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Estimates for Wainger's Singular Integrals Along Curves

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Dedicated to Alberto P. Calderón

1. Introduction

The theory of singular integrals started by Calderón and Zygmund [1] has been extended in several different ways during the last three decades. One such extension is the singular integral operator

$$Hf(x) = p \cdot v \int_{-\infty}^{\infty} f(x - \Gamma(t)) \frac{dt}{t} \tag{i}$$

where $\Gamma: \mathbb{R} \to \mathbb{R}^n$, $n \geqslant 2$, is a curve passing through the origin: $\Gamma(0) = 0$. This object, called the Hilbert transform along the curve Γ, appears in the study of parabolic differential operators as well as in the harmonic analysis of homogeneous spaces. A great deal of effort has been dedicated, principally under the influence of S. Wainger, to the understanding of their boundedness properties. See [2] for a fairly complete presentation, together with a summary of results and applications.

These objects are also multipliers of the Fourier transform and, therefore, can be defined by the formula

$$Hf(\xi) = \hat{f}(\xi) m(\xi)$$

where

$$m(\xi) = p \cdot v \int_{-\infty}^{\infty} \exp(i\xi \cdot \Gamma(t)) \frac{dt}{t}.$$

Similarly to the case of the ordinary Hilbert transform, where we have the Hardy-Littlewood maximal function as the positive operator controlling its behaviour, one can define maximal operators associated to these singular integrals

(ii) $$Mf(x) = \sup_{h>0} \frac{1}{2h} \int_{-h}^{h} |f(x - \Gamma(t))|\, dt$$

(iii) $$H^*f(x) = \sup_{\epsilon>0} \left| \int_{|t| \geq \epsilon} f(x - \Gamma(t)) \frac{dt}{t} \right|.$$

The first natural question one may ask is which are the conditions to be imposed on the curve Γ so that H, H^* and M are bounded on some, or all, spaces $L^p(\mathbb{R}^n)$. In a series of papers, A. Nagel, E. Stein, S. Wainger and others have obtained positive results involving some kind of «positiveness» of the curvature of Γ, taking advantage of the decay of the multiplier $m(\xi)$ that it implies. The proofs are completed by an ingeneous use of the method of complex interpolation, together with arguments of a tauberian type involving certain square functions. In [3], it has been observed that, for convex curves in $\mathbb{R}^2$, $\Gamma(t) = (t, \gamma(t))$, with $\gamma(0) = \gamma'(0) = 0$, there exist necessary and sufficient conditions for H to be bounded in $L^2(\mathbb{R}^2)$: when γ is even (resp. odd) the condition is that $\gamma'(t)$ (resp. $h(t) = t\gamma'(t) - \gamma(t)$) must have the doubling time property, i.e., $\gamma'(Ct) \geqslant 2\gamma'(t)$ (resp. $h(Ct) \geqslant 2h(t)$) for some fixed constant C and for every $t > 0$. This condition has been analyzed in [4] in relation to the maximal function M. The range of L^p-estimates has been extended from L^2 to L^p, $4/3 < p < 4$, in reference [5].

In this paper, we improve and extend the results mentioned above in the following manner: we consider a C^1 curve in $\mathbb{R}^2$, $\Gamma = \{(t, \gamma(t)) : -\infty < t < +\infty\}$ $\gamma(0) = \gamma'(0) = 0$, with the following properties:

a) γ is biconvex, i.e., $|\gamma'(t)|$ is decreasing in $(-\infty, 0)$ and increasing in $(0, +\infty)$.

b) γ has *doubling time*, i.e., there exists a constant $C > 1$ such that $|\gamma'(Ct)| \geqslant 2|\gamma'(t)|$.

c) γ is *balanced*, by which we mean the following: there exists $k > 1$ such that $|\gamma(k^{-1}t)| \leqslant |\gamma(-t)| \leqslant |\gamma(kt)|$ for every $t > 0$.

