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On the Two-Fold Symbol Chain of a C*-Algebra of Singular Integral Operators on a Polycylinder

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Dedicated to A. Calderón

The importance of the symbol homomorphism for the theory of singular integral equations is an old-established fact. In particular Gohberg [Go1] proved in 1960 that in a certain algebra of singular integral operators over $\mathbb{R}^n$ it is a necessary and sufficient condition for an operator to be Fredholm if its symbol does never vanish. A similar criterion holds for singular integral operators on a compact manifold, and has served as the analytical foundation of the Atiyah-Singer index theorem [ASj], $j = 1, 3, 4, 5$. The above are only two examples of a long list of papers on the subject (cf. for example [Gi1], [Mi1], [CZj], $j = 1, 2$, [Se1]).

In all cases mentionned it proves important that the corresponding operator algebra, as a subalgebra of $\mathcal{L}(\mathfrak{X})$, for some Hilbert or Banach space $\mathfrak{X}$, has compact commutators. In fact the criterion was proven to be a direct consequence of the Gelfand representation of commutative Banach algebras (cf. [Go1], [Se1], [CS], [BC1, 2], [He1], [CH1]).

On the other hand, in some cases of algebras where not all commutators are compact results were obtained involving multiple symbol chains. For example in [CS2], [CC1], it was shown that a certain algebra $\mathfrak{A}$ of singular integral operators on a half-space $\mathbb{R}^{n+1}$ (in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{n+1})$) has a two-link ideal chain

$$\mathfrak{A} \supset \mathcal{E} \supset \mathcal{K}(\mathcal{H}), \tag{0.1}$$

where $\mathcal{E}$ denotes the operator norm closed ideal generated by the commutators of $\mathfrak{A}$, while $\mathcal{K}(\mathcal{H})$ is the ideal of compact operators. The point is that both quotients $\mathfrak{A}/\mathcal{E}$ and $\mathcal{E}/\mathcal{K}(\mathcal{H})$ are function algebras, giving raise to two symbols: (i) A complex-valued symbol $\sigma_A \in C(\mathbb{M})$ is defined for all $A \in \mathfrak{A}$ over a certain explicitly given compact space $\mathbb{M}$. Also (ii) a compact operator-valued symbol $\gamma_E \in CO(\mathbb{E}, \mathfrak{k})$ with the class $\mathfrak{k}$ of compact operators on the Hilbert space $\mathfrak{h} = L^2(\mathbb{R}^+)$ is defined only for $E \in \mathcal{E}$ on a different (only locally compact) space $\mathbb{E}$. If $\sigma_A \neq 0$ then A is invertible mod $\mathcal{E}$. The equation $Au = f$ then can be reduced to two equations of the form $(1 + E)v = g$, with some $E \in \mathcal{E}$. Then the compact-valued $\mathcal{E}$-symbol serves to decide invertibility of $1 + E \bmod \mathcal{K}$, i.e., the Fredholm property.

In [CE] such criterion was used to discuss the elliptic boundary problem over the (noncompact) half-space $\mathbb{R}^{n+1}_+$. Necessary and sufficient criteria in the form of Lopatinski-Shapiro conditions were obtained (For a detailed discussion cf. [C1], V).

Similar multi-link ideal chains were obtained for certain algebras involving Wiener-Hopf operators [Dy1], [Up1], and for an algebra of singular integral operators over $\mathbb{R}$ with periodic coefficients [CMe1].

In the present paper we will consider an algebra on a non-compact Riemann space of the form

$$\Omega = \mathbb{R}^{n'} \times B, \qquad \dim B = n'', \qquad n' + n'' = n, \tag{0.2}$$

with a compact Riemannian space B of dimension n'' and metric

$$ds^2 = dt^2 + d\rho^2, \tag{0.3}$$

with the Euclidean metric dt^2 of $\mathbb{R}^{n'}$ and the metric $d\rho^2$ of B. Such space will be called a poly-cylinder.

We will analyze the simplest nontrivial case: The Hilbert space $\mathcal{H} = L^2(\Omega, dS)$, with the surface measure dS of (0.3), and an algebra $\mathcal{C}$ of operators acting on complex-valued functions, not crossections of vector bundles. We believe that extensions of our results to L^p-(Sobolev-) spaces are possible, but offer only more complications and are slightly less perfect. (For similar L^p-investigations cf. [Il1], [AN1], [LM1].) On the other hand our present L^2-results may be combined with results of [C2] to obtain straight generalizations as follows.

(*a*) (cf. [C2], VIII) Algebras on a general non-compact Riemann space Ω with a finite atlas $\{\Omega_1, \ldots, \Omega_N\}$ subordinate partition of unity $1 = \sum \omega_j$,

$\operatorname{supp}\omega_j \subset \Omega_j$, such that for each j we have either $\operatorname{supp}\omega_j$ compact and Ω_j charted on $\mathbb{R}^n$ or Ω_j charted on a polylinder Ω_j of the form (0.2) and ω_j contained in the corresponding polycylinder algebra $\mathcal{C}_j$ discussed below (with metric of Ω and Ω^j coinciding in $\operatorname{supp}\omega_j$).

(*b*) (cf. [C2], X, 3) Operators acting on crossections of vector bundles on Ω as described under (*a*), where the vector bundles have to be suitably restricted at infinity.

(*c*) (cf. [C2], IX) Subalgebras of L^2-Sobolev spaces $\mathcal{H}_s$ over Ω, using $\Lambda^s: \mathcal{H} \to \mathcal{H}_s$ as an isometry, where $\Lambda = (1 - \Delta)^{-1/2}$, with the Laplace operator Δ of the metric (0.1).

