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The Dirichlet Problem for the Degenerate Monge-Ampère Equation

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Dedicated to Alberto Calderón on his 65th birthday

Let Ω be a bounded convex domain in R^n with smooth, strictly convex boundary $\partial\Omega$, i.e. the principal curvatures of $\partial\Omega$ are all positive. We study the problem of finding a convex function u in Ω such that

$$\det(u_{ij}) = 0 \quad \text{in} \quad \Omega \tag{1}$$

$$u = \phi \quad \text{given on } \partial\Omega. \tag{2}$$

Here $u_i = \partial u/\partial x_i$, $u_{ij} = \partial^2 u/\partial x_i \partial x_j$ etc. The existence of a smooth solution in $\bar{\Omega}$ satisfying (2) of the corresponding elliptic problem

$$\det(u_{ij}) = \psi > 0 \quad \text{in} \quad \bar{\Omega} \tag{1$'$}$$

has been shown recently by N. V. Krylov [5] and the authors [3] in case ψ and ϕ are sufficiently smooth. It is of interest to treat the degenerate problem (1), (2). The corresponding question for degenerate complex Monge-Ampère equations is also of interest: find a plurisubharmonic function w in a bounded strictly pseudoconvex domain Ω in $\mathbb{C}^n$ satisfying

$$\det(w_{z_j \bar{z}_k}) = 0 \quad \text{in} \quad \Omega \tag{3}$$

and (2). In fact in [4], with J. J. Kohn, we treated the equation

$$\det(w_{z_j \bar{z}_k}) = \psi \geqslant 0 \quad \text{in} \quad \Omega \tag{3$'$}$$

and showed that there is a plurisubharmonic solution w belonging to $C^{1,1}(\bar{\Omega})$ provided $\psi \not\equiv 0$, ψ satisfies some other conditions, and ψ and ϕ are sufficiently smooth. See [4] for further references on (3)′, (2). E. Bedford and J. E. Fornaess [1] presented an example with ψ satisfying all our conditions and for which the (unique) solution of (1)′ is not in $C^2(\Omega)$.

Several authors have studied the Dirichlet problem (1), (2): J. Rauch and B. A. Taylor [6], Bedford and Taylor [2] as well as [1], and, most recently, N. S. Trudinger and J. I. F. Urbas [7]. The unique solution of (1), (2) is given by

$$(4) \qquad u(x) = \max\{v(x) \mid v \in C(\bar{\Omega}), \quad v \text{ convex and } \quad v \leqslant \phi \quad \text{on} \quad \partial\Omega\},$$

and the papers cited study the regularity of u. The best regularity result is that in [7] where it is shown that if $\partial\Omega \in C^{1,1}$ and $\phi \in C^{1,1}(\partial\Omega)$ then the function u belongs to $C^{0,1}(\bar{\Omega})$ and to $C^{1,1}$ in every compact subset of Ω. In this paper we prove an extension up to the boundary of this regularity in case ϕ is sufficiently smooth.

Theorem. *Assume $\partial\Omega$ is in $C^{3,1}$, and $\phi \in C^{3,1}(\partial\Omega)$; then the function u defined by* (4) *is of class $C^{1,1}(\bar{\Omega})$.*

We do not know if a corresponding estimate holds for the solution w of (3), (2) —which is characterized by

$$(4)' \quad w(z) = \max\{v(z) \mid v \in C(\bar{\Omega}), \ v \text{ plurisubharmonic in } \Omega \text{ and } v \leqslant \phi \text{ on } \partial\Omega\}.$$

The following examples show that the theorem is optimal; here Ω is the unit disc centred at the origin in the (x, y) plane.

Ex. 1. The function

$$u = (1 + y)^{2-\epsilon}, \qquad 0 < \epsilon \quad \text{small}$$

satisfies (1) and (2) with

$$\phi \in C^{3, 1-2\epsilon}(\partial\Omega).$$

The function u is of class $C^{1,1-\epsilon}$ but not $C^{1,1}$ in $\bar{\Omega}$.

