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Well-Posedness of the Euler and Navier-Stokes Equations in the Lebesgue Spaces $L^p_s(\mathbb{R}^2)$

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Dedicated to A. P. Calderón

Introduction

In this paper we show that the Euler equation for incompressible fluids in $\mathbb{R}^2$ is well posed in the (vector-valued) *Lebesgue spaces*

$$L^p_s = (1 - \Delta)^{-s/2} L^p(\mathbb{R}^2) \quad \text{with} \quad s > 1 + 2/p, \qquad 1 < p < \infty,$$

and that the same is true of the Navier-Stokes equation uniformly in the viscosity ν. The solutions obtained are classical since $L^p_s \subset C^1$. (The L^p_s are also called generalized Sobolev spaces, Liouville spaces, or spaces of Bessel potentials; see Calderón [4], Stein [16], Bergh-Löfström [3]. We use the notation L^p_s indiscriminately for scalar and vector valued functions.)

In well-posedness we include existence, uniqueness and persistence. (Persistence means that the state $u(t)$ of the fluid at time t belongs to the same function space X as does the initial state and describes a continuous curve in X.) The continuity of the map $u(0) \to u(t)$ in L^p_s is not considered here, however.

The standard norm for $u \in L^p_s$ is $\|(1 - \Delta)^{s/2} u\|_p$, which may admit different interpretations in the vector case. But any equivalent norms are acceptable for our purposes.

The initial value problem for the Navier-Stokes equation in space domain $\mathbb{R}^m$, $m = 2, 3, \ldots$, may be written

$$\partial_t u - \nu \Delta u + (u \cdot \partial)u + \partial \pi = f \qquad (x \in \mathbb{R}^m, t > 0),$$
$$\operatorname{div} u = 0, \qquad u(x, 0) = \phi(x),$$

where $\partial_t = \partial/\partial t$, $\partial = \text{grad} = (\partial_1, \ldots, \partial_m) = (\partial/\partial x_1, \ldots, \partial/\partial x_m)$; $u = u(x, t) = (u_1(x, t), \ldots, u_m(x, t))$ is the velocity field; $\pi = \pi(x, t)$ is the pressure; $\nu > 0$ is the kinematic viscosity; $(u \cdot \partial)u$ has j-component $u_k \partial_k u_j$ (with summation convention); and $f = f(x, t)$ is a given external force.

The initial value problem for the Euler equation is formally given by setting $\nu = 0$ in the Navier-Stokes problem.

In these equations, it is convenient to eliminate the pressure π by applying the projection operator P, which projects into solenoidal vectors and annihilates gradients, to transform them into

$$\text{(NS)} \qquad \partial_t u - \nu \Delta u + F(u) = f, \qquad Pu = u, \qquad Pf = f, \qquad u(0) = \phi.$$
$$\text{(E)} \qquad \partial_t u + F(u) = f, \qquad Pu = u, \qquad Pf = f, \qquad u(0) = \phi,$$

where

$$F(u) = F(u, u), \qquad F(u, v) = P(u \cdot \partial)v.$$

The pressure can then be recovered by

$$\partial \pi = -(1 - P)(u \cdot \partial)u,$$

and by subsequent integration of $\partial \pi$. There is no difficulty in the integration since $\partial \pi$ will be continuous and bounded in most cases we consider, but π will be determined only up to an additive constant depending on t, and π may be unbounded.

P is formally given by $P = (1 - \partial_j \partial_k \Delta^{-1})$, and is a pseudo-differential operator with a matrix symbol $(\delta_{jk} - \xi_j \xi_k / |\xi|^2)$; P is a continuous operator on L_s^p into itself, as is seen by Mihlin's theorem (see [13]). It is in fact a singular integral operator with a matrix kernel of Calderón-Zygmund type (see [5]) except for some delta-function components in diagonal elements.

The existence and uniqueness for the Navier-Stokes equation in $\mathbb{R}^2$ was proved by Leray [11]; here persistence holds with $X = PL^2$. The existence and uniqueness for the Euler equation in a domain $\Omega \subset \mathbb{R}^2$ (interior or exterior) was proved by Wolibner [18]; here it is assumed that the initial vorticity rot ϕ is Hölder continuous and bounded by const$(1 + |x|)^{-1-\epsilon}$, and persistence holds in the corresponding space for the velocity field. Analogous results were proved by Golovkin [6] and McGrath [12], in which the Navier stokes equation was also studied together with the convergence problem as $\nu \to 0$. The function spaces used by the latter authors where more restricted than Wolibner's, and the persistence property was not obious in their cases. See also Yudovic [19] and Bardos [1] for related results in bounded domains in $\mathbb{R}^2$.

It is of some interest to find other function spaces X in which one has well-posedness for the Euler equation as well as for the Navier-Stokes equation with uniformity in ν. It is shown in Kato [9] that $X = PL_s^2$ has this property if $s > 2$. In [14] Ponce proves that $X = PL_s^p \cap L^2$ has the property if $2 \leqslant p < \infty$ and $s \geqslant 2$, except the case $s = p = 2$. In the present paper we shall extend these results to all $X = PL_s^p$ with $s > 1 + 2/p$, $1 < p < \infty$. It may be noted that if $p > 2$, the solution need not have a finite energy ($= L^2$-norm squared).

The plan of this paper is as follows. In section 1 we prove that (NS) is locally (in time) well-posed in $PL_s^p(\mathbb{R}^m)$ with $1 < p < \infty$, $s > 1 + m/p$ (Theorem I). In the remainder of the paper we assume $m = 2$, $s > 1 + 2/p$, and $f = 0$ for simplicity. In section 2 we prepare some lemmas on linearized vorticity equation derived from (NS), with a view to obtaining uniform estimates in ν. Using these results, we prove in section 3 that (NS) is globally well-posed in PL_s^p, with uniform estimates for $\nu > 0$ (Theorem II). In section 4, we prove the main theorem on the global well-posedness of (E) (Theorem III), together with a convergente theorem as $\nu \to 0$ (Theorem IV). In Appendix we prove some of the lemmas used in the proofs.