(*d*) (cf. [C2], X, 6) More general L^2-Sobolev spaces, such as those using an operator $\phi\Lambda^s: \mathcal{H} \to \mathcal{H}_{s,\delta}$ as an isometry, with a function $\phi \in C^\infty(\Omega)$ such as the spaces $W^p_{s,\delta}$ of [LM1], in case of $p = 2$, (but general real s).

As in the case of the space with boundary $\mathbb{R}^{n+1}_+$ we obtain a 2-link ideal chain $\mathcal{C} \supset \mathcal{E} \supset \mathcal{K}(\mathcal{H})$ for our algebra $\mathcal{C}$. Again the two quotients $\mathcal{C}/\mathcal{E}$ and $\mathcal{E}/\mathcal{K}(\mathcal{H})$ are function algebras. However the result (thm. 3.2) appears to be more perfect, for the following reason: A major difficulty in applying the Fredholm inversion is the matter of checking (quasi-) invertibility of the $\mathcal{E}$-symbol. While the symbol σ_A of $A \in \mathcal{C}$ often is directly given as an explicit function over $\mathbb{M}$, the inversion of $(1 + E)$ can only be attempted after an $\mathcal{E}$-inverse B is obtained and E (or γ_E) is obtained explicitly.

This difficulty is avoided by obtaining an extension of the homomorphism γ from $\mathcal{E}$ to $\mathcal{C}$ again. Then the two symbols σ_A and γ_A of an operator A can be directly obtained. (σ_A coincides with the ordinary symbol of a singular integral operator, but γ_A is defined over $\mathbb{E}$, a space of infinite points, (i.e. points over infinity of a certain compactification of Ω). A necessary and sufficient condition for $A \in \mathcal{C}$ to be Fredholm is that $\sigma_A \neq 0$ and that γ_A be invertible and the inverse bounded on $\mathbb{E}$.

The result is easily applied to certain realizations of a given differential expression over Ω, for certain expressions L. One finds that $A = L(1 - \Delta)^{-N/2}$, for L of order N with suitable coefficients in an operator in $\mathcal{C}$. Thus $Z = A(1 - \Delta)^{N/2}$ defines an unbounded operator and a realization of L on $\mathcal{H}$ which is Fredholm if and only if $A \in \mathcal{C}$ is Fredholm. This clearly makes thm. 3.2 applicable to study the Fredholm property of the realization A. Also the symbols σ_A and γ_A are easily obtained explicitly, using the t-Fourier transform and the symbol of $\mathcal{L}$ as a differential operator. We get $\sigma_A \neq 0$ if and only if L is uniformly elliptic.

For a uniformly elliptic differential expression L of order N the γ-symbol of A defines a family $\{L_{t_0}(\tau): t_0 \in \partial\Omega, \tau \in \mathbb{R}^{n'}\}$ of N-th order elliptic differential expressions depending continuously on t_0 (over an infinite boundary $\partial\Omega$) but analytically on $\tau \in \mathbb{R}^{n'}$. (In fact, the $L_{t_0}(\tau)$ are polynomials in τ.)

A very simple device of Agmon and Nirenberg [AN1] (together with the (interior) Sobolev estimate on compact manifolds) may be employed to show that γ_A is invertible for $\tau \in \mathbb{R}^{n'}$ with large $|\tau|$. This (and a result by Gramsch [Gr1]) implies that the family $L_{t_0}(\tau + i\delta)$, i.e. γ_A, is invertible for all real τ whenever the real $\delta \in \mathbb{R}^{n'}$ avoids a certain countable set $\mathcal{Z}_{t_0}$ without finite cluster points. For $n' = 1$ this coincides with the result of Lockhart-McOwen regarding the operator $e^{\delta t} L e^{-\delta\tau}$ (i.e. of L relative to the weighted L^2 norm with weight $e^{\delta t}$). For $n' > 1$ the exceptional set depends on the infinite point t_0. Thus the set $\mathcal{Z}_{t_0}$ of δ to be avoided is more complicated, unless one assumes L_{t_0} independent of t_0 at each end of Ω.

The results on differential operators are discussed in more detail in [C2], IX, X.

Finally we want to point to a variety of results by Bruening and Seeley [BS], Melrose-Mendoza [MM1], Choquet-Bruhat and Christodolou [CBC], all related in general aim, but different in method and approach. In particular it is clear that results on the Fredholm index (i.e., an index formula) are implied, as we also shortly indicate in sec. 3.

1. A C*-Algebra on a Poly-Cylinder

First we look at a Laplace comparison algebra with noncompact commutator on a poly-cylinder $\Omega = \mathbb{R}^{n'} \times B$. Here B denotes a compact Riemannian space of dimension n'' with metric $d\rho^2 = g_{jk}\, dx^j\, dx^k$. Accordingly, for the metric and the Laplace operator of Ω we get

$$(1.1) \qquad ds^2 = dt^2 + d\rho^2, \qquad \Delta = \Delta_t + \Delta_x, \qquad \Delta_x = (\sqrt{g})^{-1}\partial_{x^j}\sqrt{g}g^{jk}\partial_{x^k},$$

where Δ_x is the Laplace operator on B. In (1.1) we are using the Euclidean metric $dt^2 = dt^{1^2} + \cdots + dt^{n'^2}$ of $\mathbb{R}^{n'} = \{t = (t^1, \ldots, t^{n'}) \colon t^j \in \mathbb{R}\}$, and the Euclidean Laplace operator $\Delta_t = \sum \partial_{t^j}^2$. The summation convention often will be used from 1 to n'', as will be clear from the context. We set $n = n' + n''$, so that Ω is n-dimensional.

