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Special Positions for Surfaces Bounded by Closed Braids

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A «braid» is an algebraic datum —an element of a certain group. A «closed braid» is a geometric construct from that datum— a knot or link in a certain sort of special position in the 3-sphere. By a theorem of Alexander, every (tame, oriented) link can be moved into that special position. In this way the algebra of braids has been brought to bear on various aspects of the geometrical theory of knots and links.

Now, every link is the boundary of surfaces of various kinds (e.g., embedded surfaces in S^3; ribbon-immersed surfaces in S^3; surfaces embedded in more or less restricted ways in D^4), and these surfaces are of interest not only for what they tell about their boundaries but also in themselves. It is natural to ask whether surfaces bounded by a closed braid can themselves be put into any sort of special position, which might or might not be constructible from some kind of algebraic data. These notes are concerned with various such constructions.

Here is a rough outline of the paper. In §1, I define closed braids and recall from [Rudolph 1] the notion of a *braided surface* in $D^2 \times D^2$ bounded by a closed braid in $S^1 \times D^2$. A braided surface in $D^2 \times D^2$ is essentially the same thing as a *ribbon surface* in D^4, and §2 gives a fairly detailed account of ribbon surfaces in D^4 and their relationship to *ribbon-immersed surfaces* in S^3. In §3, I use various simple branched coverings (first of $\mathbb{C}$ by $\mathbb{C}$, given by a complex polynomial of degree n with $n - 1$ distinct critical values; then of $S^1 \times \mathbb{C}$ by $S^1 \times \mathbb{C}$; finally of S^3 by S^3, branched over a trivial link of unknot-

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ted circles) to establish «representation theorems» for braids and closed braids. (Sample: Proposition 3.10 shows that every closed braid on n strings is the inverse image in S^3 of a suitable unknot in S^3, *via* the covering alluded to above.) In §4, I continue this program, constructing *braided surfaces in* S^3which are ribbon-immersed (and correspond by «pushing into D^4» to the earlier braided surfaces), from *preband representations* of braids in the free *prebraid group*. The methods throughout are geometric, emphasizing «multivalued functions», although the surfaces constructed all have convenient algebraic descriptions.

In §5, I introduce the reader to the *Markov surfaces* which [Bennequin] has recently contributed to the study of closed braids, with signal success. An important subclass of the Markov surfaces (which includes all the incompressible ones), which I have named *Bennequin surfaces*, includes within it those braided surfaces in S^3 which are embedded (rather than simply ribbon-immersed). I show that, conversely, there is a formal sense in which the theory of Bennequin surfaces can be reduced to the theory of embedded braided surfaces in S^3.

In §6, I quote without proof Markov's Theorem and the important new *Inequality of Bennequin.* Using Markov's Theorem, I show that there is also a formal reduction of the theory of general (smooth, oriented —in short, *slice*) surfaces in D^4, bounded by a closed braid $\hat{\beta}$, $\beta \in B_n$, to the theory of band representations of the various usual injections $\beta^{(k)}$ of β into B_{n+k} (i.e., adding k extra trivial strings). I end with a discussion of the possibility that Bennequin's Inequality, which he has proved for the standard (Seifert) genus of a closed braid, might hold also for the slice (or Murasugi) genus.

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CONTENTS

§1. Preliminaries; closed braids, braided surfaces, and band representations

Throughout, we will view the real plane $\mathbb{R}^2$ in its guise as the complex line $\mathbb{C}$. In particular, complex structures (plus the «outward normal» convention) orient such spaces as $D^2, D^4, \dots$.

1.1. *Notation.* The *round* 4-*ball* is $D^4 = \{(z, w) \in \mathbb{C}^2: N(z, w) \leqslant 2\}$, where $N(z, w) = |z|^2 + |w|^2$ is the *squared-norm* function. The *round* 3-*sphere* is $S^3 = \partial D^4$. (Note that D^4 and S^3 have radius $\sqrt{2}$. We will drop the adjective «round» most of the time.) We will write rD^2 for $\{z \in \mathbb{C}: |z| \leqslant r\}$, rS^1 for $\partial(rD^2)$, and drop the r if it equals 1. A *bidisk* is a product $r_1D^2 \times r_2D^2 \subset \mathbb{C}^2$ $(r_1, r_2 > 0)$. *We will always take* $r_1 = 1$, and write D_r for $D^2 \times rD^2$. The *unit bidisk* is $D = D_1$.

The boundary of a bidisk is a 3-sphere with corners. Let there be fixed, once for all, a smoothing homeomorphism $h: D \to D^4$ such that $h|S^1 \times S^1$ is the identity on $S^1 \times S^1 = S^3 \cap \partial D$, and $h|(D - S^1 \times S^1)$ is a diffeomorphism between the smooth manifolds-with-boundary $D - S^1 \times S^1$ and $D^4 - S^1 \times S^1$. We will write $\partial_1 D_r = S^1 \times rD^2$ and $\partial_2 D_r = D^2 \times rS^1$ for the two solid tori into which $S^1 \times rS^1$ splits ∂D_r. The smoothing h gives fixed product structures to the two solid tori $h(\partial_1 D)$ and $h(\partial_2 D)$ in S^3. We will use H, composed with a homothety of w, to smooth any bidisk boundary ∂D_r.

Fix an integer $n \geqslant 1$. The *n-fold symmetric product* of $\mathbb{C}$ is the quotient $\mathbb{C}^n/S_n$, where the symmetric group S_n of all permutations of $\{1, \dots, n\}$ acts on the space $\mathbb{C}^n$ of ordered n-tuples of points of $\mathbb{C}$ (i.e., maps $\{1, \dots, n\} \to \mathbb{C}$) naturally (by «permuting coordinates»). Let $E_n \subset \mathbb{C}[T]$ denote the complex affine space of dimension n consisting of the monic polynomials of degree n. By the Fundamental Theorem of Algebra, the map from $\mathbb{C}^n/S_n$ to E_n induced by $(z_1, \dots, z_n) \to (T - z_1) \dots (T - z_n)$ is a bijection (in fact, it is a homeomorphism if the symmetric product is given the quotient topology). *Henceforth we identify* $\mathbb{C}^n/S_n$ *and* E_n *by this map.* In particular, we give the symmetric product a smooth structure. We think of elements of $\mathbb{C}^n/S_n = E_n$ indifferently as unordered n-tuples of complex numbers (counting multiplicities), n-element subsets of $\mathbb{C}$ (counting multiplicities), or monic polynomials of degree n, as suits our convenience. The subset Δ_n, or simply Δ, of n-tuples with some multiplicity at least 2 (alternatively, monic polynomials with some repeated root) is called the *discriminant locus*; its complement $E_n - \Delta$ is the *configuration space* (*of n distinct points of* $\mathbb{C}$).

1.2 *Facts.* The discriminant locus is a complex algebraic hypersurface in the affine space E_n. It is irreducible (being the image of the hyperplane $\{z_1 = z_2\}$ in $\mathbb{C}^n$) but for $n \geqslant 3$ it is singular (along its subset of n-element multisubsets of $\mathbb{C}$ supported by $n - 2$ or fewer distinct points). The isomorphism class of the fundamental group of the configuration space does not depend on the choice of basepoint (because Δ is of real codimension 2 and sufficiently well-behaved), and for any choice of basepoint this group is the normal closure of a single element (namely, the boundary of a 2-disk which meets Δ transversely at precisely one point of its dense, open, connected subset of regular points). There are exactly two (mutually inverse) conjugacy classes of such elements.

1.3 Definitions. The *n-string braid group* B_n (*based at* $* \in E_n - \Delta$) is the fundamental group $\pi_1(E_n - \Delta; *)$ of the configuration space. A *band* in B_n is an element represented by a loop bounding a disk in E_n which meets Δ transversely in a single regular point; a band is *positive* or *negative* according to the sign of its linking number with Δ (which, at least along its regular points, has a natural orientation because it is a complex algebraic set). Bands will be one of our principal technical tools in what follows: the name will be justified then.

1.4 *Conventions.* E_0 is a one-point set $\{0\}$, Δ_0 is empty, and B_0 is the one-element group with identity denoted o. The identity of B_n, $n \geqslant 1$, will be denoted $o^{(n)}$. Note that $B_1 = \{o^{(1)}\}$ is isomorphic to B_0 but not identical to it; more generally, each pair of groups B_n, $B_m (n \neq m)$ is *disjoint* (they are, after all, groups of homotopy classes of paths in disjoint spaces). This apparent pedantry will, I believe, be seen to pay off later.

1.5 Definitions. An oriented closed 1-manifold L (briefly, a *link*) embedded in a bidisk boundary ∂D_r is a *closed braid on n strings* if $L \subset \partial_1 D_r$ and $pr_1|L: L \to S^1$ is an orientation-preserving covering map of degree n. Closed braids in S^3 are defined *via* the smoothing h. A compact oriented 2-manifold-with-boundary S smoothly embedded in D_r is a *braided surface of degree n* if $pr_1|S: (S, \partial S) \to (D^2, S^1)$ is an orientation-preserving branched cover of degree n. Braided surfaces in D^4 are defined *via h*. (Neither «cover» nor «branched cover» is intended to imply that the total space is connected.) Note that the boundary of a braided surface is a closed braid.

If X is any set, an *n-valued complex function* on X is a function $f: X \to E_n$. The *graph* of an n-valued function f is $grf = \{(x, z): x \in X, z \in f(x)\} \subset X \times \mathbb{C}$.

1.6 Proposition. *A closed braid L on n strings is the graph of a* (*continuous*) *n-valued function on* S^1 *with values in the configuration space* $E_n - \Delta \subset E_n$. *A braided surface S of degree n is the graph of a* (smooth) *n-valued function on* D^2.

PROOF. Clear (the smoothness of the function associated to S comes from the implicit function theorem and the Fundamental Theorem of Algebra). □

Conversely, if f is any continuous function from S^1 to the configuration space, its n values are uniformly bounded in size by some constant, so its graph is a closed braid in some bidisk boundary ∂D_r.

1.7 *Notation.* If $\beta \in B_n$ then any closed braid on n strings which is the graph of a (based) loop in the homotopy class β will be called a *closure* of β, and denoted $\hat{\beta}$.

1.8 Proposition. *The set of closed braids on n strings, modulo the equivalence relation of isotopy through closed braids, is naturally in bijection with the set of conjugacy classes in* B_n.

PROOF. Both sets are naturally in bijection with the set of free homotopy classes of loops in the configuration space. □

The situation is less simple for braided surfaces, however. If g is an arbitrary continuous (even smooth) function from D^2 to E_n, then its graph certainly lies in all sufficiently large bidisks, but it need not be a braided surface—for indeed it need not be even topologically embedded.

1.9 *Example.* The function $D^2 \to E_2: z \to T^2 - z^2$ is smooth (in fact, complex analytic); its graph is the union of two copies of D^2 with the origins identified. Note that this function is not transverse to Δ_2.

The function $D^2 \to E_3: x + iy \to T^3 - 3(x^2 + y^2)T + x(1 + iy)$ is smooth, though not transverse to Δ_3. Its graph is not smooth at (0, 0) though it is p.l. locally flat there.

The function $D^2 \to E_3: z \to T^3 - z^2$ is smooth, not transverse to Δ_3, and has a graph which is a topologically embedded disk which is not p.l. locally flat at one point.

The function $D^2 \to E_3: z \to T^3 - z$ is smooth and has a smooth graph even though it is not transverse to Δ_3.

To say precisely which functions into E_n have smooth graphs is a non-trivial problem. (It would involve having an explicit understanding of which closed braids are of the isotopy type, in the 3-sphere, of some completely split link of unknots.) However, for our purposes the generic situation suffices.

1.10 Definition. A braided surface S is *simple* if the branched covering $pr_1|S$ is simple (that is, the critical points are locally like either $z \to z^2$ or $z \to \bar{z}^2$ near $0 \in \mathbb{C}$, and the critical values are distinct).

The following is clear.

1.11 Proposition. *A simple braided surface S of degree n in D_r is the graph of a smooth n-valued function on D^2, transverse to Δ_n, which has all its values bounded in absolute value by r; and conversely. Every braided surface of degree n can be arbitrarily closely approximated by simple braided surfaces of degree n.* □

1.12 Definitions. A *band representation of length $l \geqslant 0$ in B_n* is an ordered l-tuple $\underline{\underline{b}} = (b(1), \ldots, b(l))$ where each $b(j)$ is a band, positive or negative, in B_n (Def. 1.3.). We write $l(\underline{\underline{b}})$ for l. The *braid of* $\underline{\underline{b}}$ is $\beta(\underline{\underline{b}}) = b(1)b(2)\ldots$ $\ldots b(l) \in B_n$; the *closed braid of* $\underline{\underline{b}}$ is $\hat{\beta}(\underline{\underline{b}})$, the closure of $\beta(\underline{\underline{b}})$. (If $l = 0$ then $\underline{\underline{b}}$ is the empty tuple, with braid $o^{(n)}$ and closed braid $\hat{o}^{(n)}$, the simplest closed braid representing the completely split link of n unknots.)

1.13 Proposition. *To each band representation $\underline{\underline{b}}$ of length l in B_n can be associated a simple braided surface $S(\underline{\underline{b}})$ of degree n with l branch points of $pr_1|S(\underline{\underline{b}})$ and $\partial S(\underline{\underline{b}}) = \hat{\beta}(\underline{\underline{b}})$. Up to isotopy though simple braided surfaces (covering an isotopy of D^2) every simple braided surface is some such $S(\underline{\underline{b}})$. The various band representations $\underline{\underline{b}}$, such that a given simple braided surface can be so isotoped to $S(\underline{\underline{b}})$, are all related to each other in a reasonable way.*

Outline of proof (for more details, consult [Rudolph 1]): The ordered composition of loops in the configuration space representing the bands $b(1), \ldots, b(l)$ is a loop which extends to a map of D^2 into E_n which meets Δ_n transversely in l points, each corresponding to one of the bands in the composition. A small perturbation of this map is smooth everywhere and its graph is a surface $S(\underline{\underline{b}})$ with the desired properties.

Conversely, given a smooth map $g: D^2 \to E_n$ transverse to Δ_n in l points, any system of arcs in D^2 joining the points of $g^{-1}(\Delta_n)$ to the basepoint of S^1, and disjoint except for that common basepoint, provides one with a band representation $\underline{\underline{b}}$ of the homotopy class of $g|S^1$, of length l, with gr g a particular $S(\underline{\underline{b}})$. Two different such systems of arcs differ by an autohomeomorphism of D^2 which fixes S^1 pointwise and $g^{-1}(\Delta_n)$ setwise. The group of such autohomeomorphisms (which is, as a matter of fact, isomorphic to B_l) acts on the set of band representations of length l; this group is generated by *slides* (or, in [Moishezon]'s language, *elementary transformations*) $(b(1), \ldots, b(i), b(i+1), \ldots, b(l)) \to (b(1), \ldots, b(i)b(i+1)b(i)^{-1}, b(i), \ldots, b(l))$. (Note that the conjugate of a band is of course again a band.) □

1.14 *Remarks.* The boundary of a braided surface, as we have defined it, is a smooth closed braid. We did not require that a closed braid be smooth. But no generality would be lost if we did: for pr_1 induces a normal bundle for any closed braid L, and L is therefore tame, and can be isotoped (even though closed braids) to be smooth.

