{C}$ be defined by

$$\tilde{A}(x_1, y_1, x_2, y_2) = \frac{A(x_1, x_2) + A(y_1, y_2) - A(y_1, x_2) - A(x_1, y_2)}{(x_1 - y_1)(x_2 - y_2)}$$

for all (x_1, y_1) and $(x_2, y_2) \in \Omega$. If L is antisymmetric and satisfies (9.4), then $L\tilde{A}$ has the same properties. Hence $L\tilde{A}$ defines an operator T_a in the same manner as L defines T. The first kernel $K_{a,1}$ of T_a is defined by

$$\langle g, K_{a,1}(x, y)f \rangle = \lim_{\epsilon \to 0} \iint_{|u-v| > \epsilon} g(u) L(x, y, u, v) \tilde{A}(x, y, u, v) f(v)\, dv\, du.$$

Notice now that because T is a δ-SIO, so is T_a. This is an immediate consequence of Proposition 1 and the fact that for x and y fixed, $\tilde{A}(x, y, u, v)$ is of the form $[B(u) - B(v)]/(u - v)$, with $\|B'\|_\infty \leqslant \|a\|_\infty$ and $\frac{\partial A}{\partial x}(x, y, u, v)$ is of the form $[C(u) - C(v)]/(u - v)$, with $\|C'\|_\infty \leqslant \|a\|_\infty/(x - y)$.

Proposition 2. *If T is a* CZO, *then T_a is a* CZO, *and for all $\delta \in]0, 1]$ there exists $C_\delta > 0$ such that*

$$\|T_a\|_{\mathrm{CZ}\delta} \leqslant C_\delta \|a\|_\infty \|T\|_{\mathrm{CZ}\delta}.$$

In particular the kernel $[(x_1 - y_1)(x_2 - y_2)]^{-1} \cdot [\tilde{A}]^k$ defines a CZO T_k of norm less than $C^k \|a\|_\infty^k$. The L^2-boundedness of T_1 was first proved by J. Aguirre in [1].

We now turn to the proofs and start with Proposition 1. For simplicity we shall assume $\delta = 1$. We know that it is enough to show that for $a \in L^\infty$, $T_a 1 \in \mathrm{BMO}$ and $\|T_a 1\|_{\mathrm{BMO}} \leqslant C\|a\|_\infty$. To show this inequality we are going to exhibit a CZO S such that $T_a 1 = Sa$, that is, for all $g \in C_{00}^\infty(\mathbb{R}) \langle g, Sa \rangle = \langle g, T_a 1 \rangle$. This equality determines $\langle g, Sa \rangle$ for all $g \in C_{00}^\infty(\mathbb{R})$ and $a \in C_0^\infty(\mathbb{R})$. But since $|K_a(x, y)| \leqslant C|x - y|^{-2}$ when $a \in C_0^\infty(\mathbb{R})$, $T_a 1$ acts not only on $C_{00}^\infty(\mathbb{R})$ but on $C_0^\infty(\mathbb{R})$, so that $\langle g, Sa \rangle$ is well defined for all $a \in C_0^\infty(\mathbb{R})$ and $g \in C_0^\infty(\mathbb{R})$. Moreover,

$$\langle g, Sa \rangle = \lim_{\epsilon \to 0} \iint_{|x-y| > \epsilon} g(x) K(x, y) \frac{A(x) - A(y)}{x - y}\, dx\, dy.$$

Let g and a have disjoint supports. Then if $x \in \operatorname{Supp} g$ and $|y - x| \leqslant \operatorname{dist}(\operatorname{Supp} a, \operatorname{Supp} g)$, $A(x) = A(y)$ so that the integral defining $\langle g, Sa \rangle$ is absolutely convergent. This permits us to compute the kernel of S, namely $K_s(x, u) = \int_{-\infty}^{u} K(x, y) \frac{1}{x - y}\, dy$ if $x > u$ and $K_s(x, u) = \int_u^{+\infty} K(x, y) \frac{1}{y - x}\, dy$ if $x < u$. This kernel K_s is clearly a standard kernel and because of that we can apply Theorem 1 to show that S is a CZO. We first notice that $S1 = T1$ and therefore lies in BMO. This is formally obvious, since when $a = 1$, $[A(x) - A(y)] \cdot (x - y)^{-1}\,dx\,dy$, but it can be proved rigorously using Lemma 1. Next, we compute S^*1. For $a \in C_0^\infty(\mathbb{R})$, $A(x) = 0$ for x large enough. Moreover, since $\langle g, Sa\rangle$ can be rewritten as $\frac{1}{2}\iint [g(x) - g(y)]K(x,y)\,[A(x) - A(y)](x-y)^{-1}\,dx\,dy$ an application of Lemma1 shows easily that $S^*1 = 0$. Finally, to prove that S has the WBP we choose g and $a \in C_0^\infty(\mathbb{R})$ and suppose that the supports of a and g are contained in some interval $]x_0 - t, x_0 + t[$. We decompose the integral as $I_1 + I_2 + I_3$, where