1. Local theory for the N-S equation

In this section we prove that the Navier-Stokes equation in $\mathbb{R}^m$ is locally (in time) well posed in the Lebesgue spaces PL_s^p (see introduction for the definition). In what follows we simply denote by L_s^p either $L_s^p(\mathbb{R}^m)$ or $L_s^p(\mathbb{R}^m; \mathbb{R}^m)$ (vector-valued functions), and by PL_s^p the subspace of the latter restricted by the divergence condition $\partial \cdot u = \partial_j u_j = 0$ (with summation convention). We denote the norm in L_s^p (scalar or vector-valued) by $\| \ \|_{s,p}$.

The main result of this section is given by

Theorem I. *Let*

$$s > 1 + m/p, \qquad 1 < p < \infty. \tag{1.1}$$

Let $\phi \in PL_s^p$ *and* $f \in C([0, T_0]; PL_s^p)$. *Then there is* $T > 0$, $T \leqslant T_0$, *and a unique solution to* (NS) *such that*

$$u \in C([0, T]; PL_s^p) \cap C((0, T]; PL_{s+1}^p) \cap C^1((0, T]; PL_{s-1}^p). \tag{1.2}$$

Moreover, we have

$$\|u(t)\|_{s+1,p} \leqslant Kt^{-1/2}, \qquad \|\partial_t u(t)\|_{s-1,p} \leqslant Kt^{-1/2}. \tag{1.3}$$

T and K depend on m, s, p, $\nu > 0$, $\|\phi\|_{s,p}$ *and the norm of f in* $C([0, T_0]; L_s^p)$. *In particular if* $f = 0$, *then* $u \in C^\infty((0, T); L_\infty^p)$.

Remark 1.1. (*a*) The assumptions on ϕ and f can be considerably weakened if $u(t) \in L_s^p$ is not required to be continuous up to $t = 0$.

(*b*) In this theorem only a single value of p is involved. If $p > 2$, the solution $u(t)$ need not have a finite energy ($= L^2$-norm).

For the proof of Theorem I, we shall use the method of integral equation as in previous publications [7,10] (cf. also [8]). Since the proof is almost identical with (and simpler than) those given in these papers, it will suffice to indicate only the main points.

In this section we frequently write $\| \ \|_s$ for $\| \ \|_{s,p}$, since p will be fixed throughout. The operator norm from L_s^p into $L_{s'}^p$ will be denoted by $\| \ \|_{s;s'}$. Also we assume that $\nu = 1$ without loss of generality, since we keep $\nu > 0$ fixed.

First we convert (NS) into the integral equation

$$u(t) = u_0(t) + \int_0^t U(t - \tau)F(u(\tau))\, d\tau, \tag{1.4}$$

where

$$U(t) = \exp(t\Delta), \qquad u_0(t) = U(t)\phi + \int_0^t U(t - \tau)f(\tau)\, d\tau. \tag{1.5}$$

Lemma 1.2. $(U(t); t \geqslant 0)$ forms a contractive and analytic C_0-semigroup on L_s^p as well as on PL_s^p. Moreover, $U(t)$ is bounded for $t > 0$ from PL_s^p into $PL_{s'}^p$, with $s' > s$, with the operator norm

$$\|U(t)\|_{s;s'} \leqslant c(1 + t^{-(s'-s)/2}) \qquad (t > 0), \tag{1.6}$$

where c depends on $s' - s$ (in addition to m and p, which are fixed), but it may be taken as a constant as long as s and s' vary on a finite interval.

PROOF. The semigroup properties of $U(t)$ in L^p are well known. The same properties hold in L_s^p and PL_s^p because $U(t)$ commutes with $(1 - \Delta)^{s/2}$ and with P. For the same reason, we have

$$\|U(t)\|_{s;s'} = \|(1 - \Delta)^{(s'-s)/2}U(t)\|_{0;0}.$$

If $(s' - s)/2$ is an integer, (1.6) follows immediately from this. The general case can be dealt with by interpolation. (Or one may apply Mihlin's theorem directly to the symbol $(1 + |\xi|^2)^{(s'-s)/2} \exp(-t|\xi|^2)$.) □

The basic property of the nonlinear operator $F(u, v)$ is given by

Lemma 1.3. *If $u \in L_{s-1}^p$ and $v \in L_s^p$ with $s > 1 + m/p$, then $F(u, v) \in PL_{s-1}^p$ with*

$$\|F(u, v)\|_{s-1,p} \leqslant c\|u\|_{s-1,p}\|v\|_{s,p}. \tag{1.7}$$

Proof. This is an immediate consequence of the fact that (the scalar-valued) L_{s-1}^p is a Banach algebra under pointwise multiplication (see Stein [16], Strichartz [17]). □

Lemma 1.4. *Let $G(u, v) = (1 - P)(u \cdot \partial)v$. If $u, v \in PL_s^p$ with $s > 1 + m/p$, then $G(u, v) = G(v, u) \in L_s^p$ with*

$$\|G(u, v)\|_{s,p} \leqslant c\|u\|_{s,p}\|v\|_{s,p}. \tag{1.8}$$

Proof. Since $(u \cdot \partial)v - (v \cdot \partial)u$ has divergence zero, it is annihilated by $1 - P$, showing that $G(u, v) = G(v, u)$. This vector is a gradient, being in the range of $1 - P$. Writing $G(u, v) = \partial\pi$, we have $\Delta\pi = \partial \cdot (u \cdot \partial)v = \partial u \cdot \partial v$ ($= (\partial_j v_k) \cdot \cdot (\partial_k v_j)$) $\in L_{s-1}^p$, again due to L_{s-1}^p being an algebra. Since $\partial\pi$ and $\Delta\pi$ are both in L_{s-1}^p, it follows that $\partial\pi \in L_s^p$, with the desired inequality (1.8). □

Lemma 1.5. *The free term $u_0(t)$ on the right of* (1.4) *belongs to the class* (1.2) *with T replaced with T_0, and satisfies* (1.3).