Or course, every smooth closed braid bounds simple braided surfaces. They are not unique, for indeed, one may always increase the number of branch points by two. On the level of band representations, this corresponds to replacing $(b(1), \dots, b(l))$ by the *elementary expansion* $(b(1), \dots, b(l), w, w^{-1})$, for any band w. It is shown in [Rudolph 1] that any two band representations of a given braid in B_n may be joined by a sequence of elementary expansions, slides, and *elementary contractions* (inverses, when possible, of elementary expansions).

We conclude this section with a digression—a proof in the language of multivalued functions of a well-known and interesting fact.

1.15 *Scholium. The configuration space $E_n - \Delta$ is an Eilenberg-MacLane space (that is, its higher homotopy groups $\pi_k(E_n - \Delta)$, $k \geqslant 2$, all vanish).*

PROOF BY INDUCTION ON n. Clearly $E_1 - \Delta = E_1 = \mathbb{C}$ is contractible. Let n be greater than 1, and let $f: S^k \to E_n - \Delta$ be a continuous map of a k-sphere, $k > 1$, into the configuration space. We will show that f is freely homotopic to a constant map, which will prove the theorem. Consider $\operatorname{gr} f$ in $S^k \times \mathbb{C}$. This is a covering space of S^k; because S^k is simply connected, it is a trivial covering space, i.e., $\operatorname{gr} f$ is the union of n disjoint graphs of 1-valued functions $f_1, \dots, f_n$. Clearly f is homotopic to f' in $E_n - \Delta$, where $\operatorname{gr} f'$ is the union of the graphs of the n functions $f_1 - f_n, f_2 - f_n, \dots, 0$, and for $i = 1, \dots, n-1$, the function $f_i - f_n$ is nowhere zero on S^k. Then each $f_i - f_n$ lifts to a function g_i on S^k with $\exp g_i = f_i - f_n$; since the graphs of the $f_i - f_n$ are pairwise disjoint, so are the graphs of $g_1, \dots, g_{n-1}$, and thus their union is the graph of a continuous $(n-1)$-valued function g on S^k. A homotopy of g to a constant gives a homotopy of f', and thus of f, to a constant. □

§2. Ribbon surfaces and braided surfaces

Recall that N is the squared-norm function from D^4 to $[0, 2]$.

2.1 Definitions. A smooth function ω from a compact 2-manifold-with-boundary S to $[0, 2]$ is *topless* if $\omega^{-1}(2) = \partial S$, ω has no critical points in a collar of ∂S, and no critical point of ω (in $\operatorname{Int} S$) is a local maximum of ω. (Note that no non-degeneracy assumptions are put on the critical points of ω; but if ω is, say, real-analytic—and presumably in general—then suitable arbitrarily small perturbations of ω are both topless and Morse. In particular, if S supports a topless function then S is itself *topless* in the sense that it has a handle decomposition without 2-handles; alternatively, S has no closed components.) A surface S embedded smoothly and *properly* (i.e., $\partial S = S \cap S^3$) in D^4 is *ribbon-embedded* if $N|S$ is topless, and S is *ribbon* if it is ambient isotopic to a ribbon-embedded surface.

We define ribbon-embedded and ribbon surfaces in D *via* the smoothing homeomorphism h.

2.2 *Remarks.* Ribbon-embedded surfaces arise in nature. One class of examples comes from complex analytic geometry. Let U be an open set in $\mathbb{C}^2$ containing D^4, and let $\Gamma \subset H$ be a non-singular complex analytic curve (i.e., locally in U, Γ is the zero-set of a complex-analytic function with non-zero gradient), so that Γ intersects S^3 transversely. Then the surface $S = \Gamma \cap D^4$ is ribbon-embedded. (This may be proved directly by using a local parametrization of S. [More generally, the composition of M with the resolution of a singular piece of complex curve will be topless too.] Or one may appeal to the much more general theorem in [Milnor] on Stein manifolds.)

2.3 *Example.* Let $U = \mathbb{C}^2$, and let Γ be defined by $4zw = 1$. Then S is an annulus naturally parametrized by $A = \{\zeta \in \mathbb{C}: 4 - \sqrt{15} \leqslant |\zeta|^2 \leqslant 4 + \sqrt{15}\}$ under the map $f\colon \zeta \to (\zeta/2, 1/2\zeta)$. Considering $N \circ f$, we see that the critical points of $N|S$ are all degenerate—they form a circle of local minima. Nonetheless, with our definition the surface is ribbon-embedded as it should be. Of course, an arbitrarily small linear perturbation of S will replace it with an equivalently embedded ribbon-embedded annulus on which N has a single minimum and a single saddlepoint, both non-degenerate. (This is a special case of an observation of Nomizu and Cecil.)

2.4 *Remarks* (*continued*). Another class of naturally occuring ribbon-embedded surfaces (which in fact includes the complex curves) consists of smooth minimal surfaces in D^4, in the sense of differential geometry. In fact, [Hass] proves a converse: every isotopy class of ribbon surfaces in D^4 contains minimal surfaces.

(It is important, by the way, to understand that Hass's result concerns the *round* ball—of course, the value of the radius is irrelevant—with the *flat* metric induced from $\mathbb{C}^2$. Presumably such other nice metrics as those of constant, non-zero curvature could also be used. But it is easy to find, cf. [Rudolph 4], for any smooth topless orientable S in D^4, a smooth embedding i of D^4 in $\mathbb{C}^2$ which carries S onto a surface in the non-round ball $i(D^4)$

minimal with respect to its flat metric; or, alternatively, which pulls back that flat metric to a non-flat metric on the round ball, in which S is minimal. Yet, as we shall recall shortly, there are surfaces in D^4 isotopic to no ribbon surface.)

2.5 *Question.* Of course there are non-orientable ribbon surfaces, while every complex curve is naturally oriented. But: *does every isotopy class of orientable (oriented?) ribbon surfaces in D^4 contain a piece of complex curve?* (The answer is presumably «no» but I know of no proof.)

2.6 *Remarks (concluded).* Not every smooth, properly embedded, topless surface in D^4 is ribbon. For relative Morse theory shows that if S is a ribbon with tubular neighborhood u in D^4, then the exterior $D^4 - u$ of S can be built from a collar of the exterior $S^3 - u$ of ∂S in S^3 by attaching handles of index 2, 3, and 4 only. In particular the inclusion-induced homomorphism $\pi_1(S^3 - \partial S) \to \pi_1(D^4 - S)$ is onto. Yet there are, for instance, many smooth 2-disks in D^4 bounded by an unknot and having a larger group than $\mathbb{Z}$ as fundamental group of the complement. Such disks are not ribbon disks.

2.7 Proposition. *A braided surface is a ribbon surface.*

PROOF. Comparing $M \circ h|S$ with $|pr_1|S|^2$, we see that—perhaps after an initial vertical isotopy to make $pr_2|S$ uniformly very nearly zero—the evident toplessness of the latter imposes toplessness on the former. □

If S is a simple braided surface of degree n with l branch points of $pr_1|S$, the proof shows S is isotopic to a ribbon-embedded suface on which $N \circ h$ is Morse with n local minima and l saddles.

The following theorem is proved in [Rudolph 1]; a variant on the proof (which needs only minor modifications, along the lines of the first proof, to cover the general case) is presented in [Rudolph 2].

2.8 Theorem. *Every oriented ribbon surface is ambient-isotopic to a braided surface.* □

2.9 *Remark.* The isotopy constructed in the cited proof(s) is generally «large», and cannot be expected to be «conservative»—that is, the isotopy cannot usually be taken to be relative to a part of S on which pr_1 already happens to be a branched cover of its image. For instance, [Morton] gives an example of a 4-string closed braid L which is unknotted in ∂D and thus certainly bounds ribbon disks in D; his proof that L is «irreducible» (in a certain sense) actually shows more, namely, that any braided surface bounded by L has genus at least 1 (read the proof in conjunction with Example 5.2 of [Rudolph 1]), so an isotopy of a ribbon disk bounded by L into braided position must move L quite far (across $\partial_2 D$, in fact).

Although we will give no proof of Theorem 2.8, it may be remarked that the proof is essentially 3-, rather than 4-, dimensional, and makes heavy use of the notion of ribbon-immersed surfaces in S^3, which we now introduce for other purposes.

2.10 Definition. Let S be a topless (not necessarily oriented, or orientable), surface. A mapping $f\colon S \to S^3$ is a *ribbon immersion* if it has the following properties:

1. f is a smooth immersion without triple points;
2. in the domain S of f, the double points consist of $2r$ pairwise disjoint closed arcs $A'_1, A''_1, \dots, A'_r, A''_r$ with $f(A'_k) = f(A''_k)$, $k = 1, \dots, r$, such that each A'_k is contained in $\operatorname{Int} S$ *and each* A''_k has both endpoints (and no other points) on ∂S; and
3. along the r arcs $A_k = f(A'_k)$ of double points of f in the range, the two sheets of $f(S)$ cross transversely.

Of course the arcs A_k may be quite twisted, but there is an ambient isotopy of S^3 carrying them onto short «straight» (e.g., geodesic) arcs, and after such an isotopy a ribbon immersion looks, locally in domain and range, like figure 1.

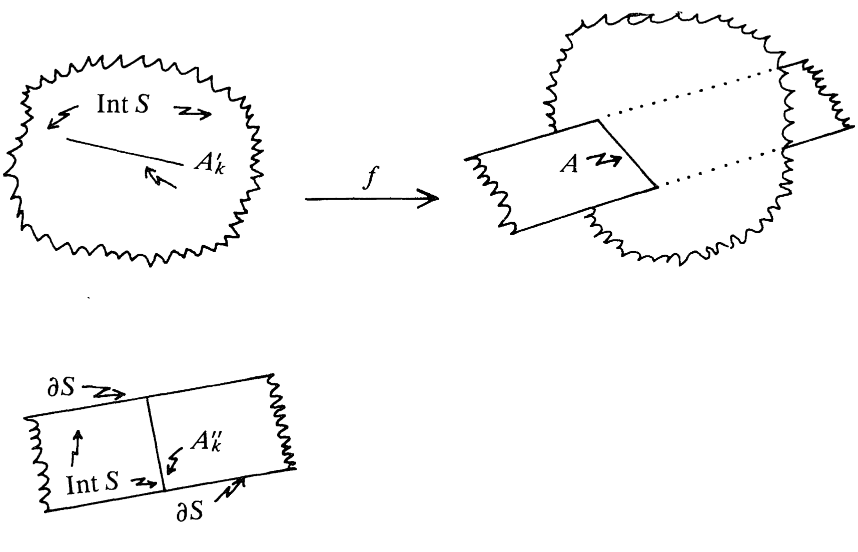

Fig. 1

2.11 Definition. Let f, S, $A'_1, \ldots, A''_r$ be as above. Then a topless Morse function ω on S is *adapted* to f if is strictly positive and for each $j = 1, \ldots, r$, $\omega(A'_j) \cap \omega(A''_j) = \varnothing$. A topless handle decomposition $S = h^0_1 \cup \ldots \cup h^0_s \cup h^1_1 \cup \ldots \ldots \cup h^1_t$ is *adapted* to f if each A'_i is interior to some h^0_j and each A''_i is proper in some h^1_k.

It is easy to see that, given f and S, there do indeed exist both adapted topless Morse functions and adapted handle decompositions, and indeed that the many-many correspondence of functions and decompositions preserves adaptation to f.

2.12 *Construction.* Let f be a ribbon immersion of S in S^3. Then for any topless Morse function ω adapted to f, the map $\frac{1}{2}\omega f$: $S \to D^4$: $x \to \frac{1}{2}\omega(x)f(x)$ is a ribbon embedding of S in D^4. (The factor $\frac{1}{2}$ is due to our convention that D^4 has radius $\sqrt{2}$.) We will call it *the push-in of f by the factor $\omega/2$.*

PROOF. The push-in is an embedding because the only possible double points in the range are separated by the radial coordinate, by definition of adaptation to f. It is ribbon because on the image the functions ω and N are essentially the same. □

2.13 Proposition. *Each ambient-isotopy class of ribbon surfaces in D^4 contains a ribbon-embedded surface which is the push-in of a ribbon immersion in S^3 by an appropriate factor. Different push-ins of the same ribbon immersion are ambient isotopic.*

Idea of proof (see, e.g., [Tristram] for more details, in a different language): Start with a ribbon-embedded surface in the given isotopy class, not containing (0, 0). By isotopy, this surface may be assumed to have all its local minima for the restriction of N in the interval $]0, 1[$ and all its saddle values in $]1, 2[$. At this stage, an application of relative Morse theory yields a topless handle decomposition of the surface, and a ribbon immersion to which that decomposition is adapted, such that the push-in of that immersion by the appropriate adapted factor (half the square root of the restriction of N to the surface) is isotopic to the surface by an isotopy leaving N invariant. The second statement is proved similarly. □

2.14 *Remark.* Given f and S, an adapted Morse function on S may well need to have more critical points than the minimal number for a topless, but not adapted, Morse function. On the other hand, an obvious construction of adapted handle decomposition —which makes each A''_i a transverse arc of a different 1-handle, and engulfs each A'_i by a different 0-handle, and uses as many more handles as necessary to get adaptation— is likely to use far too many handles.

Figure 2 depicts a ribbon-immersed disk (bounded by the square knot), together with immersions onto it from (*a*) a disk with an adapted handle decomposition displayed, (*b*) a disk with some level sets of an adapted Morse function drawn in.

In §4, we will see how to go from a band representation $\underline{\underline{b}}$ in B_n (more correctly, from a «preband» representation that maps onto $\underline{\underline{b}}$) directly to a ribbon immersion of (the abstract surface) $S(\underline{\underline{b}})$ in S^3, in such a way that a natural push-in of this immersion recovers $S(\underline{\underline{b}})$ as braided surface. (See also 5.22.)

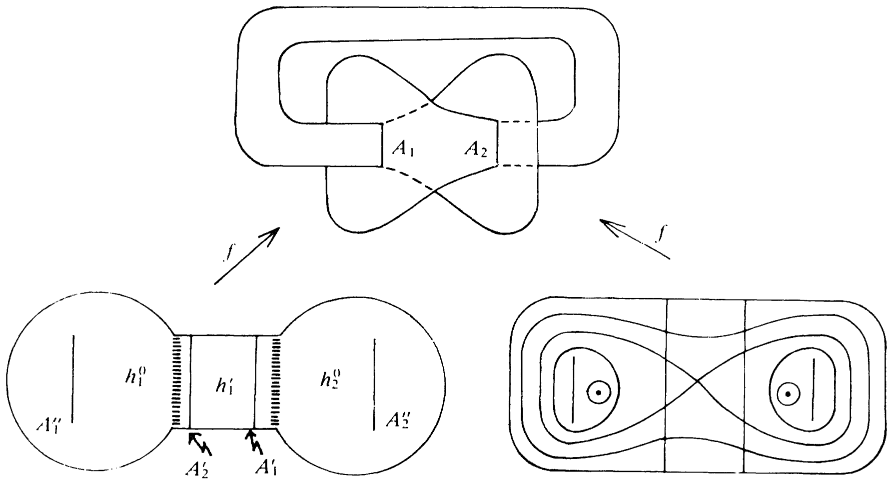

Fig. 2

§3. Prebraids and «standard generators» of the braid group

Let $V = \{v_1, \ldots, v_{n-1}\}$ be a set of $n-1$ (pairwise distinct) complex numbers, v_0 a basepoint in $\mathbb{C} - V$. Let F_{n-1} denote the group $\pi_1(\mathbb{C} - V; v_0) = \pi_1((\mathbb{C} \cup \cup \{\infty\}) - (V \cup \{\infty\}); v_0)$. Of course F_{n-1} is a free group of rank $n-1$. More specifically: there are $n-1$ arcs I'_j in $\mathbb{C}$, such that I'_j has endpoints v_0 and v_j, $I'_j \cap I'_k = \{v_0\}$ if $j \neq k$, and the counterclockwise cyclic order of the I'_j at their common endpoint v_0 is $I'_1, \ldots, I'_{n-2}, I'_{n-1}, I'_1$; and there are closed 2-cells N'_j, with $N'_j \cap N'_k = \{v_0\}$ if $j \neq k$, and I'_j embedded in N'_j like a radius in D^2. Then the elements $x_1, \ldots, x_{n-1}$ of F_{n-1}, where x_j is represented by the boundary of N_j' traversed once counterclockwise, are free generators of F_{n-1}, which we call *the standard generators of* F_{n-1} *with respect to the star* $\bigcup_{j=1}^{n-1} I_j'$. (The generators don't depend on the N_j', only on the star; in fact, only on the star up to ambient isotopy fixing $V \cup \{v_0\}$ at all times. Differently embedded stars, however, do give different sets of standard generators—which are of course related by easily understood moves. We will not be concerned with this.)

