Lifting Properties, Nehari Theorem and Paley Lacunary Inequality

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Dedicated to Alberto P. Calderón on his 65th birthday

Abstract

A general notion of lifting properties for families of sesquilinear forms is formulated. These lifting properties, which appear as particular cases in many classical interpolation problems, are studied for the Toeplitz kernels in $\mathbb{Z}$, and applied for refining and extending the Nehari theorem and the Paley lacunary inequality.

1. Introduction

There is a close connection between interpolation problems and lifting properties. Sarason [24] showed that classical interpolation results can be obtained as corollaries of a theorem about commutators of the truncated shift operator

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operator, later extended by Nagy and Foias [14] into a general lifting theorem for contractions. Furthermore, the theorem of Sarason was proved by Page [9] and Nikolskii [18] as equivalent to the vector version of the interpolation theorem of Nehari [17].

Questions concerning weighted inequalities and prediction theory lead the authors to a lifting property of the so-called generalized Toeplitz kernels (GTKs) [7], which can be considered as a generalization of the Nehari theorem (see (IV) below). This lifting property, later proved equivalent to the Nagy-Foias theorem [1], [13], is closely related to the study of subordinated reproducing kernels, used by Aronszajn, Masani, Burbea and Beatrous [3], [4], [5], [7] as a tool to solve interpolation problems.

The above-mentioned results are in fact (although generally in a implicit form) lifting properties of families of sesquilinear forms with respect to a given pair of hilbertian seminorms. This suggests a general notion of lifting properties for families of forms with respect to pairs of seminorms, which is formulated and exemplified in this section. This leads to new results in interpolation as well as to refinements of the known ones.

A summary of known results in terms of lifting properties follows as (I)-(VII), after the introduction of the necessary notation.

Fix V a vector space, $V_1, V_2 \subset V$ subspaces, and let Λ be a family of sesquilinear forms $S: V \times V \to \mathbb{C}$. Given a pair σ_1, σ_2 of seminorms in V, write

(1) $$S \leqslant (\sigma_1, \sigma_2) \qquad (\text{resp.,} \ S \prec (\sigma_1, \sigma_2))$$

if

(1a) $$|S(a, b)| \leqslant \sigma_1(a)\sigma_2(b)$$

for all $(a, b) \in V \times V$ (resp. all $(a, b) \in V_1 \times V_2$).

Given V, V_1, V_2, *a family of sesquilinear forms Λ satisfies the lifting property with respect to σ_1, σ_2* if for every $A \in \Lambda$ such that $S \prec (\sigma_1, \sigma_2)$ there is a $S' \in \Lambda$ such that $S' \leqslant (\sigma_1, \sigma_2)$ and $S'|_{V_1 \times V_2} = S|_{V_1 \times V_2}$.

In most of our examples the sesquilinear forms considered arise from kernels. More precisely, fixing a set X, and $X_1, X_2 \subset X$, and setting $V = = V(X) = \{a : X \to \mathbb{C}$, functions of finite support$\}$, $V_j = V(X_j) = \{a \in V:$ supp $a \subset X_j\}$, $j = 1, 2$, each sesquilinear form $S: V \times V \to \mathbb{C}$ is given by a kernel $K: X \times X \to \mathbb{C}$

(2) $$S(a, b) = S_K(a, b) = \sum_{x, y \in X} K(x, y)a(x)\overline{b(y)}.$$

A kernel K is positive definite, p.d., if $S_K(a, a) \geqslant 0$ for all $a \in V$. To each p.d. kernel corresponds a hilbertian seminorm in V, defined by $\sigma_K(a) = S_K(a, a)^{1/2}$.

Given two p.d. kernels K_1, K_2, write

$$(3) \qquad K \leqslant (K_1, K_2) \qquad (\text{resp.}, K \prec (K_1, K_2))$$

if

$$(3a) \qquad S_K \leqslant (\sigma_{K_1}, \sigma_{K_2}) \qquad (\text{resp.}, S_K \prec (\sigma_{K_1}, \sigma_{K_2}))$$

For $V = V(X)$, $V_1 = V(X_1)$, $V_2 = V(X_2)$, fixed, *a family* Λ *of kernels satisfies the lifting property with respect to* K_1, K_2 if for every $K \in \Lambda$ such that $K \prec (K_1, K_2)$ there is a $K' \in \Lambda$ such that $K' \leqslant (K_1, K_2)$ and $K'|_{X_1 \times X_2} = K|_{X_1 \times X_2}$.

Let us illustrate this notion in the case of the theorem of Nehari, which asserts that, for a given sequence s: $\mathbb{Z} \to \mathbb{C}$, the following are equivalent:

(a) there exists a bounded function on the circle, f, such that $|f(t)| \leqslant 1$ and $\hat{f}(n) = s(n)$ for $n > 0$.

$$(b) \qquad \left| \sum_{\substack{m \geqslant 0 \\ n < 0}} s(m - n) a(m) \overline{b(n)} \right| \leqslant \left(\sum_{m \geqslant 0} |a(m)|^2 \right)^{1/2} \left(\sum_{n < 0} |b(n)|^2 \right)^{1/2}$$

for all finite sequences $(a(n)), (b(n))$.

Take now $X = \mathbb{Z}$, $X_1 = \mathbb{Z}_1 = \{n \in \mathbb{Z}: n \geqslant 0\}$, $X_2 = \mathbb{Z}_2 = \{n \in \mathbb{Z}: n < 0\}$, $V = \{a: \mathbb{Z} \to \mathbb{C}, \text{ finite sequences}\}$, $V_j = \{a \text{ finite sequence, } \operatorname{supp} a \subset \mathbb{Z}_j\}$, $j = 1, 2$, $K_1(m, n) = K_2(m, n) = \delta_{m-n}$. If $K(m, n) = s(m - n)$, condition (b) becomes $K \prec (K_1, K_2)$. On the other hand, setting $\hat{f}(n) = s'(n)$, $K'(m, n) = s'(m - n)$, it is easy to verify that $|f| \leqslant 1$ is equivalent to $K' \leqslant (K_1, K_2)$. Thus,

(I) (Nehari [17].) If $X = \mathbb{Z}$, $X_j = \mathbb{Z}_j$, $V = V(\mathbb{Z})$, $V_j = V(\mathbb{Z}_j)$, $K_j = \delta_{m-n}$, $j = 1, 2$, and $\Lambda = \{K: \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}, K(m, n) = s(m - n)\}$ is the family of Toeplitz kernels, then Λ has the lifting property with respect to K_1, K_2.

Other examples are

(II) (Bergman-Schiffer [6].) If $X = $ a domain in $\mathbb{C}$, $X_1 = X_2 = $ an open subset of X, $K_1 = K_2 = $ the Bergman kernel of X, and $\Lambda = \{F: X \times X \to \mathbb{C}, F \text{ holomorphic}\}$, then Λ satisfies the lifting property with respect to K_1, K_2.

(III) (Beatrous-Burbea [4].) If $X_1 = X_2$, $K_1 = K_2 = $ p.d. kernel, $\Lambda = \{K: X \times X \to \mathbb{C}, K(x, y) = f(x)\overline{f(y)}\}$, then Λ satisfies the lifting property with respect to K_1, K_2 (cf. also [5]).

Next example is an equivalent version of the so-called lifting theorem for p.d. GTKs (cf. [13]).

