

Geometry of KDV (1): Addition and the Unimodular Spectral Classes

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To Alberto Calderón for his 65th birthday

1. INTRODUCTION

This is the first of three papers on the geometry of KDV. It presents what purports to be a foliation of an extensive function space into which all known invariant manifolds of KDV fit naturally as special leaves. The two main themes are *addition* (each leaf has its private one) and *unimodular spectral classes* (each leaf has a spectral interpretation), but first a bit of background.

Darboux's transformation: scattering case

Let⁽²⁾ $Q = -D^2 + q(x)$ be a Schrödinger operator on \mathbb{R} with potential of scattering class⁽³⁾ C_+^∞ . The spectrum of Q acting in $L^2(\mathbb{R})$ comprises $0 \leq g < \infty$ sim-

⁽¹⁾ The work presented in this paper was performed at the Courant Institute of Mathematical Sciences with the support of the National Science Foundation under Grant NSF-MCS-76-07039.

⁽²⁾ D signifies differentiation with respect to $x \in \mathbb{R}$.

⁽³⁾ C_+^∞ is the class of infinitely differentiable real-valued functions vanishing rapidly at $\pm\infty$.

ple *bound states* $-k_1^2 < \dots < -k_g^2 < 0$, plus the *continuum* $[0, \infty)$ of multiplicity 2. The ground state $-k_1^2$ (if present) has an eigenfunction e_1 of one sign; it is removed by the transformation $Q \rightarrow Q - 2D^2 \lg e_1 = Q^-$, the bound states of the latter being the same as for Q with $-k_1^2$ left out. Bound states can also be added: if $-k_0^2$ is any number below $\text{spec } Q$, then $Qh = -k_0^2 h$ has positive solutions $h_- \in L^2(-\infty, 0]$ with $\int_0^\infty h_-^2 = \infty$ and $h_+ \in L^2[0, \infty)$ with $\int_{-\infty}^0 h_+^2 = \infty$, and if $e_0 = (1 - c)h_- + ch_+$ with $0 < c < 1$, then $Q^+ = Q - 2D^2 \lg e_0$ has bound states $-k_0^2 < -k_1^2 < \dots < -k_g^2$. This type of transformation stems from Darboux [1882]; see also Bargmann [1949], Crum [1955], and Faddeev [1964]. It can be expressed in other ways: for example, if e is any positive solution of $Qe = -k^2 e$, if $AQ = Q - 2D^2 \lg e$, and if $p = e'/e$, then $Q = -D^2 + p' + p^2 - k^2$ while $AQ = -D^2 - p' + p^2 - k^2$, so that the Darboux transformation A is identified with the Bäcklund transformation B of KDV in the form discovered by Miura [1968], *to wit*, $B: p' + p^2 - k^2 \rightarrow -p' + p^2 - k^2$. A variant is to express Q as⁽⁴⁾ $(eDe^{-1})^\dagger (eDe^{-1}) - k^2$ and to exchange the factors: $(eDe^{-1})(eDe^{-1})^\dagger - k^2 = AQ$; see Deift [1978].

Addition defined

I make two small but important changes in the Darboux transformation: *I insist that $-k^2$ be to the left of $\text{spec } Q$ and I take for e always h_- or h_+* , which was not done before. Let $\mathfrak{p} = (\lambda, -)$ or $(\lambda, +)$ with $\lambda = -k^2$ and take $e(x, \mathfrak{p}) = h_-(x)$ or $h_+(x)$ in accordance with this choice. The map

$$A^{\mathfrak{p}}: Q \rightarrow Q - 2D^2 \lg e(x, \mathfrak{p})$$

is called *addition* of \mathfrak{p} for reasons to be explained presently; unlike the previous maps of its kind, it is always *isospectral*, and a great deal more besides, as you will see. The composition rule⁽⁵⁾

$$A^{\mathfrak{p}_1} \dots A^{\mathfrak{p}_n} = Q - 2D^2 \lg [e(x, \mathfrak{p}_1), \dots, e(x, \mathfrak{p}_n)]$$

shows that addition is *commutative*, and its specialization to $n = 2$ with $\mathfrak{p}_1 = \mathfrak{p} = (\lambda, \pm)$ and $\mathfrak{p}_2 = -\mathfrak{p} = (\lambda, \mp)$ shows that $A^{-\mathfrak{p}}$ is inverse to $A^{\mathfrak{p}}$, so that repeated additions produce a commutative group of transformations. The point now to be stressed is that addition makes sense in a very wide class of operators: the only thing you really need is that $\text{spec } Q$ is *bounded away*

⁽⁴⁾ The *dagger* signifies the *transposed*.

⁽⁵⁾ Crum [1955]. $[e_1, \dots, e_n]$ is Wronski's determinant.

from $-\infty$! The corresponding class of (smooth) potentials is the *extensive function space* alluded to before.

Addition explained: Hill’s case⁽⁶⁾

The name *addition* has a better justification than its mere commutativity. Let Q be a Hill’s operator with potential of class⁽⁷⁾ C_1^∞ . Then the spectrum of Q acting in $L^2(\mathbb{R})$ consists of bands

$$[\lambda_0^+, \lambda_1^-] \cup [\lambda_1^+, \lambda_2^-] \cup [\lambda_2^+, \lambda_3^-] \cup \dots$$

marked off by the periodic/anti-periodic spectrum of Q :

$$-\infty < \lambda_0^+ < \lambda_1^- \leq \lambda_1^+ < \lambda_2^- \leq \lambda_2^+ < \lambda_3^- \leq \dots \uparrow \infty.$$

Now the class of Hill’s operators Q having one and the same spectrum as some fixed specimen is faithfully represented by divisors in real position on the (nonclassical) multiplier curve M determined by the irrationality

$$\sqrt{\Delta^2 - 1} = \sqrt{\lambda_0^+ - \lambda} \prod_{n=1}^{\infty} n^{-2} \pi^{-2} \sqrt{(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)}.$$

I pause to explain what all that means. The points of the curve are pairs $p = (\lambda, \pm)$ comprising a *projection* $\lambda(p) \in \mathbb{C}$ and a *signature* (of the irrationality). The function $e(x, p)$ introduced before agrees, to the left of spec Q , with the *Baker-Akhiezer function* of M specified by

- a) $Qe = \lambda(p)e$;
- b) $e(x + 1) = m(p)e(x)$, $m(p)$ being the *multiplier* $\Delta - \sqrt{\Delta^2 - 1}$;

and

- c) $e(0) = 1$.

The pole divisor $p_1 + p_2 + \dots$ of $e(x, p)$ is independent of $0 < x < 1$. It is the *divisor of Q* and is in *real position* in the sense that its projections fall one into each of the spectral lacunae: $\lambda_n^- \leq \lambda(p_n) \leq \lambda_n^+ (n \geq 1)$. The association of Q to its divisor is 1:1. The latter form an ∞ -dimensional torus J which is (the

⁽⁶⁾ McKean-Trubowitz [1976] is cited for background.

⁽⁷⁾ C_1^∞ is the class of infinitely differentiable real-valued functions of period 1.

real part of) the *Jacobi variety* of M . In this language, *the addition*⁽⁸⁾ of \mathfrak{p}_0 with $\lambda(\mathfrak{p}_0) < \lambda_0^+$ to the left of $\text{spec } Q$ is effected by the following recipe

$$\begin{aligned} Q &\rightarrow \mathfrak{p}_1 + \mathfrak{p}_2 + \cdots \\ &\rightarrow -\mathfrak{p}_0 + \mathfrak{p}_1 + \mathfrak{p}_2 + \cdots \\ &\rightarrow \infty + \mathfrak{p}'_1 + \mathfrak{p}'_2 + \cdots \\ &\rightarrow Q' \end{aligned}$$

in which $-\mathfrak{p}_0$ is \mathfrak{p}_0 reckoned with the opposite signature; $\mathfrak{p}'_1 + \mathfrak{p}'_2 + \cdots$ is a new divisor in real position; the divisors of lines 2 and 3 are equal in J , meaning that the one comprises the roots and the other the poles of a function of rational character on M ; and $Q' = A^{\mathfrak{p}_0}Q$ is the new Hill's operator with divisor $\mathfrak{p}'_1 + \mathfrak{p}'_2 + \cdots$. The proof is presented below; compare McKean [1985].

Note. The special addition (not permitted by the present recipe) of the point of ramification $\mathfrak{p}_0 = \lambda_0^+$ is an involution⁽⁹⁾ of the Jacobi variety corresponding to the addition (in J) of the sum of all real half-periods; see McKean [1980].

KDV-type fields

The next item is the connection between addition and KDV. Take $\mathfrak{p} = (\lambda, +)$ and $\mathfrak{p}' = (\lambda + \Delta\lambda, +)$ for λ below $\text{spec } Q$. Then

$$A^{\mathfrak{p}'}A^{-\mathfrak{p}}Q = Q - XQ\Delta\lambda + \text{etc.}$$

with⁽¹⁰⁾

$$XQ = 2G'_{xx}(\lambda),$$

$G_{xy}(\lambda)$ being the Green's function $(Q - \lambda)_{xy}^{-1}$. The vector field X , familiar from Hill's equation,⁽¹¹⁾ appears in this way as an *infinitesimal addition*. Now XQ may be expanded as $\lambda \downarrow -\infty$ in diminishing half-integral powers of $-\lambda$:

$$XQ = \sum_{n=1}^{\infty} \lambda^{-(1/2+n)} X_n Q,$$

⁽⁸⁾ It would be more accurate to speak of *subtraction* but never mind.