Represent F_{n-1} on S_n (the permutations of $\{1, \ldots, n\}$) by sending x_j to the transposition $(j\,j+1)$, for each j. In the usual way, the regular covering space of $(\mathbb{C} \cup \{\infty\}) - (V \cup \{\infty\})$ which corresponds to this representation has a unique completion to a branched covering space $p: X \to \mathbb{C} \cup \{\infty\}$ branched over $V \cup \{\infty\}$. The map p has *simple* critical values at the points of V (two sheets come together) and complete branching (all sheets coming together in an n-cycle) over ∞, and X is a compact, connected surface.

By calculating Euler characteristics, one finds that X is homeomorphic to a sphere. The map p induces on X a unique complex structure for which p is complex analytic, so with that structure X must be biholomorphic to the Riemann sphere $\mathbb{C} \cup \{\infty\}$ itself. But now p must be a rational map. As only one point of X maps to ∞, there are coordinates for X in which p is a polynomial. Now we drop the notation X and forget about the points at infinity. We have proved the following.

3.1 Proposition. *With V as above, there is a polynomial $p: \mathbb{C} \to \mathbb{C}$ of degree n with simple critical points, and critical values V, so that p realizes the branched covering of $\mathbb{C}$ which corresponds to the given representation of F_{n-1}. (We may take p to be monic.)* □

3.2 *Remark.* A critical point of a polynomial is one where the derivative vanishes; it is simple iff the second derivative is non-zero there. Of course it is a generic property of polynomials to have all critical points simple and all critical values distinct.

3.3 *Example.* Let $n = 3$, $V = \{2, -2\}$, $I_1' = [0, 2]$, $I_2' = [-2, 0]$. Then we can take $p(z) = z^3 - 3z$. The critical points of p are 1 and -1, with corresponding critical values -2 and 2 as desired.

Consider $p^{-1}(N_j')$ in the domain of p. Since N_j' contains a single, simple critical value of p, this inverse image has $n - 1$ components. Let N_j denote that one which contains a critical point of p, so $p|N_j: N_j \to N_j'$ is a 2-sheeted cyclic branched cover of a disk by a disk. Let $I_j = N_j \cap p^{-1}(I_j')$, so I_j is embedded in N_j like a diameter in D^2. Evidently the endpoints of I_j are points of $p^{-1}(v_0)$. In fact, we can number the points of $p^{-1}(v_0)$ as $z_1, \ldots, z_n$ in such a way that I_j has endpoints z_j and z_{j+1}, $j = 1, \ldots, n-1$. The union $I = \bigcup_{j=1}^{n-1} I_j$ is an arc (which p folds onto the star used to define the standard generators x_j of F_{n-1}) with endpoints z_1 and z_n, containing all the z_j in order.

3.4 Definition. Let $J \subset \mathbb{C}$ be an arc with endpoints w_1 and w_n containing, in linear order, n distinct points $w_1, \ldots, w_n$. Let Q_j be a closed 2-cell in $\mathbb{C}$ which intersects J along J_j, its subarc with endpoints w_j and w_{j+1}, in such a way that J_j is embedded in Q_j like a diameter and $Q_j \cap Q_k$ contains either one point or none, depending on whether $|j-k| = 1$ or $|j-k| > 1$. Realize the n-string braid group B_n as $\pi_1(E_n - \Delta; \{w_1, \ldots, w_n\})$. Then for $j = 1, \ldots, n-1$, the *standard generator* σ_j *of* B_n (*with respect to the given basepoint and given arc* J) is the homotopy class of the loop l_j: $(S^1, 1) \to (E_n - \Delta, \{w_1, \ldots, w_n\})$: $\exp i\theta \to \{s(\theta), t(\theta), w_1, \ldots, \hat{w}_j, \hat{w}_{j+1}, \ldots, w_n\}$, where $s(\theta), t(\theta)$ are the preimages of $\exp i\theta$ by a fixed (for instance, by using arc length if ∂Q_j is rectifiable) double cover of S^1 by ∂Q_j such that $1 \in S^1$ is covered by $\{w_j, w_{j+1}\}$ and the cover respects orientations.

3.5 *Remarks.* The notation σ_j makes no reference to n. Since (cf. 1.4) the groups B_n are disjoint, this—though hallowed by use, justified by algebra, and undoubtedly convenient—is geometrically unfortunate. ...Nor does the notation indicate the arc J. Clearly, choices of J which differ by isotopies fixing each w_j at all times give the same «standard generators» (nor do the choices of Q_j, etc., matter); but differently embedded arcs give different sets of generators, which, however, differ by understandable moves. Abstractly, of course, all such sets of generators are identical, in that they differ by automorphisms of B_n (induced by homeomorphisms of $\mathbb{C}$ fixing each w_j).

3.6 Proposition. *Each* σ_j *is a positive band in* B_n. *The set* $\sigma_1, \ldots, \sigma_{n-1}$ *of standard generators of* B_n *is, indeed, a set of generators of* B_n.

PROOF. The first phrase follows from the observation that the double cover of S^1 by ∂Q_j used to define σ_j extends to a 2-sheeted cyclic branched cover of D^2 by Q_j, and that the 2-valued inverse to this, extended to be n-valued by the $n-2$ constants w_i $(i \neq j, j+1)$, is a map of D^2 into E_n which—with a minimal amount of care—is transverse to Δ which it meets in one point, positive by orientation arguments.

The rest of the proposition is due to [Artin]. A proof may be given along these lines: take $w_j = j$, $J = [1, n]$, Q_j the round 2-disk of radius 1/2 centered at $j + \frac{1}{2}$. With the right choice of double cover $Q_j \to S^1$, the real parts of the n values of the loop l_j look as drawn in Figure 3A (where S^1 has been cut open into $[0, 2\pi]$). By isotopies (through closed braids, respecting a basepoint) any closed braid can be first put into general position with respect to projection of its n values onto their real parts, then moved until it is a composition of (appropriately rescaled) pictures like that in Figure 3A; cf. Figures 3B, 3C. □

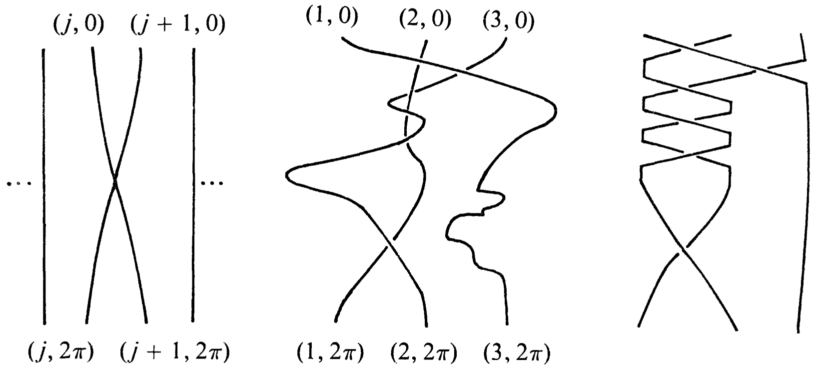

Fig. 3

3.7 *Remarks.* The standard generators satisfy some geometrically obvious relations, namely, for $i = 1, \ldots, n-2$, R_i: $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, and for $1 \leqslant i < j-1 \leqslant n-1$, R_{ij}: $\sigma_i\sigma_j = \sigma_j\sigma_i$. We will not scruple to use these. It is somewhat less obvious that all relations in B_n are consequences of these; since we will not use this fact, we refer the reader to various published proofs, in [Birman] and sources cited there (beginning with [Artin]).

(In the light of what we will do next, an alternative proof of 3.6. and 3.7. can presumably be given by carefully following through the details of a Lefschetz-style analysis of the fundamental group of an algebraic surface and its plane sections.)

We return to the polynomial map p: $\mathbb{C} \to \mathbb{C}$ of degree n.

3.8 Proposition. *The n-valued map* p^{-1}: $\mathbb{C} \to E_n$ *maps* $\mathbb{C} - V$ *into the configuration space* $E_n - \Delta$. *The induced homomorphism on* π_1, $(p^{-1})_*$: $F_{n-1} \to B_n$ (*where we use* $p^{-1}(v_0)$ *as basepoint of* B_n), *is given by* $x_j \to \sigma_j$, $j = 1, \ldots, n-1$.

PROOF. A choice of arc J is implicit in the notation $\sigma_1, \ldots, \sigma_{n-1}$. Take $J = I$ (as defined before 3.4.), $Q_j = N_j$. The composition $(p^{-1}) \circ m_j$, where m_j is a loop representing x_j, isn't quite a loop l_j of the sort required in 3.4., but it differs only by inessential activity in the $n-2$ 2-cells other than N_j which lie in $p^{-1}(N_j')$.

(The parenthesis at the end of 3.7. is based on the observation that p^{-1} is a *linear* map into E_n. Indeed, writing $w = p(z)$, we have $p^{-1}(w) = \{z \in \mathbb{C}: p(z) = = w\} = \{z: p(z) - w = 0\}$, so p^{-1} linearly parametrizes a line on which only the constant term of the monic polynomial changes. If this line is in general position with respect to Δ—and it is—then Lefschetz tells us that F_{n-1} maps onto B_n. Further, [Zariski] and [van Kampen] tell us that moving the line around appropriate loops of lines gives all relations.)

3.9 Definitions. With V, p, etc., as above, the *prebraid group* is F_{n-1}. A *closed prebraid* is the graph in $S^1 \times \mathbb{C} - V$ of a loop $f: S^1 \to \mathbb{C} - V$. (Note that a closed prebraid, as a subset of an appropriately large $S^1 \times rD^2$, is a 1-string closed braid in ∂D_r and in particular an unknot.) A *criticized closed prebraid* is the union of a closed prebraid and $S^1 \times V$; appropriately oriented, a criticized closed prebraid is a closed n-string braid of a very special sort, in sufficiently big bidisk boundaries.

3.10 Proposition. *The set in $S^1 \times \mathbb{C}$, which is the inverse image by $id_{S^1} \times p$ of a closed prebraid, is a closed n-string braid (in every sufficiently big bidisk boundary). Every isotopy class of closed braids on n strings contains such covers of prebraids.*

PROOF. The first statement is evident. The second is the geometric counterpart of surjectivity of $(p^{-1})_*$ (3.6., 3.8.). □

In §§4-5 we shall see how the use of prebraids can simplify various constructions of surfaces bounded by closed braids.

3.11 *Remark.* All the material in this section can be somewhat generalized, as follows. Instead of representing F_{n-1} in S_n by $x_j \to (j\,j+1)$, take some other representation in which each x_j goes to a transposition and the product $x_1 \dots x_{n-1}$ goes to the same n-cycle $(1 n\, n-1 \dots 3\, 2)$ as before. Then, again, the corresponding simple covering can be taken to be a polynomial p of degree n with simple critical points, and critical values $v_1, \dots, v_{n-1}$; now, however, the interesting part of the preimage of the star $\bigcup_{j=1}^{n-1} I_j$ is a tree T which is not necesarily an arc. (It is combinatorially equivalent to the tree on vertices $1, \dots, n$ with an edge for each transposition which is the image of an x_j.) As before, T can be thickened into a «cactus» on which p is 2:1 onto a neighborhood of the original star; and one can read off from the tree certain generators (represented by motions of points inside the cactus) of B_n which might be called T-standard and which are the images by $(p^{-1})_*$ of the x_j. For $n \geqslant 4$, however, these generators do *not* have to be equivalent by automorphism to the standard generators.

3.12 *Example.* If T is a triod Y, we can express a set of Y-standard generators of B_4 in terms of the standard standard generators as σ_1, $\sigma_2\sigma_3\sigma_2^{-1}$, σ_2. No automorphism of B_4 carries these three elements to σ_1, σ_2, σ_3 in any order (consider the standard relations, and the calculation $\sigma_i(\sigma_2\sigma_3\sigma_2^{-1})\sigma_i^{-1}(\sigma_2\sigma_3\sigma_2^{-1})^{-1} \neq$ $\neq o^{(4)}$ for $i = 1, 2$).

3.13 *Remark, concluded.* The construction in [Rudolph 3] of the fibration of the complement of a closed strictly positive braid works equally well for braids that are strictly positive (or more generally, strictly homogeneous) in any fixed set of T-standard generators.

§4. Prebands, tadpoles, and ribbon immersions

As in §3, $p: \mathbb{C} \to \mathbb{C}$ is a polynomial of degree n with $n-1$ distinct critical values $v_1, \ldots, v_{n-1}$ which form the set V, v_0 is a basepoint in $\mathbb{C} - V$, and F_{n-1} is the (free) prebraid group $\pi_1(\mathbb{C} - V; v_0)$, with standard generators $x_1, \ldots, x_{n-1}$.

4.1 Definition. A *positive* (resp., *negative*) *preband in* F_{n-1} is a conjugate of a standard generator (resp., the inverse of a standard generator). Note that $(p^{-1})_*: F_{n-1} \to B_n$ maps each preband to a band (of the same sign), but for $n \geqslant 3$ the preimage of any band contains both prebands and prebraids that are not prebands.

4.2 Definition. A (smooth) map τ: $(D^2, 1) \to (\mathbb{C}, v_0)$ is a *standard tadpole* if it has the following properties:

(1) for $\operatorname{Re} z$, the real part of z, non-negative, $\tau(z) = \tau(\operatorname{Re} z)$;
(2) $\tau|(\operatorname{Int} D^2 \cap \{z: \operatorname{Re} z < 0\})$ is a diffeomorphism onto an open set $U(\tau)$; the closure $U(\tau)$ is disjoint from v_0 and includes precisely one point of V, namely $v_{i(\tau)}$, which is $\tau(-1/2)$;
(3) $\tau|[0, 1]$ is an immersion in general position (i.e., it has no triple points and only finitely many double points, at which tangent directions are distinct) into the complement of $U(\tau) \cup V$, and $\tau^{-1}(v_0) = 1$;
(4) $\tau|(S^1 - \{1\})$ is an immersion.

A *tadpole* is a composition $\tau \circ \delta^{-1}$, where δ is a diffeomorphism of $(D^2, 1)$ with some other smooth 2-cell and τ is a standard tadpole.