Theorem. *Under the assumptions a*), *b*) *and c*) *on the curve the singular integral H and the maximal functions M and H^* are bounded operators in $L^p(\mathbb{R}^2)$ for every p, $1 < p < \infty$.*

Immediate consequences of the boundedness of M and H^* are:

Corollary. *For every* $f \in L^p(\mathbb{R}^2)$, $1 < p < \infty$,

$$\lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} f(x - \Gamma(t))\, dt = f(x) \qquad \text{a.e.}$$

$$\lim_{\epsilon \to 0} \int_{|t| > \epsilon} f(x - \Gamma(t)) \frac{dt}{t} = Hf(x) \qquad \text{a.e.}$$

This theorem gives a fairly complete analysis of the properties of Hilbert transforms along convex curves in the plane. For every fixed p, $1 < p < \infty$, the doubling time property is both necessary and sufficient for L^p-boundedness if the curve is even. Likewise, for a biconvex curve, the condition of being balanced is necessary for the L^2-boundedness of H, as we show in §6 below. The fact that for such curves one cannot obtain boundedness of the Hilbert transform in some $L^p(\mathbb{R}^2)$ without obtaining the full range $1 < p < \infty$, explains, in retrospect, the success of the theory, in contrast with the situation in the analysis of the spherical summation multipliers.

We believe that equally important as the theorem itself, is the fact that we treat the operator $Hf(\xi) = \hat{f}(\xi)m(\xi)$ by taking advantage of the decay of $m(\xi)$, its smoothness and the geometry of its level sets, which suggest the use of different (angular and vertical) Littelewood-Paley decompositions. Weighted and vector valued inequalities are also some of the tools used to pass information from estimates of the maximal funcion to estimates for the Hilbert transform, and vice-versa. The set of techniques is due to the contribution of a long list of people, as reference [6] clearly indicates. There are several slightly different proofs of the boundedness of the operators H and M for even or odd curves, one of which is given in [6]. Here we present our version, together with the extension to the class of balanced curves and the appropriate estimates for the maximal Hilbert transform which give the pointwise existence of the principal value singular integral.

2. Decomposition of the Operators and Sketch of Proof

All the L^p estimates will be proved for $f \in \mathcal{S}(\mathbb{R}^2)$, and then extended by continuity. We define the singular measures M_k and h_k $(k \in \mathbb{Z})$ such that

$$\hat{M}_k(\xi, \eta) = \int_{1 \le |t| \le 2} \exp\{i(2^k t\xi + \gamma(2^k t)\eta)\}\, dt$$

$$\hat{h}_k(\xi, \eta) = \int_{1 \le |t| \le 2} \exp\{i(2^k t\xi + \gamma(2^k t)\eta)\} \frac{dt}{t} \quad .$$

and consider the following basic decompositions of our operators

$$Mf(x) \simeq \sup_k |M_k * f(x)|$$

$$Hf(x) = \sum_{-\infty}^{\infty} h_k * f(x)$$

$$H^*f(x) \simeq \sup_k \left\| \sum_{n>k} h_n * f(x) \right\|$$

We shall also use the following square function

(iv) $$Gf(x) = \left(\sum_{k=-\infty}^{\infty} |\tilde{M}_k f(x)|^2 \right)^{1/2}$$

where

$$\tilde{M}_k f(\xi, \eta) = [\hat{M}_k(\xi, \eta) - \hat{\Phi}(2^k \xi)\hat{M}_k(0, \eta)]\hat{f}(\xi, \eta)$$

for a suitable function Φ in the Schwartz class satisfying $\hat{\Phi}(0) = 1$.

The proof of the boundedness of the maximal function Mf is based on the following strategy: we have the pointwise majorization $Mf(x) \leqslant Gf(x) + f^*(x)$, where f^* denotes the strong maximal function of f. We start by proving the L^2-boundedness of Gf by means of Plancherel's theorem. This implies the same result for Mf which, in turn, implies the L^p estimate, $4/3 < p < 4$, for the square function Gf by a nowadays familiar argument based on a vector valued inequality and an ad-hoc Littlewood-Paley decomposition of frequency space. In general, and L^p estimate for M implies an L^q estimate for G, with $\left|\frac{1}{q} - \frac{1}{2}\right| < \frac{1}{2p}$, and therefore, a bootstrap argument will yield the whole range of p's, $1 < p < \infty$.