⁽⁹⁾ $\mathfrak{p}_0 = -\mathfrak{p}_0$.

⁽¹⁰⁾ The *prime* signifies differentiation on diagonal.

⁽¹¹⁾ McKean-Trubowitz [1976/78].

in which $X_1Q = q'$, $X_2Q = 3qq' - (1/2)q'''$, etc. are the conventional KDV fields up to unimportant constants, the point being that, *in the full generality envisaged here, addition provides a substitute for the flows of the KDV hierarchy even when the latter have no existence.*⁽¹²⁾ This realization prompts the formation of the *additive class* produced by closing up the operators Q obtained from a fixed specimen by repeated additions of points p to the left of spec Q . *In this way, the space of operators with spectrum not extending to $-\infty$ is foliated⁽¹³⁾ by classes which fill the office of the invariant manifolds of KDV, each class having its private addition.*

Note. The vector fields $XQ = 2G'_{xx}(\lambda)$, taken for λ to the left of spec Q , may be integrated without obstruction to produce commuting, class-preserving flows. This can be done by explicit formulae involving Fredholm determinants, much as in McKean-Trubowitz [1982]; see the third paper in this series. Mumford [1983/4] has studied these flows in a special case.

Unimodular spectral classes

It is known that the invariant manifolds of KDV have a spectral basis: for example, in the scattering case, the transmission coefficient is *the* invariant specifying the manifold, while in the Hill's case, it is the periodic/anti-periodic spectrum, alias the discriminant, that is fixed. You will ask: *what is the corresponding invariant for the general additive class?* I present a conjecture verified in three examples cited below. Let $dF(\lambda)$ be the (2×2) *spectral weight*⁽¹⁴⁾ of Q . Then Q is isospectral (= unitarily equivalent) to a second such operator Q' if and only if the spectral weight of the latter is related to the former weight as⁽¹⁵⁾ $GdFG^\dagger$ with a factor $G = G(\lambda)$ taking values in $GL(2, \mathbb{R})$. Now it is easy to compute the spectral weight $dF^p = GdFG^\dagger$ of A^pQ : isospectrality prevails, the factor being

$$G(\lambda) = \frac{1}{\sqrt{\lambda - \lambda(p)}} \begin{vmatrix} c & -1 \\ \lambda - \lambda(p) - c^2 & c \end{vmatrix} \quad \text{with} \quad c = \frac{e'(0, p)}{e(0, p)},$$

and you notice that $\det G = 1!$ This prompts a definition: *two operators Q belong to the same unimodular spectral class if they are unitarily equivalent and*

⁽¹²⁾ KDV cannot be balanced with initial data x^2 , for example.
⁽¹³⁾ The usage is informal as the dimensionality of the leaf varies from 0 to ∞ !
⁽¹⁴⁾ Kodaira [1949] is cited for background; compare art. 2 below.
⁽¹⁵⁾ The *dagger* signifies the *transposed*, as before.

if the factor G takes its *value not just in $GL(2, \mathbb{R})$ but in $SL(2, \mathbb{R})$* ; evidently, the additive class of Q is part of its unimodular spectral class. I conjecture that, with the proper technical precautions, *these classes are always one and the same*. The second paper of this series verifies this conjecture in three examples:

- a) the scattering case C_{\downarrow}^{∞} ⁽¹⁶⁾;
- b) the Hill's case C_1^{∞} ;
- c) if the additive class is of finite dimension.

Under (a), the class has fixed transmission coefficient, the phase of the reflection coefficient together with the logarithmic norming constants serving as additive coordinates. Under (b), the periodic/anti-periodic spectrum is fixed and the class is identified as the real part of the Jacobi variety, the addition of the latter falling in with the addition of the class. Under (c), the class is a leaf of the Neumann system and every leaf appears in this way.⁽¹⁷⁾

Second measure class

The two measure classes determining the conventional isospectral class of Q are typified by⁽¹⁸⁾

$$sp dF = df_{11} + df_{22}$$

and⁽¹⁹⁾

$$\sqrt{\det dF} = \sqrt{f'_{11}f'_{22} - (f'_{12})^2} sp dF.$$

The latter is of special importance in its role of *unimodular spectral invariant*. It is always smaller than $d\lambda$ and the density $D = \sqrt{\det dF}/d\lambda$ can be interpreted as the modulus of a (*mean*) *transmission coefficient*; in fact, it is precisely $|s_{11}|$ in the scattering case. The evaluation of the gradient:

$$\frac{\partial \lg D}{\partial q(x)} = -\text{the real part of } G_{xx}(\lambda + \sqrt{-1} 0+) \quad \text{if } D > 0$$

hints at an attractive connection between the unimodular invariant and the vector fields $XQ = 2G'_{xx}(\lambda)$, but this has not been fully understood.

⁽¹⁶⁾ $Q = -D^2$ is the simplest instance; it is settled in item 7, art. 5.

⁽¹⁷⁾ McKean [1979] and Moser [1978] are cited for background.

⁽¹⁸⁾ $dF = [df_{ij}; 1 \leq i, j \leq 2]$.

⁽¹⁹⁾ $f'_{ij} = df_{ij}/sp dF$ ($1 \leq i, j \leq 2$).

Acknowledgement

I wish to thank L. Menezes: her comments led to several improvements of the original presentation.

2. PRELIMINARY SPECTRAL THEORY

I collect for future use a number of standard facts about Q under the sole assumption that its spectrum is bounded from $-\infty$.

Eigendifferential expansions⁽²⁰⁾

Fix a complex number λ outside $\text{spec } Q$ so that $Qh = \lambda h$ has independent solutions

$$h_- \in L^2(-\infty, 0] \quad \text{with} \quad \int_0^\infty |h_-|^2 = \infty$$

and

$$h_+ \in L^2[0, \infty) \quad \text{with} \quad \int_{-\infty}^0 |h_+|^2 = \infty.$$

The Wronskian $[h_-, h_+] = h'_- h_+ - h_- h'_+$ is always taken as 1. Then Green's function $(Q - \lambda)_{xy}^{-1} = G_{xy}(\lambda)$ is expressed as $h_-(x)h_+(y)$ if $x \leq y$. Let the bottom of $\text{spec } Q$ be placed at 0 for simplicity. The *fundamental matrix*

$$M(\lambda) = [m_{ij}(\lambda): 1 \leq i, j \leq 2] = \begin{bmatrix} 2h_- h_+ & (h_- h_+)' \\ (h_- h_+)' & 2h'_- h'_+ \end{bmatrix}$$

evaluated at $x = 0$ is analytic in the cut plane $\mathbb{C} - [0, \infty)$; it is also symmetric and of determinant -1 . It is real for $\lambda < 0$; most important, its imaginary part is positive ($-$ definite) in the open upper half-plane⁽²¹⁾ and so has the representation

$$\text{imag } M(\lambda) = \frac{b}{\pi} \int_0^\infty [(\lambda' - a)^2 + b^2]^{-1} dF(\lambda')$$

⁽²⁰⁾ Weyl [1910] but see Kodaira [1949] for the present more elegant version.

⁽²¹⁾ This is easily seen from the identity

$$\text{imag } G_{xy}(\lambda) = \text{imag } \lambda \times [(Q - \lambda)^{-1}(Q - \lambda^*)^{-1}]_{xy} = \int G_x(\lambda)G_y^*(\lambda).$$

with $\lambda = a + \sqrt{-1} b$, $b > 0$, and a 2×2 , real, symmetric, positive⁽²²⁾ spectral weight

$$dF(\lambda = a) = \lim_{b \downarrow 0} \text{imag } M(a + \sqrt{-1} b) da = [df_{ij}(\lambda): 1 \leq i, j \leq 2].$$

The spectral theorem

$$Q = \int_0^\infty \lambda d\mathfrak{P}(\lambda)$$

is implemented thereby: if $E(x, \lambda) = (e_1, e_2)$ with $Qe = \lambda e$, $e_1(0) = e_2'(0) = 1$, and $e_1'(0) = e_2(0) = 0$, then the kernel of the projection $d\mathfrak{P}(\lambda)$ is

$$d\mathfrak{P}_{xy}(\lambda) = (2\pi)^{-1} E(x, \lambda) dF(\lambda) E^\dagger(y, \lambda).$$