Let $\mathcal{H}$ be the Hilbert space $L^2(\Omega) = L^2(\Omega, dS)$, with the surface measure $dS = dS'dS'' = \sqrt{g}\, dt\, dx$ of the metric (1.1). Let $\mathcal{C} \subset \mathcal{L}(\mathcal{H})$ be the smallest C^*-subalgebra containing the (5 types of) operators

$$(1.2) \qquad a \in \mathfrak{A}_B^{\#}, \quad s_j(t) = t_j/\langle t\rangle, \quad \Lambda = (1 - \Delta)^{-1/2},$$
$$\partial_{t^j}\Lambda, \quad D_x\Lambda, \quad D_x \in \mathfrak{D}_B^{\#}, \quad j = 1, \ldots, n'.$$

Here $\langle t\rangle = (1 + t^2)^{1/2}$; we write $\mathfrak{A}_B^{\#} = C^\infty(B)$ while $\mathfrak{D}_B^{\#}$ denotes the collection of all C^∞-vector fields on B. Also Λ is the unique positive inverse square root of the (unique) self-adjoint realization $(1 - \Delta)$ of the Laplace differential expression Δ. There is a unique such self-adjoint realization because Ω is a complete Riemannian space (cf. [Ga1], [CWS], [Ch1]). The functions $a \in \mathfrak{A}_B^\#$ and $s_j(t)$ in (1.2) represent the corresponding multiplication operators on functions $u(t, x) \in \mathfrak{H}$. Correspondingly we denote by $\mathfrak{A}^\#$ and $\mathfrak{D}^\#$ the algebra finitely generated by C_0^∞ and the first two kinds of generators, and the linear space spanned (mod $\mathfrak{A}^\#$) by $\{1, \partial_{tj}, \mathfrak{D}_x\}$, respectively. The operations $\partial_{tj}\Lambda u$ and $D_x\Lambda u$ are well defined for $u \in \Lambda^{-1}C_0^\infty(\Omega)$, a dense subspace of $\mathfrak{H}$ (cf. [C2], V) and have continuous extensions to $\mathfrak{H}$, as easily seen (cf. also (1.6), below).

We notice that $\mathfrak{H} = \mathfrak{h}_t \hat{\otimes} \mathfrak{h}_x$ is the topological tensor product of the Hilbert spaces $\mathfrak{h}_t = L^2(\mathbb{R}^{n'})$, and $\mathfrak{h}_x = L^2(B)$ (cf. [C1], V, 8). Let us write $\mathfrak{L}_x = \mathfrak{L}(\mathfrak{h}_x)$, $\mathfrak{k}_x = \mathfrak{K}(\mathfrak{h}_x)$, $\mathfrak{L}_t = \mathfrak{L}(\mathfrak{h}_t)$, $\mathfrak{k}_t = \mathfrak{K}(\mathfrak{h}_t)$. It may be observed that the topological tensor products $\mathfrak{L}_t \hat{\otimes} \mathfrak{L}_x = \mathfrak{L}_{tx}$, $\mathfrak{k}_t \hat{\otimes} \mathfrak{k}_x = \mathfrak{K}(\mathfrak{H})$, $\mathfrak{L}_t \hat{\otimes} \mathfrak{k}_x = \mathfrak{K}_x$, $\mathfrak{k}_t \hat{\otimes} \mathfrak{L}_x = \mathfrak{K}_t$ are all well defined C^*-subalgebras of $\mathfrak{L}(\mathfrak{H})$, where $\mathfrak{K}(\mathfrak{H}) \subset \mathfrak{K}_x$, $\mathfrak{K}_t \subset \mathfrak{L}_{xt} \subset \mathfrak{L}(\mathfrak{H})$ all are proper inclussions. In fact, $\mathfrak{K}_x$ and $\mathfrak{K}_t$ are proper closed two-sided ideals of $\mathfrak{L}_{tx}$, and, of course, $\mathfrak{K}(\mathfrak{H})$ is a proper closed ideal of all the others. Evidently $\mathfrak{L}_{tx}$ (but not $\mathfrak{K}_x$ or $\mathfrak{K}_t$) contains the identity operator I.

Note that we may write $\mathfrak{H} = L^2(\mathbb{R}^{n'}, \mathfrak{h}_x)$ as the space of all functions over $\mathbb{R}^{n'}$ with values in $\mathfrak{h}_x = L^2(B)$ such that

$$\|u\|^2 = \int_{R^{n'}} dt \, \|u(t, \bullet)\|^2 < \infty, \tag{1.3}$$

by Fubini's theorem. Correspondingly, if $CB(\mathbb{R}^{n'}, \mathfrak{k}_x)$ and $CB(\mathbb{R}^{n'}, \mathfrak{L}_x)$ denote the classes of bounded norm continuous functions over $\mathbb{R}^{n'}$, taking values in $\mathfrak{k}_x$ and $\mathfrak{L}_x$, respectively, then a function $\phi \in CB(\mathbb{R}^{n'}, \mathfrak{L}_x)$ has a natural interpretation as an operator in $\mathfrak{L}(\mathfrak{H})$, defined by $u(t, \bullet) \to \phi(t)u(t)$. Moreover, this operator is in $\mathfrak{K}_x$ whenever $\phi \in CO(\mathbb{R}, \mathfrak{k}_x)$ (cf. [C1], V, 8). (We indicate a proof of this fact in cor. 1.4.) This establishes an isometric *-isomorphism of $CB(\mathbb{R}; L_x)$ (as a Banach algebra with norm

$$\sup\{\|A(\tau)\| : \tau \in \mathbb{R}\}, \tag{1.4}$$

where $\|A(\tau)\|$ is the norm in $\mathfrak{L}_x$) into $\mathfrak{L}(\mathfrak{H})$. In the following we will use this interpretation of functions as operators in $\mathfrak{L}(\mathfrak{H})$.