The doubling time condition of the curve will be used to define angular Littlewood-Paley decompositions. For the sake of simplicity, we shall assume that the doubling time constant is $C = 2$. Then, we consider the angular regions

$$\Delta_k^{\pm} = \left\{ (\xi, \eta): \frac{1}{2} |\gamma'(\pm 2^k)| < \frac{|\xi|}{|\eta|} < 2|\gamma'(\pm 2^{k+1})| \right\}$$

($k \in \mathbb{Z}$). It turns out that $\{\Delta_k^+\}_{-\infty}^{\infty}$ is a lacunary decomposition of $\mathbb{R}^2$ with finite overlapping: $\Delta_h^+ \cap \Delta_k^+ = \varnothing$ if $|k - h| \geqslant 3$, and the same is true for $\{\Delta_k^-\}$.

3. The Maximal Operator and the Square Function

Let $\Delta_k = \Delta_k^+ \cup \Delta_k^-$ and define the projection operators T_k by $(T_k f)^\wedge = \hat{f}\chi_{\Delta_k}$, $k \in \mathbb{Z}$.

Lemma 1. *For every p, $1 < p < \infty$, we have*

a) $$\Big\| \Big(\sum_k |T_k f|^2\Big)^{1/2} \Big\|_p \leqslant c_p \|f\|_p$$

b) $$\Big\| \sum_k T_k f_k \Big\|_p \leqslant c_p \Big\| \Big(\sum_k |f_k|^2\Big)^{1/2} \Big\|_p$$

c) $$\Big\| \sup_k \Big| \sum_{n \geq k} T_n f \Big| \Big\|_p \leqslant c_p \|f\|_p.$$

a) is nowadays a well known inequality (see [2] for references) and *b)* is its dual form; *c)* follows from *a)* in the same way that one proves the lacunary convergence of Fourier series.

Lemma 2.

$$\Gamma f(\delta) = \sup_{h>0} \frac{1}{2h} \int_{-h}^{h} |f(\delta - \gamma(t))| \, dt$$

is a bounded operator in $L^p(\mathbb{R})$, $1 < p < \infty$.

The proof is obvious due to the monotonicity of γ':

$$\frac{1}{h} \int_0^h f(\delta - \gamma(t)) \, dt = \frac{1}{h} \int_0^{\gamma(h)} f(\delta - u) \frac{du}{\gamma'(\gamma^{-1}(u))}$$
$$= \int_0^c f(\delta - u)\varphi(u) \, du \leqslant f^*(\delta),$$

(Hardy-Littlewood maximal function), because $\varphi(u)$ is nonnegative, decreasing and $\int_0^c \varphi(u)\, du = 1$. The same argument holds for the average over the interval $[-h, 0]$.

Lemma 3. *Let $\Phi \in \mathcal{S}(\mathbb{R})$ be such that $\chi_{[-1,1]} \leqslant \hat{\Phi} \leqslant \chi_{[-2,2]}$. Then*

$$|\hat{M}_k(\xi, \eta) - \hat{\Phi}(2^k \xi)\hat{M}_k(0, \eta)| \leqslant C \min(2^k|\xi|, (2^k|\xi|)^{-1}),$$

for all $(\xi, \eta) \notin \Delta_k$. The same estimates hold for $\hat{h}_k(\xi, \eta)$ in place of $\hat{M}_k(\xi, \eta)$.

PROOF. The estimate for the case $2^k|\xi| \leqslant 1$ is trivial because M_k is a measure supported in $\{(x_1, x_2): |x_1| \leqslant 2^{k+1}\}$. For $2^k|\xi| > 1$ we use Van der Corput lemma

$$\hat{M}_k(\xi, \eta) = \int_1^2 e^{ig_k(t)} \, dt + \int_1^2 e^{il_k(t)} \, dt$$

with

$$|g_k'(t)| = |2^k(\xi + \gamma'(2^k t)\eta|$$

and

$$|l_k'(t)| = |2^k(\xi + \gamma'(-2^k t)\eta)|.$$

If $|\xi| \geqslant 2|\eta| \cdot |\gamma'(2^{k+1})|$ then $|g_k'(t)| \geqslant 2^{k-1}|\xi|$. If $|\xi| \leqslant {}^1/2\,|\eta| \cdot |\gamma'(2^k)|$, then also $|g_k'(t)| \geqslant 2^{k-1}|\gamma'(2^k t)| \cdot |\eta| \geqslant 2^k|\xi|$ and the situation for l_k' is similar. Finally the complement of these four cases corresponds to points $(\xi, \eta) \in \Delta_k$. The proof for $\hat{h}_k(\xi, \eta)$ is essentially the same. □

Suppose that $U_k f = U_k * f$ is a sequence of positive operators uniformly bounded in $L^\infty(\mathbb{R}^n)$ and consider $U^*f(x) = \sup_k |U_k * f(x)|$.