We note that (1) forces τ/S^1 to hve tangent vector zero at 1, so we can't strengthen (4); nor is $\tau \mid (S^1 - \{1\})$ in general position, for again by (1) the entire right semicircle consists of (at best) double points of this immersion. From (2) and (4) we deduce that $\overline{U(\tau)}$ is a homeomorph of D^2, with boundary smooth except at the single point which is the image by τ of the diameter of D^2 on the imaginary axis; there, the boundary has a (generalized, real) «cusp», i.e., there is a one-sided tangent line.

We call $\overline{U(\tau)}$ the *head* of τ, $v_{i(\tau)}$ the *eye* of τ, and the immersed arc $\tau([0, 1])$ the *tail* of τ. The head of τ and τ itself are called *positive* or *negative* depending on whether the orientation induced on $U(\tau)$ by the range agrees or disagrees with that induced by the domain. If we abuse language slightly and call the image of τ, rather than the map τ, a tadpole, no harm is done so long as we remember orientation.

The following lemma is clear, and clearly the motivation for these definitions.

4.3 Lemma. *A prebraid is a preband if and only if it is represented by the boundary of a tadpole.* □

4.4 Definition. A tadpole is *embedded* if its tail is embedded. A preband is *embedded* if it can be represented by the boundary of an embedded tadpole.

4.5 *Construction.* Fix, once for all, a homeomorphism α: $[-1, 1] \to [0, 2]$ such that $\alpha \mid [-1, 0]$: $x \to (1 - x^2)^{1/2}$, α is C^∞ on $]-1, 1[$, and α^{-1} is C^∞ at 1 and has all derivatives 0 there. (If we are willing to work with C^1 surfaces and braids—and there is no real reason not to—one could take $\alpha \mid [0, 1]: x \to 2 - (1 - x^2)^{1/2}$.) Call the region $T = \{z \in \mathbb{C}: \operatorname{Re} z \in [-1, 1], |\operatorname{Im} z| \leqslant \alpha(\operatorname{Re} z)\}$ the *tongue*; the tongue contains D^2 and any standard tadpole τ has a unique extension, which we shall also denote τ, over T which preserves property (1). Note that T has a boundary which is smooth except for two corners, at $1 \pm 2i$, where it has (infinitely flat) cusps.

The function $\mathbb{C} - \{3\} \to S^1: z \to (z - 3)/|z - 3|$ restricts to a map η_{I_0}: $T \to S^1$ with image a closed subinterval $I_0 = \exp i\left[\frac{3}{4}\pi, \frac{5}{4}\pi\right]$, which sends the corners of T to the endpoints of the interval and has each other level set a closed line segment in T. Write η_I for the composition η_{I_0} with the direct similarity (in the affine structure of $S^1 \approx \mathbb{R}/2\pi\mathbb{Z}$) which carries I_0 onto I, any other (non-trivial) closed subinterval of S^1. A function η_I is a *height* for T.

Finally, given a (standard) tadpole τ (as extended to T), we construct the map $R(\tau, I)$: $T \to S^1 \times \mathbb{C}: z \to (\eta_I(z), \tau(z))$, and call it (or, abusively, its image) a (*standard*) *geometric preband*.

4.6 Proposition. *A geometric preband is a ribbon immersion of T into $S^1 \times \mathbb{C}$. It is an embedding if and only if the tadpole involved is embedded.*

(In 2.10, ribbon immersions were defined with target S^3, and on manifolds without corners, but it is clear what should be meant.)

PROOF. On the right half of the tongue, $R(\tau, I)$ is, essentially, the restriction of the product of an embedding η_I and τ, an immersion on each level set of η_I; so $R(\tau, I)$ is an immersion there. On the left half of the tongue, τ itself is an immersion, so $R(\tau, I)$ is there, too. Now we need only check the double points. They are all due to double points of the tail of τ. If $\tau(s) = \tau(t)$, $0 < < s < t < 1$, then each of the intervals I, $\eta_I^{-1}(\{\mathrm{Re} z = t\})$, $\eta_I^{-1}(\{\mathrm{Re}\, z = s\})$ contains the next in its interior. Write $A'' = \{z \in T: \mathrm{Re}\, z = s\}$, $A' = \{z \in T: \mathrm{Re}\, z = t, \eta_I(z) \in \eta_I(A'')\}$, $A = R(\tau, I)(A')$. Then also $A = R(\tau, I)(A'') = = \eta_I(A'') \times \{\tau(s)\}$; A'' is a proper arc in T, and A' is an interior arc; and all such pairs of arcs A', A'' exhaust the double points of $R(\tau, I)$ and enjoy the properties required in Definition 2.10. □

4.7 Definition. (Compare with 1.12.) *A preband representation of length l in F_{n-1}* is an l-tuple $\underline{\underline{x}} = (x(1), \ldots, x(l))$ of prebands in F_{n-1}. The **prebraid** of $\underline{\underline{x}}$ is $\rho(\underline{\underline{x}}) = x(1) \ldots x(l)$. The *closed prebraid of $\underline{\underline{x}}$* is $\hat{\rho}(\underline{\underline{x}}) \subset S^1 \times (\mathbb{C} - V)$.

The following lemma is easily proved, and provides one way to get around a slight technical awkwardness in Construction 4.9.

4.8 Lemma. *Let $q: \mathbb{C} \to \mathbb{C}$ be a polynomial of degree n with leading monomial $q_0(z) = az^n$, $a \neq 0$. Then for every $\epsilon > 0$, there exists a radius $r > 0$ for which the following are true.*

(1) *The critical values of q lie in $\frac{r}{4}D^2$.*
(2) $q^{-1}\left(\frac{r}{4}D^2\right) \subset q_0^{-1}\left(\frac{r}{2}D^2\right)$.
(3) *There is a smooth branched covering $\tilde{q}: \mathbb{C} \to \mathbb{C}$, uniformly within ϵ of q in the chordal metric of $S^2 = \mathbb{C} \cup \{\infty\}$, such that $\tilde{q}$ is equal to q on $q_0^{-1}\left(\frac{r}{2}D^2\right)$ and $\tilde{q}$ is equal to q_0 on $q_0^{-1}\left(\frac{3}{4}rD^2\right)$.* □

4.9 *Construction.* We use our polynomial p to construct related branched coverings. *It will now be convenient to write w for its variable, rather than z.* Let r be such as provided by Lemma 4.8., for $p = q$ (for any ϵ), and $\tilde{p}$ likewise. The map $\mathbb{C}^2 \to \mathbb{C}^2$: $(z, w) \to (z, p(w))$ is a branched covering, with $n - 1$ complex lines of critical points, and critical values $\mathbb{C} \times V$. So is its approximation $(z, w) \to (z, \tilde{p}(z, w))$, and this latter map has the advantage—since $\tilde{p}$ is a monomial «near infinity»—that what covers a sufficiently large bidisk is itself exactly a bidisk. In fact, let $\tilde{r} = |\tilde{p}^{-1}(r)|$, then this map covers D_r by $D_{\tilde{r}}$. We will use the letter P to denote, indifferently, the restriction of $(z, w) \to (z, \tilde{p}(z, w))$ to $D_{\tilde{r}}$, or to $\partial D_{\tilde{r}}$. Somewhat more abusively (but to our immense convenience) we shall *also* denote by P the branched covering $D^4 \to D^4$ induced by P *via* our standard smoothings of bidisks, as well as its restriction $S^3 \to S^3$.

4.10 *Remarks.* Of course (from (1) of Lemma 4.8, and taking minimal care with the basepoint) the induced homomorphisms $(p^{-1})_*$ and $(\tilde{p}^{-1})_*$ from F_{n-1} to B_n are identical.

On S^3, P actually decomposes—as we have set things up—into a covering of $h(\partial_1 D)$ by itself, and a covering of $h(\partial_2 D)$ by *it*self. All the action happens in the former, where each meridional disk $h(\{\exp i\theta\} \times D^2)$ covers itself by a simple n-sheeted cover; in $h(\partial_2 D)$, P is unbranched, the product of the identity on D^2 with the n-sheeted cover of S^1 by itself.

In fact (after giving them a natural orientation) the critical values of P on $D_{\tilde{r}}$ or on D^4 are a braided surface of degree $n-1$, none other than a particular $S(\phi)$ (where ϕ is the band representation of length zero in B_{n-1}); likewise, the critical values of P on $\partial D_{\tilde{r}}$ or on S^3 are a closed braid $\hat{o}^{(n-1)}$. And the same is true of the critical points (in the covering spaces). Proposition 3.10 can be sharpened to say that every isotopy class of closed n-string braids in S^3 is represented by a closed braid $P^{-1}(\hat{\rho})$, where $\hat{\rho} \cup \hat{o}^{(n-1)}$ is a closed braid (i.e., $\hat{\rho}$ is a closed prebraid criticized by the critical values of P in S^3).

4.11 *Construction.* Let the polynomial p and radius r be as above. Henceforth we demand of each tadpole τ that it satisfy these extra hypotheses:

(5) the head of τ lies in $\frac{r}{4}D^2$;

(6) the part of the tail of τ in the annulus $\frac{3r}{4}D^2 - \operatorname{Int}\frac{r}{4}D^2$ is a straight radial line segment.

Thus we have put the basepoint v_0 on $\frac{3r}{4}S^1$, which is no loss of generality. Certainly Lemma 4.3 still holds for these restricted tadpoles.

Now, let $\underline{\underline{x}}$ be a preband representation in F_{n-1}; let $I(1), \ldots, I(l(\underline{\underline{x}}))$ be disjoint closed intervals of $S^1 - \{1\}$, occurring in the order of their indices; let τ_j be a tadpole representing $x(j)$. We construct a subset of ∂D_r from this data; denoted $\Sigma(\underline{\underline{x}})$, it is the union of the (images of the) geometric prebands $R(\tau_j, I(j))(T)$, together with the annulus $S^1 \times \left[1, \frac{4}{3}\right] v_0 \subset S^1 \times rD^2$, together with the disk $D^2 \times \left\{\frac{4}{3}v_0\right\} \subset D^2 \times rS^1$. The following lemma is evident.

4.12 Lemma. *The set $\Sigma(\underline{\underline{x}})$ is a smooth, ribbon-immersed disk in ∂D_r (with corners along the corners of ∂D_r). It intersects the critical values of P transversally in $l(\underline{\underline{x}})$ points (one in each geometric preband). It intersects $\partial_2 D_r$ in a single meridional disk. The boundary $\partial\Sigma(\underline{\underline{x}})$ is a closed prebraid in ∂D_r, of type $\hat{\rho}(\underline{\underline{x}})$, and is criticized by the critical values of P. The map $\Sigma(\underline{\underline{x}}) \cap \cap \delta_1 D \to S^1$ gotten by restricting pr_1 has no critical points.* □

The reader may formulate a notion of equivalence of surfaces with the properties enunciated in the lemma, so that the various examples of $\Sigma(\underline{\underline{x}})$ produced by varying the choices of tadpoles, etc., are equivalent.

4.13 Definition. A *prebraided disk* is any such $\Sigma(\underline{\underline{x}})$. If $\Sigma(\underline{\underline{x}})$ is a prebraided disk, let $\Sigma'(\underline{\underline{x}})$ temporarily denote its «resolution», that is, the canonical smooth disk which ribbon-immerses onto $\Sigma(\underline{\underline{x}})$ (abstractly, an identification space obtained by glueing together a disk, an annulus, and several tongues).

4.14 Construction. Let $\underline{\underline{x}}$ be a preband representation. Denote by $S(\underline{\underline{x}})$ the subset of S^3 obtained as the image by smoothing of $P^{-1}(\Sigma(\underline{\underline{x}})) \subset \partial SD_{\bar{r}}$ (or alternatively, as P^{-1} of the image in S^3, by smoothing, of $\Sigma(\underline{\underline{x}}) \subset \partial D_r$). We suppose h to have been chosen sensibly so that $S(\underline{\underline{x}})$ is smoothly embedded near $S^1 \times S^1$.

4.15 Proposition. *Let $\underline{\underline{x}}$ be a preband representation in F_{n-1}, $\underline{\underline{b}} = ((p^{-1})_*$ $(x(1)), \ldots, (p^{-1})_*(x(l)))$ the corresponding band representation in B_n. Then $S(\underline{\underline{x}})$ is a ribbon-immersed surface in S^3, and there is a push-in of it into D^4* which is the braided surface $S(\underline{\underline{b}})$. *In particular, $\partial S(\underline{\underline{x}}) = \hat{\beta}(\underline{\underline{b}})$.*

Sketch of proof. The covering P induces a covering of $\Sigma'(\underline{\underline{x}})$, call it $S'(\underline{\underline{x}})$, which is a smooth surface, and evidently ribbon-immerses onto $S(\underline{\underline{x}})$.

We can actually produce a push-in back at the level of the prebraided disk $\Sigma(\underline{\underline{x}})$, which pushes it (rather, its smoothed image in S^3) into D^4 to be a braided surface of degree 1, transverse to the critical values of P. It suffices to find a topless Morse function of $\Sigma'(\underline{\underline{x}})$, with a single local minimum of value (say) 1, and constantly 2 on the boundary, which is adapted to the ribbon-immersion onto $\Sigma(\underline{\underline{x}})$. To find one, consider the geometric preband associated to a tadpole; thanks to evident properties of the height function, of the two components A', A'' of the preimage of a double arc in the range, it is always the case that the proper arc separates the interior arc from the head of the tadpole; so we can construct a Morse function that «engulfs» the interior arc before it touches the proper arc, and this is what is wanted.

When we lift such a push-in factor back to $S'(\underline{\underline{x}})$, the single local minimum becomes n local minima, and each of the $l(\underline{\underline{x}})$ intersections of $\Sigma(\underline{\underline{x}})$ with the critical values of P creates a saddlepoint. □

4.16 *Remarks.* This proposition gives a *practical* justification for the notation $S(\underline{\underline{x}})$, which in any case is not in *formal* conflict with the notation $S(\underline{\underline{b}})$ as introduced in 1.13, since $\underline{\underline{x}}$ and $\underline{\underline{b}}$ are objects of two different types.

When $\underline{\underline{x}}$ is embedded (4.4), or rather when the tadpoles chosen to represent the prebands are all embedded, $S(\underline{\underline{x}})$ is embedded in S^3. It is the (essentially) what was called an *O-braided surface in* S^3 in [Rudolph 2]: here O is the unknot $h(\{0\} \times S^1)$ thought of as a fibred knot in S^3; that component of P^{-1} of the image of an embedded geometric preband, which contains a critical point of P, is a (geometric) *band* in the sense of [Rudolph 2].

4.17 Definition. Such a surface as $S(\underline{\underline{x}})$ is a *braided surface in* S^3. (This, again, is not a formal conflict, nor should it be one in practice since the braided surfaces $S(\underline{\underline{b}})$ previously defined are in 4-dimensional ambient spaces.) This is sharpening of the usage in [Rudolph 1], where $S(\underline{\underline{b}})$ was used indiferently for $S(\underline{\underline{x}})$ and $S((p^{-1})_*(\underline{\underline{x}}))$: the new notation seems preferable because it actually indicates *the arrangement of the singularities* in S^3.

Figure 4 illustrates two prebraided disks and corresponding brainded surfaces. (Only the parts in $h(\delta_1 D)$, cut open, are shown.)