Measure classes

The function $f_1(x)$ is taken as $h_-(x)h_+(0)$ if $x < 0$ and as $h_-(0)h_+(x)$ if $x \geq 0$ for fixed $\lambda = -1$, say, to the left of $\text{spec } Q$; similarly, $f_2(x)$ is taken as $h_-(x)h'_+(0)$ if $x < 0$ and as $h'_-(0)h_+(x)$ if $x \geq 0$. Then⁽²³⁾

$$[(f_i, d\mathfrak{P}(\lambda)f_j): 1 \leq i, j \leq 2] = (2\pi)^{-1}(\lambda + 1)^{-2} dF(\lambda);$$

also, the families $d\mathfrak{P}f_1$ and $d\mathfrak{P}f_2$ together span the whole of $L^2(\mathbb{R})$. Now df_{12} is dominated by⁽²⁴⁾ df_{11} and/or df_{22} , so every spectral measure $(f, d\mathfrak{P}f)$ is dominated by

$$(f_1, d\mathfrak{P}f_1) + (f_2, \text{coprojection of } d\mathfrak{P}f_2 \text{ upon } d\mathfrak{P}f_1) = (f_1, d\mathfrak{P}f_1) + (f_2, d\mathfrak{P}f_2) - r(f_2, d\mathfrak{P}f_1)$$

with $r = df_{12}/df_{11}$, i.e., by $df_{11} + df_{22} - r df_{12}$. The corresponding *top measure class* is typified by the *trace* $sp dF = df_{11} + df_{12}$; indeed, by the positivity of dF , $|df_{12}| \leq \sqrt{df_{11} df_{22}}$,⁽²⁵⁾ so that $|r df_{12}| \leq df_{22}$, and the vanishing of $f_{11}(\Delta) + f_{22}(\Delta) - \int_\Delta r df_{12}$ implies $f_{11}(\Delta) = 0$, $|f_{12}|(\Delta) = 0$, and $f_{22}(\Delta) = 0$, in that order. The top measure class is now seen to be based upon the family

⁽²²⁾ The adjective means that $f_{11}(\Delta) \geq 0$, $f_{22}(\Delta) \geq 0$, and $f_{12}^2(\Delta) \leq f_{11}(\Delta)f_{22}(\Delta)$ for arbitrary sets Δ .

⁽²³⁾ Kodaira [1949]. The *parenthesis* is the inner product.

⁽²⁴⁾ $f_{11}f_{22} \geq f_{12}^2$.

⁽²⁵⁾ This type of expression always signifies what is must: here, the radical is $\sqrt{f'_{11}f'_{22}} \times sp dF$, the primed densities being taken relative to the trace.

$(1 - r) d\mathfrak{F}f_1 + d\mathfrak{F}f_2$: in fact, the associated spectral measure is just $df_{11} + df_{22} - r df_{12}$. The second class is based upon the perpendicular family

$$\frac{df_{22} + (1 - r) df_{12}}{df_{11} + df_{22} - r df_{12}} d\mathfrak{F}f_1 - \frac{df_{11}}{df_{11} + df_{22} - r df_{12}} d\mathfrak{F}f_2,$$

the associated measure being

$$\frac{df_{11} df_{22} - (df_{12})^2}{df_{11} + df_{22} - r df_{12}} = \frac{f'_{11} f'_{22} - (f'_{12})^2}{f'_{11} + f'_{22} - r df'_{12}} \times sp dF;$$

in particular, *the second class is typified by the measure*⁽²⁶⁾

$$\sqrt{\det dF} = \sqrt{df_{11} df_{22} - (df_{12})^2} = \sqrt{f'_{11} f'_{22} - (f'_{12})^2} sp dF.$$

Note. Masani-Wiener [1957-58] introduced $\sqrt{\det dF}$ in a different context and in a different way: it is a elementary fact that the interval function $D(I) = \sqrt{\det F(I)}$ is superadditive: $D(A \cup B) \geq D(A) + D(B)$ if $A \cap B$ is void. This permits the alternative definition:

$$\int_{\Delta} \sqrt{\det dF} = \text{the infimum of the sum of } D(I),$$

the infimum being taken over countable covers of Δ by intervals I . This will be helpful later on.

Unitary equivalence

The main fact about unitary equivalence can now be stated in a convenient form:⁽²⁷⁾ *two operators Q_1 and Q_2 are isospectral if and only if they determine the same top and the same bottom measure classes, or, what is the same, if and only if their spectral weights are related as $dF_2 = G dF_1 G^\dagger$, the 2×2 factor $G = G(\lambda)$ taking its values in $GL(2, \mathbb{R})$.* I could not find the second criterion stated in just this form, though it is an easy consequence of the first, which is standard.

PROOF. Let Q_1 and Q_2 be isospectral. The weight dF_1 can be expressed as $OD_1 O^\dagger dm$ with $O \in SO(2)$, 2×2 non-negative diagonal D_1 , and a positive

⁽²⁶⁾ Compare the preceding footnote.

⁽²⁷⁾ Stone [1932].

(numerical) measure dm . Now as dF_2 has a similar expression with the *same* dm , it suffices to produce $G = [g_{ij}: 1 \leq i, j \leq 2]$ taking its values in $GL(2, \mathbb{R})$ so as to make $D_2 = GD_1G^\dagger$ under the condition that $sp D_2$ vanishes simultaneously with $sp D_1$, and $\det D_2$ with $\det D_1$. The rest will be plain.

Note 1. dF is not intrinsic. The fact is, it depends upon the choice of origin $x = 0$, and if the latter is displaced to $x = c$, then the former weight dF is changed to $G dF G^\dagger$ with the factor

$$G = \begin{bmatrix} e_1(c) & e_2(c) \\ e'_1(c) & e'_2(c) \end{bmatrix}.$$

G is unimodular, so $\sqrt{\det dF}$ is not changed. This already commends it to special attention.

Note 2. The *unimodular spectral class* of Q_1 is the subclass of isospectral operators Q_2 having one and the same *invariant* $\sqrt{\det dF_2} = \sqrt{\det dF_1}$; equivalently, the factors G cited above take their values *not just in* $GL(2, \mathbb{R})$ *but in* $SL(2, \mathbb{R})$. The distinction is non-existent at bound states: if $dF_2 = G dF_1 G^\dagger$ with $G \in GL(2, \mathbb{R})$ at a jump of dF_1 , then the simplicity of bound states of Q implies that dF_1 and dF_2 are of rank 1. Now G may be chosen from $SL(2, \mathbb{R})$, as you will easily check.

Side operators

These are employed infrequently and can be skipped for now, as can the rest of this article. The *side operator* Q_+^0 is the restriction of Q to functions on the half-line $x \geq 0$ vanishing at $x = 0$; its spectrum is confined to $[0, \infty)$, in agreement with the form of its Green's function

$$G_{xy}^0(\lambda) = e_2(x)h_+(y)/h_+(0) \quad (x \leq y)$$

and the fact that $h_+(0)$ is *root-free off the cut* $[0, \infty)$.⁽²⁸⁾ The imaginary part of $G_{xx}^0(\lambda)$ is positive in the upper half-plane, so

$$\lim_{x \downarrow 0} x^{-2} \operatorname{imag} G_{xx}^0(\lambda) = \operatorname{imag} \frac{h'_+(0)}{h_+(0)} = \frac{b}{\pi} \int_0^\infty [(\lambda' - a)^2 + b^2]^{-1} df_+^0(\lambda'),$$

⁽²⁸⁾ $h_+(0) = 0$ off the cut means that h_+ is an eigenfunction of the side operator Q_+^0 . Then λ must be real and negative, and $\int_0^\infty h_+ Q h_+ = \lambda \int_0^\infty h_+^2 < 0$ violates $\operatorname{spec} Q \subset [0, \infty)$.

in which $\lambda = a + \sqrt{-1b}$, $b > 0$, and the spectral weight df_+^0 is non-negative. The other side operator of interest is the restriction Q_+^∞ of Q to functions on the half-line with vanishing slope at $x = 0$; its spectrum is likewise confined to $[0, \infty)$, with the possible exception of an isolated ground state to the left of 0, in agreement with the form of its Green's function

$$G_{xy}^\infty(\lambda) = -e_1(x)h_+(y)/h'_+(0) \quad (x \leq y)$$

and the fact that $h'_+(0) = 0$ has at most one (real, simple, negative) root off the cut $[0, \infty)$.⁽²⁹⁾ The imaginary part of $G_{00}^\infty(\lambda)$ is positive in the upper half-plane, so

$$\text{imag } G_{00}^\infty(\lambda) = -\text{imag} \frac{h_+(0)}{h'_+(0)} = \frac{b}{\pi} \int [(\lambda' - a)^2 + b^2]^{-1} df_+^\infty(\lambda'),$$

in which $\lambda = a + \sqrt{-1b}$, $b > 0$, and the non-negative spectral weight df_+^∞ vanishes off $[0, \infty) +$ ground state, if any. The analogous side operators Q_-^0 and Q_-^∞ for the left half-line also play a small role below.

Note. The identity for m_{22}^0 of the last footnote shows that if $h'_+(0) = 0$ has a root < 0 , then $h'_-(0) = 0$ does not, and it is easy to deduce that if the ground state of Q_+^∞ is below 0, then the bottom of $\text{spec } Q_-^\infty$ is at 0.

Inverse spectral problem

The association of Q to its 2×2 spectral weight dF is 1:1. This is made plausible by a count: 2 degrees of freedom in Q , one for $(-\infty, 0]$ and one for $[0, \infty)$, versus 3 degrees of freedom in the symmetric 2×2 weight, less 1 to account for $\det M = -1$.⁽³⁰⁾ The proof is easy, too: dF determines M and so also

$$-h'_-(0)/h_-(0) = (1/m_{11})(-1 - m_{12}) \text{ and } h'_+(0)/h_+(0) = (1/m_{11})(-1 + m_{12}),$$

⁽²⁹⁾ $h'_+(0) = 0$ off the cut means that h_+ is an eigenfunction of the side operator Q_+^∞ . Then $\lambda < 0$ as before, and

$$2[h'_-(0)h'_+(0)]^* = m_{22}^*(\lambda) = \frac{1}{\pi} \int (\lambda' - \lambda)^{-2} df_{22}(\lambda')$$

does the rest, the *spot* signifying differentiation with regard to λ .