Lemma 4. *If $U^*f(x) = \sup_k |U_k \cdot f(x)|$ is bounded in $L^q(\mathbb{R}^n)$ then*

$$\Big\| \Big(\sum_k |U_k f_k|^2\Big)^{1/2} \Big\|_p \leqslant c_p \Big\| \Big(\sum_k |f_k|^2\Big)^{1/2} \Big\|_p$$

for $\left|\frac{1}{p} - \frac{1}{2}\right| = \frac{1}{2q}$.

This is a well known result and we have stated it here only to clarify the presentation of the proof of the theorem. Our proof of the estimates of the maximal function depends on the following proposition of a conditional nature:

Lemma 5. *Suppose that $\|Mf\|_q \leqslant C\|f\|_q$ for some q. Then for all p such that*

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{2q}$$

we have the inequality

$$\|Gf\|_p \leqslant C_p \|f\|_p$$

where Gf is defined by (iv).

Proof. Recall that $T_k f = \chi_{\Delta_k} \cdot \hat{f}$, and let us write $f^k = f - T_k f$ and

$$S_k f = \hat{f} \cdot \chi_{\{(\xi,\eta):\, 2^{-k} \leq |\xi| \leq 2^{-k+1}\}}.$$

Then

$$Gf \leqslant \Big(\sum_k |\tilde{M}_k * T_k f|^2\Big)^{1/2} + \Big(\sum_k |\tilde{M}_k * f^k|^2\Big)^{1/2} = I + II.$$

Now, observe that $\sup_k (|\tilde{M}_k| * f)$ is a bounded operator in L^q due to the hypothesis and lemma 2. Therefore

$$\|I\|_p \leqslant C_p \left\| \left(\sum_k |T_k f|^2 \right)^{1/2} \right\|_p \leqslant C_p \|f\|_p$$

by lemmas 4 and 1.

$$II = \left(\sum_k \left| \tilde{M}_k * \left(\sum_{j=-\infty}^{+\infty} S_{j+k} f^k \right) \right|^2 \right)^{1/2} \leqslant \sum_j \left(\sum_k |\tilde{M}_k * S_{j+k} f^k|^2 \right)^{1/2} = \sum_j II_j.$$

We shall prove:

$$\|II_j\|_2 \leqslant C 2^{-|j|} \|f\|_2 \quad \text{and} \quad \|II_j\|_{p_0} \leqslant C \|f\|_{p_0} \quad \text{for} \quad \left| \frac{1}{2} - \frac{1}{p_0} \right| = \frac{1}{2q}.$$

Interpolating, we shall obtain

$$\|II\|_p \leqslant \sum_j \|II_j\|_p \leqslant \sum_j C 2^{-\theta|j|} \|f\|_p \leqslant C_p \|f\|_p \quad \text{with} \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{2}.$$

We use Plancherel's theorem for the L^2-inequality

$$\|II_j\|_2^2 = \sum_k \int_{\substack{|\xi| \sim 2^{-j-k} \\ (\xi,\eta) \notin \Delta_k}} |\hat{M}_k(\xi,\eta) - \hat{\Phi}(2^k \xi) \hat{M}_k(0,\eta)|^2 |\hat{f}(\xi,\eta)|^2 \, d\xi \, d\eta$$

$$\leqslant (\text{by lemma 3}) \leqslant C \int |\hat{f}(\xi,\eta)|^2 (2^{-|j|})^2 \, d\xi \, d\eta = C 2^{-2|j|} \|f\|_2^2.$$