(IV) (Cotlar-Sadosky [8].) If $X = \mathbb{Z}$, $X_j = \mathbb{Z}_j$, $j = 1, 2$, K_1, K_2 p.d. Toeplitz kernels, $\Lambda = \{K: \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}, K(m, n) = s(m - n)\}$, then Λ satisfies the lifting property with respect to K_1, K_2.

Clearly, the Nehari theorem (I) is a special case of (IV), which also provides refinements to classical results of Helson, Szegö, Sarason and others [8], [2].

The following is a special case of the Nagy-Foias lifting theorem.

(V) (Nagy-Foias [16].) If V is a Hilbert space, U a unitary operator on V, $V_1 = V_2 =$ subspace of V semiinvariant with respect to U, $\sigma_1 = \sigma_2 =$ the hilbertian norm in V, and Λ is the family of sesquilinear forms commuting with U, then Λ satisfies the lifting property with respect to σ_1, σ_2.

(VI) (Cotlar-Sadosky [13].) If X is an arbitrary set, $X_1, X_2 \subset X$, $\zeta\colon X \to X$ is a bijection such that $\zeta(X_1) \subset X_1$, $\zeta^{-1}(X_2) \subset X_2$, $\Lambda = \{K\colon X \times X \to \mathbb{C}, K(\zeta x, \zeta y) = K(x, y)\}$, $K_1, K_2 \in \Lambda$ p.d., then Λ satisfies the lifting property with respect to K_1, K_2.

This generalization of (IV) was obtained combining (V) with the arguments used to prove (III). Conversely, the operator-valued version of (IV) implies (V) (cf. [1]). Thus, the framework in which the above results are presented naturally includes the Nagy-Foias lifting theorem. A more general concept of norm majoration allows a further inclusion. Writing $K \leqslant \sigma_0$ (resp., $K \quad \sigma_0$) if $|S_K(a, b)| \leqslant \sigma_0(a, b)$ for all $(a, b) \in V \times V$ (resp., $(a, b) \in V_1 \times V_2$), where $\sigma_0\colon V \times V \to \mathbb{R}^+$, the following result relates to (I) – (VI).

(VII) (Cotlar-Sadosky [10].) If $X = \mathbb{Z}, X_j = \mathbb{Z}_j, j = 1, 2, \Lambda =$ the Toeplitz kernels, $\sigma_0(a, b) = \rho(|\check{a} + \check{b}|^2)$, where ρ is a seminorm in $C(\mathbb{T})$ under certain natural assumptions, and $\check{a}(t) = \sum_n a_n e^{\mathrm{int}}$, then Λ satisfies the lifting property with respect to σ_0.

This is a special case of the so-called lifting theorem for majorized GTKs [10], that yields L^p versions of the Helson —Szegö and Nehari theorems. But it is to be remarked that (VII) neither generalizes (IV) to the case when $\sigma_1 \neq \sigma_2$ are non hilbertian, nor extends (I) for general σ_1, σ_2. Some extensions can be obtained from (VII) when $\sigma_1 = \sigma_2$ is a non-deterministic seminorm (see (5) below and Corollary 2). Extensions of (I) and (IV) for pairs σ_1, σ_2 are given in Section 2 and 3.

Section 2 deals with lifting properties of the family of Toeplitz kernels in $\mathbb{Z}$ with respect to pairs of seminorms σ_1, σ_2; Section 3, with several refinements of the Nehari theorem, and Section 4, with generalizations of the Paley lacunary inequality. Some of the results in Section 4 were announced in [12].

2. Lifting properties of Toeplitz kernels in $\mathbb{Z}$

Fix $X = \mathbb{Z}, X_j = \mathbb{Z}_j, V = V(\mathbb{Z}), V_j = V(\mathbb{Z}_j), j = 1, 2$. Let $\mathcal{P} = \{\check{a}(t) = \sum a(n)e^{\mathrm{int}}\colon a \in V\}$ be the trigonometric polynomials, $\mathcal{P}_1 = \{\check{a}(t)\colon a \in V_1\}$, the analytic polynomials, $\mathcal{P}_2 = \{\check{a}(t)\colon a \in V_2\}$. If $\check{a} = f \in \mathcal{P}$ then $a(n) = \hat{f}(n), n \in \mathbb{Z}$. $\mathcal{P}$ is dense in $C(\mathbb{T})$, the continuous functions in $\mathbb{T}$. For $1 \leqslant p \leqslant \infty$, $H^p(\mathbb{T}) = \{f \in L^p(\mathbb{T})\colon \hat{f}(n) = 0 \text{ for } n < 0\}$. In what follows, all integrals are taken over $\mathbb{T}$.

Consider the seminorms ρ defined in $C(\mathbb{T})$ which satisfy

(4a) $|f| \leq |g|$ implies $\rho(f) \leq \rho(g), \rho(f) = \rho(|f|)$

(4b) $\rho(f) \leq c\|f\|_\infty$, $\forall f \in C(\mathbb{T})$, c fixed constant

(4c) $\rho'(f) = \rho(|f|^2)^{1/2}$ is also a seminorm.

Seminorm ρ is said to be *absolutely continuous* if $f_n \downarrow 1_A$, $|A| = 0$, implies $\rho(f_n) \to 0$.

Seminorm ρ is said to be *non deterministic* if there is a fixed constant c such that, if ρ' is given by (4c),

(5) $\rho'(f_1) + \rho'(f_2) \leq c\rho'(f_1 + f_2)$, $\forall (f_1, f_2) \in \mathcal{P}_1 \times \mathcal{P}_2$.

Examples of non-deterministic norms are $\rho_p(f) = \left(\int |f|^p \, dt\right)^{1/p}$, $1 < p < \infty$, and also $\rho_{p\mu}(f) = \left(\int |f|^p \, d\mu\right)^{1/p}$ for $\mu \in A_p$, since the Hilbert transform is then bounded in the corresponding spaces.

To each seminorm ρ in $C(\mathbb{T})$ that satisfies $(4a) - (4c)$ it is associated a seminorm σ in V by

(6) $\sigma \sim \rho$ whenever $\sigma(a) = \rho'(\breve{a}) = \rho(|\breve{a}|^2)^{1/2}$.

For a pair of such seminorms, σ_1, σ_2, and a kernel $K: \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$, consistently with the notations of Section 1, $K \leq (\sigma_1, \sigma_2)$ (resp., $K \prec (\sigma_1, \sigma_2)$) means that

(1b) $|S_K(a, b)| \leq \rho_1(|\breve{a}|^2)^{1/2}\rho_2(|\breve{b}|^2)^{1/2}$

for all $(a, b) \in V \times V$ (resp., all $(a, b) \in (V_1 \times V_2)$).

Lemma 1. *For every sesquilinear form* $S: V \times V \to \mathbb{C}$ *the following conditions are equivalent*:

(i) $S \leq (\sigma_1, \sigma_2)$ *(resp.,* $S \prec (\sigma_1, \sigma_2)$*)*;

(ii) $S(a, b) + \overline{S(a, b)} = 2\mathrm{Re}\, S(a, b) \leq \sigma_1(a)^2 + \sigma_2(b)^2$, $\forall$ $(a, b) \in V \times V$ *(resp.,* $(a, b) \in V_1 \times V_2$*)*;

(iii) $2|S(a, b)| \leq \sigma_1(a)^2 + \sigma_2(b)^2$ *for all such* (a, b).