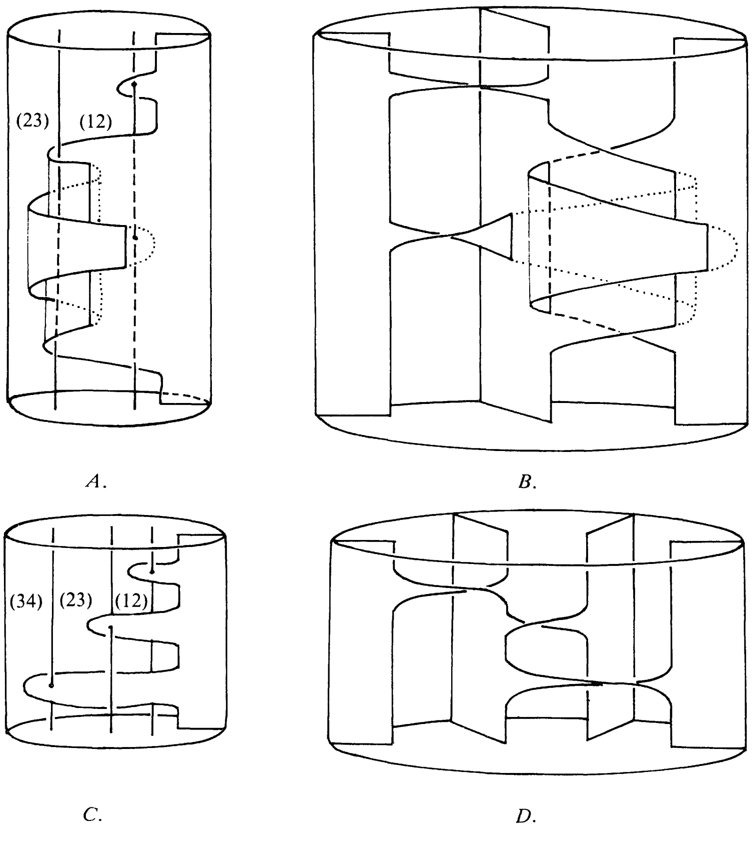

Fig. 4 *A* & *B*: $\Sigma(\underline{\underline{x}})$ and $S(\underline{\underline{x}})$, $\underline{\underline{x}} = (x_1^{-1}, x_2^2 x_1^{-1} x_2^{-2})$ in F_2 (immersed).
C & *D*: $\Sigma(\underline{\underline{x}})$ and $S(\underline{\underline{x}})$, $\underline{\underline{x}} = (x_1, x_2^{-1}, x_{23})$ in F_3 (embedded).

4.18 Remark. The reader is referred to [Rudolph 1] for an exposition, and examples, of the «calculus of (pre-)bands» (use of slides, expansions, and contractions, cf. 1.13-1.14, above) and the techniques of picturing braided surfaces in S^3.

§5. Markov surfaces and Bennequin surfaces

Recently, [Bennequin] proved some new and interesting facts about closed braids, and obtained new proofs of some old results, as a preliminary stage in his investigation of exotic contacts structures on S^3. His proofs involved the introduction and exploitation of a class of specially positioned Seifert surfaces. I propose to reinterpret these surfaces in the context of braided surfaces which we have established.

5.1 *Reminder*. A surface S in S^3 is a **Seifert surface** (*for* the link ∂S) if it is smooth, compact, oriented, and topless. A surface S in S^3 is *incompressible* if, for every simple closed curve $C \subset S$ such that C is the boundary of a smooth disk with interior disjoint from S, there is a disk contained in S with boundary C. It is a fact that every (smooth) link in S^3 is the boundary of some incompressible Seifert surface. In particular, if S is a Seifert surface which has maximal Euler characteristic among all Seifert surfaces with the same boundary, then S is incompressible. (There are, however, links with incompressible Seifert surfaces of arbitrarily high genus.)

5.2 Notation. If L is a link in S^3, let $X(L)$ be the maximum of the integers $\chi(S)$, S a Seifert surface for L.

For L a knot, or more generally a link which admits only connected Seifert surfaces (e.g., a fibred link), the *genus* $g(L)$ is unambiguously defined, and of course $g(L) = 1 - \frac{1}{2}(X(L) + rkH_0(L))$. But X is easier to calculate with, here.

Recall (1.1) that S^3 contains $S^1 \times S^1$ and is split by this torus into the two solid tori we have called $h(\partial_1 D)$ and $h(\partial_2 D)$, where $h(\partial_1 D) = \{(z, w) \in \mathbb{C}^2: |z|^2 + |w|^2 = 2, |z| \geqslant |w|\}$. These solid tori are equipped, *via* h, with fixed product structures. In particular, we can identify the universal cover of $h(\partial_1 D)$ with $\mathbb{R} \times D^2$, and treat projection on the first factor as a multivalued function θ: $h(\partial_1 D) \to \mathbb{R}$ (that is, θ is some fixed branch of $\frac{1}{i} \log \circ pr_1 \circ h^{-1}|h(\partial_1 D)$). We write D_t^2 for the meridional disk $\theta^{-1}(t) \subset h(\partial_1 D)$. Similarly, write ϕ for the «angular coordinate» in $h(\delta_2 D)$.

5.3 Definition. A Seifert surface $S \subset S^3$ is a *Markov surface* if it has the following properties (1) – (4):

(1) ∂S is a closed braid in $\operatorname{Int} h(\partial_1 D)$;
(2) $S \cap h(\partial_2 D)$ is the union of finitely many meridional disks $\phi^{-1}(s_j)$ (disregarding orientations);
(3) θ_S, the restriction of θ to $S \cap h(\partial_1 S)$, has no degenerate critical points, and distinct critical points of θ_S have distinct critical values (modulo 2π);
(4) each critical point of θ_S is a saddlepoint.

5.4 *Remark.* Taken together, (3) and (4) mean, geometrically, that all tangencies of S with leaves of the product foliation of Int $h(\partial_1 D)$ by fibres of θ are non-degenerate saddles. Now, in [Bennequin] (which after all appears in a volume dedicated to Georges Reeb), Markov surfaces are defined similarly but in terms not of this foliation but rather of a Reeb foliation of Int $h(\partial_1 D)$. But I claim the definitions are essentially equivalent: for, granting non-degeneracy of the tangencies, there are only finitely many; they all happen, therefore, inside some *closed* solid torus interior to $h(\partial_1 D)$, on which the foliation induced by the Reeb foliation and the foliation by meridional disks are isotopic. Thus the two species of Markov surface differ only by an inessential change of coordinates.

Thus (see also 5.7) all the pictures in [Bennequin] can be understood in terms of the present definition: in fact, more readily than for the original, since what is drawn is a disk but on the original interpretation it has to be understood as a plane (a leaf of the Reeb foliation) together with a circle at infinity.

5.5 Definition. Let S be a Seifert surface satisfying properties (1), (2), and (3) of Definition 5.3. Then both the level set $\theta_S^{-1}(t)$ and the pair $(D_t^2, \theta_S^{-1}(t))$ will be called *the t-section of S*. If t is not a critical value of θ_S, the t-section is a smooth, oriented 1-manifold-with-boundary, containing some number (perhaps zero) of simple closed curves as components, together with n arcs joining a point of S_t^1 to a point of Int D_t^2 and k arcs joining two points of S_t^1 to each other: following [Bennequin], we call the former arcs *free* and the latter *tied*. (The integer $n \geqslant 1$ is of course the number of strings of the closed braid ∂S; and $n + 2k$ is the number of meridional disks $\phi^{-1}(s_j)$ of $h(\partial_2 D)$ contained in S.) A *critical section*, $\theta_S^{-1}(t)$ for t a critical value, has precisely one singular point, which is either an isolated point of Int D_t^2 (if it is a local extremum of θ_S) or a point at which the section is an immersed 1-manifold with two transverse branches (a saddlepoint). The critical sections also have well-defined oriented boundaries. A saddle section may have zero, one, or two simple closed curves (through the singular point) which are properly contained in a component of $\theta_S^{-1}(t)$.

5.7. We leave to the reader to formulate and prove converses to the assertions of 5.6, to the effect that any (suitably smoothly changing) family of «abstract» t-sections actually fits together into the part of a surface with properties (1), (2), and (3) which lies in $h(\partial_1 D)$. (Be cautious: it might not be a Seifert surface without extra hypotheses, i.e., a closed component could appear.) Also, note that, given S, if $I \subset S^1$ is an interval without critical values of θ_S, then $S \cap \theta_S^{-1}(I)$ is the trace of an isotopy between the two sections of S at the endpoints of I, which is suitably unique, and that therefore a surface with properties (1), (2), and (3) can be adequately pictured (as in [Bennequin, pp. 109-110 ff.], or at the lower corners of the pages in [Douady]) by drawing sections «just on each side of» the singular sections (and, if there are closed components, making it clear how they move about in sections with more than one).

5.8 Theorem. *Let $\hat{\beta} \subset h(\partial_1 D) \subset S^3$ be a closed braid, and $S \subset S^3$ an incompressible Seifert surface for $\hat{\beta}$. Then S is ambient isotopic, with $\hat{\beta}$ fixed, to a Markov surface.*

(This slightly strengthens the statement of Théorème 4 of [Bennequin]; the proof is basically the same.)

PROOF. We achieve properties (1) – (4) of Definition 5.3 by a sequence of isotopies, each of which leaves intact those properties already acquired; we keep calling the surface S.

Property (1) is a hypothesis.

An arbitrarily small isotopy not only puts S transverse to the core circle $h(\{0\} \times S^1)$ of $h(\partial_2 D)$, but makes its intersections with an infnitesimal solid torns $h(\epsilon D^2 \times S^1)$ all into (not necessarily correctly oriented) meridional disks. A radial expansion of this torus achieves property (2).

Again, an arbitrarily small isotopy (supported in $\operatorname{Int} h(\partial_1 D)$) achieves property (3).

We are left with the task of eliminating local extrema of θ_S. First, without changing their number, we rearrange them, as follows. Let the t_1-section of S be non-singular. Among its simple closed curves (if any), consider one of those (if any) which were born at a local minimum of θ_S with value $t_0 \in]t_1 - 2\pi, t_1[$: that is, if possible, take $C \subset \beta_S^{-1}(t_1)$ which bounds a disk (its life history) on S with interior disjoint from $D^2_{t_1}$ and on which θ_S has a single critical point (a local minimum). Such a curve C also bounds a disk in $D^2_{t_1}$ (possibly with interior points in S). The two disks together make a 2-sphere, which bounds a 3-cell, which guides an isotopy («pushing from bottom to top») which decreases by at least one the number of simple closed curves in the t_1-section of S which were born at minima —so we may assume there are none such.

Now let $t_0 \in]t_1 - 2\pi, t_1[$ be the greatest minimum value of θ_S less that t_1 (if there are any local minima). The simple closed curve born at the isolated point of $\theta_S^{-1}(t_0)$ does not survive to the t_1-section, so it dies at some intermediate critical level. It doesn't die at a local maximum (or S would contain a 2-sphere), so it dies at a saddlepoint. If it dies by absorption (i.e., the number of simple closed curves in the sections decreases by one as you pass up through the saddle section), the life history of the 2-cell it bounds in D_t is a 3-cell with interior disjoint from S, in position for an «embedded handle cancellation» —an isotopy which reduces the total number of critical points of θ_S by two (the minimum and the saddlepoint). If, on the contrary, the curve dies by splitting into two disjoint simple closed curves C_1 and C_2 (in the regular section $\theta_S^{-1}(t_2)$ «just above» the saddle), then certainly one of C_1 or C_2 (and perhaps both) bounds a 2-cell in $D^2_{t_2}$ with interior disjoint from S (for —numbering so C_1 does not enclose C_2— nothing but C_1 could possibly get inside C_2, and nothing at all could get inside C_1 since we have controlled births): now we use incompressibility to conclude that C_1 (say) also bounds a disk contained in S, on which θ_S necessarily has at least one local *maximum*. Now an isotopy cancels (at least) two critical points of θ_S (namely, the saddle-point where the minimum died, and the maximum).

Thus after isotopies which don't create new critical points, we can assume θ_S has no local minima. Turning the procedure upside down, we can eliminate local maxima the same way, achieving (4), and S is now Markov. □

5.9 Definition. A *Bennequin* surface is a Markov surface for which no section contains a simple closed curve.

5.10 Theorem (Bennequin). *Let $\hat{\beta}$ be a closed braid in S^3. Then $\hat{\beta} = \partial S$ for some Bennequin surface S. If F is a Seifert surface for $\hat{\beta}$ with $\chi(F) = X(\hat{\beta})$ then F is ambient isotopic, with $\hat{\beta}$ fixed, to a Bennequin surface.*

Proof. Let S be a Markov surface. If $C \subset \theta_S^{-1}(t)$ is a simple closed curve then C does not bound on S (for if it did, θ_S would have to have a local extremum on the subsurface of S bounded by C), much less bound a disk on S. Yet if C is an innermost such curve in D^2_t, it does bound a disk with interior disjoint from S. So an incompressible Markov surface is a Bennequin surface. In particular, an incompressible Seifert surface (for instance, one of maximal Euler characteristic for its boundary) is ambient isotopic, with boundary fixed, to a Markov surface which is *ipso facto* Bennequin. □

5.11 Definition. Let S be a Markov surface, with boundary $\hat{\beta}$ a closed braid on n strings. Each meridional disk $\phi^{-1}(s_j)$ in $S \cap h(\partial_2 D)$ inherits an orientation from S, which it passes on to its boundary (oriented counterclockwise), a circle $h(S^1 \times \{\exp is_j\}) \subset S^1 \times S^1$. If this orientation agrees with the orientation by increasing t on S^1, call the circle and the disk *positive*, otherwise *negative*: so there are $n + k$ positive and k negative disks. Let S^+ be the (essentially unique) Markov surface obtained from S by removing from S each of its negative disks $\phi^{-1}(s_j)$ together with a collar of $\partial(\phi^{-1}(s_j))$ in $h(\partial_1 D)$ on which θ_S has no critical points (either interior to the collar or as restricted to its boundary circles).

5.12 Theorem. *Let $\underline{\underline{x}}$ be a preband representation in F_{n-1} with each preband $x(i)$ embedded. Then the braided surface $S(\underline{\underline{x}}) \subset S^3$ is a Bennequin surface* *with no negative disks. Up to isotopy through surfaces of the same sort, every Bennequin surface with no negative disks is a braided surface* $S(\underline{\underline{x}})$.

PROOF. *Via* h, we can consider the prebraided disk $\Sigma(\underline{\underline{x}}) \subset S^3$ as a very special Bennequin surface, containing a single (positive) meridional disk of $h(\partial_2 D)$, and such that $\theta_{\Sigma(\underline{\underline{x}})}$ has no critical points —so each section is a single free arc. Then the preimage $S(\underline{\underline{x}}) = P^{-1}(\Sigma(\underline{\underline{x}}))$ in S^3 is manifestly a Bennequin surface with n positive disks (and no negative ones); the section $\theta^{-1}_{S(\underline{\underline{x}})}(t)$ is singular if and only if $\theta^{-1}_{\Sigma(\underline{\underline{x}})}(t)$ passes through a critical value of P in D^2_t; then, as the branching of P is simple, the singular point of $\theta^{-1}_{S(\underline{\underline{x}})}(t)$ is a crossing of two transverse branches, i.e., t is a saddle value for $\theta_{S(\underline{\underline{x}})}$, and we have derived as a bonus that the number of critical values of $\theta_{S(\underline{\underline{x}})}$ is the length of $\underline{\underline{x}}$.