Finally

$$\begin{aligned} \|II_j\|_{p_0} &\leqslant C_{p_0} \left\| \left(\sum_k |S_{j+k} f^k|^2 \right)^{1/2} \right\|_{p_0} \\ &\leqslant C_{p_0} \left\{ \left\| \left(\sum_k |S_{j+k} f|^2 \right)^{1/2} \right\|_{p_0} + \left\| \left(\sum_k |S_{j+k}(T_k f)|^2 \right)^{1/2} \right\|_{p_0} \right\} \\ &\leqslant C_{p_0} \left\{ \|f\|_{p_0} + \left\| \left(\sum_k |T_k f|^2 \right)^{1/2} \right\|_{p_0} \right\} \leqslant C_{p_0} \|f\|_{p_0}. \quad \square \end{aligned}$$

To finish the proof of the theorem for the maximal operator, we use the majorization

$$Mf(x) \leqslant 2 \sup_k |M_k * f(x)| \leqslant 2 \sup_k |\tilde{M}_k * f(x)| + 2Nf(x) \leqslant 2Gf(x) + 2Nf(x)$$

where N is the hardy-Littlewood operator in the x_1-variable and the operator Γ of lemma 2 in the x_2—varible. Now, the bootstrap argument sketched in section 2 can be applied and yields $\|Mf\|_p \leqslant C_p \|f\|_p$, $1 < p \leqslant \infty$.

4. The Singular Integral

The estimate for Hf will be based on an argument similar to that of lemma 5 together with the fact that the operators $h_k * f$ are acontrolled by the maximal operator Mf.

We first consider the case of even $\gamma(t)$, so that $\hat{h}_k(0, \eta) = 0$. The notation used is as in lemma 5.

$$Hf = \sum_k h_k * T_k f + \sum_k h_k * f^k = H_I f + H_{II} f.$$

By using lemmas 1 and 4, and the fact that $\sup_k |h_k| * f \leqslant Mf$

$$\|H_I f\|_p = \Big\| \sum_k T_k (h_k * T_k f) \Big\|_p \leqslant C_p \Big\| \Big(\sum_k |h_k * T_k f|^2 \Big)^{1/2} \Big\|_p$$
$$\leqslant C_p \Big\| \Big(\sum_k |T_k f|^2 \Big)^{1/2} \Big\|_p \leqslant C_p \|f\|_p.$$

On the other hand

$$|H_{II} f| = \Big| \sum_k h_k * \sum_{j=-\infty}^{+\infty} S_{j+k} f^k \Big| \leqslant \sum_j \Big| \sum_k h_k * S_{j+k} f^k \Big| = \sum_j |II_j|.$$

Since $\hat{h}_k(0, \eta) = 0$, lemma 3 and Plancherel's theorem yield

$$\|II_j\|_2^2 = \sum_k \int_{\substack{|\xi| \sim 2^{-j-k} \\ (\xi, \eta) \notin \Delta_k}} |\hat{f}(\xi, \eta)|^2 |\hat{h}_k(\xi, \eta)|^2 \, d\xi \, d\eta$$
$$\leqslant C \int |\hat{f}(\xi, \eta)|^2 2^{-2|j|} \, d\xi \, d\eta \leqslant C 2^{-2|j|} \|f\|_2^2$$

In L^p, $1 < p < \infty$, we have a uniform estimate

$$\|II_j\|_p = \Big\| \sum_k S_{j+k} (h_k * S_{j+k} f^k) \Big\|_p$$
$$\leqslant C_p \Big\| \Big(\sum_k |h_k * (S_{j+k} f^k)|^2 \Big)^{1/2} \Big\|_p$$
$$\leqslant C_p \Big\| \Big(\sum_k |S_{j+k} f^k|^2 \Big)^{1/2} \Big\|_p \leqslant C_p \|f\|_p$$

and the proof ends by interpolating and summing a geometric series.