Proof. This follows easily from (1.5), (1.6); note that u_0 is the solution of the heat equation $\partial_t u_0 = \Delta u_0 + f$ with the initial condition $u_0(0) = \phi$. □

Lemma 1.6. *Under the assumptions of Theorem I, there is $T > 0$ and a unique solution of the integral equation* (1.4) *in the class*

$$u \in C([0, T]; PL_s^p). \tag{1.9}$$

Proof. (1.4) is a regular integral equation of Volterra type, and can be solved by successive approximation, starting with $u_0(t)$ as the first approximation. Note that F maps $u(\tau) \in PL_s^p$ into PL_{s-1}^p, which is mapped back into PL_s^p by $U(t - \tau)$; the associated operator norm does not exceed $c(t - \tau)^{-1/2}$ by (1.6), which is integrable. The successive approximation converges on a certain interval $[0, T]$ and gives a unique solution to (1.4) in the class (1.9). For details cf. [10]. □

Lemma 1.7. *A solution u of the integral equation* (1.4) *in the class* (1.9) *is a solution of the differential equation* (NS) *in the class* (1.2). *The converse is also true.*

PROOF. The converse is easily proved by integrating

$$(\partial/\partial\tau)(U(t-\tau)u(\tau)) = U(t-\tau)(F(u(\tau)) + f(\tau))$$

in $\tau \in [0, t]$.

To prove the first part, let u be a solution of (1.4) considered and let $v(t)$ denote the second term on the right of (1.4). Then $\|F(u(t))\|_{s-1} \leqslant K = \text{const}$ and so $\|v(t)\|_{s+1/2} \leqslant Kt^{1/4} \leqslant K$ because $\|U(t-\tau)\|_{s-1;s+1/2} \leqslant ct^{-3/4}$. Since $\|u_0(t)\|_{s+1/2} \leqslant Kt^{-1/4}$, as is seen from (1.5) and (1.6), the same is true for $u(t)$ and therefore $\|F(u(t))\|_{s-1/2} \leqslant Kt^{-1/4}$ by (1.7). Hence $\|v(t)\|_{s+1} \leqslant \leqslant K\int_0^t (t-\tau)^{-3/4}\tau^{-1/4}\,d\tau \leqslant K$, so that $\|u(t)\|_{s+1} \leqslant Kt^{-1/2}$. With a little argument about continuity, this shows that v, and hence u too, belongs to $C((0,T]:PL^p_{s+1})$. Since the validity of the differential equation (NS) in a lower level, say in PL^p_{s-2} is obvious, it follows that $\partial_t u \in C((0,T];L^p_{s-1})$, completing the proof of Lemma 1.7. □

PROOF OF THEOREM I. The main part of Theorem I follows from Lemmas 1.6 and 1.7. To prove the last assertion for $f = 0$, we repeat the bootstrap argument given above to show that $u \in C([0,T];PL^p_s) \cap C((0,T);L^p_\infty)$. The differentiability of $u(t)$ in $t > 0$ then follows by repeated differentiation of (NS) in t.

2. The linearized vorticity equation in $\mathbb{R}^2$

From now on we assume $m = 2$. We use the following notations. $\|\ \|_{s,p}$ for the L^p_s-norm; $\|\ \|_p$ for $\|\ \|_{0,p}$; $\|\ \|_{s,p;r,q}$ for the operator norm from L^p_s to L^q_r; $\|\ \|_{p;q}$ for $\|\ \|_{0,p;0,q}$.

In this section we prove some lemmas on the linearized vorticity equation

$$\partial_t\zeta - \nu\Delta\zeta + a(t,x)\cdot\partial\zeta = 0, \qquad \zeta(0) = \omega, \tag{2.1}$$

where a is a given vector function such that

$$a \ \text{ and } \ \partial a \in C([0,T];L^\infty), \qquad \partial\cdot a = 0. \tag{2.2}$$

Lemma 2.1. *Let $1 < p < \infty$ and $0 \leqslant k \leqslant 1$. If $\omega \in L^p_k$, then* (2.1) *has a unique solution $\zeta \in C([0,T];L^p_k) \cap C((0,T];L^p_{k+1}) \cap C^1((0,T];L^p_{k-1})$, with*

$$\|\zeta(t)\|_p \leqslant \|\omega\|_p, \tag{2.3}$$

$$\|\zeta(t)\|_{k,p} \leqslant c\|\omega\|_{k,p}\exp\left(c\int_0^t \|\partial a(\tau)\|_\infty\,d\tau\right). \tag{2.3'}$$

Here and in what follows c denotes various constants that may depend on s, p, but not on ν or ω.

PROOF. (2.1) is a simple linear scalar equation of parabolic type and is easily solved. One may use the method of integral equation similar to the one used in the proof of Theorem I, to establish existence and uniqueness of the solution required by the theorem. The proof is simpler than before inasmuch as the equation is linear, the unknown ζ is scalar-valued, and there is no projection P involved. Due to the linearity, on the other hand, the smoothness of the solution depends on a. But $a(t)$ maps L_k^p into itself (from vectors to scalars), and this suffices for the proof.