The same u shows that if we flatten $\partial\Omega$ at $(0, -1)$ to make the curvature of $\partial\Omega$ vanish here to high order, ϕ may be very smooth while u is not in $C^{1,\delta}(\bar{\Omega})$ for $\delta > 1 - \epsilon$.

Ex. 2. (This is due to John Urbas). Let

$$\phi(y) \equiv \left(y^2 - \frac{1}{4}\right)^2$$

this is convex in $y > \frac{1}{2}$. It is easily verified that the function u defined by (4) satisfies the conditions in Ω: $u \equiv 0$ in the inscribed rectangle in Ω bounded on top and bottom by $|y| = 1/2$, and $u \equiv \phi(y)$ above the rectangle.

Here $\phi \in C^\infty(\partial\Omega)$ but u is not in C^2 near $\left(0, \frac{1}{2}\right)$.

In [5], [3], a priori estimates are established for the C^2 norms of solutions of (1)′, (2). It seems natural to try to establish such estimates (independent of ϵ for $0 < \epsilon \leqslant 1$) of convex solutions of, say,

$$(1)_\epsilon \qquad \det(u_{ij}) = \epsilon \quad \text{in} \quad \Omega$$

satisfying (2). However we have not done this. Instead we work directly with the characterization (4), and this paper is quite independent of [3].

Section 1. Beginning of proof

To prove the theorem we will establish the following estimate: there is constant A depending on Ω and the $C^{3,1}$ norm of ϕ such that

$$(5) \qquad |u(y) - u(x) - (y - x) \cdot \nabla u(x)| \leqslant A|x - y|^2$$

holds for every pair of points x, y in Ω. It is easy to see that (5) then yields the conclusion of the Theorem. We have, namely, to show that for some constant B depending only on Ω and on A,

$$(6) \qquad |\nabla u(x) - \nabla u(y)| \leqslant B|x - y| \quad \text{for all} \quad x, y \in \Omega.$$

Observe first that there are fixed positive numbers ϵ, δ depending only on Ω such that for every $y \in \Omega$, Ω contains a truncated cone

$$K(y) = \{z \neq y \mid |y - z| < \epsilon, \text{ and the angle } (z - y) \text{ makes with some unit vector } \xi(y) \text{ is less than } \delta\}.$$

It follows that there is a positive number α depending only on δ (and so only on Ω) such that for any vector $\eta \neq 0$ there is a point z in $K(y)$ such that

$$|(z - y) \cdot \eta| \geqslant \alpha |z - y| \cdot |\eta|.$$

Clearly there is such a z in $K(y)$ with $|z - y| =$ any positive number $t < \epsilon$.

To establish (6), since by [6] and [7] we have a bound $|\nabla u| \leqslant$ constant, we may suppose $|x - y| < \epsilon$ (the ϵ determined above). After subtraction of an affine function we may assume

$$u(x) = |\nabla u(x)| = 0,$$

and we have then to show

$$|\nabla u(y)| \leqslant B|x-y| \quad \text{if} \quad |x-y| < \epsilon. \tag{6)'$$

By (5), we have for every $z \in \Omega$,

$$|u(z)| \leqslant A|z-x|^2$$

and

$$|u(z) - u(y) - (z-y) \cdot \nabla u(y)| \leqslant A|z-y|^2,$$

and of course

$$|u(y)| \leqslant A|y-x|^2.$$

Hence

$$|(z-y) \cdot \nabla u(y)| \leqslant A(|z-x|^2 + |z-y|^2 + |y-x|^2). \tag{7}$$

Using what we have asserted above, we may choose z in Ω with $|z-y| =$ $= |x-y| < \epsilon$ and such that

$$|(z-y) \cdot \nabla u(y)| \geqslant \alpha|z-y| \cdot |\nabla u(y)|.$$

(for we may suppose $\nabla u(y) \neq 0$). Combining this with (7) we obtain

$$\alpha|x-y| \cdot |\nabla u(y)| \leqslant 6A|x-y|^2,$$

i.e., (6)', with $B = 6A/\alpha$.