In order to find the commutator ideal $\mathfrak{E}$ of the unital C^*-algebra $\mathfrak{C}$ with generators (1.2) we conjugate the generators with the Fourier transform in the t-direction. In detail we have

$$F_t u(\tau) = (2\pi)^{-n'/2} \int_{\mathbb{R}^{n'}} e^{-it\tau} u(t)\, dt, \tag{1.5}$$

which defines a unitary operator of $\mathfrak{h}_t$. By conjugation with F_t we, of course, mean conjugation with $F_t \otimes I_x$, where I_x denotes the identity operator. (However, we will write this as $F_t^{-1} A F_t$, for $A \in \mathfrak{L}(\mathfrak{H})$.) First consider the

F_t-conjugations of Λ, $D_{tj}\Lambda$, ${}_{D_x}\Lambda$:

$$(1.6) \qquad \Lambda^\sim(\tau) = (\langle\tau\rangle^2 - \Delta_x)^{-1/2}, \qquad \tau_j\Lambda^\sim(\tau), \qquad D_x\Lambda^\sim(\tau),$$

with $D_{t,j} = -i\partial_{t,j}$ and above interpretation of a function as operator.

The manifold B is compact, therefore $\Lambda^\sim(\tau)$ of (1.5), for fixed τ, as operator in $\mathcal{L}(\mathfrak{h}_x)$, is compact (cf. [C2], III, cor. 3.9.). Moreover, this defines an operator function $\Lambda^\sim(\tau) \in CO(\mathbb{R}, \mathfrak{h}_x)$, which even is analytic, in norm topology of $\mathcal{L}(\mathfrak{h}_x)$, and we have $\|\Lambda^\sim(\tau)\| \leqslant \langle\tau\rangle^{-1}$ as well as $\|\Lambda^\sim(0)^{-1}\Lambda^\sim(\tau)\| \leqslant 1$, so that $\tau_j\Lambda^\sim(\tau)$, $D_x\Lambda^\sim(\tau) = (D_x\Lambda_x)(\Lambda_x^{-1}\Lambda^\sim(\tau))$ are functions in $CB(\mathbb{R}^{n'}, \mathcal{L}_x) \subset \mathcal{L}(\mathcal{H})$, confirming the boundedness of the last two types (1.2). Also, writing $\Lambda_x = \Lambda^\sim(0)$, $T(\tau) = (1 + \tau^2\Lambda_x^2)^{-1/2}$, the generators (1.2) correspond to

$$(1.7) \qquad a(x), \qquad \mu_j*, \qquad \Lambda_x T(\tau), \qquad (i\tau_j\Lambda_x)T(\tau), \qquad (D_x\Lambda_x)T(\tau),$$

with $\mu_j = F_t^{-1}s_j$ and $\mu_t^* =$ convolution in $\mathfrak{h}_t$.

Proposition 1.1. *The operator functions $\Lambda_x^\epsilon T(\tau)$ and $|\tau|^{1-\epsilon}\Lambda_x T(\tau)$, for each fixed ϵ, $0 \leqslant \epsilon < 1$, are in $CO(\mathbb{R}, \mathfrak{h}_x)$.*

Proof. Just note that $\Lambda^\sim = \Lambda_x T(\tau) \in CO(\mathbb{R}, \mathfrak{h}_x)$, while $T(\tau)$ and $|\tau|\Lambda_x T(\tau)$ belong to CB. All operators are positive self-adjoint, so that $\Lambda_x^\epsilon T(\tau) = \Lambda^\epsilon T(\tau)^{1-\epsilon} \in CO$, $|\tau|^{1-\epsilon}\Lambda_x T(\tau) = \Lambda^\epsilon(|\tau|\Lambda_x T(\tau))^{1-\epsilon} \in CO$, as products of a bounded function and a function with limit 0, q.e.d.

Now we first will describe the algebra generated by the commutators of the F_t-conjugated generators (1.7).

Proposition 1.2. *All commutators of the F_t-conjugated generators and their adjoints are contained in the algebra*

$$(1.8) \qquad \mathcal{G}^\wedge = CO(\mathbb{R}^{n'}, \mathcal{K}(\mathfrak{h}_x)) + \mathcal{K}(\mathcal{H}) \subset \mathcal{K}_x.$$

Here a function $C(\tau) \in CO(\mathbb{R}^{n'}, \mathcal{K}(\mathfrak{h}_x))$ must be interpreted as an operator in $\mathcal{K}_x$ in the manner described above.

Proof. We will use the resolvent integral technique used in [C2], V. Let the operators (1.7) be denoted by $G_1, \ldots, G_5$, in the order listed (we write $t = t^j$, $\partial_t = \partial_{t,j}$, $\tau = \tau_j$, etc.). Clearly $[G_1, G_2] = [G_3, G_4] = 0$.

We get $(D_x\Lambda^\sim)^* - D_x^*\Lambda^\sim = [\Lambda^\sim, D_x^*] \in CO(\mathbb{R}^{n'}, \mathfrak{h}_x)$, by (1.14) below, and adjoint invariance of $\mathfrak{D}_B^\#$. All other generator (classe)s are self-adjoint. Hence the adjoint generators need no special attention.