⁽³⁰⁾ $\det M = -1$ looks like the loss of 2 degrees of freedom but, as the real part is conjugate to the imaginary, the count is only 1.

which determine the side operators Q_-^0 and Q_+^0 by the recipe of Gelfand-Levitan [1951]. Newton [1983] proves this directly in the 2×2 format, of which the only drawback is that one does not know what $\det M = -1$ signifies for dF , i.e., one cannot (to date) satisfactorily describe what are the 2×2 spectral weights.

3. ADDITION

The letter ν_0 denotes a pair (λ_0, \pm) comprising a *projection* λ_0 to the left of $\text{spec } Q \subset [0, \infty)$ and a *signature* indicating which function $e(x, \nu_0) = h_-$ or h_+ is to be employed. The operation of *adding* ν_0 to (the divisor of) Q is the map

$$A^{\nu_0}: Q \rightarrow Q - 2D^2 \lg e(x, \nu_0) = Q^{\nu_0}.$$

Its simplest properties will be elicited below.

Item 1. $e_0(x) = e(x, \nu_0)$ cannot vanish so that *addition is well-defined*.

Item 2. $Q = P^\dagger P + \lambda_0$ in which $P = e_0 D e_0^{-1}$ and P^\dagger is its transpose $-e_0^{-1} D e_0$. Q^{ν_0} is produced by exchange of factors: $Q^{\nu_0} = P P^\dagger + \lambda_0$; in particular, $P: h \rightarrow e_0^{-1}[h, e_0]$ maps solutions of $Qh = \lambda h$ into solutions of $Q^{\nu_0}h = \lambda h$. The proof is routine.

Item 3. Ph_- belongs to $L^2(-\infty, 0]$ if λ is not on the cut $[0, \infty)$; similarly, Ph_+ belongs to $L^2[0, \infty)$.

PROOF. h_+ is typical:⁽³¹⁾

$$\begin{aligned} \int_0^x |Ph_+|^2 &= \int_0^x [h_+, e_0]^* D(h_+/e_0) = \frac{h_+}{e_0} [h_+, e_0]^* \Big|_0^x - \int_0^x \frac{h_+}{e_0} D[h_+, e_0]^* \\ &= \frac{h_+}{e_0} [h_+, e_0]^* \Big|_0^x + (\lambda - \lambda_0)^* \int_0^x |h_+|^2 \equiv r e^{\sqrt{-1}\theta} + O(1) \end{aligned}$$

so that *either* $Ph_+ \in L^2[0, \infty)$ *or else* $(h_+/e_0)[h_+, e_0]^* = r e^{\sqrt{-1}\theta}$ tends to ∞ as $x \downarrow \infty$ in such a way that $r \rightarrow +\infty$ and $\theta \rightarrow 0$. In the second case,

⁽³¹⁾ The *star* means complex conjugate. D stands for differentiation with regard to x .

$$\begin{aligned} |h_+|^{-2} &= r^{-2}|Ph_+|^2 = r^{-2}[Dre^{\sqrt{-1}\theta} + (\lambda - \lambda_0)^*|h_+|^2] \\ &= \left[\frac{r'}{r^2} + \frac{\sqrt{-1}\theta'}{r} \right] e^{\sqrt{-1}\theta} + \frac{(\lambda - \lambda_0)^*}{r^2} |h_+|^2 \\ &= \frac{r'}{r^2} \cos \theta - \frac{\theta'}{r} \sin \theta + \text{a summable function,} \end{aligned}$$

upon taking the real part. Now the imaginary part of the first formula reveals that $r \sin \theta$ is monotone and bounded so that $(r \sin \theta)'$ is summable. It follows that

$$\frac{\cos \theta}{|h_+|^2} = \frac{r'}{r^2} - \frac{\sin \theta}{r^2} (r \sin \theta)' + \text{a summable function}$$

is itself summable, so $\int_0^\infty |h_+|^{-2} < \infty$, contradicting $\int_0^\infty |h_+|^2 < \infty$. The proof is finished.

Item 4. *The addition of p_0 is a unimodular isospectral transformation.*

PROOF. By item 3, Ph_- and Ph_+ play the role of h_- and h_+ for Q^{p_0} , after accounting for the fact that $[Ph_-, Ph_+] = \lambda - \lambda_0$ is not unity; in particular, $\text{spec } Q^{p_0}$ is confined to $[0, \infty)$, and

$$M^{p_0} = \frac{1}{\lambda - \lambda_0} \begin{bmatrix} 2Ph_- Ph_+ & (Ph_- Ph_+)' \\ (Ph_- Ph_+)' & 2(Ph_-)'(Ph_+)' \end{bmatrix} \text{ taken at } x = 0$$

is the fundamental matrix of Q^{p_0} , permitting the evaluation of its 2×2 spectral weight as

$$dF^{p_0}(\lambda = a) = \lim_{b \downarrow 0} \text{imag } M^{p_0}(a + \sqrt{-1} b) da = G dF G^\dagger$$

with the unimodular factor

$$G = \frac{1}{\sqrt{\lambda - \lambda_0}} \begin{bmatrix} c & -1 \\ \lambda - \lambda_0 - c^2 & c \end{bmatrix}, \quad c = \frac{e'_0(0)}{e_0(0)}.$$

The computation is routine in view of $M^{p_0} = GMG^\dagger$.

EXAMPLE 1. If $Q = -D^2$, then $dF(\lambda)$ is the diagonal matrix $[\lambda^{-1/2}, \lambda^{+1/2}] d\lambda$ and $G dF G^\dagger = dF$, in agreement with $D^2 \lg e_0 = 0$: in brief, *addition has no effect.*

EXAMPLE 2. The proviso that $\lambda(p_0)$ lies to the left of $\text{spec } Q$ may be essential for isospectrality, as the example $Q = -D^2 + x^2 - 1$ shows. $Qe_0 = 0$ with $e_0(x) =$

$\exp(-x^2/2)$ = the ground state, h_- and h_+ being coincident, and the addition of $p_0 = (0, \pm)$ shifts the whole spectrum 2 units: $Q \rightarrow Q - 2D^2 \lg e_0 = -D^2 + x^2 + 1$. Contrariwise, if $Q = -D^2 +$ a positive compact function, then $\text{spec } Q = [0, \infty)$, $Qh = 0$ has 2 independent positive solutions $h_- = 1$ near $-\infty$ and $h_+ = 1$ near $+\infty$, and both additions $(0, \pm)$ are unimodular isospectral, as in item 4.

Item 5. The effect of repeated additions is described next. Let p_1 and p_2 be distinct points. The function $e(x, p_2)$ associated to Q^{p_1} is proportional to $e_1^{-1}[e_2, e_1]$ in which $e_1 = e(x, p_1)$ and $e_2 = e(x, p_2)$ are now formed for Q itself. It follows that

$$A^{p_2}A^{p_1}Q = Q - 2D^2 \lg e_1 - 2D^2 \lg e_1^{-1}[e_2, e_1] = Q - 2D^2 \lg [e_1, e_2].$$

The more general formula⁽³²⁾

$$A^{p_1} \dots A^{p_n}Q = Q - 2D^2 \lg [e(x, p_1), \dots, e(x, p_n)]$$

is obtained by induction; in particular, addition is *commutative* and *invertible*, the inverse to A^p being A^{-p} formed with the point $-p$ having the opposite signature to p but the same projection.

Item 6. A^p approximates the identity as the projection $\lambda(p)$ tends to $-\infty$.

PROOF. Let $Q = -D^2 + q(x)$ and take $p = (\lambda, +)$ for instance. Then $D^2 \lg e = q(x) - \lambda - (h'_+/h_+)^2$, so what is needed is the development

$$-\frac{h'_+(x)}{h_+(x)} = (-\lambda)^{1/2} + \frac{1}{2}q(x)(-\lambda)^{-1/2} + \text{etc.} \quad (\lambda \downarrow -\infty).$$

The idea is to write the Green's function G_{00}^∞ in the form of a Brownian integral:

$$G_{00}^\infty(\lambda) = -\frac{h_+(0)}{h'_+(0)} = \int_0^\infty e^{\lambda t} (\pi t)^{-1/2} E_{00}(e^{-\mathfrak{D}}) dt,$$

in which E_{00} is the expectation for the reflecting Brownian motion $\mathfrak{r}(t)$: $t \geq 0$ with infinitesimal operator D^2 , conditional on $\mathfrak{r}(0) = 0$ and $\mathfrak{r}(t) = 0$, and \mathfrak{D} is the integral $\int_0^t q[\mathfrak{r}(t')] dt'$. The computation is localized by checking that paths

⁽³²⁾ $[e_1, \dots, e_n]$ is Wronski's determinant.

with $1 \leq \max [r(t'): t' \leq t]$ do not contribute to the development. I omit the routine details.⁽³³⁾

Item 7. Let p be any point $(\lambda, +)$ to the left of $\text{spec } Q$ and let $p' = (\lambda + \Delta\lambda, +)$. Then $A^{p'}A^{-p}Q = Q - XQ\Delta\lambda + \text{etc.}$ with $XQ = 2G'_{xx}(\lambda)$.