We will derive (5) from the following local form, the heart of our proof:

Proposition. *There is a constant C depending only on Ω and the $C^{3,1}$ norm of ϕ, such that for every point x^0 in Ω, $\exists\epsilon(x^0) > 0$ so that for every x in Ω with $|x-x^0| \leqslant \epsilon(x^0)$, we have*

$$|u(x) - u(x^0) - (x-x^0) \cdot \nabla u(x^0)| \leqslant C|x-x^0|^2. \tag{8}$$

We claim that the proposition yields (5) with any constant A greater than C (and hence in the limit for $A = C$). Fix a constant $A > C$. If (5) does not hold, there are points x, y in Ω such that

$$|u(y) - u(x) - (y-x) \cdot \nabla u(x)| > A|x-y|^2.$$

By (8) there is a closest point z in $\bar{\Omega}$ to x such that

$$|u(z) - u(x) - (z-x) \cdot \nabla u(x)| = A|z-x|^2.$$

On the closed segment L joining x to z consider the function

$$f(y) = A|y - x|^2 - [u(y) - u(x) - (y - x) \cdot \nabla u(x)].$$

By (8), this is positive for y near x, $y \neq x$, but it is zero at x and z. So f achieves a positive maximum at some point w in the interior of L. At w the second difference quotient

(9) $\Delta_h^2 f = f(w + h(z - x)) + f(w - h(z - x)) - 2f(w) \leqslant 0 \quad \text{for} \quad |h| \quad \text{small.}$

But

$$\Delta_h^2 f = 2Ah^2|z - x|^2 - [u(w + h(z - x)] + u[w - h(z - x)) - 2u(w)]$$

and by (8) with x^0 replaced by w, the expression in the square bracket satisfies

$$|[\quad]| \leqslant 2Ch^2|z - x|^2 \quad \text{for} \quad |h| \quad \text{small.}$$

Thus

$$\Delta_h^2 f \geqslant 2(A - C)h^2|z - x|^2 \quad \text{for} \quad |h| \quad \text{small,}$$

contradicting (9). □

To complete the proof of the theorem we have to prove the Proposition. First some preliminary simple lemmas.

Lemma 1. *Let S be a straight segment with endpoints on $\partial\Omega$. Suppose one of these is the origin and that S makes an angle $\pi/2 - \alpha$ with the interior normal to $\partial\Omega$ there, $0 < \alpha$ small. Then length of $S = 0(\alpha)$.*

PROOF. We may assume the positive x_n axis is interior normal to $\partial\Omega$ at the origin and that S is of the form

$$x_n = \beta x_1, \qquad 0 \leqslant x_1 \leqslant t, \qquad \beta = \tan\alpha, \qquad x_2 = \cdots = x_{n-1} = 0.$$

At the other end point of S we have

$$x_n = c_0 t^2 + O(|t|^3) = \beta t,$$

$c \geqslant c_0$ a fixed positive constant. It follows that

$$t = O(\beta) = O(\alpha)$$

and also $x_n = O(\alpha^2)$. □

In the following lemma, u is the function defined by (4).

Lemma 2. *Let x^0 be any point in Ω. Subtracting from u a linear plane of support there we may suppose*

$$u \geqslant 0, \qquad u(x^0) = 0.$$

Then x^0 is a convex combination of $(n+1)$ points $x^1, \ldots, x^{n+1}$ in $\partial\Omega$ with $u(x^i) = 0$ for $i = 1, \ldots, n+1$.