For the case of a general curve, we decompose

$$h_k = \tilde{h}_k + m_k$$

where, with Φ as above

$$\hat{m}_k(\xi, \eta) = \hat{\Phi}(2^k \xi) \cdot \hat{h}_k(0, \eta).$$

Then, the Fourier transform of $\tilde{h}_k$ satisfies the same estimates as in the case on an even curve, namely

$$|(\tilde{h}_k)^\wedge(\xi, \eta)| \leqslant C \min(2^k|\xi|, (2^k|\xi|)^{-1}) \quad \text{if} \quad (\xi, \eta) \notin \Delta_k$$

and $\sup |\tilde{h}_k| * f \leqslant Mf + Nf$ is bounded in $L^p(\mathbb{R}^2)$ for all $p > 1$. Therefore, the previous argument gives

$$\left\| \sum_k \tilde{h}_k * f \right\|_p \leqslant C_p \|f\|_p, \qquad 1 < p < \infty.$$

Next, we need to estimate the decay of $\hat{h}_k(0, \eta)$. It is precisely here where the hypothesis of having a balanced curve plays a role. Let us assume for simplicity that the balancing constant is $k \leqslant 2$. Define $\lambda_k = |\gamma(2^k)|$, $k \in \mathbb{Z}$, so that $\{\lambda_k\}_{-\infty}^{\infty}$ is (by convexity) a lacunary sequence, and the balancing condition implies $\lambda_{k-1} \leqslant |\gamma(-2^k)| \leqslant \lambda_{k+1}$, $k \in \mathbb{Z}$. Now, for small η we have

$$\begin{aligned} |\hat{h}_k(0, \eta)| &= \left| \int_{1 \leq |t| \leq 2} [e^{i\eta\gamma(2^k t)} - 1] \frac{dt}{t} \right| \\ &\leqslant (\log 2)|\eta|(|\gamma(2^{k+1})| + |\gamma(-2^{k+1})|) \leqslant \text{Const}\, \lambda_{k+2} \cdot |\eta|. \end{aligned}$$

For large η we once again use Van der Corput's lemma; since

$$\left| \frac{d}{dt}(\eta\gamma(2^k t)) \right| = |2^k \eta \gamma'(2^k t)| \geqslant \lambda_{k-1}|\eta|, \qquad 1 \leqslant |t| \leqslant 2$$

we obtain

$$|\hat{h}_k(0, \eta)| \leqslant \text{Const}\,(\lambda_{k-1}|\eta|)^{-1}.$$

We now apply the same argument of the even case alter replacing the operators S_j by $\tilde{S}_j$ defined by

$$(\tilde{S}_j f)^\wedge(\xi, \eta) = \hat{f}(\xi, \eta) \cdot \chi_{I_j}(\eta)$$

where $I_j = [\lambda_{j+1}^{-1}, \lambda_j^{-1}] \cup [-\lambda_j^{-1}, -\lambda_{j+1}^{-1}]$. Then

$$\sum_k m_k * f = \sum_{j,k} m_k * \tilde{S}_{j+k} f = \sum_{j=-\infty}^{+\infty} A_j f.$$

The uniform estimate $\|A_j f\|_p \leqslant C_p \|f\|_p$, $1 < p < \infty$, is as in the even case. In L^2 we have

$$\|A_j f\|_2^2 = \sum_k \int_{\mathbb{R}} \int_{I_{j+k}} |\hat{f}(\xi, \eta)|^2 |m_k(\xi, \eta)|^2 \, d\xi \, d\eta.$$

If $j > 1$ and $\eta \in I_{j+k}$,

$$|m_k(\xi, \eta)| \leqslant \frac{\lambda_{k+2}}{\lambda_{k+j}} \leqslant \rho^{2-j},$$

where $\rho > 1$ is the lacunarity constant of the sequence $\{\lambda_k\}_{-\infty}^{-\infty}$. If $j < -1$ and $\eta \in I_{j+k}$,

$$|m_k(\xi, \eta)| \leqslant \frac{\lambda_{j+k+1}}{\lambda_{k-1}} \leqslant \rho^{2+j}.$$

Therefore

$$\|A_j f\|_2 \leqslant \mathrm{Const}\, \rho^{-|j|} \|f\|_2 \qquad (-\infty < j < \infty)$$

and this ends the proof.