It remains to prove the estimates (2.3) and (2.3′). First we prove (2.3) for $2 \leqslant p < \infty$. Let $J\zeta = \zeta^{p-1} \equiv |\zeta|^{p-2}\zeta$ be the duality map on L^p to its dual $L^{p'}$, normalized in such a way that $\|J\zeta\|_{p'} = \|\zeta\|_p^{p-1}$. It is easy to see that J not only maps L^p into $L^{p'}$ but also maps L_1^p into $L_1^{p'}$, continuously in both cases. Since $\partial_t \zeta(t) \in L_{-1}^p$ for $t > 0$, we obtain

$$\begin{aligned}\partial_t \|\zeta(t)\|_p^p &= p\langle J(\zeta(t)), \partial_t \zeta(t)\rangle \\ &\leqslant -\nu p\langle \partial\zeta(t)^{p-1}, \partial\zeta(t)\rangle - p\langle \zeta(t)^{p-1}, a(t)\cdot\partial\zeta(t)\rangle \\ &\leqslant -p(p-1)\nu \int |\zeta(t)|^{p-2}|\partial\zeta(t)|^2\,dx \leqslant 0,\end{aligned}$$

where integrations by parts have been made using the property $\partial \cdot a = 0$. Thus $\|\zeta(t)\|_p$ is monotone decreasing, proving (2.3) for $2 \leqslant p < \infty$.

This result may be conveniently expressed by using the evolution operator $U_\nu(t, \tau)$ associated with the linear equation (2.1). Thus

$$\|U_\nu(t,\tau)\|_{p;p} \leqslant 1. \tag{2.4}$$

Then we obtain $\|U_\nu(t,\tau)\|_{p';p'} \leqslant 1$ by duality; indeed, the adjoint equation to (2.1) has the same form, with the sign of a reversed, due to the condition $\partial \cdot a = 0$. This proves (2.4) and (2.3) for $1 < p < \infty$.

We note in passing that (2.3) had been proved in [12] using the fundamental solution of (2.1).

Next we prove (2.3′) for $k = 1$. On differentiating (2.1) in x, we obtain a system of equations for $\partial\zeta$:

$$\partial_t\partial\zeta - \nu\Delta\partial\zeta + a\cdot\partial\partial\zeta + \partial a\cdot\partial\zeta = 0, \qquad \partial\zeta(0) = \partial\omega.$$

Application of Duhamel's formula then leads to the integral equation

$$\partial\zeta(t) = U_\nu(t,0)\partial\omega - \int_0^t U_\nu(t,\tau)\partial a\cdot\partial\zeta(\tau)\,d\tau. \tag{2.5}$$

Using (2.4), we thus obtain the integral inequality

$$\|\partial\zeta(t)\|_p \leqslant \|\partial\omega\|_p + c\int_0^t \|\partial a(\tau)\|_\infty \|\partial\zeta(\tau)\|_p\,d\tau. \tag{2.6}$$

Combined with the previous estimate for $k = 0$, this leads to the estimate (2.3′) for $k = 1$. The constant c in (2.3′) depends on the way $\|\zeta\|_{1,p}$ depends on $\|\zeta\|_p$ and $\|\partial\zeta\|_p$.

Finally the case $0 < k < 1$ can be dealt with by interpolation (see [3]).

The result can again be expressed in terms of the evolution operator:

$$\|U_\nu(t,\tau)\|_{k,p;k,p} \leqslant c\exp\left(c\int_\tau^t \|\partial a(\rho)\|_\infty d\rho\right), \qquad 0 \leqslant k \leqslant 1. \quad \square \tag{2.6a}$$

Lemma 2.2. *In Lemma* 2.1 *assume that* $a \in C([0,T]; L_r^q)$, $1 < q < \infty$, $r > 1 + 2/q$. Let $1 < p \leqslant q$, $0 \leqslant k \leqslant [r]$. *If* $\omega \in L_k^p$, *then the solution* ζ *given by Lemma* 2.1 *belongs to* $C([0,T]; L_k^p)$ *with*

$$\|\zeta(t)\|_{k,p} \leqslant c\|\omega\|_{k,p}\exp\left(c\int_0^t \|a(\tau)\|_{r,q}\,d\tau\right). \tag{2.7}$$

PROOF. For $k = 0$, this is contained in Lemma 2.1; note that the assumption implies (2.2). Next consider the case when $k = [r]$. We differentiate (2.1) k-times, obtaining

$$\partial_t\partial^k\zeta - \nu\Delta\partial^k\zeta + a\cdot\partial\partial^k\zeta = -\sum_{j=1}^k c_j\partial^j a\cdot\partial^{k-j+1}\zeta$$

with $\partial^k\zeta(0) = \partial^k\omega$. Corresponding to (2.5) and (2.6) in the proof of Lemma 2.1, this leads to

$$\partial^k\zeta(t) = U_\nu(t,0)\partial^k\omega - \int_0^t U_\nu(t,\tau)\sum_{j=1}^k c_j\partial^j a(\tau)\cdot\partial^{k-j+1}\zeta(\tau)\,d\tau$$

and

$$\|\partial^k\zeta(t)\|_p \leqslant \|\partial^k\omega\|_p + c\int_0^t\sum_{j=1}^k \|\partial^j a(\tau)\cdot\partial^{k-j+1}\zeta(\tau)\|_p\,d\tau$$

$$\leqslant \|\omega\|_{r,p} + c\int_0^t \|a(\tau)\|_{r,q}\|\zeta(\tau)\|_{k,p}\,d\tau. \tag{2.8}$$

Here the Sobolev embedding theorem is used at the last stage (for details see Appendix). Since the same estimate holds if $\partial^k\zeta$ on the left is replaced by lower derivatives, we have a linear integral inequality for $\|\zeta(t)\|_{k,p}$ and obtain the desired result (2.7).