For the commutators $[G_j, G_l]$, $j \neq 2$, we use the well known resolvent integral representation of $\Lambda^{\sim} = G_3$: Let $R(s) = (s + \langle\tau\rangle^2 - \Delta_x)^{-1}$. Then we have

$$(1.9) \qquad G_3 = \Lambda^{\sim}(\tau) = \Lambda_x T(\tau) = (\langle\tau\rangle^2 - \Delta_x)^{-1/2} = 1/\pi \int_0^\infty R(s)\, ds/\sqrt{s},$$

(cf. [C2], IX, (5.13)), with a norm convergent improper Riemann integral, in the algebra $\mathcal{L}(\mathfrak{k}_x)$. We get

$$(1.10) \qquad [G_3, G_5] = [\Lambda^{\sim}, D_x]\Lambda^{\sim}, \qquad [G_4, G_5] = [\Lambda^{\sim}, D_x](\tau\Lambda^{\sim}),$$

where $\Lambda^{\sim}, \tau\Lambda^{\sim} \in CB$, so that it suffices to show that $[\Lambda^{\sim}, D_x] \in CO(\mathbb{R}, \mathfrak{k}_x)$.
Similarly,

$$(1.11) \qquad \begin{aligned} &[G_1, G_3] = [a_x, \Lambda^{\sim}], \qquad [G_1, G_4] = (\tau\Lambda^{\sim})(\Lambda^{\sim\,-1}[a_x, \Lambda^{\sim}]), \\ &[G_1, G_5] = p_x\Lambda^{\sim} + (D_x\Lambda^{\sim})(\Lambda^{\sim\,-1}[a_x, \Lambda^{\sim}]), \end{aligned}$$

with $p_x = [a_x, D_x] \in C^\infty(B)$. Accordingly we also must show that $\Lambda^{\sim\,-1}[a_x, \Lambda^{\sim}] \in$ $\in CO(\mathbb{R}, \mathfrak{k}_x)$. Both these facts are consequences of prop. 1.3, below. Before we discuss it we turn to the commutators $[G_2, G_1]$. There we find it practical to work without the Fourier transform (1.5), writing

$$(1.12) \qquad \Lambda = 1/\pi \int_0^\infty S(r)\, dr/\sqrt{r}, \qquad S(r) = (r + 1 - \Delta_t - \Delta_x)^{-1}.$$

Instead of «diagonalizing the t-variable by using the Fourier transform» we will consider the x-variable diagonalized later on. Let $A_j = F_t G_j F_t^{-1}$ be the generators (1.2). Then we have

$$(1.13) \qquad \begin{aligned} &[A_2, A_4] = p_t\Lambda + (\partial_t\Lambda)V, \qquad [A_2, A_5] = (D_x\Lambda)V, \\ &V = \Lambda^{-1}[A_2, A_3], \qquad p_t \in C^\infty. \end{aligned}$$

Note that $p_t \in CO(\mathbb{R}^{n'})$ hence $p_t\Lambda \in \mathcal{K}(\mathcal{H})$, by a well known result (cf. [C2], III, thm. 3.7). We claim that all 3 commutators $[A_2, A_l]$, $l = 3, 4, 5$, are in $\mathcal{K}(\mathcal{H})$. Again this is a trivial consequence of prop. 1.3 below, so that all of prop. 1.2 has been reduced to prop. 1.3.

Proposition 1.3. *For ϵ with $0 \leqslant \epsilon < 1$ we have*

$$(1.14) \qquad \Lambda^{\sim\,-1-\epsilon}[a_x, \Lambda^{\sim}] \in CO(\mathbb{R}^{n'}, \mathfrak{k}_x), \qquad \Lambda^{\sim\,-\epsilon}[D_x, \Lambda^{\sim}] \in CO(\mathbb{R}^{n'}, \mathfrak{k}_x),$$

$$(1.15) \qquad \Lambda^{-1-\epsilon}[s_j(t), \Lambda] \in \mathcal{K}(\mathcal{H}), \qquad s_j(t) = t_j/\langle t\rangle$$

Remark. As a consequence of prop. 1.3 the algebra $\mathcal{C}$ satisfies condition (m_4) of [C2], VIII, 3, as required later on (cf. thm. 2.3).

For the proof of prop. 1.3 we use (1.9) for

$$(1.16) \qquad \Lambda^{\sim -1-\epsilon}[G_1, G_3] = 1/\pi \int_0^\infty \Lambda^{\sim -1-\epsilon} R(s) L_1 R(s)\, ds/\sqrt{s},$$

$L_1 = [a(x), \Delta_x]$. The integrand in (1.16) is a norm-continuous (even analytic) function $F(s, \tau)$, with values in k_x. Indeed, one may write

$$(1.17) \qquad F(s, \tau) = (\Lambda^{\sim -1-\epsilon} R(s))(L_1 \Lambda_x)(\Lambda_x^{-1} R(s)/\sqrt{s}),$$

and use the estimates

$$(1.18) \qquad \|\Lambda_x^{-\eta} R(\sigma)\| \leqslant (1 + s + \tau^2)^{\eta/2-1}, \qquad \|\Lambda^{\sim -\eta} R(s)\| \leqslant (1 + s)^{\eta/2-1},$$