5. The Maximal Hilbert Transform

We shall prove here the L^p-boundedness of the operator $H^*f(x)$ defined by (iii) for all $1 < p < \infty$. Since

$$H^*f(x) \leqslant Mf(x) + \sup_k \left| \sum_{n \geq k} h_n * f(x) \right|$$

it suffices to consider the maximal operator:

$$\sup_k \left| \sum_{n \geq k} h_n * f \right|.$$

For simplicity, we shall only do this in the case of an even curve; the necessary modifications for dealing with general balanced curves are essentially contained in the last computations of §4. Define $\Phi \in \mathcal{S}(\mathbb{R})$ as in lemma 3, write $\hat{\Phi}_k(\xi) = \hat{\Phi}(2^k \xi)$, and denote by $\overset{(1)}{*}$ convolution in the first variable. Then, we decompose the truncated Hilbert transform as follows:

$$\sum_{n \geq k} h_n * f = \Phi_k \overset{(1)}{*} \left\{ Hf - \sum_{n < k} h_n * f \right\} + (\delta - \Phi_k) \overset{(1)}{*} \sum_{n \geq k} h_n * T_n f +$$

$$+ (\delta - \Phi_k) \overset{(1)}{*} \sum_{n \geq k} h_n * f^n = A_k f + B_k f + C_k f$$

and we need to estimate $\sup_k |A_k f|$, $\sup_k |B_k f|$ and $\sup_k |C_k f|$. In doing so, we shall use the following maximal operators:

a) $g_1^*(x)$ = Hardy-Littlewood maximal function of $g(x_1, x_2)$ in the x_1-direction.

b) $Ng(x)$ = the Hardy-Littlewood operator acting on the x_1 varible and the operator Γ of lemma 2 acting on x_2 (this operator appears at the end of §3).

c) $T^*g(x) = \sup_k |T_k g(x)|$.

d) $S^*g(x) = \sup_k \left|\sum_{n \geq k} T_n g(x)\right|$ (see lemma 1).

e) $Mf(x)$ = maximal operator along the curve Γ.

All of these are bounded in $L^p(\mathbb{R}^2)$ for every p, $1 < p < \infty$. Now, to estimate $A_k f$, we observe first that the action of the kernel $\Phi_k \overset{(1)}{*} \sum_{n<k} h_n$ is pointwise dominated by the action of the maximal operator N. It suffices to prove this at the point $x = 0$ and for $k = 0$; given a test function g, we have

$$\left|\left\langle \Phi \overset{(1)}{*} \sum_{n<0} h_n, g \right\rangle\right| = \left| \int_{-\infty}^{\infty} \left\{ p \cdot v \int_{-1}^{1} g(s+t, \gamma(t)) \frac{dt}{t} \right\} \Phi(s)\, ds \right|$$

$$= \left| \int_{-\infty}^{\infty} \int_{-1}^{1} [\Phi(s-t) - \Phi(s)] g(s, \gamma(t)) \frac{dt}{t}\, ds \right|$$

$$\leqslant \text{Const} \int_{-1}^{1} \int_{-\infty}^{\infty} \frac{1}{1+s^2} |g(s, \gamma(t))|\, ds\, dt \leqslant Ng(0)$$

(we have used the fact that γ is even in the second equality). Therefore

$$\sup_k |A_k f(x)| \leqslant \text{Const}\, [(Hf)_1^*(x) + Nf(x)].$$

For the second term, since $\{T_n\}_{n \in \mathbb{Z}}$ are orthogonal projections, we can write $h_n * T_n f = T_n\left(\sum_j h_j * T_j f\right)$. Remember from §4 that Hf was decomposed as $Hf = H_I f + H_{II} f$, and both pieces were bounded in L^p for all $1 < p < \infty$. Then,

$$\sup_k |B_k f(x)| = \sup_k \left| (\delta - \Phi_k) \overset{(1)}{*} \sum_{n \geq k} T_n(H_I f)(x) \right| \leqslant \text{Const}\, (S^* H_I f)_1^*(x).$$