$$I_1 = \frac{1}{2}\iint_{x,y\in]x_0-2t,x_0+2t[} [g(x) - g(y)]K(x,y)\frac{A(x) - A(y)}{x-y}\,dx\,dy,$$

$$I_2 = \frac{1}{2}\iint_{\substack{x\in]x_0-2t,x_0+2t[\\ y\notin]x_0-2t,x_0+2t[}} g(x)K(x,y)\frac{A(x) - A(y)}{x-y}\,dx\,dy,$$

and $I_3 = I_2$ because of the antisymmetry of K. Clearly $|I_1| \leqslant C\|g'\|_\infty\|a\|_\infty t^2$ and $|I_2| \leqslant C\|g\|_\infty\|a\|_\infty t$. These estimates imply that S has the WBP. Theorem 1 can be applied to S, which is a CZO. This proves Proposition 1.

We shall denote by U the linear mapping that sends a CZO T defined by an antisymmetric kernel to the operator S we have just considered. From the proof of Proposition 1 it follows that (9.6) $\|U(T)\|_{CZ\delta} \leqslant C\|T\|_{CZ\delta}$.

The proof of Proposition 2 follows the same lines as the proof of Proposition 1. Notice first that an SIO T defined from an antisymmetric kernel by (9.5) has the WBP. This can be seen exactly as in the classical situation. Moreover, such an operator satisfies $T1 = T^*1 = -\tilde{T}1 = -\tilde{T}^*1$. Hence, to prove that it is bounded on L^2, it is enough to show that $T1 \in \mathrm{BMO}$, by Theorem 4, and this is necessary by Theorem 3.

We now wish to prove that if an antisymmetric kernel L defines a CZO, T then the SIO T_a defined by $L\tilde{A}$ satisfies $T_a 1 \in \mathrm{BMO}$. To do this we consider the operator W: $L^\infty(\mathbb{R}\times\mathbb{R}) \to [C_{00}^\infty(\mathbb{R}) \otimes C_{00}^\infty(\mathbb{R})]'$ defined by $Wa = T_a 1$. If $a \in C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})$ then $T_a 1$ is actually an element of $[C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})]'$ because of the decay properties of $L\tilde{A}$ at ∞. Hence $\langle g_1 \otimes g_2, Wa_1 \otimes a_2\rangle$ can be defined for $g_1, g_2, a_1, a_2 \in C_0^\infty(\mathbb{R})$ by

$$\lim_{\epsilon_1\to 0\,\epsilon_2\to 0}\iiiint_{\substack{|x_1-y_1|>\epsilon_1\\ |x_2-y_2|>\epsilon_2}} g_1(x_1)g_2(x_2)L(x_1,y_1,x_2,y_2)\tilde{A}(x_1,y_1,x_2,y_2)\,dx_1\,dy_1\,dx_2\,dy_2.$$

We are left with proving that W is a CZO. To compute the kernels X_1 and X_2 of W we notice that if $a = a_1 \otimes a_2$.

$$\tilde{A}(x_1, y_1, x_2, y_2) = \frac{A_1(x_1) - A_1(y_1)}{x_1 - y_1} \frac{A_2(x_2) - A_2(y_2)}{x_2 - y_2},$$

where $A_1' = a_1$ and $A_2' = a_2$. The computation we did to compute K_s shows that if $x_1 > u_1$ and $g_2, a_2 \in C_0^\infty(\mathbb{R})$, then

$$\langle g_2, X_1(x_1, u_1)a_2 \rangle =$$

$$= \lim_{\epsilon_2 \to 0} \int_{-\infty}^{u_1} \iint_{|x_2 - y_2| > \epsilon_2} g_2(x_2) \frac{L(x_1, y_1, x_2, y_2)}{x_1 - y_1} \frac{A_2(x_2) - A_2(y_2)}{x_2 - y_2} dx_2 \, dy_2 \, dy_1.$$

This is also equal to

$$\left\langle g_2, \left(U\left[\int_{-\infty}^{u_1} K_1(x_1, y_1) \frac{1}{x_1 - y_1} dy_1 \right] \right) a_2 \right\rangle.$$