PROOF. It is enough to observe that (ii) holds iff the quadratic form in λ_1, λ_2, $\sigma_1(a)^2 \lambda_1\bar{\lambda}_1 - S(a, b)\lambda_1\bar{\lambda}_2 - \overline{S(a, b)}\bar{\lambda}_1\lambda_2 + \sigma_2(b)^2\lambda_2\bar{\lambda}_2$ is nonnegative for all $\lambda_1, \lambda_2 \in \mathbb{C}$ (for all pertinent (a, b)), and that (i) expresses that the determinant of this form is nonnegative. $\square$

Lemma 2. *If* $\sigma \sim \rho$ *as in* (6), $\rho(1) = 1$ *and* $K(m, n) = s(m - n)$, *then*

(i) $K \leq (\sigma, \sigma)$ *is equivalent to*

(ii) $|\sum_n s(n)a(n)| \leq \rho'(a)$, $\forall a \in V$.

Proof. If $K \leqslant (\sigma, \sigma)$, setting $b(n) = \delta_0$, or $\breve{b} \equiv 1$, in (1b), it becomes (ii). Conversely, if (ii) holds, then $|S_K(a, b)| = \left| \sum_{m,n} s(m - n) a(m) \overline{b(n)} \right| = = \left| \sum_n \left(\sum_m a(m) \overline{b(m - n)} \right) s(n) \right| \leqslant \rho(\breve{a} \cdot \breve{b}) \leqslant 1/2(\rho(|\breve{a}|^2) + \rho(|\breve{b}|^2)) = 1/2(\sigma(a)^2 + \sigma(b)^2)$, and (i) follows from Lemma 1. $\square$

A kernel $K: \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ is *Toeplitz* if $K(m, n) = s(m - n)$ for some $s: \mathbb{Z} \to \mathbb{C}$.

Theorem 1. *For $X = \mathbb{Z}$, $X_j = \mathbb{Z}_j$, $V = V(\mathbb{Z})$, $V_j = V(\mathbb{Z}_j)$, seminorms $\sigma_j \sim \rho_j$ as in (6), $j = 1, 2$, the family Λ of Toeplitz kernels in $\mathbb{Z}$ satisfies the lifting property with respect to σ_1, σ_2.*

More precisely, if $K \in \Lambda$, $K(m, n) = s(m - n)$, $K \prec (\sigma_1, \sigma_2)$, then there exists $K' \in \Lambda$, $K'(m, n) = s'(m - n)$, such that $K' \leqslant (\sigma_1, \sigma_2)$ and $K'|_{\mathbb{Z}_1 \times \mathbb{Z}_2} = K_{\mathbb{Z}_1 \times \mathbb{Z}_2}$. Moreover, $s'(n) = \hat{\mu}(n)$, where μ is a finite complex measure in $\mathbb{T}$.

If in addition $\rho_1 = \rho_2$ is an absolutely continuous seminorm or $s(n) = \hat{\omega}(n)$ for $\omega \in L^1$, then $d\mu = \theta \, dt$ for some $\theta \in L^1$.

Proof. The main difficulty of this proof appears where $\sigma_1 \neq \sigma_2$. Therefore it is divided in two cases.

(*Case* $\sigma_1 = \sigma_2 \sim \rho$). Define in $\mathcal{P}$ a linear functional by $l(f) = \sum_n a(n) s(n)$ if $f = \breve{a}$, $a \in V$, so that if $g = \breve{b}$ then $|l(f\bar{g})| = \left| \sum_{m,n} s(m - n) a(m) \overline{b(n)} \right) \right| \leqslant \leqslant \rho'(f) \rho'(g)$. Taking $f(t) \equiv \|g\|_\infty$, it follows from (4b) that $\|g\|_\infty |l(\bar{g})| \leqslant \leqslant c \|g\|_\infty \cdot c \|g\|_\infty$. Thus, $|l(\bar{g})| \leqslant c^2 \|g\|_\infty$, $\forall \bar{g} \in e^{it} \mathcal{P}_1 = \mathcal{P}_{10}$, and there exists a finite complex measure ν in $\mathbb{T}$ such that $l(f) = \int f \, d\nu$, $\forall f \in \mathcal{P}_{10}$ and $s(n) = \hat{\nu}(n)$ for $n \geqslant 0$. Thus

$$(7) \qquad \left| \int f\bar{g} \, d\nu \right| \leqslant \rho'(f) \rho'(g), \qquad \forall (f, g) \in \mathcal{P}_1 \times \mathcal{P}_2$$

and from (4b) follows that (7) holds also for $(f, g) \in \overline{\mathcal{P}_1} \times \overline{\mathcal{P}_2}$, $\overline{\mathcal{P}_1}$, $\overline{\mathcal{P}_2}$ the closures in $C(\mathbb{T})$ of $\mathcal{P}_1, \mathcal{P}_2$. Since every analytic polynomial $F \in e^{it} \mathcal{P}_1$ can be written as $F = f\bar{g}$ with $(f, g) \in \overline{\mathcal{P}_1} \times \overline{\mathcal{P}_2}$, $|f|^2 = |g|^2 = |F|$, then

$$\left| \int F \, d\nu \right| = \left| \int f\bar{g} \, d\nu \right| \leqslant \rho'(f) \rho'(g) \leqslant 1/2(\rho'(f)^2 + \rho'(g)^2)$$
$$= 1/2(\rho(|f|^2) + \rho(|g|^2)) = \rho(|F|).$$

Thus, $\left| \int F \, d\nu \right| \leqslant \rho(F)$, $\forall F \in e^{it} \mathcal{P}_1$, and since $\rho(G) \leqslant c \|G\|_\infty$, $\forall G \in C(\mathbb{T})$, it follows easily that there exists a finite measure in $\mathbb{T}$ such that $\int F \, d\mu = \int F \, d\nu$ for $F \in e^{it} \mathcal{P}_1$ and $\left| \int G \, d\mu \right| \leqslant \rho(G)$, $\forall G \in C(\mathbb{T})$. Hence

$$\left| \int f_1 f_2 \, d\mu \right| \leqslant \rho(f_1 f_2) \leqslant 1/2(\rho(|f_1|^2) + \rho(|f_2|^2)), \qquad \forall (f_1, f_2) \in \mathcal{P} \times \mathcal{P}.$$

Setting $s'(n) = \hat{\mu}(n)$, and using Lemma 1, the assertion on s' follows. The last part of the thesis is proved as Proposition 3 in [23].

(*Case* $\sigma_1 \neq \sigma_2$). As in the previous case, there exists a measure ν such that $s(n) = \hat{\nu}(n)$ for $n \geq 0$, so that $S_K(a, b) = \int f\bar{g} \, d\nu$ for $f = \breve{a} \in \mathcal{P}_1$, $g = \breve{b} \in \mathcal{P}_2$, i.e., for $(a, b) \in V_1 \times V_2$. Thus, by hypothesis and Lemma 1,

$$2\left| \int f\bar{g} \, d\nu \right| \leq \rho_1(|f|^2) + \rho_2(|g|^2),$$

or

(7a) $$\int f\bar{g} \, d\nu + \int \bar{f}g \, \overline{d\nu} \leq \rho_1(|f|^2) + \rho_2(|g|^2), \qquad \forall (f, g) \in \mathcal{P}_1 \times \mathcal{P}_2.$$

Again we want to prove that there is a measure μ such that $\int F \, d\mu = \int F \, d\nu$ for $F \in e^{it}\mathcal{P}_1$, and such that (7a) holds for all $(f, g) \in \mathcal{P} \times \mathcal{P}$ when $d\nu$ is replaced by $d\mu$. We use an argument from [9].