PROOF. The composite addition produces $Q - 2H'$ with⁽³⁴⁾

$$\begin{aligned} H &= D \lg [h_-(x, \lambda), h_+(x, \lambda + \Delta\lambda)] = D \lg (1 + [h_-, h_+] \Delta\lambda + \text{etc.}) \\ &= (h''_+ h'_+ - h_- h''_-) \Delta\lambda = (h_- Q h'_+ - h'_+ Q h_-) \Delta\lambda = h_- h_+ \Delta\lambda, \end{aligned}$$

to leading order, by item 5.

Note 1. The vector fields $X: Q \rightarrow 2G'_{xx}(\lambda)$ inherit the commutativity of addition.

Note 2. X appears in item 7 as an *infinitesimal addition*. This is of particular interest in view of the fact that, as in the scattering case, $G_{xx}(\lambda)$ can be developed, as in item 6, in diminishing half-integral powers of $\lambda \downarrow -\infty$, with the conventional KDV fields as coefficients. It follows that *addition commutes with, and shares the invariant manifolds of, the KDV flows when these have any existence; in particular, addition can be viewed as a substitute for the KDV hierarchy under the sole condition that $\text{spec } Q$ is bounded from $-\infty$.*

Note 3. X can be expressed in commutator format as $XQ = [K, Q]$ in which K is the infinitesimal skew operator⁽³⁵⁾ $[(3/2)G'_{xx} - DG_{xx}]G$. The skewness of K is equivalent to the vanishing of

$$(Q - \lambda)(K + K^\dagger)(Q - \lambda) = qD + Dq - (1/2)D^3 - 2\lambda D \text{ acting on } G_{xx},$$

which is well known and easily checked. This facilitates the evaluation of the commutator:

$$[K, Q] = K(Q - \lambda) + (Q - \lambda)K^\dagger = \frac{3}{2}G'_{xx} - DG_{xx} + \frac{3}{2}G'_{xx} + G_{xx}D = 2G'_{xx}.$$

⁽³³⁾ McKean-van Moerbeke [1975] will serve as a model.

⁽³⁴⁾ The *spot* means differentiation with respect to λ .

⁽³⁵⁾ McKean-van Moerbeke [1975] misstated this without proof. G is the Green's operator $(Q - \lambda)^{-1}$. DG_{xx} is now the operator $G'_{xx} + G_{xx}D$ not the function G'_{xx} .

Note 4. The gradient $\partial G_{xx}(\lambda)/\partial q(y) = -G_{xy}^2(\lambda)$ is easily computed; it is symmetric in x and y so that *in the small*, $2G_{xx}(\lambda)$ is itself a gradient $\partial H/\partial q(x)$, by Poincaré’s lemma, and the field X has the conventional KDV form $XQ = (\partial H/\partial q)'$. H is termed an *integral*. The discussion indicates but does not prove that the additive class of Q should be determined by fixing the values of these integrals for every λ to the left of $\text{spec } Q$. Unfortunately, the application of Poincaré’s lemma is only formal in the present very wide generality. The integrals are morally equivalent to the *additive invariant* $D = \sqrt{\det dF/d\lambda}$, or rather to its logarithm, as attested by item 6, art. 4:

$$\partial D/\partial q(x) = -\text{the real part of } G_{xx}(\bullet + \sqrt{-1} 0+) \times D,$$

but this looks more satisfactory than it really is. For example, in Hill’s case, $G_{xx}(\bullet + \sqrt{-1} 0+)$ is imaginary on $\text{spec } Q$ where D lives, and the formula is without content.

Item 8. Concerns an *additive duality*⁽³⁶⁾ which exchanges the spectral weights of the side operators Q_+^0 and Q_+^∞ , and likewise the weights of the side operators Q_-^0 and Q_-^∞ for the left half-line. The weights are recalled from art. 2: df_+^0 represents $\text{imag } h'_+(0)/h_+(0)$, df_+^∞ represents $-\text{imag } h_+(0)/h'_+(0)$, and so on. I take⁽³⁷⁾ the ground state $\lambda_0 = \text{bottom spec } Q_+^\infty$ to the left of $0 = \text{bottom spec } Q_-^\infty$ to fix ideas and consider the effect of *adding the point* $\mathfrak{p}_0 = (\lambda_0, +)$. Then $h'_-(0)$ vanishes so $h'_-(0)$ does not, and it is the result of a brief computation that the addition produces a *duality of spectral weights*: with $m = -h'_-(0)/h_-(0)$ evaluated at the ground state, you find

$$\begin{aligned} df_+^\infty &\rightarrow (\lambda - \lambda_0)^{-1} df_+^0 \\ df^0 &\rightarrow (\lambda - \lambda_0) df_+^\infty && \text{knocking out the weight of the ground state} \\ df_-^\infty &\rightarrow (\lambda - \lambda_0)^{-1} df_-^0 && \text{plus the positive mass } m \text{ at the ground state} \\ df_-^0 &\rightarrow (\lambda - \lambda_0) df_-^\infty. \end{aligned}$$

The ground state migrates from Q_+^∞ to Q_-^∞ . A further addition of $\mathfrak{p}_0 = (\lambda_0, -)$ reproduces Q : in short, *duality is involutive*. A similar duality holds if $\text{bottom spec } Q_+^\infty = \text{bottom spec } Q_-^\infty = 0$. Then $e_1(x, 0)$ does not vanish and may be used in place of the former ground state. The computation is the same.

⁽³⁶⁾ McKean [1985] treats Hill’s case; see also Isaacson-McKean-Trubowitz [1984].

⁽³⁷⁾ The spectra of Q_-^∞ and Q_+^∞ cannot both extend to the left of 0; see art. 2 under *side operators*.

4. UNIMODULAR CLASSES

The present article investigates the unimodular spectral class with special attention to the invariant $\sqrt{\det dF}$. The fundamental matrix M is written $A + \sqrt{-1} B$ with $A = [a_{ij}; 1 \leq i, j \leq 2]$ and $B = [b_{ij}; 1 \leq i, j \leq 2]$.

Item 1. $\sqrt{\det dF}$ is independent of the choice of origin $x = 0$, as noted in art. 2; in particular, the unimodular class is closed under translation.

Item 2. $\sqrt{\det B}$ is a superharmonic function in the open upper half-plane.

PROOF. Let do be the uniform distribution on the perimeter of a circle C in the open half-plane. The interval function

$$D(I) = \sqrt{\det \int_I B do}$$

is defined for circular arcs I : it is superadditive, B being symmetric and positive, whence⁽³⁸⁾

$$\sqrt{\det B(\text{center})} = D(C) \geq \lim_{N \rightarrow \infty} \sum_{n=1}^N D(I_n) = \int_C \sqrt{\det B} do.$$

Item 3.

$$H(\lambda = a + \sqrt{-1} b) = \frac{b}{\pi} \int [(\lambda' - a)^2 + b^2]^{-1} \sqrt{\det dF}(\lambda')$$

is the greatest harmonic minorant of $\sqrt{\det B}$; in particular,

$$\sqrt{\det dF}(a) = \lim_{b \downarrow 0} \sqrt{\det B}(a + \sqrt{-1} b) da.$$

PROOF. The superadditivity employed in item 2 is valid for horizontal lines as well, whence the interval function

$$D(I) = \sqrt{\det \frac{b}{\pi} \int_I [(\lambda' - a)^2 + b^2]^{-1} dF(\lambda')}$$

formed for fixed $a + \sqrt{-1} b$, satisfies⁽³⁹⁾

⁽³⁸⁾ B is harmonic. C is divided into N equal arcs $I_n; n \leq N$.

⁽³⁹⁾ \mathbb{R} is divided into small intervals I_n . The final step is by routine approximation.

$$\sqrt{\det B} = D(R) \geq \Sigma D(I_n) \downarrow \frac{b}{\pi} \int [(\lambda' - a)^2 + b^2]^{-1} \sqrt{\det dF}(\lambda') = H,$$

so that H is overestimated by the harmonic minorant m of $\sqrt{\det B}$: $H \leq m$. But also, and for the same reason,

$$\int_I m da \leq \int_I \sqrt{\det B} da \leq \sqrt{\det} \int_I B da = \sqrt{\det F(I)} + o(1) \quad \text{as } b \downarrow 0$$

for most intervals I , and as the left side is additive while the right side is superadditive, so $m \leq H$. The rest is routine.

Item 4.

$$\det B = [h^*(0), h_-(0)][h_*(0), h_+(0)] = 1 - |[h^*(0), h_+(0)]|^2 \leq 1;$$

in particular, $\sqrt{\det dF} \leq d\lambda$, and the density $D = \sqrt{\det dF}/d\lambda$ is the limiting value $\sqrt{\det B}(\lambda + \sqrt{-1}0+)$ at almost every point of $[0, \infty)$. The computation is routine.