Finally, the last term can be decomposed as follows:

$$\sup_k |C_k f| = \sup_k \left| \sum_{j=0}^{\infty} (\delta - \Phi_k) \overset{(1)}{*} h_{j+k} * f^{j+k} \right| \leqslant \sum_{j=0}^{\infty} P_j f$$

where

$$P_j f(x) = \sup_k |(\delta - \Phi_k) \overset{(1)}{*} h_{j+k} * f^{j+k}(x)|.$$

The operators P_j are uniformly bounded in $L^p(\mathbb{R}^2)$, $1 < p < \infty$, because

$$P_j f(x) \leqslant \text{Const}\, (Mf + MT^*f)_1^*(x).$$

On the other hand, a more favourable estimate holds for $p = 2$, since

$$\|P_j f\|_2^2 \leqslant \sum_k \iint_{(\xi,\eta)\notin\Delta_{j+k}} |1 - \hat{\Phi}(2^k\xi)|^2 |\hat{h}_{j+k}(\xi,\eta)|^2 |\hat{f}(\xi,\eta)|^2 \, d\xi \, d\eta$$

$$\leqslant \text{Const} \iint \sum_{k:2^{-k}\leq|\xi|} |2^{j+k}\xi|^{-2} |\hat{f}(\xi,\eta)|^2 \, d\xi \, d\eta$$

$$\leqslant \text{Const}\, 2^{-2j} \|f\|_2^2.$$

Again, interpolating and summing a geometric series shows that

$$\left\| \sum_{j=0}^{\infty} P_j f \right\|_p \leqslant C_p \|f\|_p, \qquad 1 < p < \infty$$

finishing the proof.

6. Balance is Necessary

Suppose that $\Gamma = \{(t, \gamma(t))\}_{t\in\mathbb{R}}$ is a biconvex curve, i.e., $\gamma(0) = \gamma'(0) = 0$ and $|\gamma'(t)|$ increases in $(0, +\infty)$ and decreases in $(-\infty, 0)$, and let H be the Hilbert transform along Γ. We wish to prove that, if H is bounded in any L^p space, then Γ must be a balanced curve. In fact, since H is a translation invariant operator corresponding to the multiplier

$$m(\xi,\eta) = p \cdot v \int_{-\infty}^{\infty} \exp(i\xi t + i\eta\gamma(t)) \frac{dt}{t},$$

we have H bounded in $L^p(\mathbb{R}^2) \Rightarrow H$ bounded in $L^2(\mathbb{R}^2) \Leftrightarrow m \in L^\infty(\mathbb{R}^2) \Rightarrow \Gamma$ is balanced.

Only the last implication must be proved, but this follows from

Lemma 6. *If Γ is biconvex and $|\gamma(-t_0)| = |\gamma(t_1)|$ with $t_0, t_1 > 0$, then $|\log t_1/t_0| \leqslant 8 + |m(0,\eta)|$, where $\eta = 1/|\gamma(t_1)|$.*

Proof. Suppose that $t_0 < t_1$. Then

$$m(0,\eta) = \int_{t_1}^{\infty} e^{i\gamma(t)\eta} \frac{dt}{t} + \int_{-\infty}^{t_0} e^{i\gamma(t)\eta} \frac{dt}{t}$$

$$+ \int_{-t_0}^{t_1} [e^{i\gamma(t)\eta} - 1] \frac{dt}{t} + \int_{t_0}^{t_1} \frac{dt}{t}$$

$$= I_1 + I_2 + I_3 + \log \frac{t_1}{t_0}.$$

Let $F(t) = \int_{t_1}^{t} e^{i\gamma(s)\eta}\, ds$, $t_1 < t < \infty$. For $s > t_1$, we have $|\gamma'(s)|\eta \geqslant |\gamma'(t_1)|\eta \geqslant$ $\geqslant (|\gamma(t_1)|/t_1)\eta = 1/t_1$, and γ' is monotone. Therefore, $F(t) \leqslant 3/t_1$, and integrating by parts yields

$$|I_1| = \left| \int_{t_1}^{\infty} F(t) t^{-2}\, dt \right| \leqslant 3$$

Similarly, $|I_2| \leqslant 3$. On the other hand

$$|I_3| \leqslant \int_{-t_0}^{t_1} |\gamma(t)|\eta \frac{dt}{|t|} \leqslant \eta \int_{-t_0}^{t_1} |\gamma'(t)|\, dt = \eta(|\gamma(t_1)| + |\gamma(-t_0)|) = 2.$$

Putting everything together, $|I_1 + I_2 + I_3| \leqslant 8$, q.e.d.

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