Let $\mathfrak{M}$ be the dual of $C(\mathbb{T})$, and set $E = \mathfrak{M} \times \mathfrak{M} \times \mathfrak{M} \times \mathfrak{M}$, $E' = C(\mathbb{T}) \times \overline{\mathcal{P}_{10}} \times \overline{\mathcal{P}_2} \times C(\mathbb{T})$, where the closures are taken in $C(\mathbb{T})$. For each quadruple of measures $(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}) \in E$ and $(\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22}) \in E'$, set

(8) $$\langle (\mu_{jk}), (\phi_{jk}) \rangle = \sum_{j,k=1,2} \int \phi_{jk} \, d\mu_{jk}.$$

Endow the space E with the topology given by this coupling, so that E' is the dual of E, and E the dual of E'. Let $\Pi \subset E$ be the set of all quadruples $(\mu_{11}, \nu, \bar{\nu}, \mu_{22})$ where

(9) $$\left| \int f \, d\mu_{jj} \right| \leq \rho_j(f), \qquad \forall f \in C(\mathbb{T}), \qquad j = 1, 2.$$

and let $\Gamma \subset E$ be the set of all quadruples $(\lambda_{11}\delta_t, \lambda_{12}\delta_t, \lambda_{21}\delta_t, \lambda_{22}\delta_t)$, where δ_t is the Dirac measure at $t \in \mathbb{T}$ and (λ_{jk}) is a positive definite 2×2 numerical matrix. In particular, all $(\lambda_{jk} = \lambda_j\bar{\lambda}_k)_{j,k=1,2}$ are such matrices, Π is a compact convex set in E, and $\Gamma_1 =$ convex hull of Γ, is a cone in E. Let us show that Π intersects the closure of Γ_1. Assuming the contrary there will be, by the polar theorem (cf. [21] p. 168) an element $(\phi_{jk}) \in E'$ such that $\mathrm{Re}\langle(\lambda_{jk}\delta_t),$ $(\phi_{jk})\rangle \geq 0$, $\forall(\lambda_{jk}\delta_t) \in \Gamma$, while $\mathrm{Re}\langle(\mu_{jk}), (\phi_{jk})\rangle \leq a < 0$, $\forall(\mu_{jk}) \in \Pi$. It is easily seen that since $\mu_{12} = \overline{\mu_{21}}$ for all elements in both Π and Γ, we can modify (ϕ_{jk}) as to have

(10) $$\begin{aligned} \langle(\lambda_{jk}\delta_t), (\phi_{jk})\rangle &\geq 0, \qquad \forall(\lambda_{jk}\delta_t) \in \Gamma \\ \langle(\mu_{jk}), (\phi_{jk})\rangle &\leq a < 0, \qquad \forall(\mu_{jk}) \in \Pi \end{aligned}$$

Taking $\lambda_{jk} = \lambda_j\bar{\lambda}_k$ in (10), it follows that $(\phi_{jk}(t))$ is a positive definite matrix for all $t \in \mathbb{T}$. Hence

(11) $$\phi_{11}(t) \geq 0, \qquad \phi_{22}(t) \geq 0, \qquad |\phi_{12}(t)|^2 \leq \phi_{11}(t)\phi_{22}(t), \qquad \forall t \in \mathbb{T}.$$

By the Corollary of the Hahn-Banach theorem, we can take $\int \phi_{jj} d\mu_{jj} = \rho_j(\phi_{jj})$, $j = 1, 2$ and by (10),

(12) $$\rho_1(\phi_1) + \int \phi_{12} \, d\nu + \int \phi_{21} \, \overline{d\nu} + \rho_2(\phi_{22}) \leqslant a < 0.$$

By (11) we can take (ϕ_{jk}) so that

(13) $\phi_{11} = \phi_+ \bar{\phi}_+$, $\phi_{22} = \phi_- \bar{\phi}_-$, $|\phi_{12}|^2 \leqslant \phi_+ \bar{\phi}_- \bar{\phi}_+ \phi_-$, $\phi_+ \in \overline{\mathcal{P}_1}$, $\phi_- \in \overline{\mathcal{P}_2}$.

Since ϕ_{12} and $\phi_+ \bar{\phi}_-$ both belong to $\overline{\mathcal{P}_{10}}$, (13) gives

(13a) $$\phi_{12} = \gamma \phi_+ \bar{\phi}, \qquad |\gamma| \leqslant 1, \qquad \gamma \in \overline{\mathcal{P}_1}.$$

Thus, $(\phi_{jk}) = (F_1 \overline{F_1} + G, F_1 \overline{F_2}, \overline{F_1} F_2, F_2 \overline{F_2})$, where

(13b) $F_1 = \gamma \phi_+ \in \overline{\mathcal{P}_1}$, $F_2 = \phi_- \in \overline{\mathcal{P}_2}$, $G = \phi_+ \bar{\phi}_+ (1 - |\gamma|^2) \geqslant 0$.

Hence, by (13b), (7a) we get

$$\rho_1(\phi_{11}) + \int \phi_{12} \, d\nu + \int \phi_{21} \, \overline{d\nu} + \rho_2(\phi_{22})$$
$$\geqslant \rho_1(F_1 \overline{F_1}) + \rho_2(F_2 \overline{F_2}) + \int F_1 \overline{F_2} \, d\nu + \int \overline{F_1} F_2 \, \overline{d\nu} \geqslant 0 > a$$

which contradicts (12).

Therefore, there is some quadruple $(\mu_{11}, \nu, \bar{\nu}, \mu_{22}) \in \Pi$ which is in the closure of Γ_1, hence there is a net of elements $(\nu_{jk}^\gamma) \in \Gamma_1$ such that

(14) $$\int f_{jj} \, d\nu_{jj}^\gamma \to \int f_{jj} \, d\mu_{jj}, \qquad j = 1, 2, \qquad \int f_{12} \, d\nu_{12}^\gamma \to \int f_{12} \, d\nu$$

for every $(f_{jk}) \in E'$ such that $f_{12} = \overline{f_{21}} \in \overline{\mathcal{P}_{10}}$.