Item 5. Interprets the density $D = \sqrt{\det dF}/d\lambda$ of item 4 as the modulus of a (mean) transmission coefficient, as advertised in art. 1.

DISCUSSION. $D = \sqrt{\det dF}/d\lambda = |s_{11}|$ in the scattering case; see example 1 of art. 5, below. The Jost functions f_- and f_+ and the scattering matrix $[s_{ij}: 1 \leq i, j \leq 2]$ figuring in that computation depend for their definition upon the possibility of standardizing eigenfunctions at $\pm\infty$, as in

$$f_+(x) = s_{11} \exp(\sqrt{-1}kx) + o(1) \quad \text{at } x = +\infty,$$

but this can be side-stepped, even in the most general case. The trick is to standardize the functions h_- and h_+ by $h_-(0) = h_+(0) > 0$ for $\lambda < 0$, keeping $[h_-, h_+] = 1$; then $h_-(0) = h_+(0)$ everywhere off the cut $[0, \infty)$. Now the harmonic functions m_{11} , $-1/m_{11}$, and m_{12} have finite limiting values at almost every point of $[0, \infty)$, so the same is true of $h_-(0) = h_+(0)$, $h'_-(0) + h'_+(0) = m_{12}/h_+(0)$, and $h'_-(0) - h'_+(0) = 1/h_+(0)$, and so also of $h_-(x)$ and $h_+(x)$, independently of $x \in \mathbb{R}$, $[h_-, h_+] = 1$ being maintained on the cut. The values h_- and h_+ from the upper bank of the cut and likewise the values h^* and h_* from the lower bank provide a base of solutions of $Qh = \lambda h$, so you can patch them across the cut:

$$\begin{aligned} h_+ &= r_{11}h^* + r_{12}h_* \\ h_- &= r_{21}h^* + r_{22}h_* \end{aligned}$$

with a matrix $[r_{ij}: 1 \leq i, j \leq 2]$ reminiscent of the scattering matrix in its role of patcher of Jost functions. You read off

$$r_{11} = [h_+, h_+^*], \quad r_{12} = [h_-^*, h_+] = r_{21}^*, \quad r_{22} = -[h_-, h_-^*],$$

and $-r_{11}r_{22} + |r_{12}|^2 = 1$ by item 4, so that the density

$$\sqrt{\det dF/d\lambda} = \lim_{b \downarrow 0} \sqrt{\det B} = \sqrt{-r_{11}r_{22}} = \sqrt{1 - |r_{12}|^2}$$

is seen as the modulus of a (mean) transmission coefficient.

EXAMPLE. In the scattering case,

$$h_- = c_- f_-, \quad h_+ = c_+ f_+,$$

and with $c_- = c$ for short, you find $-2\sqrt{-1}ks_{11} = 1/c_-c_+$ and⁽⁴⁰⁾

$$\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} -1/2\sqrt{-1}k|c|^2 & s_{21}^*c^*/c \\ s_{21}c/c^* & 2\sqrt{-1}k|s_{11}|^2|c|^2 \end{bmatrix};$$

incidentally, $(4k)^{-1} \leq |c|^2 \leq 4|s_{11}|^2$ on $[0, \infty)$, so that $(1 + k^2)^{-1} \lg |c|$ is summable and c extends to a non-vanishing Hardy function off the cut.

Item 6. There is a close but untransparent connection between $\sqrt{\det dF}$ and the vector fields $X: Q \rightarrow 2G'_{xx}(\lambda)$ of arts. 1 and 3: with $D = \sqrt{\det dF/d\lambda}$ as before,

$$\frac{\partial D(\lambda)}{\partial q(x)} = -\text{the real part of } G_{xx}(\lambda + \sqrt{-1}0+) \times D \text{ on the cut};$$

compare note 4, art. 3.

PROOF. D is insensitive to translation so it suffices to compute at $x = 0$. Let $Q^\#$ be the operator $Q + c \times$ the unit mass at $x = 0$, with variable $-1 < c < 1$. This falls outside the class of operators permitted before but never mind. Now if the origin is taken at $x = 0+$, then with $H = 1 + ch_-(0)h_+(0)$,

$$\begin{aligned} h_-^\#(0) &= H^{-1}h_-(0), & h_+^\#(0) &= h_+(0), \\ h_-^{\#'}(0) &= H^{-1}[h_-'(0) + ch_-(0)], & h_+^{\#'}(0) &= h_+'(0), \end{aligned}$$

while if it is taken at $x = 0-$, then

$$\begin{aligned} h_-^\#(0) &= h_-(0), & h_+^\#(0) &= H^{-1}h_+(0), \\ h_-^{\#'}(0) &= h_-'(0), & h_+^{\#'}(0) &= H^{-1}[h_+'(0) - ch_+(0)], \end{aligned}$$

⁽⁴⁰⁾ $\lambda = k^2 \geq 0$.

and it turns out (as it must) that the 2 determinations of $\det B^\#$ are the same: $\det B^\# = H^{-2} \det B$. It follows that

$$D^\# = |1 + (c/2)m_{11}|^{-1} D$$

with $m_{11} = m_{11}(\bullet + \sqrt{-1}0+)$ and⁽⁴¹⁾

$$\begin{aligned} \frac{\partial D}{\partial q(0)} &= \frac{\partial D^\#}{\partial c} \quad \text{evaluated at } c = 0 \\ &= \frac{\partial}{\partial c} \left[\left(1 + \frac{c}{2} a_{11} \right)^2 + \frac{c^2}{4} b_{11}^2 \right]^{-1/2} D = (a_{11}/2) D, \end{aligned}$$

as promised. The computation is a bit formal but reliable, so I leave it at that.

Item 7 is a test case. I proposed in art. 1 that the unimodular spectral class of Q and its additive class must be one and the same thing. I prove it now for $Q^0 = -D^2$ whose additive class is the singleton Q^0 itself.

PROOF. The spectral weight of Q^0 is $dF^0 = \text{diag} [\lambda^{-1/2}, \lambda^{+1/2}] d\lambda$. Let dF be the spectral weight of an operator Q from the same unimodular class as Q^0 so that $sp dF$ belongs to the Lebesgue class on $[0, \infty)$ and $D = \sqrt{\det dF/d\lambda}$ is the indicator thereof. Then⁽⁴²⁾

$$p * D = \frac{b}{\pi} \int_0^\infty [(\lambda' - a)^2 + b^2]^{-1} d\lambda', \quad \text{taken at } \lambda = a + \sqrt{-1} b,$$

is the harmonic minorant of $\sqrt{\det B}$, and for any interval $I \subset [0, \infty)$, $\int_I [h_-, h_+]^*$ taken at $x = 0$ and

$$\begin{aligned} a + \sqrt{-1} b]^2 da &\leq 2 \int_I (1 - \sqrt{\det B})^{(43)} \\ &\leq 2 \int_I (1 - p * D) \\ &= \frac{2b}{\pi} \int_I da \int_{-\infty}^0 [(\lambda' - a)^2 + b^2]^{-1} d\lambda' \\ &= \frac{2b}{\pi} \int_I da \int_a^\infty (c^2 + b^2)^{-1} dc \\ &\leq \frac{2b}{\pi} \int_I a^{-1} da, \end{aligned}$$

⁽⁴¹⁾ $m_{11} = a_{11} + \sqrt{-1} b_{11}$.

⁽⁴²⁾ p stands for $(b/\pi)[(\lambda' - a)^2 + b^2]^{-1}$. The *star* signifies *convolution*.

⁽⁴³⁾ $1 - r^2 = (1 - r)(1 + r) \leq 2(1 - r)$ if $0 \leq r \leq 1$.

whence

$$\begin{aligned} & \left| \operatorname{imag} \int_I \left[\frac{h'_-}{h_-} - \frac{h^{*'}_+}{h^*_+} \text{ taken at } x = 0 \text{ and } a + \sqrt{-1} b \right] da \right|^2 \\ & \leq \int_I |[h_-, h^*_+]|^2 da \int_I |h_- h_+|^{-2} da \\ & \leq \frac{2}{\pi} \int_I \frac{da}{a} \int_I \frac{4b_{11}}{|m_{11}|^2} \frac{b}{b_{11}} da^{(41)} \\ & = \frac{2}{\pi} \int_I \frac{da}{a} 4 \int_I \operatorname{imag}(-1/m_{11}) \times \left[\frac{1}{\pi} \int_0^\infty [(c-a)^2 + b^2]^{-1} df_{11}(c) \right]^{-1}. \end{aligned}$$