$0 \leqslant \eta < 2$, easily derived from the spectral decomposition of the self-adjoint operator $-\Delta_x \geqslant 0$ of $\mathfrak{k}_x$. Note that L_1 is a folpde on B independent of s, τ, so that the second factor in (1.17) is a constant in $\mathcal{L}(\mathfrak{k}_x)$. Analyticity of the first and third factor is a consequence of analyticity of the resolvent $R(s)$. These factors are $O((1 + s)^{(\epsilon-1)/2})$ and $O(s^{-1/2}(1 + s)^{-1/2})$, respectively. Thus $F(s, \tau) = O((1 + s)^{-1+\epsilon/2}/\sqrt{s})$ uniformly for all $\tau \in \mathbb{R}^{n'}$. Also both factors are in $\mathfrak{k}_x$ since B is compact, insuring the compactness of the resolvent $R(s)$ of the Laplace operator Δ_x ([C2], III, thm. 3.1). This implies existence of the improper Riemann integral (1.16) in norm convergence of $\mathcal{L}(\mathfrak{k}_x)$ and uniformly so, for $\tau \in \mathbb{R}^{n'}$. Thus the integral is in $CB(\mathbb{R}^{n'}, \mathfrak{k}_x)$. (For more detail in such a proof cf. [C2], V, 3). Moreover, since $0 < \epsilon < 1$ is arbitrary, one may use this for $\epsilon + \delta < 1$, with some $\delta > 0$. This gives a factor $\Lambda^{\sim\delta} \in CO$, whence the first (1.14) is CO, not only CB.

Similarly we may use (1.9) for

$$(1.19) \quad \Lambda^{\sim -\epsilon}[D_x, \Lambda^{\sim}] = 1/\pi \int_0^\infty (\Lambda^{\sim -\epsilon} \Lambda_x^{-1} R(s))(\Lambda_x [D_x, \Delta_x] \Lambda_x)(\Lambda_x^{-1} R(s))/\sqrt{s},$$

where the integral exists for analogous reason. (Now the second factor contains the second order operator $[D_x, \Delta_x]$ over B so that again the factor in a constant in $\mathcal{L}(\mathfrak{k}_x)$.)

For (1.15) we use the other resolvent integral (1.12), for

$$(1.20) \quad \Lambda^{-1-\epsilon}[s(t), \Lambda] = 1/\pi \int_0^\infty \Lambda^{-1-\epsilon} S(r) M_1 S(r)\, dr/\sqrt{r}, \qquad M_1 = [s(t), \Delta_t].$$

Now, with respect to an orthonormal base of the Hilbert space $\mathfrak{k}_x$ consisting of eigenfunctions of the self-adjoint operator Δ_x, the operator $S(r)$ is diagonalized, with respect to $\mathfrak{k}_x$, in the tensor product decomposition of $\mathcal{H}$. Its diagonal components are

$$(1.21) \qquad S_j(r) = (r + \langle \lambda_j \rangle^2 - \Delta_t)^{-1},$$

with the eigenvalues λ_j^2 of the positive self-adjoint operator Λ_x on B. In analogy to (1.18) we conclude that, with $\Lambda_t = (1 - \Delta_t)^{-1/2}$,

$$(1.22) \qquad \|\Lambda_t^{-\eta} S_j(r)\| \leqslant (1 + r + \lambda_j^2)^{\eta/2 - 1}, \qquad \|\Lambda_t^{j-\eta} S_j(r)\| \leqslant (1 + r)^{\eta/2 - 1},$$

for $j = 1, 2, \ldots$, and $\Lambda_t = (1 - \Delta_t)^{-1/2}$, $\Lambda_t^j = (1 + \Lambda_j^2 + \Delta_t)^{-1/2}$.

Notice that the entire relation (1.20) is «x-diagonalized», i.e., decomposes into a set of countably many relations, involving Λ_t^j and S_j instead of Λ, $j = 1, 2, \ldots$. Again one may write the integrand $F_j(r)$ as a product of three factors:

$$(1.23) \qquad F_j(r) = (\Lambda_t^{j-1-\epsilon} S_j(r))(M_1 \Lambda_t)(\Lambda_t^{-1} S_j(r)/\sqrt{r}),$$

where the second term is a constant in $\mathcal{L}(\mathcal{H})$. For the first and third term we get estimates $O((1 + r)^{(\epsilon - 1)/2})$ and $O((1 + r + \lambda_j^2)^{-1/2}/\sqrt{r}) = O(\langle\lambda_j\rangle^{-\delta}(1 + r)^{(\delta - 1)/2} r^{-1/2})$, for any $0 < \delta < 1$.
Thus

$$(1.24) \qquad \|F_j(s)\| = O(\langle\lambda_j\rangle^{-\delta} r^{-1/2}(1 + r)^{(\epsilon + \delta)/2 - 1}).$$

The right hand side is integrable, as long as $\epsilon + \delta < 1$. The integral $\int F_j(r)\,dr = C_j$ is an operator in $\mathcal{K}(\mathfrak{h}_t)$, and we get

$$(1.25) \qquad \|C_j\| = O(\langle\lambda_j\rangle^{-\delta}).$$

Accordingly the operator (1.20) corresponds to a diagonal matrix

$$((C_j \delta_{kl}))_{k, l = 1, 2, \ldots}$$

with diagonal components converging to 0 in norm. This indeed implies that (1.20) belongs to $\mathcal{K}(\mathcal{H})$. (It is limit in $\mathcal{L}(\mathcal{H})$ of the sequence of diagonal matrices T_k obtained by setting all C_j, $j > k$, equal to zero, while the matrices T_k are in $\mathcal{K}(\mathfrak{h}_t) \otimes \mathcal{F}(\mathfrak{h}_x) \subset K(\mathfrak{h}_t) \otimes \mathcal{K}(\mathfrak{h}_x) \subset \mathcal{K}(\mathcal{H})$, with the class $\mathcal{F}(\mathfrak{h}_x)$ of bounded operators of finite rank over $\mathfrak{h}_x$.) This completes the proof of proposition 1.3.