Similar formulas hold for the case $x_1 < u_1$ and for X_2. From (9.6) it follows that X_1 and X_2 are both 1CZ-1-standard kernels. The WBP of W can also be checked easily. Indeed, for $g_1, a_1 \in C_0^\infty(\mathbb{R})$,

$$\langle g_1, W^1 a_1 \rangle = U\left[\lim_{\epsilon \to 0} \iint_{|x_1 - y_1| > \epsilon} g_1(x_1) K_1(x_1, y_1) \frac{A_1(x_1) - A_1(y_1)}{x_1 - y_1} dx_1 \, dy_1 \right],$$

so that the proof in the classical case extends immediately. Moreover, this equality implies that for $a_1, g_1 \in C_0^\infty(\mathbb{R}) \langle g_1, W^1 a_1 \rangle^* 1 = 0$, or equivalently, $W^{2*}1 = 0$. Similarly $W^{1*}1 = 1$. By Lemma 3 this implies $W^*1 = \tilde{W}1 = \tilde{W}^*1 = 0$. Finally, $W1 = T1$ can be proved using Lemma 3. Therefore W is a CZO and Proposition 2 is proved.

Of course this result extends to an arbitrary finite product of copies of $\mathbb{R}$, by a simple induction on the number of factors. We omit the details.

10. A Littlewood-Paley inequality for arbitrary rectangles

We wish to prove the following extension of Theorem 2.

Theorem. 5. *Let $\{R_k\}_{k \in \mathbb{N}}$ be a collection of disjoint rectangles in $\mathbb{R}^n$ with sides parallel to the axes and let S_k be the Fourier multiplier of symbol χ_{R_k}. Let $\bar{\Delta}$ be defined on L^2 by $|\bar{\Delta} f| = \left[\sum_k (S_k f)^2\right]^{1/2}$. Then $\bar{\Delta}$ is bounded on L^p for all $p \in [2, +\infty[$.*

We shall assume the reader to be familiar with [15] where this theorem is proved in the case $n = 1$. In this paper it is shown that the theorem for $n = 1$ is a consequence of the following.

Lemma 15. *Let ψ be fixed in $S(\mathbb{R})$ son that $\chi_{[-2,2]} \leqslant \hat{\psi} \leqslant \chi_{[-3,3]}$ and let Ψ_k^j be the convolution operator of symbol $\hat{\psi}\left(\frac{\xi}{2^k} - j\right)$. Let χ: $\mathbb{Z} \times \mathbb{Z} \to \{0, 1\}$ be such that the operator T_χ is bounded on L^2 where $T_\chi f(x) = \chi(j, k)[\Psi_k^j f(x)]_{j,k}$ takes its values in $l^2(\mathbb{Z} \times \mathbb{Z})$. Then T_χ is bounded from L^∞ to* $\mathrm{BMO}_{l^2(\mathbb{Z}\times\mathbb{Z})}$.

Observe that, by Plancherel's theorem, the L^2-boundedness of T_χ is equivalent to

$$\sum_{j,k} \chi(j, k)|\hat{\psi}(\xi 2^{-k} - j)|^2 \in L^\infty(d\xi).$$

The reduction of Theorem 2 to Lemma 15 is done by means of standard Littlewood-Paley theory, A_2 weights and interpolation between $L^2 \to L^2$ and $L^\infty \to \mathrm{BMO}$. All these arguments go through in the n-dimensional setting without any problem. Finally the main ingredient in the proof of Lemma 15 is the following.

Lemma 16. *Let $(x, t) \in \mathbb{R}_+^2$ and $a \in L^2{}_{\mathrm{loc}}$ be supported out of $]x - 2t, x + 2t[$. Then for all $\eta > 0$, there exists $C_\eta > 0$ such that*

$$\sum_{k,j} |(Q_t \Psi_k^j a)(x)|^2 \leqslant C_\eta \int a^2(z) \left(\frac{t}{|x - z|}\right)^{5/3 - \eta} \frac{dz}{t}. \tag{10.1}$$

Note that the summation is over all $(k, j) \in \mathbb{Z} \times \mathbb{Z}$. This lemma is actually a reformulation of the Lemma 4.1 of [15] and we leave the translation to the reader. From (10.1) it follows by a standard argument that if $a \in L^\infty$ then

$$\sum_{k,j} \chi(j, k)|(Q_t \Psi_k^j a)^2(x)|^2 \frac{dt}{t} dx$$

is a Carleson measure, or equivalently that $[\chi(j, k)\Psi_k^j(a)]_{k,j}$ lies in $\mathrm{BMO}_{l^2(\mathbb{Z}\times\mathbb{Z})}(\mathbb{R})$. Thus Lemma 16 implies Lemma 15.