Now introduce the representing measures df^0_- of $-\operatorname{imag} h'_-(0)/h_-(0)$, df^0_+ of $\operatorname{imag} h'_+(0)/h_+(0)$, and df_{00} of $-1/m_{11}$, and make $b \downarrow 0$ in the preceding display to obtain a) the bound

$$|f^0_+(I) - f^0_-(I)|^2 \leq \frac{2}{\pi} \int_I \frac{da}{a} 4 \int_I \left[\frac{1}{\pi} \int_0^\infty (c-a)^{-2} df_{11}(c) \right]^{-1} df_{00}(a),$$

b) an estimate $|df^0_+ - df^0_-| \leq a$ multiple of $a^{-1/2} \sqrt{da df_{00}}$ in the small, and c) the conclusion that df^0_- and df^0_+ have the same singular part, except perhaps for different jumps at 0. Now df^0_- and df^0_+ have the same non-singular parts as well: indeed, at almost every point of the cut $[0, \infty)$, the densities $f'_- = df^0_-/da$ and $f'_+ = df^0_+/da$ satisfy

$$f'_+ - f'_- = \lim_{b \downarrow 0} \operatorname{imag} \left[\frac{h'_-}{h_-} - \frac{h^{*'}_+}{h^*_+} \right] = \lim_{b \downarrow 0} \frac{[h_-, h^*_+]}{h_- h^*_+} = 0$$

in view of the fact that $m_{11} = 2h_- h_+$ has a non-vanishing limiting value and the estimate at the start of the proof:

$$\int_I |[h_-, h^*_+]|^2 \frac{da}{a} \leq \frac{2b}{\pi} \int_I \frac{da}{a} = o(1)$$

as $b \downarrow 0$ if I does not extend to 0. Note, finally, that neither df^0_- nor df^0_+ can jump at 0; otherwise, the representing measure df_{00} of $(1/2) \operatorname{imag} [h'_+/h_+ - h'_-/h_-] = \operatorname{imag}(-1/m_{11})$ has a jump (at 0) so that

$$-1/m_{11} \geq a \text{ constant} + (-\lambda)^{-1} f_{00}[0] \uparrow \infty \text{ as } \lambda \uparrow 0,$$

contradicting the fact that m_{11} is positive there. The upshot is that $df^0_- = df^0_+$, which is to say that Q is symmetrical about $x = 0$. But the unimodular class is invariant under translation, so the same is true for any choice of origin,

whence $Q = -D^2 +$ a constant function, and the constant vanishes because $\text{spec } Q$ starts at 0. The proof is finished.

5. EXAMPLES

EXAMPLE 1. Scattering case.⁽⁴⁴⁾ $[s_{ij}(k): 1 \leq i, j \leq 2]$ is the scattering matrix defined for real $k = \sqrt{\lambda}$, with values in $U(2)$; it is related to the behaviour at infinity of the Jost functions f_- and f_+ , as in the table. For $k \geq 0$, f_- and f_+ are the values of h_- and h_+ along the upper bank of the cut $[0, \infty)$, except that $[f_-, f_+] = -2\sqrt{-1}ks_{11}(k)$ is not unity.

	$x \downarrow -\infty$	$x \downarrow +\infty$
f_-	$s_{22}e^{-\sqrt{-1}kx}$	$e^{-\sqrt{-1}kx} + s_{21}e^{\sqrt{-1}kx}$
f_+	$e^{\sqrt{-1}kx} + s_{12}e^{-\sqrt{-1}kx}$	$s_{11}e^{\sqrt{-1}kx}$

The 2×2 spectral weight is comprised of jumps at the bound states and a pure Lebesgue part on $[0, \infty)$ with density

$$\frac{dF}{d\lambda} = \text{the real part of } \frac{1}{ks_{11}} \begin{bmatrix} 2f_-f_+ & (f_-f_+)' \\ (f_-f_+)' & 2f'_-f'_+ \end{bmatrix} \text{ taken at } x = 0.$$

The additive invariant (second measure class) $\sqrt{\det dF}$ is readily computed: it is unchanged by translation, so the entries of the table may be used in place of f_- and f_+ , with the outcome

$$\begin{aligned} \frac{\sqrt{\det dF}}{d\lambda} &= \sqrt{\det} \begin{bmatrix} k(1 + \text{real } s_{21}e^{2\sqrt{-1}kx}) & -\text{imag } s_{21}e^{2\sqrt{-1}kx} \\ -\text{imag } s_{21}e^{2\sqrt{-1}kx} & k^{-1}(1 - \text{real } s_{21}e^{2\sqrt{-1}kx}) \end{bmatrix} \\ &= \sqrt{1 - |s_{21}|^2} \\ &= |s_{11}|. \end{aligned}$$

The KDV *invariant manifold* is determined by fixing the transmission coefficient s_{11} , which is to say *by the fixing the bound states* $-k_1^2 < \dots \leq -k_g^2$ and

⁽⁴⁴⁾ Faddeev [1966] and/or Deift-Trubowitz [1979] are cited for background.

the modulus $|s_{12}| = \sqrt{1 - |s_{11}|^2}$ because s_{11} encodes just this information:

$$s_{11}(k) = \exp \left[\frac{1}{\pi \sqrt{-1}} \int_{-\infty}^{\infty} \frac{\lg |s_{11}(k')|}{k' - k} dk' \right] \prod_{i=1}^g \frac{k + \sqrt{-1} k_i}{k - \sqrt{-1} k_i};$$

also, it is known that the phase s_{12} together with the logarithms of the norming constants

$$c_i^2 = \int_{-\infty}^{\infty} |f_+ / s_{11}|^2 \text{ taken at } k = \sqrt{-1} k_i \quad (i \leq g)$$

serve as (additive) coordinates on the manifold. Now to elucidate the effect of addition, take $-k_0^2$ to the left of $\text{spec } Q$ and let $p_0 = (-k_0^2, +)$ to fix the ideas. Then $e_0(x) = e(x, p_0) = f_+(x, \sqrt{-1} k_0) = s_{11}(\sqrt{-1} k_0) \exp(-k_0 x)$ at $+\infty$ with a similar exponential behavior at ∞ : in fact, $[\lg e_0(x)]'$ vanishes rapidly at $\pm\infty$, so *adding p_0 not only keeps you in the scattering case but also preserves the KDV manifold* in view of note 2, item 7 of art. 3. The effect of addition upon the coordinates is found by comparison of new eigenfunctions $e_0^{-1}[e_0, f_+]$ to old:

$$e_0^{-1}[e_0, f_+] \approx e^{k_0 x} [e^{-k_0 x}, s_{11} e^{\sqrt{-1} k x}] = -(k_0 + \sqrt{-1} k) s_{11} e^{\sqrt{-1} k x}$$

at $+\infty$; similarly,

$$e_0^{-1}[e_0, f_+] \approx (-k_0 + \sqrt{-1} k) s_{12} e^{-\sqrt{-1} k x} - (k_0 + \sqrt{-1} k) e^{\sqrt{-1} k x}$$

at ∞ , so

$$s_{12} \rightarrow \frac{k_0 - \sqrt{-1} k}{k_0 + \sqrt{-1} k} s_{12}.$$

A similar rule applies to the bound states, the i th norming constant being multiplied by $(k_0 + k_i)(k_0 - k_i)^{-1}$ ($i \leq g$); in particular, *repeated additions followed by closure produce the whole invariant manifold*, as you will easily check.

EXAMPLE 2. Hill's case. Now h_- and/or h_+ is proportional to the so-called *Baker-Akhiezer function*

$$e(x, p) = e_1(x, \lambda) + e_2^{-1}(1, \lambda)[m(p) - e_1(1, \lambda)]e_2(x, \lambda)$$

taken at the point $p = (\lambda, \pm)$ of the *multiplier curve* of Q . The latter is the Riemann surface of the 2-valued *multiplier*⁽⁴⁵⁾ $m(p) = \Delta(\lambda) - \sqrt{\Delta^2(\lambda) - 1}$; the

⁽⁴⁵⁾ Δ is the discriminant $(1/2) [e_1(1, \lambda) + e_2(1, \lambda)]$.

former is the solution of $Qe = \lambda(p)e$ with $e(x + 1) = m(p)e(x)$ and $e(0) = 1$. The point is that, for λ to the left of $\text{spec } Q$, the number Δ exceeds $+1$, so that the multipliers satisfy $0 < m_+ < 1$ and $1 < m_- < \infty$, with the result that $e(x, p)$ belongs to $L^2[0, \infty)$ if $m(p) = m_+$ and to $L^2(-\infty, 0]$ if $m(p) = m_-$. Now the KDV invariant manifold may be described *either* as the class of Hill's operators with fixed periodic/anti-periodic spectrum *or* as (the real part of) the Jacobi variety of the multiplier curve; moreover, $[\text{lg } e(x, p)]''$ is of period 1, *so addition not only keeps you in the Hill's case but also preserves the KDV invariant manifold*, just as in example 1. Kodaira [1949] computed the 2×2 spectral weight: it is pure Lebesgue with density

$$\frac{dF}{d\lambda} = \frac{\pm 1/2}{\sqrt{1 - \Delta^2}} \begin{bmatrix} 2e_2(1, \lambda) & e_2'(1, \lambda) - e_1(1, \lambda) \\ e_2'(1, \lambda) - e_1(1, \lambda) & -2e_1'(1, \lambda) \end{bmatrix}$$

on the bands of $\text{spec } Q$; the signatures alternate starting with $+1$. The additive invariant $\sqrt{\det dF}$ is easily elicited:

$$\frac{\sqrt{\det dF}}{d\lambda} = \frac{-4e_2e_1 - (e_2' - e_1)^2}{4(1 - \Delta^2)} = 1 \text{ on spec } Q;$$

it determines $\text{spec } Q$ and so also the discriminant Δ , the multiplier curve M , and its Jacobian J . The use of the word *addition* can now be fully justified; it will be a by-product of the discussion that *repeated additions (followed by closure) are transitive on the invariant manifold*, just as in the scattering case.