As to the converse, we prove (what appears to be) more than stated (see the Remark following), namely, that up to isotopy through Bennequin surfaces with no negative disks, every such Bennequin surface is $S(\underline{\underline{x}})$ for some preband representation $\underline{\underline{x}}$ in which each $x(j)$ is an *elementary* embedded preband: one of the $n(n-1)$ prebands $x^{\pm 1}_{uv} = (x_u x_{u+1} \ldots x_{v-1}) x^{\pm 1}_v (x_u \ldots x_{v-1})^{-1}$, $1 \leqslant u \leqslant v \leqslant n-1$ (so $x_{uu} = x_u$ is a standard generator).

In fact, by Constructions 4.9, 4.11, and 4.13, all $S(\underline{\underline{x}})$, $\underline{\underline{x}}$ a preband representation in F_{n-1}, meet $h(\partial_2 D)$ in precisely the same set of n positive meridional disks, and these may be naturally numbered (cf. 3.3) from 1 to n so that they appear in that cyclic order and so that, for instance, $S((x_j))$ has $n-1$ components, one of which contains disks j and $j+1$. Let S be a given Bennequin surface with no negative disks. By isotopy we may assume its n positive disks are these n canonical disks. Let t_j, $0 < t_1 < \ldots < t_l$ be the critical values of θ_S. The component of $\theta^{-1}_S(t_j)$ containing the singular point has two boundary points on ∂S and two others on two of the n canonical positive disks, let them be numbered $u(j)$ and $v(j)+1$, $u(j) \leqslant v(j)$. Also, there is a sign $\epsilon(j) = \pm 1$ naturally associated to the critical point of θ_S in the t_j-section, determined not just by the section but by local behavior on either side of it; the sign of the critical point of $S((x_j))$ is $+1$. Put $x(j) = x^{\epsilon(j)}_{u(j), v(j)}$, $\underline{\underline{x}} = (x(1), \ldots, x(l))$. I claim that S is isotopic (by an isotopy through Bennequin surfaces, supported in $h(\partial_1 D)$) to $S(\underline{\underline{x}})$. This claim is a refinement of the claim in 5.7, and like that, is left to the reader to prove. (See also 5.22). □

5.13 *Remarks*. A consequence of Theorem 5.12 is that there is a «retraction», call it $x \to x'$, of the set of all embedded prebands onto its subset of elementary embedded prebands, for which $(p^{-1})_*(x) = (p^{-1})_*(x')$; and this latter band is always one of the bands $\sigma^{\pm 1}_{uv} = (\sigma_u \sigma_{u+1}, \ldots, \sigma_{v-1}) \sigma^{\pm 1}_v (\sigma_u, \ldots, \sigma_{v-1})^{-1}$ which were called embedded bands in [Rudolph 1]. In that paper's construction of models of braided surfaces $S(\underline{\underline{b}})$ in S^3 (as ribbon immersions), only these bands —*expressed as given as words in the standard generators*— could be used if double points were not to be created (essentially because not only the positive disks, but also their collars in $h(\partial_1 D)$, were held fixed). The present construction, using surfaces $S(\underline{\underline{x}})$ associated to preband representations, is thus at once more precise and less general, for *not every ribbon-immersed surface* called $S(\underline{\underline{b}})$ in the earlier paper occurs as $S(\underline{\underline{x}})$ in the current terminology (the point being that $S(\underline{\underline{x}})$ is, as it were, p-equivariant). However, all embedded surfaces do so occur.

For $n \geqslant 3$ there are always non-elementary embedded prebands. One source of many such (perhaps all?) is the following. Let f: $\frac{r}{4}D^2 \to \frac{r}{4}D^2$ be an autohomeomorphism fixing $\frac{r}{4}S^1$ pointwise and V setwise. Then f_* is an automorphism of the prebraid group F_{n-1}. If f_* leaves invariant the representation on S_n associated to p (equivalently, if f is covered by an autohomeomorphism g of $p^{-1}\left(\frac{r}{4}D^2\right)$, then f_* will carry any embedded preband, in particular an elementary one, onto an embedded preband that will generally not be elementary.

For example: with V, N_j', etc., as at the beginning of §3, write W_{ij} for some 2-cell which is the union (not disjoint) of N_i', N_j', and a small disk centered at v_0 which is disjoint from V. Let σ: $W_{ij} \to W_{ij}$ be an autohomeomorphism which is conjugate, *via* a homeomorphism of $(W_{ij}, \{v_i, v_j\})$ with $(2D^2, \{-1, 1\})$, to $r\exp\sqrt{-1}\theta \to r\exp\sqrt{-1}\,(\theta + (2-r)\pi)$. Then if $j > i+1$, the two-fold composition $f = \sigma \circ \sigma$ is such an f as described above, while if $j = i+1$, the three-fold composition $f = \sigma \circ \sigma \circ \sigma$ is. In the first case, from the standard generators x_i and x_j are produced embedded prebands $(x_i x_j) x_i (x_i x_j)^{-1}$ and $x_i x_j x_i^{-1}$, respectively; in the second case we get $(x_i x_{i+1} x_i) x_{i+1} (x_i x_{i+1} x_i)^{-1}$ and $(x_i x_{i+1}) x_i (x_i x_{i+1})^{-1}$; other generators aren't touched.

It seems likely that further analysis of this situation would pay off. Perhaps it would lead to a geometric derivation of the relations in the standard presentation of the braid group. But we will not pursue the topic further at this time.

To continue our study of the connection between braided surfaces and Bennequin surfaces, we must hark back to the braid groups.

5.14 Definition The *usual injection* $\bar{u}_{n,n+k}$ of the prebraid group F_{n-1} into F_{n+k-1}, $k \geqslant 0$, takes the standard generator x_j of $F_{n-1}(j = 1, \ldots, n-1)$ to the standard generator of the same name in F_{n+k-1}. The *usual injection* $u_{n,n+k}$: $B_n \to B_{n+k}$ likewise is defined on standard generators, taking $\sigma_j \in B_n$ to $\sigma_j \in B_{n+k}$ $(j = 1, \ldots, n-1)$, cf. 1.4. That $u_{n,n+k}$ is a homomorphism follows from the fact that it actually is the homomorphism induced by such a map of configuration spaces $E_n - \Delta_n \to E_{n+k} - \Delta_{n+k}$ as $\{z_1, \ldots, z_n\} \to$ $\to \{z_1, \ldots, z_n, 1 + \sum_1^n |z_j|, 2 + \sum_1^n |z_j|, \ldots, k + \sum_1^n |z_j|\}$. (But note the adhockery of this map, which in any case could never be chosen to be a complex polynomial.) That $u_{n,n+k}$ is injective is less evident. But that (and the fact that it is a homomorphism) follows from the known presentations of the groups (cf. [Birman]), so we take it for granted. It must be emphasized again, however, that (for $k > 0$) the usual injection is NOT an inclusion, for the domain and range are disjoint groups; nor is the injection canonical, which is why I have named it merely «usual».

We also define F_{-1} and F_0 to be (distinct) 1-element groups, and $\bar{u}_{0,k}$ and $u_{0,k}$ in the only possible way. Then for $n, m, k \geqslant 0$, always $u_{n,n+k+m} = u_{n+k,n+k+m} \circ u_{n,n+k}$ and similarly with $\bar{u}$ for u. Also the canonical surjections $F_{n-1} \to B_n$ and the usual injections make up various commutative squares.

Now, the notations $u_{n,n+k}$ and $\bar{u}_{n,n+k}$ are cumbersome. We intend to avoid them, whenever practical, as follows.

5.15 *Notation.* If $\beta \in B_n$, let $\beta^{(k)} = u_{n,n+k}(\beta) \in B_{n+k}$. Similarly for prebraids. Thus $o^{(k)}$ is the identity of B_k (recall that o is the unique element of B_0), consistent with the earlier notation 1.4. (We could also resolve the ambiguous notation for standard generators by a convention that, for $n \geqslant 2$, $\sigma_{n-1} \in B_n$, thus making the standard generators of B_n carry the names $\sigma_1^{(n-2)}, \ldots, \sigma_j^{(n-1-j)}, \ldots, \sigma_{n-1}^{(0)} = \sigma_{n-1}$. We will not carry out this plan in this paper, however.)

Further, we denote the closure of $\beta^{(k)}$ by $\hat{\beta}^{(k)}$, rather than trying to stretch the roof over the whole complex symbol. Not only is this kind to typographers, but it is consistent with the following useful convention: if L is a link, then $L^{(k)}$ denotes the *split sum* of L with k unknots (that is, the union of L with the boundary of k smooth disks, pairwise disjoint and disjoint from L-out of the context of closed braids, this could also be denoted by $L \# O$, adapting the notation for boundary connected sum of pairs to the case of submanifolds without boundary). In particular (and this is a bit of misfortune), $\hat{o} = \varnothing$ is the empty link, $\hat{o}^{(1)}$ is the unknot.

We amend our conventions in the case of prebraids: by $\hat{x}^{(k)}$ we will denote the *criticized* closure of the prebraid $\bar{u}_{n,n+k}(x)$ (without the critical points of the covering, we couldn't see any difference between the two prebraids), where $x \in F_{n-1}$. So $\hat{x}^{(0)} \neq \hat{x}$ (the first is an n-string closed braid, the latter a 1-string closed braid).

Note that the consistency of the usual injections means that always $\beta^{(k)(m)} = \beta^{(k+m)}$.

5.16 Theorem. *Let S be a Bennequin surface with boundary $\hat{\beta}$, $\beta \in B_n$, and k negative disks. Then S^+ is a Bennequin surface with $\partial(S^+) = \hat{\beta}^{(k)}$.*

Proof. Obvious. □

5.17 *Warning*. The conclusion of 5.16 must be interpreted with care. Suppose one has chosen a way to cut open $h(\partial_1 D)$ along a meridional disk, and project it onto a rectangle, in such a way that ∂S projects onto a braid diagram in the usual way. Thus the crossings in the diagram correspond to standard generators (and their inverses) of B_n (with some basepoint of the configuration space implicit). Then, very likely, $\partial(S^+)$ will *not project to (what you think ought to be) a braid diagram for* $\beta^{(k)}$. For the choice of projection (of the solid cylinder onto a rectangle) imposes (by the conventions for reading braid diagrams) an injection of B_n into B_{n+k} which is probably not $u_{n,n+k}$. By moving the k «new» points «behind and to the right of» the n «old» points (at the top and bottom of the cylinder, and then straight all the way down) —a move which can be effected by a diffeomorphism of D^2 (times the identity on S^1)— the boundary of S^+ can, indeed, be made to look right. But generally the standard, narrow collars of the negative disks on S^+ will be carried into broad, flop ones. (See 5.19, for an example.)

5.18 Theorem. *Up to isotopy through Bennequin surfaces, every Bennequin surface bounded by a closed n-string braid* $\hat{\beta}$ *is obtained from a braided surface* $S(\underline{\underline{x}})$, *where* $\underline{\underline{x}}$ is an embedded preband representation in F_{n+k-1} *which maps to* $\beta^{(k)}$ *in* B_{n+k}, *by attaching k collared negative disks to* $\hat{\beta}^{(k)} - \hat{\beta}$.

PROOF. Immediate from 5.12 and 5.16. □

5.19 *Warning*. It is *not* true that, if $\underline{\underline{x}}$ is an embedded preband representation in F_{n+k-1} mapping to $\beta^{(k)} \in B_{n+k}$, $k \geqslant 1$, then necessarily it is possible to attach k collared negative disks to $\hat{\beta}^{(k)} - \hat{\beta}$ in the complement of $S(\underline{\underline{x}})$, to obtain a Bennequin surface for $\hat{\beta}$. Two distinct problems arise.

First, if $\underline{\underline{x}}$ is already the image by (the obvious map on preband representations associated to) the usual injection $\bar{u}_{n+m,n+k}$ of a preband representation in F_{n+m-1}, for some $1 \leqslant m < k$, then attaching collared negative disks to *all* the components of $\hat{\beta}^{(k)} - \hat{\beta}$ would produce $k - m > 1$ 2-spheres in the resulting surface. (For instance, if $\underline{\underline{x}} = (x_{12})$ in F_3, then the obvious projection of $\partial S(\underline{\underline{x}})$ doesn't look like $\hat{o}_1^{(2)}, \sigma_1^{(2)} \in u_{2,4}(B_2) \subset B_4$, *viz.* 5.17; yet it is, and attempting to attach two disks to the last two components brings trouble.) Of course this could be handled by convention.

More seriously, there are cases like $\underline{\underline{x}} = (x_1, x_1, x_1^{-1}, x_1^{-1})$ in F_1, a preband representation for the braid $o^{(2)} = o^{(1)(1)}$ in B_2. Here, *each* component of $\hat{o}^{(2)}$ has non-zero linking number with a suitable (simple closed) curve on $S(\underline{\underline{x}})$, so no disk at all (let alone a collared negative disk) can be attached to $S(\underline{\underline{x}})$ along either boundary component.

Note, however, that the surface just constructed is (very) compressible. In fact, we have the following converse to 5.18.

5.20 Theorem. *Let $S(\underline{\underline{x}})$ be an incompressible embedded braided surface with boundary $\hat{\beta}^{(k)}$. Then there is a smoothly embedded surface S in S^3 with boundary $\hat{\beta}$ which is the union along $\hat{\beta}^{(k)} - \hat{\beta}$ of $S(\underline{\underline{x}})$ and k 2-disks, such that the topless components of S are a Bennequin surface.*

PROOF. Let us say that an oriented 2-disk G^- embedded in S^3 is a *floppily collared negative disk* if $G = -G^-$, the same disk with opposite orientation, is a Bennequin surface for a 1-string closed braid and G has only one (positive) meridional disk of $h(\partial_2 D)$ in it. It is clear that if we can find k floppily collared negative disks, pairwise disjoint and with boundaries the k components of $\hat{\beta}^{(k)} - \hat{\beta}$, and interiors disjoint from $S(\underline{\underline{x}})$, then (suitably smoothed along $\hat{\beta}^{(k)} - \hat{\beta}$) the union of $S(\underline{\underline{x}})$ and these disks is such an S as we require.

We begin by finding F, a union of pairwise disjoint floppily collared negative disks with $\partial F = \hat{\beta}^{(k)} - \hat{\beta}$ such that $\operatorname{Int} F$ is disjoint from $\hat{\beta}$ and transverse to $\operatorname{Int} S(\underline{\underline{x}})$. (For instance, one can realize $S(\underline{\underline{x}})$ so its boundary really looks like $\hat{\beta}^{(k)}$ in a braid diagram, then take F to be the union of obvious collared, and *a fortiori* floppily collared, negative disks; naturally the transversality is no problem.) We will modify the given F, staying in the class of unions of k floppily collared negative disks, until $\operatorname{Int} F \cap S(x) = \emptyset$, at which point we will be done.