From (11) it follows easily that, for each γ,

$$\left| \int \phi \, d\nu_{12}^\gamma \right|^2 \leqslant \left(\int \phi \, d\nu_{11}^\gamma \right) \left(\int \phi \, d\nu_{22}^\gamma \right), \qquad \forall \phi \in C(\mathbb{T}).$$

Since the ν_{jj}^γ are positive measures converging to μ_{jj}, all the ν_{jk}^γ are in a bounded set of $\mathfrak{M}$. Hence, passing to a subnet, there are measures (ν_{jk}) such that $\int \phi \, d\nu_{jk}^\gamma \to \int \phi \, d\nu_{jk}$ for all $\phi \in C(\mathbb{T})$. From (14) follows that $\nu_{jj} = \mu_{jj}, j = 1, 2$, and that $\int \phi \, d\nu_{12} = \int \phi \, d\mu_{12}$ for $\phi \in e^{it}\mathcal{P}_1$. Now (10) implies that $\sum_{j,k} \int f_j \overline{f_k} \, d\nu_{jk} \geqslant 0$ whenever $(f_1, f_2) \in \mathcal{P} \times \mathcal{P}$. Hence, also

$$\sum_{j,k} \int f_j \overline{f_k} \, d\mu_{jk} = \int f_1 \overline{f_2} \, d\mu_{11} + \int f_1 \overline{f_2} \, d\mu + \int \overline{f_1} f_2 \, \overline{d\mu} + \int f_2 \overline{f_2} \, d\mu_{22} \geqslant 0$$

for all $(f_1, f_2) \in \mathcal{P} \times \mathcal{P}$. Since $\mu_{jj} \geqslant 0$ and $\left| \int f_j \overline{f_j} \, d\mu_{jj} \right| \leqslant \rho_j(f_j \cdot \overline{f_j})$ by (9), and since $\mu = \nu$ on $e^{it}\mathcal{P}_1$, the thesis follows. $\square$

It $\hat{\mu}(n) = s'(n)$ and $\hat{f}(n) = a(n)$ for $a \in V$, then $\sum_n s'(n) a(n) = \int f \, d\mu$. From Lemma 2 and Theorem 1 for $\sigma_1 = \sigma_2$ follows

Corollary 1. *Let* $K(m, n) = s(m - n)$, $\sigma \sim \rho$ *as in* (6). *Then* $K \prec (\sigma, \sigma)$ *is a necesary and sufficient condition for the existence of a measure* ν *such that* $\hat{\nu}(n) = s(n)$ *for* $n > 0$, *and*

$$(15) \qquad \left| \int f \, d\nu \right| \leqslant \rho(f), \qquad \forall f \in C(\mathbb{T}).$$

Remark 1. When $\sigma_j \sim \rho_j$ non deterministic seminorms, $j = 1, 2$, a transference property to vector-valued sequences, or Grothendieck type inequality similar to that for GTKs majorized in the sense of (VII), proved in [11], is valid for the relation $K \prec (\sigma_1, \sigma_2)$. This will be developed elsewhere.

Let us introduce now a weaker notion of majoration. Write $K \prec (\sigma_1, \sigma_2)$ if

$$(16) \qquad |S_K(a, b)| \leqslant \sigma_1(a)\sigma_2(b), \qquad \forall(a, b) \in V_1 \times V_2, \qquad a(n) = \overline{b(-n)}$$

i.e.: for all $(a, b) \in \mathcal{P}_1 \times \mathcal{P}_2$ such that $b(t) = \overline{a(t)}$.

If $K(m, n) = \mu(m - n)$, μ a finite measure in $\mathbb{T}$, then $K \prec (\sigma_1, \sigma_2)$ means

$$(16a) \qquad \left| \int f^2 \, d\mu \right| \leqslant \sigma_1(\hat{f})\sigma_2(\hat{f}) = \rho_1'(f)\rho_2'(f), \qquad \forall f \in e^{it}\mathcal{P}_1.$$

In the case $\sigma_1 = \sigma_2$, this reduces to

$$(16b) \qquad \left| \int f^2 \, d\mu \right| \leqslant \rho(|f|^2), \qquad \forall f \in e^{it}\mathcal{P}_1.$$

Theorem 2. *For* $X = \mathbb{Z}$, $X_j = \mathbb{Z}_j$, *seminorms* $\sigma_j \sim \rho_j$ *as in* (6), $j = 1, 2$, *let* $K(m, n) = \hat{\mu}(m - n)$ *where* μ *is a finite complex measure in* $\mathbb{T}$. *If* $K \prec (\sigma_1, \sigma_2)$, *then there exists an analytic function* $h \in H^1(\mathbb{T})$ *such that* $d\nu = d\mu + \bar{h} \, dt$ *satisfies*

$$(17) \qquad \left| \int f\bar{g} \, d\nu \right| \leqslant \rho_3'(f)\rho_3'(g), \qquad \forall(f, g) \in \mathcal{P} \times \mathcal{P},$$

where $\rho_3 = \rho_1 + \rho_2$.

Therefore, if $\rho_1 = \rho_2$, *then* $K'(m, n) = \hat{\nu}(m - n)$ *satisfies* $K'|_{X_1 \times X_2} = K|_{X_1 \times X_2}$ *and* $\frac{1}{2}K' \prec (\sigma_1, \sigma_1)$.

PROOF. By condition (4b) on ρ_1, ρ_2, the hypothesis (16a) holds also for $f \in \overline{\mathcal{P}_{10}} = $ = the closure of $e^{it}\mathcal{P}_1$ in $C(\mathbb{T})$. Again, every $F \in e^{it}\mathcal{P}_1$ can be decomposed as $F = f\bar{g}$ with $(f, g) \in \overline{\mathcal{P}_1} \times \overline{\mathcal{P}_2}$, and $|f|^2 = |g|^2 = |F|$, so that $4F = (f + \bar{g})^2 + (if - i\bar{g})^2$, where $f + \bar{g}$ and $if - ig$ are in $\overline{\mathcal{P}_1}$. By (16a),

$$4 \int F \, d\mu = \int (f + \bar{g})^2 \, d\mu + \int (if - i\bar{g})^2 \, d\mu$$

$$\leqslant \rho_1(|f + \bar{g}|^2)^{1/2}\rho_2(|f + \bar{g}|^2)^{1/2} + \rho_1(|f - \bar{g}|^2)^{1/2}\rho_2(|f - \bar{g}|^2)^{1/2}$$

$$\leqslant 8\rho_1(F)^{1/2}\rho_2(F)^{1/2} \leqslant 4\rho_3(F), \qquad \forall F \in \overline{\mathcal{P}_{10}},$$

since $|f + \bar{g}|^2 = |f|^2 + |g|^2 + fg + \overline{fg} \leqslant 4|F|$ and $|f - \bar{g}|^2 \leqslant 4|F|$.

As $K(m, n) = \hat{\mu}(m - n)$, it follows as in Theorem 1 that there is a measure $dv = d\mu + h\,dt$ satisfying (17). $\square$

3. Refinements of the Theorem of Nehari

The theorem of Nehari ((I) of Section 1) gives a condition for the existence of a function θ belonging to the dual of $L^1(dt)$ and having prescribed moments $\hat{\theta}(n)$ for $n > 0$. The theorems of Section 2 give the following refinements of that theorem, where θ is required to belong to the «dual of a more general space».

Corollary 2. *Let* s: $\mathbb{Z} \to \mathbb{C}$ *be a given sequence*, $K(m, n) = s(m - n)$ *and* $\sigma \sim \rho$ *as in* (6). *If* ρ *is absolutely continuous or if* $s(n) = \hat{\omega}(n)$, $\omega \in L^1$, *then* $K \prec (\sigma, \sigma)$ *is a necessary and suficient condition for the existence of a function* $\theta \in L^1$ *such that*

$$(18) \qquad \left| \int f(t)\theta(t)\,dt \right| \leqslant \rho(f), \qquad \forall\, f \in C(\mathbb{T})$$

(i.e., for θ *to belong to the «dual of* $L(\rho)$*»*) *and* $\hat{\theta}(n) = s(n)$ *for* $n > 0$. *Another necessary and sufficient condition (up to multiplicative constant* 2*) is* $K \prec\!\prec (\sigma, \sigma)$. *If* ρ *is non deterministic, as in* (5), *then* (18) *can be replaced by the stronger inequality*

$$(18a) \qquad \int |f(t)| \frac{|\theta(t)|^2}{\Re_e\,\theta(t)}\,dt \leqslant c\rho(f), \qquad \forall\, f \in C(\mathbb{T})$$

with $\operatorname{Re}\theta \geqslant 0$.