PROOF. By Caratheodory's theorem it suffices to show that x^0 is in the convex hull of

$$S = \{x \in \partial\Omega \mid u(x) = 0\}.$$

If not, there is a hyperplane separating them; i.e. every point x in S satisfies (after rotation and traslation of coordinates)

$$x_n < -\epsilon < \epsilon < x_n^0.$$

Thus at points on $\partial\Omega$ where $x_n \geqslant 0$ we have $u \geqslant a > 0$ for some positive constant a. But then the function

$$v = \delta x_n \quad \text{for} \quad 0 < \delta \quad \text{small}$$

satisfies

$$v \leqslant u \quad \text{on} \quad \partial\Omega.$$

Consequently, by (4), $u(x^0) \geqslant v(x^0) > \delta\epsilon$ —contradicting the fact that $u(x^0) = 0$. □

Section 2. Proof of the Proposition

We will say that a constant is under control if it depends only on Ω and the $C^{3,1}$ norm of ϕ. Fix $x^0 \in \Omega$. After subtraction of an affine function we may suppose $u \geqslant 0$, $u(x^0) = 0$. According to Lemma 2, x^0 lies in a n-dimensional simplex S with vertices on $\partial\Omega$, and on which $u \equiv 0$. It may be that x^0 lies in a lower, say k, dimensional simplex with this property. Using induction on k we will prove (8).

(i) Postponing the case $k = 1$, suppose we have proved the result for all x^0 lying in any $(k-1)$-dimensional simplex with the stated properties, and with constant $C = C_{k-1}$ under control. We wish to prove it for x^0 in such a k-dimensional simplex S with some constant C_k also under control. Let x^1 be the closest point to x^0 on any $(k-1)$-dimensional face of S. Then $y^0 = 2x^0 -$

$-x^1$ lies in S. By induction, in the ball B (around x^1 with radius $\epsilon(x^1)$, we have

$$|u(x)| \leqslant C_{k-1}|x-x^1|^2.$$

It follows by convexity of u (recall $u(y^0)=0$) that in the ball around x^0 with radius $\epsilon(x^0) := \frac{1}{2}\epsilon(x^1)$

$$|u(x)| \leqslant 2C_{k-1}|x-x^0|^2.$$

Thus we have established (8) with a constant $C_k = 2C_{k-1}$.

Consequently (8) holds for any x^0 in Ω with

$$C = 2^{n-1}C_1$$

and the proof is finished —once we have treated the case $k=1$.

(ii) Turning to that case, we suppose x^0 lies in a segment S with end points x^1, x^2 on $\partial\Omega$, on which $u=0$, and that $u \geqslant 0$ in Ω. Of the two end points, suppose that x^2 is the closer to x^0. We may suppose $x^2 = 0$ and that the positive x_n-axis is interior normal to $\partial\Omega$ there. In addition we may suppose $x^0 = (x_1^0, 0, \dots, 0, x_n^0)$ with $x_1^0 \geqslant 0$, $x_n^0 = x_1^0 \tan\alpha$, $0 < \alpha \leqslant \frac{\pi}{2}$, i.e., S makes angle $\frac{\pi}{2} - \alpha$ with the interior normal to $\partial\Omega$ at 0.

By Lemma 1,

$$|x^1| = O(\alpha). \tag{10}$$

We will distinguish two cases

$\alpha \leqslant \alpha_0$, a positive small constant to be chosen,

$\alpha > \alpha_0$.

Consider first the case $\alpha \leqslant \alpha_0$ small. We have $x^1 = (x_1^1, 0, \dots, 0, x_1^1 \tan\alpha)$. The orthogonal projection on the plane $x_n = 0$ of the segment joining the origin to x^1 is the segment L on the x_1 axis from the origin to $(x_1^1, 0, \dots, 0)$. If we think of the boundary values ϕ near 0 as a nonnegative function of $(x_1, \dots, x_{n-1})$, then at the end points of the segment L we have $\phi = 0$ and hence $\phi_1 = 0$, and $\phi_{11} \geqslant 0$ there. It follows that

$$\int_L \phi_{11}\, dx_1 = 0. \tag{11}$$

Hence ϕ_{11} has an interior minimum in L, where necessarily $\phi_{111} = 0$. Consequently on L, $|\phi_{111}| \leqslant \text{constant} \cdot x_1^1$; *here, at last, is where we use the fact that the $C^{3,1}$ norm of ϕ is finite.* By (10), we have $|\phi_{111}| \leqslant \text{constant} \cdot \alpha$, and since ϕ_{11} necessarily vanishes somewhere on L, see (11), it follows that

$$|\phi_{11}(0)| \leqslant A\alpha^2 \tag{12}$$

with A under control.