If $\operatorname{Int} F \cap S(\underline{\underline{x}}) \neq \emptyset$, by transversality it is a union of simple closed curves. Let C_1 be one of them which is «innermost» on F (that is, bounds a 2-cell in F with interior disjoint from $S(\underline{\underline{x}})$). By incompressibility, C_1 bounds a 2-cell on $S(\underline{\underline{x}})$. Let C_2 be an innermost curve in this 2-cell (possible C_1 itself). Then C_2 is not necessarily innermost on F, but we don't care. Let $E \subset S(\underline{\underline{x}})$ and $E' \subset F$ be the 2-cells bounded by C_2. I claim that if we remove a 2-cell slightly larger than E' from F, and replace it by a 2-cell with the same boundary which lies parallel and close to E in the complement of $S(\underline{\underline{x}})$, then the revised F is still a union of pairwise disjoint floppily collared negative disks with boundary $\hat{\beta}^{(k)} - \hat{\beta}$. In fact, by the transversality of the intersection and property (2) of the Markov surfaces $S(\underline{\underline{x}})$ and $-F$, either E' contains a negative disk of $-F$, or it lies in $h(\partial_1 D)$; since $\partial E' = \partial E = C_2$, consideration of linking numbers shows that whichever alternative holds for E' also holds for E. In each case, we see that «replacing E' by E» (as we essentially have done) preserves the desired properties of F. The operation also, of course, decreases the number of intersections of $\operatorname{Int} F$ and $S(\underline{\underline{x}})$. When this reaches zero we are done. □

5.21 *Remark.* Theorems 5.12, 5.16, 5.18, and 5.20 in a sense reduce Bennequin surface theory to braided surface theory, and thence (*via* the calculus alluded to in 4.18) to the algebra and combinatorics of band and preband representations. It might, for instance, be possible to prove Bennequin's Inequality (see the next section) purely within the context of braided surfaces, though to date I have not succeeded in doing so.

An interesting practical question that arises when one considers Theorem 5.20 (and one which might have an answer of independent interest, given the interest in incompressible surfaces among 3-manifold topologists) is, *How can one tell from* $\underline{\underline{x}}$ whether or not $S(\underline{\underline{x}})$ *is incompressible?*

5.22 *Addendum.* Here is a direct path from a Bennequin surface $S \subset S^3$ without negative disks to a braided surface $S(\underline{\underline{b}}) \subset D$ which is essentially a push-in of S. (Presumably, with a suitable definition of «Bennequin ribbon-immersed surface in S^3» one could obtain all $S(\underline{\underline{b}})$ this way.) First, by isotopy, arrange S so that in each critical section the arc-lengths of the four arms of the singular component of $S_t \subset D_t^2$ are of equal arc-length. Next, define r: $S \cap h(\partial_1 D) \to [1/2, 1]$ by requiring $r|\partial S$ to be identically equal to 1, $r|(S \cap S^1 \times S^1)$ to be identically 1/2, and $r|A$ to be an affine function of arc-length from ∂S for A a component of any section S_t. Clearly r is smooth and well-defined (and takes the value 3/4 at the singular points of the critical sections). Map $S \cap h(\partial_1 D)$ into $\left(D^2 - \operatorname{Int}\frac{1}{2}D^2\right) \times D^2$ by sending a point $x = h(\exp it, w)$ to $(r(x) \exp it, w)$. Then $S \cap S^1 \times S^1$ maps to the union of n circles $\frac{1}{2}S^1 \times \{\exp is_j\}$, and we extend our map to $S \cap h(\partial_2 D)$ by sending $x = h(z, \exp is_j)$ to $\left(\frac{z}{2}, \exp is_j\right)$. Evidently, the map constructed embeds S in D with image a braided surface $S(\underline{\underline{b}})$ of degree n, and $h(\partial S(\underline{\underline{b}})) = \partial S$ by construction.

§6. Markov's Theorem, Bennequin's Inequality, and some conjectural generalizations

In this section I will state, without proof, two major results in the application of braids to knot theory: the reader is referred to [Bennequin] for proofs of both (or to [Birman] for Markov's Theorem: however, the differential-topological approach of [Bennequin] is perhaps closer to the spirit of the present paper than the combinatorial-topological approach of [Birman]). I will then discuss various generalizations, all conjectural, which are suggested when one thinks in terms of braided surfaces.

6.1 Markov's Theorem. *Let $\beta \in B_n$, $\gamma \in B_p$ be two braids such that the closed braids $\hat{\beta}, \hat{\gamma} \subset S^3$ are ambient isotopic in S^3. Then there is a finite sequence $\beta(j) \in B_{n(j)}$ of braids, $j = 1, \ldots, N$ with $\beta(1) = \beta$, $\beta(N) = \gamma$, such that for each $j = 1, \ldots, N-1$, one of the following three cases holds:*

(1) *$n(j+1) = n(j)$ and for some $w(j) \in B_{n(j)}$, we have $\beta(j+1) = w(j)\beta(j)w(j)^{-1}$; or,*

(2) *$n(j+1) = n(j) + 1$ and for $\epsilon = +1$ or $\epsilon = -1$, we have $\beta(j+1) = \beta(j)^{(1)}\sigma_{n(j)}$; or,*

(3) *$n(j+1) = n(j) - 1$ and for $\epsilon = +1$ or $\epsilon = -1$, we have $\beta(j) = \beta(j+1)^{(1)}\sigma_{n(j+1)}$.*

Conversely, if two braids are joined by such a sequence, then their closures are of the same ambient isotopy type. □

6.2 *Definition.* In case (1) [resp., (2); (3)] of Markov's Theorem, we say that $\beta(j+1)$ is obtained from $\beta(j)$ by a *Markov move* of *type* (1) *with conjugator* w [resp., of *type* (2^{ϵ}); of *type* (3^{ϵ})].

6.3 Bennequin's Inequality. *Let $\beta \in B_n$. Let $e(\beta) \in \mathbb{Z}$ denote its exponent sum (see Remark* 6.4), *$X(\hat{\beta})$ the maximum Euler characteristic of a Seifert surface for $\hat{\beta}$* (cf. 5.2). *Then we have*

$$IB(\beta): \quad n - |e(\beta)| \geqslant X(\hat{\beta})$$

(*which I will call «Bennequin's Inequality for β»*). □

6.4 *Remark* Recall that e: $B_n \to \mathbb{Z}$ is abelianization, normalized to send a positive band to $+1$. Consequently $|e(\beta)|$ is certainly a lower bound for the number of bands needed to represent β in B_n, so $n - |e(\beta)|$ is an upper bound for the Euler characteristic of a braided surface (in D) with boundary $\hat{\beta}$.

6.5 Definition. A *slice surface* in D^4 is a compact, topless, smooth 2-manifold-with-boundary properly embedded in D^4. If L is a smooth, oriented link in S^3, define invariants $X_r(L)$, $X_s(L)$ by putting $X_r(L) = \max\{\chi(S)$: $S \subset D^4$ is an oriented ribbon surface with $\partial S = L\}$, $X_s L = \max\{\chi(S)$: $S \subset D^4$ is an oriented slice surface with $\partial S = L\}$. Then (since any Seifert surface in S^3 is, in particular, a ribbon-immersed surface without singularities, and can thus be pushed into D^4 to become a ribbon; and any ribbon surface is slice) we have, for every L, $X(L) \leqslant X_r(L) \leqslant X_s(L)$. It is well-known that the first inequality can be strict (existence of non-trivial «ribbon knots», e.g., $\partial S((\sigma_1, \sigma_2^3\sigma_1\sigma_2^{-3})))$; it is an open question whether the second inequality is ever strict, even in the case $X_s(L) = 1$, L a knot.

6.6 Ribbon-Bennequin Conjecture. *For every n and every $\beta \in B_n$, we have*

$$rIB(\beta): \quad n - |e(\beta)| \geqslant X_r(\hat{\beta}).$$

(*which I will call the «ribbon-Bennequin inequality for β»*).

6.7 Slice-Bennequin Conjecture. *For every n and every $\beta \in B_n$, we have*

$$sIB(\beta): \quad n - |e(\beta)| \geqslant X_s(\hat{\beta})$$

(which I will call the «slice-Bennequin inequality for β»).

6.8 *Remarks.* (1) Of course, for every β, $sIB(\beta) \to rIB(\beta) \to IB(\beta)$.

(2) There are various β for which $sIB(\beta)$ is known to hold with equality. For example, the various positive braids $\sigma_1^k \in B_2$ $(k \geqslant 1)$, $(\sigma_1\sigma_2)^k \in B_3 (1 \leqslant k \leqslant 5)$, $\sigma_1^{2k+1}(\sigma_2\sigma_1\sigma_3\sigma_2)^3 \in B_4$ all have «total signature», i.e., positive-definite Siefert form (they occur as links of so-called «simple» singularities of complex plane curves), so the embedded braided surfaces $S((\sigma_1, \ldots, \sigma_1))$, etc., corresponding to the given braid words (read as band representations), of Euler characteristic $n - e(\beta)$ in each case, actually are of maximal Euler characteristic among all slice surfaces for the closed braids.

(3) I know of no counterexample to the Slice-Bennequin Conjecture. On the other hand, suppose one defines a *topological slice surface* in D^4 to be a compact, topless 2-manifold with boundary properly embedded in D^4 which, though not necessarily smooth, has a neighborhood in D^4 which is homeomorphic to the product of the surface and $\operatorname{Int} D^2$. Then (using a deep result of [Freedman] on knots with Alexander polynomial 1) I have shown that the natural «topological-slice-Bennequin Conjecture» is *false*: for every $n \geqslant 5$, there are braids $\beta \in B_n$ such that $\hat{\beta}$ bounds some topological slice surface of Euler characteristic strictly greater than $n - |e(\hat{\beta})|$, cf. [Rudolph 6]. (Though not remarked in that paper, it is in fact the case that «many» such braids exist —e.g., any positive braid with «summit power» at least 2, cf. [Birman]). Of course, such a topological slice surface must be expected (if it is not smoothable) to have horrible behavior, somewhere, with respect to those smooth functions (N, for the round ball D^4; pr_1, for the bidisk D) in terms of which we have gained some understanding of ribbon surfaces, braided surfaces, and even (as we shall shortly see) smooth slice surfaces.

6.9 Definition. Let $S \subset D^4$ be a compact, smoothly embedded 2-manifold-with-boundary with $\partial S = S \cap S^3$ (but not necessarily topless), in general position with respect to the squared-norm function N. Let the Morse function $N|S$ have exactly $m \geqslant 0$ local maxima in $\operatorname{Int} S$, and let $G_1, \ldots, G_m$ be disjoint closed smooth 2-disks embedded in $\operatorname{Int} S$ such that $N|G_j$ is constant on ∂G_j and has a single critical point in $\operatorname{Int} G_j$, a local maximum, $j = 1, \ldots, m$. There is an isotopy of S in $\mathbb{C}^2$ which fixes the points of S outside the G_j and replaces S by $S' = \left(S - \bigcup_{j=1}^{m} G_j\right) \cup \left(\bigcup_{j=1}^{m} G'_j\right)$, where G'_j $(j = 1, \ldots, m)$ is a disk on which $N|G'_j$ has a single interior critical point, a local maximum with value *greater than* 2 (the value of N on S^3). Let $S^{(m)}$ denote $S' \cap D^4$.

By construction, $S^{(m)}$ is a ribbon-embedded surface in D^4, and $\partial(S^{(m)}) = (\partial S)^{(m)}$ (Notation 5.15), so we may unambiguously write $\partial S^{(m)}$. It may be seen that $S^{(m)}$ is well-defined up to isotopy. We call $S^{(m)}$ the *decapitation* of S.

More generally, with S as above, let $q \geqslant m$. By a small isotopy, S may be perturbed to a surface S_q with $N|S_q$ Morse, having $2(q-m)$ more critical points than $N|S$ in pairs of cancelling saddlepoints and local maxima. (If S is connected, S_q is essentially well-defined; in general, one should specify the partition of the q new maxima among the components.) We let $S^{(q)}$ denote $S_q^{(q)}$. If, in particular, S is ribbon-embedded and $q \geqslant 0$, a surface $S^{(q)}$ will be called the result of *punching q holes in S.*

We use the smoothing h to transfer all these notions and notations to the bidisk.

6.10 Example. Let $\underline{\underline{b}}$ be a band representation in B_n. Then (extending Notation 5.15) by $\underline{\underline{b}}^{(q)}$ we denote the band representation in B_{n+q} with $b^{(q)}(j) = b(j)^{(q)}$, $j = 1, \ldots, l(\underline{\underline{b}})$. Of course $\beta(\underline{\underline{b}}^{(q)}) = \beta(\underline{\underline{b}})^{(q)}$. Denote concatenation of lists by C (e.g., $(A_1, A_2)C(A_3, A_4) = (A_1, A_2, A_3, A_4)$). Then for $q \geqslant 0$ one easily sees that $S(\underline{\underline{b}})^{(q)}$ (rather, the particular type of $S(\underline{\underline{b}})^{(q)}$ obtained by punching all q holes in a certain single component of $S(\underline{\underline{b}})$) can be braided as $S(\underline{\underline{b}}^{(q)}C(\sigma_n, \sigma_n^{-1}, \sigma_{n+1}, \sigma_{n+1}^{-1}, \ldots, \sigma_{n+q-1, n+q-1}^{-1}))$. (If $S(\underline{\underline{b}})$ isn't connected, the various types of $S(\underline{\underline{b}})^{(q)}$ could all be represented similarly, using suitable embedded bands in place of standard generators.)

6.11 Proposition. *Let $\hat{\beta} \subset \partial D$ be a closed braid. If $S \subset D^4$ is an oriented slice surface with boundary $h(\hat{\beta})$, then there is some $q \geqslant 0$ and some band representation $\underline{\underline{b}}$ of $\beta^{(q)}$ such that $h(S(\underline{\underline{b}})) = S^{(q)}$. (In words: any oriented slice surface for a closed braid can have holes punched in it until it can be realized as a braided surface for the original closed braid with trivial strings added.)*

PROOF. By Theorem 2.8 (proved in [Rudolph 1]), the ribbon surface $S^{(m)}$ obtained by decapitating S is isotopic to a braided surface $h(S(\underline{\underline{c}}))$, for some band representation $\underline{\underline{c}}$. Now, as an oriented link in ∂D, $\partial S(\underline{\underline{c}})$ is of the same isotopy type as $\hat{\beta}^{(m)}$. Then the proposition is a consequence of the following lemma (take $q = m + k$, k as provided by the lemma). □

6.12 Lemma. *Let α and δ be braids with $\hat{\alpha}$ ambient isotopic to $\hat{\delta}$. Then there is an integer $k \geqslant 0$ such that, for any band representation $\underline{\underline{d}}$ of δ, there is a band representation $\underline{\underline{a}}$ of $\alpha^{(k)}$ with $S(\underline{\underline{a}})$ ambient isotopic to $S(\underline{\underline{d}})^{(k)}$.*

PROOF. Let $\underline{\underline{d}}$ be a band representation of δ.

If δ is obtained from α by a Markov move of type (1) with conjugator w, then $w^{-1}\underline{\underline{d}}\,w$ (in the obvious sense) will do for $\underline{\underline{a}}$, with $k = 0$. If δ is obtained from α by a Markov move of type (2^ϵ), $\delta = \alpha^{(1)}\sigma_n^\epsilon(\alpha \in B_n)$, then let $k = 1$ and put $\underline{\underline{a}} = \underline{\underline{d}}\,C(\sigma_n^{-\epsilon})$. If δ is obtained from α by a Markov move of type (3^ϵ), $\alpha = \delta^{(1)}\sigma_n^\epsilon(\delta \in B_n)$, again let $k = 0$, $\underline{\underline{a}} = \underline{\underline{d}}^{(1)}C(\sigma_n^\epsilon)$. Each time, $S(\underline{\underline{a}})$ is isotopic to $S(\underline{\underline{d}})^{(k)}$.