Corollary 1.4. *The C*-algebra $\mathcal{G}^\wedge$ and its F_t^{-1}-conjugate $\mathcal{G}$ are subalgebras of $\mathcal{K}_x = \mathcal{L}_t \hat{\otimes} \mathfrak{h}_x$.*

Proof. It is sufficient to show that $CO(\mathbb{R}^{n'}, \mathfrak{h}_x) \subset \mathcal{K}_x$. For any orthonormal basis $\{\phi_j : j = 1, 2, \ldots\}$, and the orthogonal projection P_n onto the span of $\{\phi_1, \ldots, \phi_n\}$ we get uniform convergence $P_n C(\tau) P_n \to C(\tau)$, $\tau \in \mathbb{R}$, for every $C(\tau) \in CO(\mathbb{R}^{n'}, \mathcal{K}(\mathfrak{h}_x))$. But the operator $P_n C(\tau) P_n$ is a finite sum of operators $(\phi_j\rangle\langle\phi_l)c_{jl}(\tau)$, $c_{jl}(\tau) \in CO(\mathbb{R})$. Thus $P_n C(\tau) P_n \in \mathfrak{h}_x$, and the limit $C(\tau)$ as well, q.e.d.

Proposition 1.5. *The commutator ideal $\mathcal{GO}$ of the C^*-subalgebra of $\mathcal{L}(\mathcal{H})$ generated by the operators of the form G_1, G_3, G_5 contains the algebra $CO(\mathbb{R}^{n'}, \mathfrak{k}_x)$.*

PROOF. For a moment consider only the C^*-algebra $\mathcal{B}$ generated by G_1, G_3, and G_5. By virtue of the isometric isomorphism mentionned initially in this section, the generators of $\mathcal{B}$ belong to the function algebra $CO(\mathbb{R}^{n'}, \mathcal{L}_x)$. Hence the algebra $\mathcal{B}$ may be interpreted as a subalgebra of $CO(\mathbb{R}^{n'}, \mathcal{L}_x)$. For a fixed $\tau = \tau_0 \in \mathbb{R}^{n'}$ the values $\mathcal{B}_{\tau_0} = \{A(\tau_0): A(\tau) \in \mathcal{B}\}$ form a *-subalgebra of $\mathcal{L}_x$. It is clear that $\mathcal{B}_{\tau_0}$ is a *-subalgebra of the C^*-subalgebra $\mathcal{C}_{\tau_0}$ of $\mathcal{L}_x$ generated by the values of the functions G_j, $j = 1, 3, 5$ at τ_0, i.e., by the operators

$$(1.26) \qquad a_x, \qquad \Lambda^{\sim}(\tau_0) = (-\Delta_x + \langle \tau_0 \rangle^2)^{-1/2}, \qquad D_x \Lambda^{\sim}(\tau_0),$$

where a_x and D_x run through all the functions (folpdes) over B. Moreover, $\mathcal{C}_{\tau_0}$ evidently is the closure of $\mathcal{B}_{\tau_0}$. Also $\mathcal{C}_{\tau_0}$ is just the minimal comparison algebra, in the sense of [C2], V, 1 generated by the triple $\{B, -\Delta_x + \langle \tau_0 \rangle^2, ds\}$, on the compact manifold B. By [C2], V, lemma 1.1, it follows that $\mathcal{C}_{\tau_0}$ and even its commutator ideal contain all of $\mathfrak{k}_x$. But commutators in $\mathcal{C}_{\tau_0}$ are compact, since B is compact. Therefore the commutator ideal of $\mathcal{C}_{\tau_0}$ equals $\mathfrak{k}_x$. Since that commutator ideal is the closure of the commutator ideal $\mathcal{E}^0_{\tau_0}$ of the finitely generated algebra, we must have $\mathcal{E}^0_{\tau_0}$ dense in $\mathfrak{k}_x$. On the other hand $\mathcal{E}^0_{\tau_0}$ clearly is contained in the commutator ideal of the algebra $\mathcal{B}_{\tau_0}$, and even in the localization $\mathcal{GO}_{\tau_0}$ at τ_0 of the commutator ideal $\mathcal{GO}$. Thus we conclude that the algebra $\mathcal{GO}_\tau$ of «values» of $\mathcal{GO}$ at τ is dense in $\mathfrak{k}_x$, for all $\tau \in \mathbb{R}^{n'}$.

We also find that the algebra $\mathcal{B}$ contains $\Lambda^{\sim}(\tau)$, hence also contains every $f(\tau, \Lambda_x)$, for a general $f \in CO(\mathbb{R}^{n'} \times [0, 1])$, by the spectral theorem and the Stone-Weierstrass theorem. Hence $\mathcal{B}$ contains all functions $\psi(\tau)E_\lambda$, with $\psi \in CO(\mathbb{R}^{n'})$ and the projection operators E_λ of the spectral family of Λ_x^{-1}. Note that E_λ are of finite rank, and that $E^N \to 1$, strongly, as $N \to \infty$. Since $\mathcal{GO}$ is an ideal of $\mathcal{B}$, it follows that $\mathcal{GO}$ contains

$$(1.27) \qquad \mathcal{GO}_N = \{\psi(\tau)E_N A(\tau) E_N: \ A(\tau) \in \mathcal{GO}, \psi(\tau) \in CO(\mathbb{R}^n)\}.$$