Since the same result holds for $k = 0$, the required result for $0 \leqslant k \leqslant [r]$ follows by interpolation. The result may again be expressed in terms of the evolution operator:

$$\|U_\nu(t,\tau)\|_{k,p;k,p} \leqslant c\exp\left(\int_\tau^t \|a(\rho)\|_{r,q}\,d\rho\right). \tag{2.7a}$$

3. Global theory for space domain $\mathbb{R}^2$

In this section we prove that (NS) is globally well-posed, uniformly for $\nu > 0$, with persistence in $PL_s^p = PL_s^p(\mathbb{R}^2)$. For simplicity we assume that $f = 0$, but the general case can be handled similarly.

Theorem II. *If $m = 2$, the solution u given by Theorem I can be continued to all $t > 0$, with the estimate*

$$\|u(t)\|_{s,p} \leqslant K(t). \tag{3.1}$$

Here and in what follows $K(t)$ denotes a real-valued continuous function defined for $0 \leqslant t < \infty$, depending on s, p, and $\|\phi\|_{s,p}$ but not on $\nu > 0$.

For the proof we need a lemma, which partially generalizes analogous results given in [2, 6, 9, 15]. (All these results contain a logarithmic function.)

Lemma 3.1. *Let $u \in PL_s^p$, $1 < p < \infty$, $s > 1 + 2/p$. Let $\zeta = \operatorname{rot} u \equiv \partial_1 u_2 - \partial_2 u_1$. Then*

$$\|\partial u\|_{s-1,p} \leqslant c\|\zeta\|_{s-1,p'} \tag{3.2a}$$

$$\|\partial u\|_\infty \leqslant c\|\zeta\|_p + c\|\zeta\|_\infty \log_+ (\|\zeta\|_{s-1,p}/\|\zeta\|_\infty), \tag{3.2b}$$

with c depending only on s and p. (Here we use the notation $\log_+(\lambda) = \max\{1, \log(\lambda)\} \geqslant 1$, slightly deviating from the commons usage.)

The proof of this lemma will be given in Appendix, in a generalized from valid in $\mathbb{R}^m$.

PROOF OF THEOREM II. First we assume that $2/p < s - 1 \leqslant 1$. As is well known, $\zeta = \operatorname{rot} u$ satisfies the vorticity equation (2.1) in which $a = u$; note that $u(t)$ and $\partial u(t)$ are in L^∞ because $u(t) \in L_s^p$ with $s > 1 + 2/p$. Thus we have by Lemma 2.1

$$\|\zeta(t)\|_{s-1,p} \leqslant c\|\omega\|_{s-1,p} \exp\left(c\int_0^t \|\partial u(\tau)\|_\infty d\tau\right). \tag{3.3}$$

Now we apply (3.2*b*) to the solution $u = u(t)$, $\zeta = \zeta(t) = \operatorname{rot} u(t)$. Here we may replace $\|\zeta(t)\|_p$ and $\|\zeta(t)\|_\infty$ respectively by their majorants $\|\omega\|_p$ and $\|\omega\|_\infty \leqslant c\|\omega\|_{s-1,p}$, since $\mu \log_+(\lambda/\mu)$ is monotone increasing in μ. ($\|\zeta(t)\|_\infty \leqslant \|\omega\|_\infty$ follows from (2.3) in the limit $p \to \infty$.) Thus

$$\|\partial u(t)\|_\infty \leqslant c\|\omega\|_p + c\|\omega\|_{s-1,p} \log_+(\|\zeta(t)\|_{s-1,p}/\|\omega\|_{s-1,p}).$$

Substitution from (3.3) then gives

$$\|\partial u(t)\|_\infty \leqslant c\|\omega\|_p + c\|\omega\|_{s-1,p}\left(1 + \int_0^t \|\partial u(\tau)\|_\infty d\tau\right).$$

In view of the fact that $\|\omega\|_p$ and $\|\omega\|_{s-1,p}$ are dominated by $c\|\phi\|_{s,p}$, application of the Gronwall inequality then shows that

$$\|\partial u(t)\|_\infty \leqslant K(t). \tag{3.4}$$

Another application of (3.3) then shows that $\|\zeta(t)\|_{s-1,p} \leqslant K(t)$. Then, using (3.2*a*) we conclude that

$$\|\partial u(t)\|_{s-1,p} \leqslant K(t). \tag{3.5}$$

In order to deduce (3.1) from (3.5), we need a uniform estimate for $\|u(t)\|_p$. This may be obtained from (1.4) by noting that (see (2.4)) $\|U_\nu(t)\|_{p;p} \leqslant 1$, where we have restored the parameter ν, writing $U_\nu(t) = \exp(\nu t\Delta)$. Since $\|F(u(\tau))\|_p \leqslant c\|u(\tau)\|_p\|\partial u(\tau)\|_\infty \leqslant K(\tau)\|u(\tau)\|_p$ by (3.4), we obtain

$$\|u(t)\|_p \leqslant \|\phi\|_p + \int_0^t K(\tau)\|u(\tau)\|_p d\tau,$$

which gives

$$\|u(t)\|_p \leqslant K(t). \tag{3.6}$$

Now (3.1) follows from (3.5) and (3.6) by virtue of the following lemma, which will be proved in Appendix in the general case of $\mathbb{R}^m$.