For other second derivatives of ϕ at the origin we have

$$|\phi_{ij}(0)| \leqslant A \tag{13}$$

while from (12) it follows that

$$|\phi_{1\beta}(0)| \leqslant A\alpha, \qquad 1 < \beta \leqslant n-1, \tag{14}$$

since $\phi \geqslant 0$ and, hence, $\sum \phi_{ij}(0)x_i x_j \geqslant 0$.

Consider now a ball B in Ω with centre x^0 and radius $\epsilon \leqslant \alpha^3$. Let Γ be the cone with vertex x^1 generated by B and set $K = \partial\Omega \cap \Gamma$; recall that $|x^1 - x^0| \geqslant |x^0|$. For $\alpha \leqslant \alpha_0$ sufficiently small, depending only on Ω, it is not difficult to verify that for ϵ sufficiently small (depending possibly on the point x^0), the orthogonal projection of K onto the plane $x_n = 0$ is contained in an ellipsoid with axes

$$C\frac{\epsilon}{\alpha} \quad \text{in the } x_1\text{-direction,} \quad C\epsilon \quad \text{in the other directions,}$$

for a fixed constant C depending only on Ω. Let us now fix such α_0.

For any point x in the ϵ-ball B let $\bar{x}$ represent the point on $\partial\Omega$ where the ray from x^1 to x strikes $\partial\Omega$. Since $u(x^1) = 0$ it follows from convexity that

$$u(x) \leqslant \phi(\bar{x}). \tag{15}$$

Set $\delta = |x - x^0|$. As described above, $(\bar{x}_1, \ldots, \bar{x}_{n-1})$ lies in the ellipsoid

$$\alpha^2 x_1^2 + x_2^2 + \cdots + x_{n-1}^2 \leqslant C^2\delta^2. \tag{16}$$

Consequently

$$\phi(\bar{x}) = \sum_{i,j=1}^{n-1} \phi_{ij}(0)\bar{x}_i\bar{x}_j + O(|\bar{x}|^3).$$

Using (12), (13), (14), and (16) it follows that

$$\phi(\bar{x}) \leqslant CA\delta^2 + C\frac{\delta^3}{\alpha^3}$$

with C a (different) constant under control. By (15), and the relations $|x - x^0| = \delta \leqslant \epsilon \leqslant \alpha^3$, we find

$$u(x) \leqslant C_1|x - x^0|^2.$$

We have proved (8) for $\alpha \leqslant \alpha_0$.

The other case to consider is $\alpha \geqslant \alpha_0$; this case is simple. There is a positive ϵ such that if $|x - x^0| < \epsilon$, then the point $\bar{x}$ on $\partial\Omega$ where the ray from x^1 to

x strikes $\partial\Omega$ satisfies

$$|\bar{x}| \leqslant C|x - x^0|$$

with C depending only on Ω. Here we use also the fact that $|x^1 - x^0| \geqslant |x^0|$. Since $\phi \geqslant 0$ and $\phi(0) = 0$ we have $\phi(\bar{x}) \leqslant A|\bar{x}|^2$ where A depends on the C^2 norm of ϕ. By convexity of u, and the fact that $u(x^1) = 0$ we conclude that

$$u(x) \leqslant \phi(\bar{x}) \leqslant A|\bar{x}|^2 \leqslant AC^2|x - x^0|^2$$

i.e. (8) holds again in this case.

The proofs of the Proposition and of the Theorem are complete.

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