As we said, Theorem 5 follows from an appropriate analogue of Lemma 15 in the n-dimensional context.

Lemma 17. *Let Ψ and Ψ_k^j be as in Lemma 15 and let χ: $[\mathbb{Z}^n]^2 \to \{0, 1\}$ be such that the operator T_χ: $L^2(\mathbb{R}^n\, dx) \to L^2_{l^2(\mathbb{Z}^{2n})}(\mathbb{R}^n\, dx)$ is bounded, where $T_\chi f(x)$ takes its vlalues in $L^2(\mathbb{Z}^{2n})$ and is given by $[\chi(j, k)([\Psi_{k_1}^{j_1} \otimes \ldots \otimes \Psi_{k_n}^{j_n}]f)(x)]_{(j,k)}$. Then T_χ is bounded from L^∞ to $\mathrm{BMO}_{l^2(\mathbb{Z}^{2n})}$, or equivalently,*

$$\sum_{j,k} \chi(j, k)[Q_{t_1}\Psi_{k_1}^{j_1} \otimes \ldots \otimes Q_{t_n}\Psi_{k_n}^{j_n} a(x)]^2\, dx \frac{dt}{t} \tag{10.2}$$

is a Carleson measure on $[\mathbb{R}_+^2]^n$ if $a \in L^\infty$.

To avoid any convergence problems we suppose that χ is finitely supported but we shall obtain estimates independent of this assertion. We shall use a variant of Lemma 5. It can be shown that (5.3) remains true if $\|[Q_{t_I}T]_{x_I z_I}\|_{CZ\epsilon}$ is replaced by $[t_I^\epsilon/(x_I - z_I)^{1+\epsilon}]$, the point being that Lemma 8 remains true if in (5.4) $\|Q_{t_1}T)_{x_1 z_1}\|_{CZ\epsilon}$ is replaced by $[t_1^\epsilon/(x_1 - z_1)^{1+\epsilon}]$. Let us rewrite the variant of (4.3):

$$\int |E_{x_I t_I z_I}| \frac{t_I^{\epsilon-1}}{(x_1 - z_I)^{1+\epsilon}} dz_I\, dt_I\, dx_I \leqslant C_{n,\epsilon}|\Omega|. \tag{10.3}$$

For technical reasons we need to assume that the T_i's constructed in Section 5 take their values in the set $\{2^k, K \in \mathbb{Z}\}$. This is of course not a restriction. Replacing T_i by $\inf\{2^k, T_i \leqslant 2^k\}$ yields (4.3) and (10.3) a fortiori and (4.2) with the constant $2^n C_n$. We shall use this familly of functions $\{T_i, i \in [1, n]\}$ with various sets playing the role of Ω and even various dimensions. Let I be a set of indices in $[1, n]$ and ω a bounded open subset of $\mathbb{R}^I$. Then $\{T_i(x, t, \omega), i \in I, (x, t) \in S(\omega)\}$ will refer to the family of functions constructed in dimension $|I|$ with ω playing the role of Ω.

Let Ω be an open subset of $\mathbb{R}^n$ and let Ω_n, $T_i(x, t, \Omega)$, $1 \leqslant i \leqslant n$ be as in Sections 4 and 5. By the same argument as in Section 4, and with the same notations, we are reduced to proving an estimate similar to (4.5), namely for $a \in \mathbb{L}^\infty(\mathbb{R}^n)$ and $\|a\|_\infty < 1$

$$\int_{S(\Omega)} \Sigma\chi(k, j)\, |Q_t \Psi_k^j a_{x,t,I}(x)|^2 \frac{dx\, dt}{t} \leqslant C_n|\Omega|. \tag{10.4}$$

This inequality will be a consequence of the following.