Lemma 3.2. *Let $v \in L^p$ with $\partial v \in L^p_{s-1}$, where $s \geqslant 1$, $1 < p < \infty$. Then $v \in L^p_s$, with $\|v\|_{s,p} \leqslant c(\|v\|_p + \|\partial v\|_{s-1,p})$.*

Of course the various estimates given above hold only in the interval on which u is known to exist. But the $K(t)$ are functions defined for $0 \leqslant t < \infty$, depending on s, p, $\|\phi\|_{s,p}$ but not on ν; in fact they can be given explicitly in terms of an exponential function exponentiated again and again. Thus a standard argument based on the local existence theorem shows that $u(t)$ can be extended step by step to all $t > 0$ satisfying (3.1). This proves Theorem II when $2/p < s - 1 \leqslant 1$.

Next we consider the case $2 < s \leqslant 3, s > 1 + 2/p$. Let q be a number such that $q > 2$ and $0 \leqslant 2/p - 2/q \leqslant s - 2$. (We may set $q = p$ if $p > 2$; and $2/q = 2/p - (s-2)$ if $p \leqslant 2$ and $s < 3$; in the exceptional case $p = 2$, $s = 3$, any $q > 2$ will do.) Then we have $\phi \in L^p_s \subset L^q_2$ by the Sobolev embedding theorem. Since $2/q < 2 - 1 = 1$, the preceding result shows that $u(t) \in L^q_2$ exists for all time, with $\|u(t)\|_{2,q} \leqslant K(t)$. Application of Lemma 2.2 with

$a = u$, $r = 2$, $k = s - 1 \leqslant r$, $q \geqslant p$ shows that $\|\zeta(t)\|_{s-1,p} \leqslant K(t)$, which leads to (3.5) by (3.2a). Using Lemma 3.2 again, we obtain (3.1).

Next consider the case $3 < s \leqslant 4$. Then $\phi \in L_s^p \subset L_3^p$ with $3 > 1 + 2/p$. Hence the preceding result shows that $\|u(t)\|_{3,p} \leqslant K(t)$. Application of Lemma 2.2 with $a = u$, $q = p$, $r = 3$, $k = s - 1 \leqslant r$ shows that $\|\zeta\|_{s-1,p} \leqslant K(t)$, which again leads to (3.5), and then to (3.1) in the same way as above. The case with larger s can be dealt with similarly. This completes the proof of Theorem II. □

4. Vanishing viscosity and the Euler equation in $\mathbb{R}^2$

We now prove the main theorem for the Euler equation in $\mathbb{R}^2$, together with the convergence of viscous solutions to the ideal one. As before we assume that $f = 0$ for simplicity.

Theorem III. *Let $s > 1 + 2/p$, $1 < p < \infty$. If $\phi \in PL_s^p$, the Euler equation* (E) *has a unique solution $u \in C([0, \infty); PL_s^p)$ with $u(0) = \phi$.*

Theorem IV. *Let ϕ be as in Theorem III. Let u^ν be the solution of* (NS) *given by Theorem II. As $\nu \to 0$, u^ν converges to the solution u given by Theorem III in the following sense: $u^\nu(t) \to u(t)$ weakly in PL_s^p for all $t \geqslant 0$, locally uniformly in t.*

Remark 4.1. Theorem IV is rather weak inasmuch as it gives only weak convergence. A stronger result could be proved, but we shall not pursue the problem here. Note, however, that weak convergence of a sequence u_n in $L_s^p(\mathbb{R}^m)$ implies (spatially pointwise) local uniform convergence of $\partial^k u_n$ for $k < s - m/p$.

We prove these theorems in several steps.

Lemma 4.2. *As $\nu \to 0$, $u^\nu(t)$ converges along a subsequence weakly in L_s^p, (locally) uniformly in t, to a solution $u(t)$ of* (E) *such that*

$$(4.0) \qquad u \in \tilde{C}([0, \infty); L_s^p),$$

which means that $u(t)$ is weakly continuous in L_s^p and strongly continuous in $L_{s'}^p$ for all $s' < s$.

PROOF. The proof is standard, and will be only sketched. It depends on the fact that the map F is weakly continuous from L_s^p into L_{s-1}^p, as is easily verified.

Since by Theorem II the $u^\nu(t)$ are locally (in time) uniformly bounded in L^p_s, which is reflexive, we can pick a subsequence of u^ν that converges weakly on a countable dense subset of t.

Since $\partial_t u^\nu(t) = \nu\Delta u^\nu - F(u^\nu(t))$ is (locally) uniformly bounded in L^p_{s-2}, $\partial_t\langle u^\nu(t), \varphi\rangle$ is uniformly bounded for each test function $\varphi \in \mathfrak{D}(\mathbb{R}^2)$. Hence $\langle u^\nu(t), \varphi\rangle$ is (locally) equicontinuous, so that it converges for all t (locally) uniformly. In view of the (local) uniform boundedness of $u^\nu(t)$ in L^p_s, it follows that $u^\nu(t)$ converges weakly in L^p_s, (locally) uniformly in t. The limit $u(t) \in L^p_s$ is weakly continuous, and it is easy to see that u is a weak solution of (E).

Then $\partial_t u(t) = -F(u(t))$ is weakly continuous in L^p_{s-1} by the remark given above. It follows that $u(t)$ is strongly continuous in L^p_{s-1}, hence strongly continuous in each $L^p_{s'}$ with $s' < s$.