For general α and δ with isotopic closures, Markov's Theorem 6.1 says that a finite sequence of Markov moves joins α and δ. We see that the lemma is true, with k the minimum number (over all such sequences) of moves of type (2^ϵ) required. □

6.13 *Remark.* Proposition 6.11 is a strong form of the observation made in the final paragraph of the body of [Rudolph 1], pp. 30-31. It allows a partial answer (Theorem 6.15) to the question raised there, whether the method of band representations can give any information about «slice genus» (essentially, X_s). It also suggests a method of attack on the problem of whether every slice knot is a ribbon knot, or more generally, whether $X_s = X_r$: namely, find very well controlled braided surfaces isotopic to arbitrary oriented slice surfaces with holes punched in them; then manipulate these surfaces until, along the lines of 5.20, the holes can be filled back in to produce *ribbon* surfaces. The problem is to find the right manipulations....

Similarly, if we apply 6.11 to the empty closed braid, we see that in some sense the whole theory of smooth, oriented, compact surfaces in D^4 without boundary «reduces» to the study of band representations of the trivial braids $o^{(k)}$, $k = 1, 2, 3, \ldots$; however it remains to be seen whether this «reduction» is useful.

6.14 Examples. Let $\beta = \sigma_3^2\sigma_2^{-1}\sigma_3\sigma_2^{-1}\sigma_1^{-3}\sigma_2\sigma_1^{-1}\sigma_2 \in B_4$. This braid, found by [Morton], has closure an unknot, yet $\hat{\beta}$ is the boundary of no braided disk in D. There is, of course, a sequence of Markov moves connecting $\beta \in B_4$ to $o^{(1)} \in B_1$, and [Morton] gives an explicit and straightforward such sequence in which there figures a single move of type (2). Then Lemma 6.12 says that there must be a once-punctured braided disk (i.e., braided annulus) in D with boundary $\hat{\beta}^{(1)}$. In fact, one discovers the quite complicated surface $S(\underline{\underline{b}})$, where $\underline{\underline{b}}$ is the band representation in B_5 given by $w(\sigma_4, \sigma_2^{-1}, \sigma_3^{-1}, u\sigma_1u^{-1})w^{-1}C(\sigma_4^{-1})$, where $w = \sigma_3\sigma_4^{-1}\sigma_2^{-1}\sigma_1\sigma_3$ and $u = \sigma_3^{-2}\sigma_2^2$. This can be simplified somewhat by the calculus of slides, but (to date) I have not succeeded in putting into a nice form. Note that this braided annulus certainly does not *appear* to be the push-in to D^4 of an embedded braided annulus $S(\underline{\underline{x}})$ in S^3 (though a calculation of $\pi_1(D^4 - S(\underline{\underline{b}}))$, along the lines of [Rudolph 1], yields $\mathbb{Z}$ and so does not rule out the possibility). Since $\hat{\beta}$ is an unknot, certainly Theorem 5.18 says that there is an embedded k-punctured disk $S(\underline{\underline{x}}) \subset S^3$ with boundary $\hat{\beta}^{(k)}$ for some $k \geqslant 1$; but the difficulties of putting the theorem on a constructive, practical, footing seem insurmountable.

In [Rudolph 1], Example 4.3 is a braided annulus of degree 4 which is the decapitation of a knotted 2-sphere (the 2-twist spun trefoil). The reader is urged to find a band representation of some $o^{(k)}$ which yields this annulus with $k - 2$ more punctures.

6.15. There is an interesting consequence of Proposition 6.11. To state it conveniently, we recall that a braid in B_n is *quasipositive* [Rudolph 1, 2, 3, 5] if it is a product of positive bands. The quasipositive braids in B_n form a subsemigroup, strictly larger (for $n \geqslant 3$) than the subsemigroup of *positive* braids (a braid is positive if it is a product of standard generators). The property of being (quasi)positive is preserved by the usual injections. A particularly important positive braid in B_n is $(\sigma_1\sigma_2 \ldots \sigma_{n-1})^n$; it is usually called Δ^2 (cf. [Birman]), but I will call it ∇ here, or ∇_n for greater precision. It is easy to prove the following.

Lemma. (1) *For every n, for every $\beta \in B_n$, there is an integer $Q \geqslant 0$ such that the product $\beta \nabla_n^Q$ is quasipositive, and an integer $P \geqslant Q$ such that the product $\beta\nabla_n^P$ is positive.* (2) *For every n, for every quasipositive $\pi \in B_n$, there is an integer $k \geqslant 0$ and a factorization $\nabla_{n+k} = \pi^{(k)}\varkappa$ in B_{n+k} with $\varkappa$ quasipositive.* (3) *The closure $\hat{\nabla}_n$ is the link of n consistently oriented fibres of the Hopf fibration $S^3 \to S^2$, e.g., the intersection of $S^3 \subset \mathbb{C}^2$ with $\{(z, w): z^N = w^n\}$.* □

6.16 Proposition. *The slice-Bennequin Conjecture 6.7 is true if (and only if) the Slice-Bennequin inequality $sIB(\beta)$ holds for every quasipositive braid β, if (and only if) $sIB(\nabla_n)$ holds for all sufficiently large n.*

PROOF. We show that, starting from any counterexample to the conjecture, one can produce n such that ∇_n is a counterexample.

Thus, assume $sIB(\beta)$ fails for some $\beta \in B_n$, that is, $X_s(\hat{\beta}) > n - |e(\beta)|$. Then also $sIB(\beta^{-1})$ fails, so we can assume that $e(\beta)$ is non-negative. Let S be a slice surface for $\hat{\beta}$, oriented, with $\chi(S) = X_s(\hat{\beta})$. By Proposition 6.11, there is some $q \geqslant 0$ and a band representation $\underline{\underline{b}}$ of $\beta^{(q)}$ in B_{n+q} with $S(\underline{\underline{b}})$ isotopic to $S^{(q)}$. By Lemma 6.14, there is $Q \geqslant 0$ with $\beta\nabla_n^Q \in B_n$ quasipositive. Let $\underline{\underline{c}}$ be the band representation of $(\beta\nabla_n^Q)^{(q)}$ which is $\underline{\underline{b}}$ followed by Qn repetitions of the braid word (band representation with each band a standard generator) $(\sigma_1, \sigma_2, \ldots, \sigma_{n-1})^{(q)}$. Then $S(\underline{\underline{c}})$ is $S(\underline{\underline{b}})$ with $n(n-1)Q$ extra 1-handles attached. Also, $S(\underline{\underline{c}})$ is isotopic to $F^{(q)}$ for some F, a slice surface with boundary the closure of $\beta\nabla_n^Q$. I claim that $sIB(\beta\nabla_n^Q)$ fails: for we have $X_s((\beta\nabla_n^Q)\hat{}\,) \geqslant \chi(F) = = q + \chi(S(c)) = q + \chi(S(b)) - n(n-1)Q = \chi(S) - n(n-1)Q = S_s(\hat{\beta}) - n(n -$

$-1)Q > n - |e(\beta)| - n(n-1)Q = n - e(\beta) - n(n-1)Q = n - e(\beta\nabla_n^Q) =$
$= n - |e(\beta\nabla_n^Q)|$. We have shown that if there is a counterexample in B_n, then there is a quasipositive counterexample in B_n.

But using (2) of Lemma 6.15, we see that if there is a quasipositive counterexample $\pi \in B_n$, then for some k there is a factorization $\nabla_{n+k} = \pi^{(k)}\varkappa$ in B_{n+k} with $\varkappa$ quasipositive. The same trick as was just used (attaching positive bands —in this case, those of a quasipositive band representation of k in B_{n+k}— to a counterexamplifying slice surface for $\hat{\pi}$) shows that $sIB(\nabla_{n+k})$ doesn't hold.

Of course, the «only if» statements are trivial. □

6.17 Corollary. *If the slice-Bennequin conjecture is false, then so is the «Thom Conjecture».*

Proof. The so-called «Thom Conjecture» asserts that a non-singular complex algebraic curve in $\mathbb{CP}^2$ has the minimal possible genus among all smoothly embedded 2-manifolds in its homology class in $H_2(\mathbb{CP}^2; \mathbb{Z})$. (Cf. [Boileau-Weber]). If the slice-Bennequin conjecture is false, let $S \subset D^4$ be a slice surface of unexpectedly high Euler characteristic for the d-component Hopf link $\tilde{\nabla}_d$. Using (3) of Lemma 6.15, one can replace, on an algebraic curve $\{(z, w)\colon z^d = w^d - \epsilon\}$, ϵ sufficiently small and nonzero, a piece bounded by this link (and having the expected Euler characteristic) by S. The resulting smooth «surgered» surface is homologous to the curve and has smaller genus. □

6.18 *Remark*. The corollary (and its proof) may be summarized in the slogan, «If you can't slice Bennequin, you can surger Thom».

Note that the Thom Conjecture might be false, but not by surgery; then there would be no reason to conclude that the slice-Bennequin conjecture is false too.

Index of notations

A_k, A'_k, A''_k	Double arcs of a ribbon immersion (Def. 2.10).
$B_n (n \geqslant 0)$	Braid group on n strings ($B_0 = \{o\}$) (Def. 1.3).
$\underline{\underline{b}}$; $b(j)$	A band representation; the j^{th} band in $\underline{\underline{b}}$ (Def. 1.12).
$\beta(\underline{\underline{b}})$; $\hat{\beta}(\underline{\underline{b}})$	The braid of $\underline{\underline{b}}$; the closed braid of $\underline{\underline{b}}$ (Def. 1.12).
$\beta^{(k)}$; $\hat{\beta}^{(k)}$	The usual injection of $\beta \in B_n$ into B_{n+k}; the closure of $\beta^{(k)}$, which is the split sum of $\hat{\beta}$ and $\hat{o}^{(k)}$ (Def. 5.14, Notation 5.15).

$D; D_r; D^4$	The unit bidisk; the bidisk of biradius $(1, r)$; the round ball of radius $\sqrt{2}$, all in $\mathbb{C}^2$ (Def. 1.1).
$D^2; rD^2; D^2_t$	The unit disk in $\mathbb{C}$; the disk of radius r in $\mathbb{C}$; the disk $\{\exp it\} \times D^2$ (Definitions 1.1, 5.2).
Δ_n $(n \geqslant 0)$; Δ	The discriminant locus in E_n ($\Delta_0 = \varnothing$); any Δ_n (Def. 1.1).
$\partial_i D$ $(i = 1, 2)$	Half the boundary of D (a solid torus) (Def. 1.1).
E_n $(n \geqslant 0)$	Complex affine space of monic polynomials in T of degree n ($E_0 = \{0\}$), identified with the n-fold symmetric product of $\mathbb{C}$ (Def. 1.1).
e	Exponent sum of a braid (Rmk. 6.4).
F_{n-1} $(n \geqslant 0)$	The prebraid group, a free group of rank $n - 1$ (F_0 and F_{-1} are distinct trivial groups) (Def. 3.0).
gr	Graph (of a multi-valued function) (Def. 1.5).
$h; h(\partial_i D)$	A fixed smoothing of the bidisk boundary; a fixed solid torus in S^3 (Definitions 1.1, 5.2)
η_I	A height for the tongue T (Construction 4.5).
$I'_j; I_j; I$	An arc used to define a standard generator of the prebraid group (Def. 3.0); a related arc (after Ex. 3.3); the union of the I_j.
J	Any arc (the end-to-end union of $J_1, \ldots, J_{n-1}$) in $\mathbb{C}$ used to define a set of «standard generators» of the braid group B_n (for example, I) (Def. 3.4).
$l(\underline{\underline{b}}); l(\underline{\underline{x}})$	Length of a band representation $\underline{\underline{b}}$ or preband representation $\underline{\underline{x}}$ (Defs. 1.12, 4.7).
N	The squared-norm function (Def. 1.1)
$N'_j; N_j$	Certain 2-cells containing I'_j and I_j (Def. 3.0).
O	The unknot (*passim*).
$o; o^{(k)}, k > 0$	The identity of B_0; the identity of B_k (Convention 1.4).
P	A certain simple branched cover $S^3 \to S^3$ (Def. 4.9).
$p; p^{-1}; \tilde{p}$	A certain simple branched cover of $\mathbb{C}$ by $\mathbb{C}$, given by a complex polynomial of degree n; the inverse of p, as an n-valued map, or a map $\mathbb{C} \to E_n$; an approximation to p which is a monomial near ∞ (Props. 3.1 and 3.8, Construction 4.9).
$(p^{-1})_*$	The surjection of F_{n-1} onto B_n induced by p^{-1} (Prop. 3.8).
Q_j	A certain 2-cell containing $J_j \subset J$ (Def. 3.4).
$R(\tau, I)$	A geometric preband (Construction 4.5).
$r; \tilde{r}$	Radii related to p, $\tilde{p}$ (Construction 4.9).
$\rho(\underline{\underline{x}}); \hat{\rho}(\underline{\underline{x}})$	The prebraid, resp., closed prebraid, of a preband representation $\underline{\underline{x}}$ (Def. 4.7).
S^+	A Markov surface with its negative disks removed.
$S(\underline{\underline{b}}); S(\underline{\underline{x}})$	The braided surface in D or D^4 constructed from a band representation $\underline{\underline{b}}$ (Prop. 1.13); the braided surface in S^3

	constructed from a preband representation $\underline{\underline{x}}$ (Construction 4.14).
S^3	The round sphere of radius $\sqrt{2}$ (Def. 1.1).
S_n	The symmetric group on n letters (Def. 1.1).
$\Sigma(\underline{\underline{x}})$	The prebraided disk in S^3 constructed from a preband representation $\underline{\underline{x}}$ (Def. 4.13).
σ_j; $\sigma_{uv}^{\pm 1}$	The standard generators ($j = 1, \ldots, n-1$) of B_n (Def. 3.4); the embedded bands ($1 \leqslant u \leqslant v \leqslant n-1$) in B_n (Rmk. 5.13).
T	The tongue (Construction 4.5).
τ	A tadpole (Def. 4.2).
$\overline{U(\tau)}$	The head of the tadpole τ (Def. 4.2).
$\bar{u}_{m,n}$; $u_{m,n}$	The usual injection of F_{m-1} into F_{n-1}, or of B_m into B_n ($m \geqslant n$) (Def. 5.14).
V	The critical values of p (Prop. 3.1).
v_0; $v_i, i > 0$	The basepoint of F_{n-1}; the elements of V (Def. 3.0).
$X(L)$	The maximal Euler characteristic of a Seifert surface for the link L (Notation 5.2).
X_r, X_s	the ribbon and slice analogues of X (Def. 6.5).
x_i ($1 \leqslant i \leqslant n-1$)	The standard generators of F_{n-1} (Def. 3.0).
$\underline{\underline{x}}$; $x(i)$	A preband representation; the i^{th} preband in $\underline{\underline{x}}$ (Def. 4.7).
$x_{uv}^{\pm 1}$	The elementary embedded prebands (Thm. 5.12).
ϕ	The angular coordinate in $h(\partial_2 D)$ (Notation 5.2).
θ; θ_S	The angular coordinate in $h(\partial_1 D)$; its restriction to the part of a surface S in $h(\partial_1 D)$.
$\hat{}$	Operation of closure applied to a braid or prebraid (Defs. 1.12, 4.7).
$\_^{(k)}$	When applied to a braid, shorthand for a usual injection; when applied to a link, split sum with a trivial link of k unknotted components (Notation 5.15).
∇_n	An element of B_n usually denoted Δ_n^2 (Def. 6.14).

When a space is given explicitly as a Cartesian product of two or more factors, the notation pr_i denotes projection onto the i^{th} factor. The restriction of a mapping f to a subset M of its domain is denoted $f|M$.

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