Lemma 18. *Suppose that the function a_{xt} is of the form $a\chi_{E_{xt}}$ where $E_{xt} \subseteq \mathbb{R}^n$ is defined by the following set of conditions.*

Let $i \in [1, n]$. For all $j \in [1, n]$, x_j, z_j, let $l_j \in \mathbb{Z}$, be such that $2^{l_j} \leqslant |x_j - z_j| < 2^{l_j+1}$. Let $S_1(x, t), S_2(x, t, l_1), \ldots, S_i(x, t, l_1, l_2, \ldots, l_{i-1})$ be i functions taking their values in the set $\{2^k, k \in \mathbb{Z}\}$ and larger than $2t_1, 2t_2, \ldots, 2t_i$ respectively. Let $F_{x_1 t_1 x_2 t_2 \ldots x_i t_i}$ be a subset of $\mathbb{R}^n$. Then $(z_1, \ldots, z_n) \in E_{xt}$ if and only if

$$(z_1, z_2, \ldots, z_n) \in F_{x_1 t_1 \ldots x_i t_i} = F_{x_I t_I}$$

$$H \begin{cases} 2^{l_1+1} > |x_1 - z_1| \geqslant 2^{l_1} \geqslant S_1(x, t) \geqslant 2t_1 \\ 2^{l_2+1} > |x_2 - z_2| \geqslant 2^{l_2} \geqslant S_2(x, t, l_1) \geqslant 2t_2 \\ 2^{l_i+1} > |x_i - z_i| \geqslant 2^{l_i} \geqslant S_i(x, t, l_1, \ldots, l_{i-1}) \geqslant 2t_i. \end{cases}$$

Let $I = [1, i]$ and x_I, t_I, l_I be fixed, and let $D_{x_I t_I l_I} = \bigcup \prod_{q>i}]x_q - t_q, x_q + t_q[$, where the union is extended to those $(x_{i+1}, t_{i+1}, \ldots, x_n, t_n)$ such that

$2^{l_1} \geqslant S_1(x, t), \ldots, 2^{l_i} \geqslant S_i(x, t, l_1, \ldots, l_{i-1})$. *If for all* $\epsilon > 0$

$$\iiint_{|x_I - z_I| \geq 2t_I} |D_{x_I t_I l_I}| \frac{dx_I\, dt_I\, dz_I}{t_I^{1-\epsilon} |x_I - z_I|^{1+\epsilon}} \leqslant C_\epsilon |\Omega|, \tag{10.5}$$

then

$$\int_{S(\Omega)} \Sigma \chi(k, j) \, |Q_t \Psi_k^j a_{x,t}(x)|^2 \frac{dx\, dt}{t} \leqslant C|\Omega|. \tag{10.6}$$

Let us see first why Lemma 18 implies (10.4). Observe that it is enough to prove (10.4) when I is of the form $[1, i]$. Indeed, if the construction of the T_i's is non-symmetric in the various indices, the properties of the T_i's which are used are expressed symmetrically and therefore we can reorder the coordinates in such a way that I is of the form $[1, i]$. Now we apply Lemma 18 with $S_1(x, t) = T_1(x, t, \Omega), S_2(x, t, \Omega_1) = T_2(x, t, \Omega) \ldots S_i(x, t, l_1 \ldots l_{i-1}) = T_i(x, t, \Omega)$ and $F_{x_1 t_1 \ldots x_n t_n} = \mathbb{R}^n$. In this case $D_{x_I t_I l_I} = E_{x_I t_I z_I}$ (with the notations of Lemma 5). Indeed if the T_j's take their values in $\{2^k, k \in \mathbb{Z}\}$, then $|x_j - z_j| \geqslant T_j \Leftrightarrow 2^{l_j} \geqslant T_j$. Thus (10.5) coincides with (10.3) and (10.6) coincides with (10.4). We are left with showing Lemma 18.

The proof of Lemma 18 uses a backward induction on i. We start with the case $i = n$. Then we can use the following.

Lemma 19. *Let* $(x, t) \in \mathbb{R}_+^2$ *and* $b_{x,t} \in L^2_{\text{loc}}$ *be such that* $b_{x,t}(z) = 0$ *if* $|x_i - z_i| \leqslant 2t_i$ *for some* $i \in [1, n]$. *Then for all* $\epsilon > 0$ *there exists* $C_\epsilon > 0$ *such that*

$$\sum_{(l,j) \in [\mathbb{Z}^n]^2} |(Q_t \Psi_k^j b_{x,t})(x)|^2 \leqslant \int [b_{x,t}(z)]^2 \frac{\left[\prod_{1 \leq i \leq n} t_i\right]^{2/3 - \eta}}{\left[\prod_{1 \leq i \leq n} |x_i - z_i|\right]^{5/3 - \eta}} dz. \tag{10.7}$$

This lemma is the n-dimensional analogue of Lemma 16 and its proof is nothing but the n-th fold application of Lemma 16 successively in each coordinate.