Lemma 4.3. *The solution* (4.0) *of* (E) *is unique.* (*Hence the original sequence u^ν converges to u weakly, without going over to a subsequence.*)

PROOF. Suppose there are two solutions u, u' to (E). On subtraction and setting $w = u' - u$, we obtain

$$\partial_t w + F(u', w) + F(w, u) = 0.$$

Hence, suppressing the variable t for simplicity,

$$\partial_t \|w\|_p^p = -p\langle Jw, F(u', w) + F(w, u)\rangle; \tag{4.1}$$

here we use the norm $\|w\|_p = (\|w_1\|_p^p + \|w_2\|_p^p)^{1/p}$, and $Jw = (Jw_1, Jw_2)$ is the vector-valued version of the duality map used in the proof of Lemma 2.1. In (4.1) we have $\|F(w, u)\|_p \leqslant K\|w\|_p$ because $\|\partial u\|_\infty \leqslant K$, contributing a term $K\|w\|_p^p$ to the right of (4.1). For the other term, we have

$$F(u', w) = (u' \cdot \partial)w - G(u', w) = (u' \cdot \partial)w - G(w, u'), \tag{4.2}$$

by Lemma 1.4. The contribution of $(u' \cdot \partial)w$ to (4.1) is zero, as in seen by the standard integration by parts using $\partial \cdot u' = 0$. On the other hand, $G(w, u')$ is dominated in L^p-norm by $K\|w\|_p$ because $\|\partial u'\|_\infty \leqslant K$. Altogether we obtain

$$\partial_t \|w\|_p^p \leqslant K\|w\|_p^p,$$

from which $w = 0$ follows if $w(0) = 0$, completing the proof of uniqueness.

To complete the proof of the theorems, we have to strengthen $\tilde{C}([0, T]; L^p_s)$ in (4.0) into $C([0, T]; L^p_s)$. For this purpose, we consider the linear vorticity equation without viscosity:

$$\partial_t \zeta + a \cdot \partial\zeta = 0, \qquad \zeta(0) = \omega. \tag{4.3}$$

Lemma 4.4. *Let* $a \in \tilde{C}([0, T]; L_r^q)$, *where* $r > 1 + 2/q$, $1 < q < \infty$. *If* $\omega \in L_k^p$ *with* $0 \leqslant k < [r]$ and $p \leqslant q$, *then there is a unique solution to* (4.3) *such that*

$$\zeta \in C([0, T]; L_k^p). \tag{4.4}$$

PROOF. Since the assumption implies that $a \in C([0, T]; C^1(\mathbb{R}^2))$, the characteristic curves for the first-order equation (4.3) are well defined. The uniqueness of the solution of (4.3) follows immediately from this.

To prove the existence, we regard (4.3) as the limit of the vorticity equation (2.1ν), which is the equation (2.1) with a replaced by a function $a^\nu \in C([0, T]; L_r^q)$ that depends on ν, in such a way that $a^\nu \to a$ in $\tilde{C}([0, T]; L_r^q)$ (which means weak convergence in L_r^q and strong convergence in $L_{r'}^q$ for $r' < r$). Such a sequence a^ν is easily constructed by regulazation of a in time.

Let ζ^ν be the solution of (2.1ν), with $\zeta^\nu(0) = \omega \in L_k^p$. Lemma 2.2 shows that ζ^ν belongs to $C([0, T]; L_k^p)$ and satisfies (2.7ν), which is the inequality (2.7) with a replaced by a^ν. Using the same argument given above to construct u from u^ν, we obtain a solution

$$\zeta \in \tilde{C}([0, T]; L_k^p) \tag{4.5}$$

as the weak limit of ζ^ν. (Recall that $a^\nu \to a$ strongly in L_{k-1}^p.)

Since ζ^ν satisfies (2.7ν), ζ satisfies an inequality of the form

$$\|\zeta\|_{k,p} \leqslant c\|\omega\|_{k,p} \exp\left(c \int_0^t \alpha(\tau)\, d\tau\right), \tag{4.6}$$

where $\alpha(t) = \limsup_{\nu \to 0} \|a^\nu(t)\|_{q,r}$ is an L^∞-function.

It remains to replace $\tilde{C}([0, T]; L_s^p)$ by $C([0, T]; L_s^p)$ in (4.5). To this end let k' be a number such that $k < k' < [r]$, and let $\omega_n \in L_{k'}^p$ be a sequence such that $\omega_n \to \omega$ in L_k^p. If ζ_n is the solution of (4.3) with $\zeta_n(0) = \omega_n$, (4.6) is true with ζ and ω replaced by $\zeta_n - \zeta$ and $\omega_n - \omega$, respectively. It follows that $\zeta_n(t)$ tends to $\zeta(t)$ strongly in L_k^p, uniformly in t. Since ζ_n belongs to $\tilde{C}([0, T]; L_{k'}^p) \subset C([0, T]; L_k^p)$ by $k' > k$, it follows that $\tilde{C}$ can be replaced by C in (4.5).

COMPLETION OF THE PROOF OF THE THEOREMS. We have to show that in (4.0), $\tilde{C}$ can be replaced by C. Since u is a solution of (E), $\zeta = \operatorname{rot} u$ satisfies the vorticity equation (4.3) in which $a = u$, $\omega = \operatorname{rot} \phi \in L_{s-1}^p$. In view of (4.0), we can set $r = s$, $q = p$, $k = s - 1$ in Lemma 4.2, obtaining the result that $\zeta \in C([0, \infty); L_{s-1}^p)$. As in the proof of Theorem II, this implies $\partial u \in C([0, \infty); L_{s-1}^p)$ and, using the known fact that $u \in C([0, \infty); L^p)$, we obtain the desired result $u \in C([0, \infty); L_s^p)$ by Lemma 3.2.

Remark 4.5. Lemma 4.4 could be proved by integrating the vector field a to yield a family of diffeomorphisms Φ_t such that $\Phi_t - id$ is in the class

$C([0, T]; L_r^q)$, and then showing that the composition $\omega \circ \Phi_t$ is in $C([0, T]; L_k^p)$. It seems to the authors that such results are not readily available in the literature.