Proof. The first two assertions follow from Theorem 1, Corollary 1 and Theorem 2. To deduce (18a), observe first that if $\hat{f}(n) = a(n)$, $\hat{g}(n) = b(n)$, $(a, b) \in V_1 \times V_2$, and $s(n) = \hat{\mu}(n)$ for $n \geqslant 0$, then $\sum s(m - n)\,a(m)\overline{b(n)} = \int f\bar{g}\,d\mu$, so that $K \prec (\sigma, \sigma)$ is equivalent to $\left| \int f\bar{g}\,d\mu \right| \leqslant \rho'(f)\rho'(g), \forall\, (f, g) \in \mathcal{P}_1 \times \mathcal{P}_2$. If ρ is non deterministic then $\rho'(f)\rho'(g) \leqslant 1/2(\rho'(f)^2 + \rho'(g)^2) \leqslant c^2/2\rho(|f + g|^2)$. Thus $\left| \int f\bar{g}\,d\mu \right| \leqslant c^2/2\rho(|f + g|^2)$ for $(f, g) \in \mathcal{P}_1 \times \mathcal{P}_2$, which is $K \leqslant \sigma_0$ for $\sigma_0(a, b) = \rho(|a + b|^2)$ in the sense of (VII). Now (18a) follows as a special case of Theorem 3 in [23]. $\square$

Remark 2. For $\rho(f) = \int |f|\,dt$, $\rho'(f) = \left(\int |f|^2\,dt \right)^{1/2}$, Corollary 2 implies the Nehari theorem. Nevertheless, even in this case, conditions $K \prec\!\prec (\sigma, \sigma)$ and (18a) provide stronger versions of the Nehari theorem.

It $\hat{\omega}(n) = s(n)$, then $\hat{\theta}(n) = s(n)$ for $n > 0$ is equivalent to $\theta = \omega + \bar{h}$, $h \in H^1(\mathbb{T})$. Thus, from Corollary 2 and the remark in its proof,

Corollary 3. *If $\omega \in L^1$, then the condition*

$$(19) \qquad \left| \int f\bar{g}\omega \, dt \right| \leqslant \rho'(f)\rho'(g), \qquad \forall (f, g) \in \mathcal{P}_1 \times \mathcal{P}_2$$

is necessary and sufficient for the existence of $\theta \in L^1$ such that $\theta = \omega + \bar{h}$, $h \in H^1(\mathbb{T})$, and

$$(18) \qquad \left| \int f\theta \, dt \right| \leqslant \rho(f), \qquad \forall f \in C(\mathbb{T}).$$

Let us finally observe that (VI) of Section 1 also leads, in the case $X = \mathbb{Z}$ considered here, to a refinement of the Nehari theorem of the following type.

Fix $p \in \mathbb{Z}_1$, take $X = \mathbb{Z}$, $X_1 = p\mathbb{Z}_1 = \{0, p, 2p, \dots\}$, $X_2 = p\mathbb{Z}_2 = \{-p, , -2p, \dots\}$, $\tau(n) = n + p$, and let $K_j(m, n) = s_j(m - n)$ be two fixed p.d. kernels, clearly τ-invariant. In this case, theorem (VI) asserts that if

$$|S_K(a, b)| \leqslant S_{K_1}(a, a)^{1/2} S_{K_2}(b, b)^{1/2}$$

for all finitely supported a, b such that $\operatorname{supp} a \subset p\mathbb{Z}_1$, $\operatorname{supp} b \subset p\mathbb{Z}_2$, then there exists another τ-invariant kernel K', such that $K'(m, n) = s(m - n)$ for $(m, n) \in p\mathbb{Z}_1 \times p\mathbb{Z}_2$ and $K' \leqslant (K_1, K_2)$.

Since K' is τ-invariant, $K'(m, n) = K'(m - n, 0)$ for $n \in p\mathbb{Z}$, i.e., $K'(m, n) = = s'(m - n)$ for $n \in p\mathbb{Z}$. Set $l(f) = l\left(\sum_n a(n)e^{\mathrm{int}}\right) = \sum a(n)s'(n)$ for $f \in \mathcal{P}$, $\hat{f} = a$. If $g = \sum_{n \in p\mathbb{Z}} b(n)e^{\mathrm{int}}$, then

$$|l(f\bar{g})| = \left| l\left(\sum_{\substack{m \in \mathbb{Z} \\ n \in p\mathbb{Z}}} a(m)\overline{b(n)}e^{i(m-n)t} \right) \right| = \left| \sum s'(m - n)a(m)\overline{b(n)} \right|$$

$$= \left| \sum K'(m, n)a(m)\overline{b(n)} \right| \leqslant \left(\int |f|^2 \, d\mu_1 \right)^{1/2} \left(\int |g|^2 \, d\mu_2 \right)^{1/2}.$$

Hence, $|l(|g|^2)| \leqslant \left(\int |g|^2 \, d\mu_1 \right)^{1/2} \left(\int |g|^2 \, d\mu_2 \right)^{1/2}$ for all $g \in \mathcal{P}(p) = \{ \sum_{n \in p\mathbb{Z}} b(n) e^{\mathrm{int}} \}$.

Taking $K_1 = K_2$ and letting $L^1(p)$ be the closed cone in $L^1(d\mu)$ generated by $\mathcal{P}(p)$, we have that $l \in L^\infty(p) = \{$the linear functionals in L^1 bounded in $L^1(p)\}$. Moreover, $l(e^{\mathrm{int}}) = s'(n)$, and $s'(m - n) = s(m - n)$ if $n \in p\mathbb{Z}$, so that $l(e^{i(m-n)t}) = s(m - n) = s'(m - n)$ if $(m, n) \in p\mathbb{Z}_1 \times p\mathbb{Z}_2$. Thus we have proved

Corollary 4. *If $\left| \sum s(m - n)a(m)\overline{b(n)} \right| \leqslant \|a\|_2 \|b\|_2$ for all sequences a, b finitely supported on $p\mathbb{Z}_1$, $p\mathbb{Z}_2$ respectively, then there exists $l \in L^\infty(p)$ such that $s(n) = \hat{l}(n) = l(e^{\mathrm{int}})$ for all $n \in p\mathbb{Z}_1$.*

Since $L^1(1) = L^1$ and $L^\infty(1) = L^\infty$, for $p = 1$ Corollary 4 reduces to the Nehari theorem.

4. Some Generalizations of Paley Inequality

A sequence $\{n_k\}$ of integers in λ-lacunary if $n_{k+1}/n_k \geqslant \lambda > 1$ for all k. A classical theorem of Paley asserts that a lacunary sequence of coefficients of an H^1 function belongs to l^2 [20]. More precisely, for every λ-lacunary sequence $\{n_k\}$ there is a constant $c = c(\lambda)$ such that if $f = \sum_{n \geq 0} c_n e^{int} \in H^1(\mathbb{T})$,

$$(20) \qquad \left(\sum_k |c_{n_k}|^2 \right)^{1/2} \leqslant c \int |f| \, dt.$$

We say that $\{n_k\}$ is a ρ-q-*Paley sequence* with constant c_q, where ρ is a seminorm as in Section 1, and $2 \leqslant q < \infty$, if for every $f = \sum_{n \geq 0} c_n e^{int} \in C(\mathbb{T})$,

$$(21) \qquad \left(\sum_k |c_{n_k}|^q n_k^{2-q} \right)^{1/q} \leqslant c_q \rho(f)$$

holds.