As for Lemma 5, (10.5) has to be interpreted differently when $I = [1, n]$. In this case it reads

$$\iiint_{|x_I - z_I| \geq 2t_I,\, H \text{satisfied}} \frac{dx_I\, dt_I\, dz_I}{t_I^{1-\epsilon} |x_I - z_I|^{1-\epsilon}} \leqslant C_\epsilon |\Omega|, \tag{10.8}$$

the restriction $z \in F_{x_I t_I}$ being irrelevant. Now to obtain (10.6), choose $\eta = \frac{1}{3}$, $b_{x,t} = a_{xt}$ and apply (10.7). Then integrate against $dx_I\, dt_I / b_I$ and (10.8) with $\epsilon = \frac{1}{3}$.

We turn to the general case and choose $i < n$.

Let $(a_{xt}, (x,t) \in S(\Omega))$ be a family of functions satisfying the hypothesis of the lemma. We decompose a_{xt} as $\sum_{l_I} a_{xtl_I}$ where $l_I \geqslant 2t_I$ and

$$a_{xtl_I} = a_{xt} \prod_{s \in I} \chi_{2^{l_s} \leq |x_s - z_s| < 2^{l_s+1}}.$$

Now for x_I, t_I, l_I fixed and $D_{x_I t_I l_I}$ as defined in Lemma 18 we apply Lemma 5 in dimension $(n - |I|)$. This yields $(n - |I|)$ functions $\tilde{T}_j$, $j > 1$, where $\tilde{T}_j(x_J, t_J) = T_j(x_J, t_J, D_{x_I t_I l_I})$ for $(x_J, t_J) \in S(D_{x_I t_I l_I})$. Let

$$\tilde{D}_{x_I t_I l_I} = \bigcup_{x_J, t_J} \prod_{j > i}]x_j - \tilde{T}_j, x_j + \tilde{T}_j[.$$

From (4.2) we conclude $|\tilde{D}_{x_I t_I l_I}| \leqslant C|D_{x_I t_I l_I}|$. We define $\tilde{a}_{xtl_I} = a_{xtl_I}\chi_{\tilde{D}_{x_I t_I l_I}}(z_{i+1} \ldots z_n)$ and $\tilde{a}_{xt} = \sum_{l_I} \tilde{a}_{xtl_I}$.

Let $(x,t) \in S(\Omega)$, $(k,j) \in [\mathbb{Z}^n]^2$ and $\alpha > 0$ be given. By Cauchy-Schwarz,

$$|Q_t \Psi_k^j \tilde{a}_{xt}(x)|^2 \leqslant \left[\sum_{l_I} 2^{-\Sigma_{1 \leq s \leq i} l_s \alpha}\right]\left[\sum_{l_I} 2^{\Sigma_s l_s \alpha} |Q_t \Psi_k^j \tilde{a}_{xtl_I}(x)|^2\right]$$

which is less than

$$\left[\prod_{1 \leq s \leq i} \frac{1}{t_s}\right]^\alpha \left[\sum_{l_I} 2^{\Sigma_{1 \leq s \leq i} l_s \alpha} |Q_t \Psi_k^j \tilde{a}_{xtl_I}(x)|^2\right]. \tag{10.9}$$

We wish to show (10.6) with $\tilde{a}_{xt}$ instead of a_{xt}. Observe that the L^2-boundedness of T_χ is equivalent to

$$\sum_{j,k} \chi(j,k)\left[\prod_{1 \leq r \leq n} \hat{\psi}(\xi_r 2^{-k_r} - j_r)\right]^2 \leqslant C.$$

Fix $(j_I, k_I) \in (\mathbb{Z}^I)^2$ and set $\xi_I = j_I 2^{k_I}$. Since $\hat{\psi}(0) = 1$, we obtain

$$\sum_{j_J, k_J} \chi(j,k)\left[\prod_{i < r \leq n} \hat{\psi}(\xi_r 2^{-k_r} - j_r)\right]^2 \leqslant C,$$

where $j = (j_I, j_J)$ and $k = (k_I, k_J)$. This implies that for (j_I, k_I) fixed, the operator $T_{j_I k_I}$ defined by $T_{j_I k_I} f(x) = [\chi(j,k)\Psi_{kJ}^{j} f(x)]_{j_J, k_J}$, is bounded from $L^2(\mathbb{R}^J)$ to $L^2_{l^2(\mathbb{Z}^J \times \mathbb{Z}^J)}(\mathbb{R}^J)$.