Appendix

1. *Proof of Lemma* 3.1.

We prove Lemma 3.1 in $\mathbb{R}^m$, assuming that $s > 1 + m/p$. In this case $\zeta = \operatorname{rot} u$ is a skew-symmetric tensor given by $\zeta_{jk} = \partial_j u_k - \partial_k u_j$. If $u \in PL_s^p$, ∂u can be recovered from $\zeta \in L_{s-1}^p$ formally by

$$\partial_j u_k = \partial_j \partial_i \Delta^{-1} \zeta_{ik}. \tag{A1}$$

Here $\partial_j \partial_i \Delta^{-1}$ is a pseudo-differential operator with symbol $\xi_j \xi_i |\xi|^{-2}$, so that the operation $\zeta \to \partial u$ is continuous in L^p, hence in L_{s-1}^p, for any s and $1 < p < \infty$ (Mihlin's theorem). This proves (3.2*a*).

Incidentally, the operator considered here is formally identical with G (see Lemma 1.4), with the only difference that it acts on tensors to tensors, while G acts on vector to vectors.

The proof of (3.2*b*) requires a deeper analysis. We can write (A1) in the form

$$\partial_j u_k = m^{-1} \zeta_{jk} + g_{ji} * \zeta_{ik} \qquad (* = \text{convolution}), \tag{A2}$$

where the $g_{jk} = |x|^{-m} \Omega_{jk}$ are Kernels of Calderón-Zygmund type (see [4]), with $\Omega_{jk}(x)$ homogeneous of degree zero, smooth in x and having zero average on the unit sphere. Thus it suffices to prove (3.2*b*) for

$$v(x) = g * \zeta(x) = \lim_{\epsilon \to 0} \int_{|y| > \epsilon} g(y) \zeta(x - y)\, dy, \tag{A3}$$

where g is one of the g_{jk} and ζ is one of the ζ_{jk}. To this end we split the integral into three parts, according to (i) $|y| > 1$, (ii) $\delta < |y| \leqslant 1$, (iii) $\epsilon < |y| \leqslant \delta$.

The contribution to the integral from (i) is estimated by $c \|\zeta\|_p$, since $|g(y)| \leqslant c|y|^{-m}$ and $p < \infty$. Using the same estimate for g, the contribution from (ii) is seen to be dominated y $c\|\zeta\|_\infty \log(\delta^{-1})$. To deal with the part (iii), we replace $\zeta(x - y)$ with $\zeta(x - y) - \zeta(x)$ without affecting the integral, since $g(y)$ has average zero on any sphere. But $\zeta \in L_{s-1}^p$ implies that ζ is Hölder continuous:

$$|\zeta(x - y) - \zeta(x)| \leqslant c|y|^\theta \|\zeta\|_{s-1,p}, \qquad 0 < \theta < 1.$$

Therefore the contribution from (iii) does not exceed $c\delta^\theta \|\zeta\|_{s-1,p}$. Altogether we obtain

$$\|v\|_\infty \leqslant c(\|\zeta\|_p + \log(\delta^{-1})\|\zeta\|_\infty + \delta^\theta \|\zeta\|_{s-1,p}) \tag{A4}$$

for any $\delta \in (0,1]$. The desired result (3.2*b*) then follows on setting $\delta^{-\theta} = \max(1, \|\zeta\|_{s-1,p}/\|\zeta\|_\infty)$. This proves (3.2*b*). □

2. *Proof of Lemma* 3.2

$\partial v \in L_{s-1}^p$ implies that $\partial^k v \in L_r^p \subset L^p$, $r = s - k$, for $1 \leqslant k \leqslant [s]$. Since $v \in L^p$ too by hypothesis, we have $v \in L_{[s]}^p \subset L_{s-1}^p$. Thus v and ∂v are both in L_{s-1}^p, from which $v \in L_s^p$ follows together with the required estimate.

3. *Proof of* (2.8)

We have to show that

$$(\partial^j a)(\partial^{k-j+1}\zeta) \in L^p, \qquad 1 \leqslant j \leqslant k \leqslant [r]. \tag{A5}$$

under the assumptions

$$a \in L_r^q, \qquad \zeta \in L_k^p, \quad \text{where} \quad r > 1 + 2/q, \qquad 1 < p \leqslant q < \infty. \tag{A6}$$

First we note that (A6) implies

$$\partial^j a \in L_{r-j}^q, \qquad \partial^{k-j+1}\zeta \in L_{j-1}^p. \tag{A7}$$

(*a*) If $j < r - 2/q$, then (A7) implies that $\partial^j a \in L^\infty$ and $\partial^{k-j+1}\zeta \in L^p$, from which (A5) follows immediately.

(*b*) If $j \geqslant r - 2/q$, then $j > 1$. Let $\epsilon > 0$ be such that $r - 1 = (2 + \epsilon)/q$. Then

$$1/p \geqslant 1/q = (r - j)/(2 + \epsilon) + (j - 1)/(2 + \epsilon). \tag{A8}$$

Thus (A7) implies

$$\partial^j a \in L^\tau \quad \text{with} \quad 1/\tau = 1/q - (r - j)/(2 + \epsilon) > 0,$$
$$\partial^{k-j+1}\zeta \in L^\sigma \quad \text{with} \quad 1/\sigma = 1/p - (j - 1)/(2 + \epsilon) > 0,$$

except when $p = q, j = r = k$. Since $1/\tau + 1/\sigma = 1/p$ by (A8), we obtain (A5) in the nonexceptional case. In the exceptional case we have $\partial^j a \in L^p$ and $\partial^{k-j+1}\zeta \in L_{r-1}^p \in L^\infty$ because $r - 1 > 2/q = 2/p$. Thus we again obtain (A5).

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