Following the notation of Section 2, given a sequence $a = \{a_n\}$, we write $a(t) = \sum a(n) e^{int}$ and $\sigma(a) = \rho(|\check{a}|^2)^{1/2}$, $\sigma \sim \rho$, for a fixed seminorm ρ.

Lemma 3. *Let ρ be an absolutely continuous seminorm satisfying* (4a) − (4c), $\sigma \sim \rho$, $2 \leqslant q < \infty$ *and* $\{n_k\}$ *a given sequence. For* $\{n_k\}$ *to be a* $\rho - q$ —*Paley sequence it is sufficient that, whenever* $\phi = \sum_{k > 0} v_k e^{in_k t} \in H^2(\mathbb{T})$, *s(n) defined for* $n > 0$ *as* v_k *if* $n = n_k$ *and zero if* $n \neq n_k$, *and* $K(m, n) = s(m - n)$,

$$(22) \qquad K \prec (r\sigma, r\sigma)$$

is satisfied for

$$(23) \qquad r^2 = r_q^2 = C_q \left(\sum_k |v_k|^{q'} n_k^{(q-2)/(q-1)} \right)^{1/q'}.$$

PROOF. Given $f = \sum_{n \geq 0} c_n e^{int} \in C(\mathbb{T})$, consider $\phi = \sum |c_{n_k}|^{q-1} n_k^{2-q} e^{in_k t}$. Then by the hypothesis on K and Corollary 3, there exists $\theta \in L^1$ such that $\left| \int f\theta \, dt \right| \leqslant r\rho(f)$, with $\bar{\theta}(n) = s(n)$ for $n > 0$. Thus we can write $\theta = \phi + \bar{h}$ for some $h \in H^1$. Now,

$$A = \sum_k |c_{n_k}|^q n_k^{2-q} = \int f\bar{\phi} \, dt = \int f(\bar{\phi} + h) \, dt$$

and

$$|A| \leqslant r\rho(f) = C_q \left(\sum_k |n_k|^{(q-1)q'} n_k^{(2-q)q'} n_k^{(q-2)/(q-1)} \right)^{1/q'} \rho(f)$$

$$= C_q |A|^{1/q'} \rho(f).$$

Therefore,

$$|A|^{1/q} \leqslant C_q \rho(f)$$

which is (21). □

Paley's theorem follows as a corollary of the Lifting Theorem 1, since it is not difficult to check that condition (22) holds for $\{n_k\}$ λ-lacunary, $q = 2$, $\rho(f) = \int |f| \, dt$, for a suitable constant C. This is part of a more general property of lacunary sequences proved in Lemma 4 below.

Lemma 3 suggests that the lifting theorems of Section 2 provide generalizations of Paley theorem. The connection between these two types of results is not surprising, since Theorem 1 provides refinements of Nehari theorem, and Paley inequality is equivalent by duality to a theorem of Rudin [22] (which says that if $\{n_k\}$ is λ-lacunary and $\{v_k\} \in l^2$ then there exists a $g \in L^\infty$ such that $\hat{g}(n) = v_k$ for $n = n_k$ and zero for $n \neq n_k$, $n > 0$, and such that $\|g\|_\infty \leqslant c\|v\|_2$) that can be derived from that of Nehari.

Given a λ-lacunary sequence $\{n_k\}$, setting the block $B_k = \{[n_k/2], \ldots, n_k\}$,

$$(24) \qquad \sum_k \sum_{j \in B_k} |b_j| \leqslant \sum_j \left(1 + \frac{1}{\log_2 \lambda}\right)|b_j|$$

for every sequence $\{b_j\}$.

This suggests to call a sequence $\{n_k\}$ (γ_j)-lacunary if γ_j is the number of blocks B_k containing the index j, so that

$$\sum_k \sum_{j \in B_k} |b_n| \leqslant \sum_j \gamma_j |b_j|$$

for every sequence $\{b_j\}$.

Lemma 4. Let $\{n_k\}$ be a (γ_j)-lacunary sequence, $2 \leqslant q < \infty$, $\{v_k\}$ a given sequence. If $s(n) = v_k$ if $n = n_k$, 0 if $n \neq n_k$, $n > 0$ and $K(m, n) = s(m - n)$, then $K \prec (r\sigma, r\sigma)$, for $\sigma \sim \rho_q$ with

$$(25) \qquad \rho_q(f) = \left(\int |f|^{q/2} |t|^{q-2} \, dt\right)^{2/q}$$

and

$$(23a) \qquad r^2 = r_{q\gamma}^2 = C_q\left(\sum_k |v_k|^{q'} n_k^{(q-2)/(q-1)} \left|\max_{j \in B_k} \gamma_j\right|^{q'}\right)^{1/q'}.$$

PROOF. By definition

$$S_K(a, b) = \sum_{\substack{m \geq 0 \\ n < 0}} s(m - n)a_m\overline{b_n} = \sum_{k > 0} v_n(a_0\overline{b_{n_k}} + a_1 b_{n_k - 1} + \cdots + a_{n_k}\overline{b_0})$$

$$\leq \sum_{k > 0} v_k \sum_{m = 0}^{[n_k/2]} a_m\overline{b_{n_k - m}} + \sum_{k > 0} v_k \sum_{m = 0}^{[n_k/2]} a_{n_k - m}\overline{b_m}$$

$$= A + B.$$

$$|A| \leq \sum_{k > 0} |v_k|(|a_0|^{q'} + \cdots + |a_{[n_k/2]}|^{q'})^{1/q'}(|b_{n_k}|^q + \cdots + |b_{[n_k/2]}|^q)^{1/q}$$

where $1/q + 1/q' = 1$. By applying again Hölder's inequality with exponents $r = q/q' \geq 1$ and $r' = (q - 1)/(q - 2)$,

$$|A| \leq \sum_{k > 0} |v_k|(|a_0|^q + \cdots + |a_{[n_k/2]}|^q)^{1/q}n_k^{(q - 2)/q}(|b_{n_k}|^q + \cdots + |b_{[n_k/2]}|^q)^{1/q}$$

$$\leq \|a\|_q \sum_{k > 0} |v_k| n_k^{(q - 2)/q}\left(\max_{j \in B_k}\gamma_j\right)(|b_{n_k}|^q/\gamma_{n_k} + \cdots + |b_{[n_k/2]}|^q/\gamma_{[n_k/2]})^{1/q}$$

Writing $w = \{w_k\}$, $w_k = v_k n_k^{(q - 2)/q}(\max_{j \in B_k}\gamma_j)$,

$$|A| \leq \|a\|_q\|w\|_{q'}\left[\sum_{k > 0}(|b_{n_k}|^q/\gamma_{n_k} + \cdots + |b_{[n_k/2]}|^q/\gamma_{[n_k/2]})\right]^{1/q}$$

$$\leq \|a\|_q\|w\|_{q'}\left(\sum_{n > 0}|b_n|^q\right)^{1/q}$$

since each b_{n_k} appears in at most γ_{n_k} blocks.
By repeating the argument for B,

$$|S_K(a, b)| \leq 2\|w\|_{q'}\|a\|_q\|b\|_q.$$

Since $q \geq 2$, by Hardy-Littlewood-Paley weighted inequality,

$$|S_K(a, b)| \leq C_q\|w\|_{q'}\left(\int |\breve{a}|^q|t|^{q - 2}\,dt\right)^{1/q}\left(\int |\breve{b}|^q|t|^{q - 2}\,dt\right)^{1/q} =$$

$$= r^2\rho(|\breve{a}|^2)^{1/2}\rho(|\breve{b}|^2)^{1/2} = r^2\sigma(a)\sigma(b)$$

where ρ is as in (25) and $\sigma \sim \rho$, for all finite sequences a, b, $\operatorname{supp} a \subset \mathbb{Z}_1$, $\operatorname{supp} b \subset \mathbb{Z}_2$, i.e., $K \prec (r\sigma, r\sigma)$.