In order to estimate the $l \cdot h \cdot s$ of (10.6) with $\tilde{a}_{xt}$ we rewrite it as

$$\int \sum_{j_I k_I} \left[\int \sum_{j_J, k_J} \chi(k,j) |Q_t \Psi_k^j \tilde{a}_{x,t}(x)|^2 \frac{dx_J\, dt_J}{t_J}\right] \frac{dx_I\, dt_I}{t_I}, \tag{10.10}$$

and estimate first the part between brackets. By (10.9) this is less than

$$\frac{1}{(t_I)^\alpha} \sum_{l_I} 2^{\Sigma_{1 \leq s \leq i} l_s \alpha} \int \sum_{k_J, j_J} \chi(k,j) \, |Q_t \Psi_k^j \tilde{a}_{xtl_I}(x)|^2 \frac{dx_J\, dt_J}{t_J}.$$

Observe that $\tilde{a}_{xtl_I}(z)$ is of the form $a(z)\chi_{G_{x_I t_I l_I}}(z)$ where $G_{x_I t_I l_I}$ is some subset of $\mathbb{R}^n$ depending only on x_I, t_I, and l_I. Now for x_I, t_I, l_I, j_I, k_I fixed we can apply the boundedness of the operator $T_{j_I k_I}$ to the function of z_J

$$[Q_{t_I}\Psi^{j_I}_{k_I}\tilde{a}_{xtl_I}](x_I, z_J)$$

since this does not depend on x_J and t_J. We obtain a majorization of the previous integral by

$$\frac{1}{(t_I)^\alpha}\sum_{l_I} 2^{\Sigma_{1\leq s\leq i} l_s\alpha}\int_{\mathbb{R}^J} |[Q_{t_I}\Psi^{j_I}_{k_I}\tilde{a}_{xtl_I}](x_I, z_J)|^2\, dz_J.$$

To estimate (10.10) we must sum in (j_I, k_I), then in l_I, and finally integrate against $dx_I\, dt_I/t_I$. We fix x_I, t_I, l_I and z_J. observe that the function $\tilde{a}_{xtl_I}(z_I, \bar{z}_J)$ vanishes if $|x_s - z_s| \leqslant 2t_s$ for some $s \in [1, i] = I$. Therefore we can apply Lemma 19 in dimension i and obtain:

$$\sum_{j_I, k_I} |[Q_{t_I}\Psi^{j_I}_{k_I}\tilde{a}_{xtl_I}](x_I, \bar{z}_J)|^2 \leqslant \int_{\mathbb{R}^I} |\tilde{a}_{xtl_I}(z)|^2 \frac{\left[\prod_{1\leq s\leq i} t_s\right]^{2/3-\eta}}{\left[\prod_{1\leq s\leq i} |x_s - z_s|\right]^{5/3-\eta}}\, dz_I.$$

Now we integrate in z_J and sum over l_I keeping in mind that $2^{l_I} \leqslant |x_I - z_I| \leqslant 2^{l_I+1}$. We are then reduced to integrating the following against $dx_I\, dt_I/t_I$:

$$\int_{\mathbb{R}^n} |\tilde{a}_{xtl_I}(z)|^2 \frac{[t_I]^{2/3-(\eta+\alpha)}}{[|x_I - z_I|]^{5/3-(\eta+\alpha)}}\, dz.$$

But this is less than

$$\|a\|_\infty^2 \int_{\mathbb{R}^I} |\tilde{D}_{x_I t_I l_I}| \frac{[t_I]^{2/3-(\eta+\alpha)}}{[|x_I - z_I|]^{5/3-(\eta+\alpha)}}\, dz_I.$$

Now we use $|\tilde{D}_{x_I t_I l_I}| \leqslant C|D_{x_I t_I l_I}|$, then integrate against $dx_I\, dt_I/t_I$ using (10.5) with $\eta = \alpha = \epsilon = \frac{2}{9}$, and we obtain the desired estimate for the expression (10.10).

To complete the proof of (10.6) we must prove it also when $a_{x,t}$ is replaced by $a_{x,t} - \tilde{a}_{x,t}$ in the $l \cdot h \cdot s$. This is where we are going to use the induction hypothesis, namely that Lemma 18 is true for $k \in]i, n]$. Recall that $a_{xt} - \tilde{a}_{xt}$ is given by

$$[a_{xt} - \tilde{a}_{xt}](z_I, z_J) = \sum_{l_I} a_{xtl_I}(z_I, z_J)\chi_{\tilde{D}^c_{x_I t_I l_I}}(z_J).$$

By the definition of $\tilde{D}_{x_I t_I l_I}$, we can write, if $z_J \in \tilde{D}^c_{x_I t_I l_I}$,

$$1 = \sum_{\substack{K \subseteq J \\ K \neq \phi}} \prod_{r \in K} \chi_{|x_r - z_r| > \tilde{T}r}.$$