DISCUSSION. Let $p_1 + p_2 + \dots$ be the divisor of Q introduced in art. 1, the points being the poles of $e(x, p)$ determined by the vanishing of $e_2(1, \lambda)$ and the choice of multiplier $m(p) = e_2'(1, \lambda)$ instead of the other possibility $e_1(1, \lambda)$, one such to each nontrivial gap $[\lambda_n^-, \lambda_n^+]$ ($n \geq 1$). Now the operator $Q - 2D^2 \text{lg } e(x, p_0)$ produced by addition of a point p_0 with projection to the left of $\text{spec } Q$ has also such a divisor $p_1' + p_2' + \dots$, and *the recipe of addition states that the old (unprimed) divisor with⁽⁴⁶⁾ $-p_0$ adjoined is linearly equivalent to the new (primed) divisor with ∞ adjoined:*

$$-p_0 + p_1 + p_2 + \dots = \infty + p_1' + p_2' + \dots \text{ in } J.$$

The proof is divided into several steps.

⁽⁴⁶⁾ $-p_0$ is the point on the sheet opposite to that of p_0 .

Step 1. The points $p_1, p_2, \text{ etc.}$ are the poles of the function

$$f(p) = e_1^{-1}(1, \lambda)[m(p) - e_1(1, \lambda)],$$

aside from an extra pole at ∞ which is detected by the development of $f(p)$ at $\lambda(p) = -\infty$:

$$f(p) = \frac{e^k - \text{ch } k}{\text{sh } k/k} [1 + o(1)] = k[1 + o(1)]$$

if $k = +\sqrt{-\lambda}$.

Step 2. The preceding development of $f(p)$ is applied to the operator Q^{p_0} . The corresponding function is⁽⁴⁷⁾

$$f_0(p) = \frac{m(p) - e_1^{p_0}(1, \lambda)}{e_2^{p_0}(1, \lambda)} = \frac{m(p) - e_0^{-1}[e_2, e_0] \text{ taken at } x = 1}{e_2^{p_0}(1, \lambda)} - c;$$

its poles are $p'_1, p'_2, \text{ etc.}$ and ∞ , by definition.

Step 3. $f_0(p)$ takes the value $-c$ at the points $p = p_1, p_2, \text{ etc.}$ of the divisor of Q and also at $-p_0$ in view of

$$e_0^{-1}[e_2, e_0] \text{ at } x = 1 = e'_2(1) - ce_2(1) = m(p)$$

in the first case and

$$\frac{[e_2, e_0] \text{ at } x = 1}{e_0(1)} = \frac{[e_2, e_0] \text{ at } x = 0}{e_0(1)} = \frac{1}{e_0(1)} = \frac{1}{m(p_0)} = m(-p_0)$$

in the second.

Step 4 is to notice that $f_0(p) = -c$ has no other roots. This follows from the evaluation $(2\pi\sqrt{-1})^{-1} \int d \lg f_0(p) = 1$ for small circles about $p = \infty$; compare step 1. The upshot is that $-p_0 + p_1 + p_2 + \text{ etc.}$ and $\infty + p'_1 + p'_2 + \text{ etc.}$ are the roots and poles of the function $f_0(p) + c$, which is what linear equivalence is all about, anyhow.

EXAMPLE 3. Bohr's case.⁽⁴⁸⁾ I cannot do so much if Q is only almost periodic in the sense of Bohr [1932]. Fix λ off $\text{spec } Q$. Then $G = G_{xx}(\lambda)$ is an almost

⁽⁴⁷⁾ $e_1^{p_0} = ce_2^{p_0} + e_0^{-1}[e_2, e_0]$ with $c = e'_0/e_0$ taken at $x = 0$.

⁽⁴⁸⁾ Johnson-Moser [1982] and Moser-Pöschl [1984] are cited for background.

periodic function with the same frequency module as Q , and

$$h_{\mp} = \sqrt{G} \exp \left[\pm(1/2) \int_0^x G^{-1} dx' \right],$$

by elementary computation⁽⁴⁹⁾ from which it follows that *addition preserves*:

- 1) *the almost periodicity,*
- 2) *the frequency module,*

and

- 3) *the rotation number*

$r(\lambda) = \lim_{x \uparrow \infty} x^{-1} \operatorname{imag} \lg h_+(x, \lambda) =$ the mean value of $\operatorname{imag} -1/2G$ off $\operatorname{spec} Q$.

PROOF OF 3. Addition of p_0 changes h_+ into $e_0^{-1}[h_+, e_0]$ so that $\operatorname{imag} \lg h_+$ is changed by the addition of

$$\operatorname{imag} \lg \left[\frac{h'_+}{h_+} - \frac{e'_0}{e_0} \right],$$

and this is bounded off $\operatorname{spec} Q$ because $\operatorname{imag} h'_+/h_+$ is of one signature while e'_0/e_0 is real.

The values of $r(\bullet + \sqrt{-1}0+)$ on the line are known to determine $\operatorname{spec} Q$ and the frequency module,⁽⁵⁰⁾ whence it is natural to conjecture that *r and the bound states (if any) determine the additive class*, but this may be naive.

References

- Bargmann, V. On the connection between phase shifts and scattering potentials. *Rev. Mod. Phys.* **21** (1949), 488-493.
- Bohr, H. *Fastperiodische Funktionen*, Springer-Verlag, Berlin, 1932.
- Crum, M. Associated Sturm-Liouville systems. *Quart J. Math. Oxford*, **6** (1955), 121-127.
- Darboux, L. Sur une proposition relative aux équations linéaires. *CR Paris*, **94** (1882), 1456.
- Deift, P. Applications of a commutation formula. *Duke Math. J.* **45** (1978), 267-310.
- Deift, P. and Trubowitz, E. Inverse scattering on the line. *CPAM*, **32** (1979), 121-251.

⁽⁴⁹⁾ $KG = 2\lambda G'$ with $K = qD + Dq - (1/2)D^3$, whence $(G')^2 - 2G''G + 4(q - \lambda)G^2 = 1$.

⁽⁵⁰⁾ Johnson-Moser [1982].

- Faddeev, L. On the relation between the S-matrix and the potential of the one-dimensional Schrödinger equation. *Tr. Mat. Inst. V. A. Steklova*, **73** (1964), 314-336; *AMST*, **65** (1966), 139-166.
- Gelfand, I. M. and Levitan, B. On the determination of a differential equation from its spectral function. *Izvest. Akad. Nauk*, **15** (1951), 309-360; *AMST*, **1** (1955), 253-304.
- Isaacson, E., McKean, H. P. and Trubowitz, E. The inverse Sturm-Liouville problem (2). *CPAM*, **37** (1984), 1-12.
- Johnson, R. and Moser, J. The rotation number of almost periodic potentials. *Comm. Math. Phys.* **84** (1982), 403-438.
- Kodaira, K. The eigenvalue problem for ordinary differential equations of second order and Heisenberg's theory of S-matrices. *Amer. J. Math.* **71** (1949), 921-945.
- McKean, H. P. Integrable systems and algebraic curves. *Global Analysis, Lecture Notes in Math.* **755** (1979), 83-200.
- Units of Hill's curve. *Geometry and Analysis* (papers dedicated to the memory of V. K. Patodi) Indian Academy of Sciences (1980), 81-94, Bombay.
- Variation on a theme of Jacobi. *CPAM*, **38** (1985), 669-678.
- McKean, H. P. and van Moerbeke, P. The spectrum of Hill's equation. *Invent. Math.* **30** (1975), 217-274.
- McKean, H. P. and Trubowitz, E. Hill's operator and hyper-elliptic function theory in the presence of infinitely many branch points. *CPAM*, **29** (1976), 143-226.
- Hill's surfaces and their theta functions. *BAMS*, **84** (1978), 1042-1085.
- The spectral class of the quantum-mechanical oscillator. *Comm. Math. Phys.* **82** (1982) 471-495.
- Masani, P. and Wiener, N. The prediction theory of multivariate stochastic processes. *Acta Math.* **98** (1957), 111-150; **99** (1958), 93-137.
- Miura, R. Korteweg-de Vries equation and generalizations (1). A remarkable explicit nonlinear transformation. *J. Math. Phys.* **9** (1968), 1202-1204.
- Moser, J. Various aspects of integrable Hamiltonian systems. *CIME*. Bressanone, 1978.
- Moser, J. and Pöschl, J. An extension of a result by Dinaburg and Sinai on quasi-periodic potentials. *Comm. Math. Helv.* **59** (1984), 39-85.
- Mumford, D. *Tata Lectures on Theta*. Birkhäuser-Verlag, Boston/Basel/Stuttgart, 1983/4.
- Newton, R. The Marcenko and Gelfand-Levitan methods in the inverse scattering problem in 1 and 3 dimensions. *SIAM Conf. Inverse Scattering* (1983), 1-74, Philadelphia.
- Stone, M. *Linear Transformations in Hilbert Space*. AMS Colloquium Publ. **15** (1932), New York.
- Weyl, H. Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörige Entwicklungen willkürlicher Funktionen. *Math. Ann.* **68** (1910), 220-269.

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