Since for a λ-lacunary sequence, $\gamma_j = (\log_2 \lambda)^{-1} + 1$ for all j, Lemmas 1 and 2 immediately imply the following result, which reduces to the classical Paley inequality when $q = 2$.

Theorem 3. *Every λ-lacunary sequence is a ρ_q-q-Paley sequence with constant depending only on λ and q, for ρ_q as in (25) and $2 \leq q < \infty$. More generally, the same holds for every (γ_j)-lacunary sequence with $|\gamma_j| \leq$ cst. for all j.*

Let us consider now some non-lacunary sequences, for which the γ_j's are not uniformly bounded. With the same notation as before, it is easy to check that

(a) for $n_k = k^r$, r fixed, γ_j (= # of blocks containing the index j) $= c_r j^{1/r}$, and $\max_{j \in B_k} (\gamma_j) = c_r k$;

(b) for $n_k = 2^{\sqrt{k}}$, $\gamma_j = 2 \log_2 j$, and $\max_{j \in B_k} \gamma_j = 2 \sqrt{k}$.

In both cases, Lemma 4 applies as before and, with it, the proof of Lemma 3, suitably modified, yields alternate inequealities to (21). Before stating these, let us observe that for ρ_q as in (25), $\rho_q(f) < \infty$ does not imply $f \in L^1(\mathbb{T})$.

In order to give a meaning to $\{c_n\} \sim f$ for all f such that $\rho_q(f) < \infty$, we may consider them as boundary values of analytic functions defined in the disc, in the following way. Let $D = \{z \in \mathbb{C}: |z| < 1\}$, and the Hardy and Nevanlinna classes in D be given by

$$H^p(D) = \left\{ F \text{ holomorphic in } D: \sup_{0 \le r < 1} \int |F(re^{it})|^p \, dt = \|F\|_{H^p}^p \le \infty \right\},$$

$$0 < p < \infty,$$

$$N(D) = \left\{ F \text{ holomorphic in } D: \sup_{0 \le r < 1} \int \log^+ |F(re^{it})| \, dt < \infty \right\}.$$

For $N^+ = \left\{ F \in N: \lim_{r \to 1} \int \log^+ |F(re^{it})| \, dt = \int \log^+ |F(e^{it})| \, dt \right\}$, the inclusion $N \supset N^+ \supset H^p$, $0 < p < \infty$, holds. A function $f \in H^p(\mathbb{T})$ can be identified with $F(e^{it})$ for $F \in H^p(D)$, with $\|f\|_p = \|F\|_{H^p}$.

Let $2 \le q < \infty$ and $F \in N^+$ with boundary values $F(e^{it}) = f(t) \in L^{q/2}(|t|^{q-2} dt)$. Then, by Hölder's inequality, $\int |F(e^{it})|^{1/2} \, dt < \infty$, which together with $F \in N^+$, imply $F \in H^{1/2}(D)$ (cf. [14], p. 28). Thus $F(z) = \sum_{n=0}^{\infty} c_n z^n$ and $f(t) = \sum_{n=0}^{\infty} c_n e^{int}$. Form the previous remarks follows

Proposition 1. *Let* $2 \le q < \infty$, *and let* $F \in N^+$ *be an holomorphic function in the disc with boundary values* $F(e^{it}) = f(t) \in L^{q/2}(|t|^{q-2} dt)$, $F(x) = \sum_0^{\infty} c_n z^n$.

(a) *If* $\{n_k\}$ *is a* λ-*lacunary sequence, then*

$$\left(\sum_k |c_{n_k}|^q n_k^{2-q} \right)^{1/q} \le c_q \left(\int |f|^{q/2} |t|^{q-2} \, dt \right)^{2/q}.$$

(b) *If* $\{n_k = k^r\}$, r *fixed,*

$$\left(\sum_k |c_{k^r}|^q k^{2-(r+1)q} \right)^{1/q} \le c_q \left(\int |f|^{q/2} |t|^{q-2} \, dt \right)^2.$$

(c) *If* $\{n_k = 2^{\sqrt{k}}\}$,

$$\left(\sum_k |c_{n_k}|^q n_k^{2-q} (\log_2 n_k)^{-q} \right)^{1/q} \le c_q \left(\int |f|^{1/2} |t|^{q-2} \, dt \right)^{2/q}.$$

The norm ρ_q is non deterministic (see Section 2) since the Hilbert transform in continuous on every $L^q(\omega\,dt)$ for $1 < q < \infty$, $\omega \in A_q$ (the Muckenhoupt class), and, for $2 \leqslant q < \infty$, $\omega = |t|^{q-2} \in A_q$ (cf. [15]). A result dual to Theorem 3 and Proposition 1 is

Theorem 4. *Let* $\{n_k\}$ *be a* (γ_j)-*lacunary sequence,* $\{v_k\}$ *a given sequence,* $2 \leqslant q < \infty$. *There exists a function* θ, $\mathrm{Re}\,\theta \geqslant 0$, $\theta^2/\mathrm{Re}\,\theta \in L^{q/(q-2)}(dt/t^2)$, *such that*

$$\hat{\theta}(n) = v_k \quad \text{if} \quad n = n_k, \ = 0 \quad \text{if} \quad n \neq n_k, \quad \text{for} \quad n > 0,$$

and

$$(26) \qquad \left| \int f \frac{\theta^2}{\mathrm{Re}\,\theta}\,dt \right| \leqslant c_q \Big(\sum_k |v_k|^{q'} n^{(q-2)/(q-1)} \Big| \max_{j \in B_k} \gamma_j \Big|^{q'} \Big)^{1/q'} \rho_q(f)$$

for all $f \in C(\mathbb{T})$, ρ_q *as in* (25).

PROOF. As in previous proofs, define for $n > 0$, $s(n) = v_k$ if $n = n_k$, $s(n) = 0$ if $n \neq n_k$, and $K(m,n) = s(m-n)$. By Lemma 4, $K \prec (r\sigma, r\sigma)$ with r, σ as in (23a), (25). By Corollary 3, there exists a function θ defined on $\mathbb{T}$, $\mathrm{Re}\,\theta \geqslant 0$, such that $\hat{\theta}(n) = s(n)$ for $n > 0$ and that (26) holds. From (26) it follows that $\theta(t)|t|^{2-q}$ belongs to the dual of $L^{q/2}(|t|^{q-2}\,dt)$, which is equivalent to $\theta \in L^{q/(q-2)}(dt/t^2)$. $\square$

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