Therefore

$$a_{xt} - \tilde{a}_{xt} = \sum_{\substack{K \subseteq J \\ K \neq \phi}} \bar{a}_{x,t,K},$$

where

$$\bar{a}_{x,t,K} = \left[\sum_{l_I} a_{xtl_I} \chi_{\tilde{D}^c_{x_I t_I l_I}}(z_J) \prod_{r \in K} \chi_{|x_r - z_r| > \tilde{T}_r} \right].$$

Now we apply the induction hypothesis to each function $\bar{a}_{x,t,K}$. It is enough to show that we can do so when K is of the form $[i+1, k]$, the general case being deduced by a reordering of the coordinate indices. Let $k \geqslant i+1$ be fixed and $K = [i+1, k]$. Then $\bar{a}_{x,t,K}$ satisfies the assumptions of Lemma 18 for k, with $S_1 \ldots S_i$ as before, $S_{i+1} = T_{i+1}(x_J, t_J, D_{x_I t_I l_I}) \ldots$ $S_k = T_k(x_J, t_J, D_{x_I t_I l_I})$. The set $F_{x_{I\cup K} t_{I\cup K}}$ is equal to

$$F_{x_I t_I} \cap \cup_{l_I} \left[\left[\prod_{s \in I} \{z_s, 2^{l_s} \leqslant |x_s - z_s| < 2^{l_s+1}\} \right] \times \tilde{D}^c_{x_I t_I l_I} \right].$$

Finally (10.5) with $I \cup K$ instead of I is a consequence of Lemma 5 applied to $D_{x_I t_I l_I}$ and more particularly of (10.3) which in this case says that

$$\int |D_{x_{I\cup K} t_{I\cup K} l_{I\cup K}}| \frac{t_k^{\epsilon-1}}{|x_K - z_K|} dz_K \, dt_K \, dx_K \leqslant C_{n,\epsilon} |D_{x_I t_I l_I}|,$$

$D_{x_{I\cup K} t_{I\cup K} l_{I\cup K}}$ being defined as in Lemma 18. The conclusion is that we can indeed apply the induction hypothesis to all the functions $\bar{a}_{x,t,K}$ for $K \subseteq [1, n] \backslash I$ and $K \neq \phi$ and therefore we obtain (10.6) for $a_{xt} - \tilde{a}_{xt}$. Thus Lemma 18 is proved, from which follows Lemma 17. Theorem 5 can now be proved by the same arguments as developed in [15] to deduce Theorem 2 from Lemma 15. We omit the details.

This proof shows the limits of the underlying philosophy of this paper, also implicitly contained in [11]: take a good class of operators, look at the tensor products of them, write all the quantitatives properties you can about those tensor products and look at the class of all operators that satisfy the same quantitative properties; then you can work on this new class. From what we just did, it seems that working with the class obtained by starting from vector-valued singular integral operators satisfying (1.11) is not so simple. Indeed to prove Theorem 5 we had to use the very special structure of the operator under consideration, in particular that the summation in Lemma 16 could be taken over all $(k, j) \in \mathbb{Z}^2$ independently of the function χ and that the operator

T_X is a «local tensor product», which corresponds to the fact that it could be written under the form $(\Psi_{k_I}^{j_I} \otimes T_{k_I j_I})_{k_I, j_I}$, with the $T_{k_I j_I}$ essentially of the same form that T_X.

Let us conclude with a remark along the same lines. Starting with a class of symbols $S^0_{\rho\delta}$ on $\mathbb{R} \times \mathbb{R}$, one can do the same «tensor product manipulation» and define a class of symbols $[S^0_{\rho\delta}]^n$ on $\mathbb{R}^n \times \mathbb{R}^n$ by the conditions

$$\left| \frac{\partial^{\alpha+\beta}}{\partial \xi^\alpha \, \partial x^\beta} \sigma(x, \xi) \right| \leqslant C_{\alpha,\beta} \prod_{1 \leq i \leq n} [1 + |\xi_i|]^{\delta\beta_i - \rho\alpha_i}.$$

Are the corresponding ψdO*'s* bounded when $0 \leqslant \delta < \rho \leqslant 1$ or when $0 \leqslant \delta = \rho < 1$, for instance on L^2 or on some L^p? A partial answer is the following: if $\rho = 1$ then the corresponding ψdO*'s* are CZO's in our sense. This can be seen by the same arguments as in [12]. Otherwise the problem seems entirely open.

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