But $\mathcal{GO}_N$ is a self-adjoint algebra of (finite) $j_N \times j_N$-matrix-valued functions, separating points in the following sense. For every $\tau_1, \tau_2 \in \mathbb{R}^{n'}$, $\epsilon > 0$, and $j_N \times j_N$-matrix P there exists $K(\tau) \in \mathcal{GO}_N$ such that $K(\tau_2) = 0$ and $|K(\tau_1) - P| < \epsilon$. (This follows from the above.) By the matrix version of the Stone Weierstrass theorem this implies that $\mathcal{GO}_N = CO(\mathbb{R}^{n'}, \mathcal{L}(\mathbb{C}^{j_N}))$. Since this holds for all N we find that $\mathcal{GO}$ contains all these matrix algebras. But for a general $A(\tau) \in CO(\mathbb{R}^{n'}, \mathfrak{k}_x)$ we get $A_N(\tau) = E_N A(\tau) E_N \in CO(\mathbb{R}^{n'}, \mathcal{L}(\mathbb{C}^{j_N}))$. Also $A_N(\tau) - A(\tau) \to 0$, in $CO(\mathbb{R}^{n'}, \mathfrak{k}_x)$ (with the norm (1.4)), since $A(\tau)$ is compact and E_N converges strongly to 1. since $\mathcal{GO}$ is closed we conclude that $A(\tau) \in \mathcal{GO}$, so that indeed $CO(\mathbb{R}, \mathfrak{k}_x) \subset \mathcal{GO}$, q.e.d.

2. Commutator Ideal and Symbol Spaces of the Cylinder Algebra

Returning to our task of describing the commutator ideal $\mathcal{E}$ of the cylinder algebra $\mathcal{C}$ we conclude from prop. 1.5 and [C2], V, lemma 1.1 that $\mathcal{E}$ contains the C^*-algebras $\mathcal{GO}^\vee = F_t \mathcal{GO} F_t^{-1}$ and $\mathcal{K}(\mathcal{H})$. It is convenient again to work with the F_t-conjugated ideal $\mathcal{E}^\wedge = F_t^{-1} \mathcal{E} F_t$, containing the sum $\mathcal{GO} + \mathcal{K}(\mathcal{H}) = \mathcal{SO}$ (which in turn contains all the commutators of the generators $G_j, j = 1, \ldots, 5$).

Notice that $\mathcal{SO}$ is invariant under left and right multiplication with the (functions in $CB(\mathbb{R}, \mathfrak{k}_x)$) G_1, G_3, G_4, G_5, but not under multiplication with G_2. Accordingly $\mathcal{E}^\wedge$ must be properly larger than $\mathcal{SO}$.

Specifically $G_2 = s_j(D_t)$ is a singular convolution operator with Cauchy-type singular integral and kernel $\mu_j = s_j^\vee$, so that a product $K(\tau)G_2$, for $K(\tau) \in CO(\mathbb{R}^{n'}, \mathfrak{k}_x)$, appears as an infinite matrix of singular integral operators on $\mathbb{R}^{n'}$, if we introduce some orthonormal base of $\mathfrak{k}_x$.

Theorem 2.1. *Let $\mathcal{SQ}$ be the C^*-algebra of singular integral operators over $\mathfrak{k}_t$ generated by the multiplications in $CO(\mathbb{R}^{n'})$, and the operators $a(M)s_j(D_t)$, $a \in CO(\mathbb{R}^{n'})$, $j = 1, \ldots, n'$. Then the ideal $\mathcal{E}^\wedge$ coincides with the topological tensor product $\mathcal{SQ}_\infty = \mathcal{SQ} \hat{\otimes} \mathfrak{k}_x$.*

PROOF. Notice that $\mathcal{SQ}$ coincides with the minimal comparison algebra of the triple $\{\mathbb{R}^{n'}, dt, 1 - \Delta_t\}$ (cf. [C2], VI). Thus it contains the compact ideal $\mathfrak{k}_t$ of $\mathcal{L}(\mathfrak{k}_t)$, and $\mathcal{SQ}_\infty$ contains $\mathfrak{k}_t \hat{\otimes} \mathfrak{k}_x = \mathcal{K}(\mathcal{H})$. Therefore it is trivial from the above that $\mathcal{SQ}_\infty$ contains all the commutators $[G_j, G_l], j, l = 1, \ldots, 5$ and that it is a closed *-ideal of $\mathcal{C}$. Hence we have $\mathcal{E}^\wedge \subset \mathcal{SQ}_\infty$. To show equality we introduce a fixed orthonormal base $\phi_1, \phi_2, \ldots$ of the space $\mathfrak{k}_x$ and first consider $K(\tau) \in \mathcal{GO}$ such that $K(\tau)$ takes span $\{\phi_1, \ldots, \phi_N\}$ into itself and its orthogonal complement to 0, for all τ. Thus the infinite matrix vanishes outside its first N rows and columns. Let CO_N denote the subalgebra of $CO(\mathbb{R}^{n'}, \mathfrak{k}_x)$ of all such finite matrices, for a given fixed N. Clearly CO_N is isometrically isomorphic to $CO(\mathbb{R}^{n'}, \mathcal{L}(\mathbb{C}^N))$.

Now we observe that $K(\tau)$ and $L(\tau)s(D_t)$, with $K, L \in CO_N$, belong to $\mathcal{E}^\wedge$, and generate the algebra $\mathcal{SQ}_N = \mathcal{SQ} \otimes \mathcal{L}(\mathbb{C}^N)$, for each $N = 1, 2, \ldots$. Also we again find that $\mathcal{SQ}_\infty$ is the norm closure of $\bigcup \mathcal{SQ}_N$. Therefore indeed $\mathcal{SQ}_\infty \subset \mathcal{E}^\wedge$, q.e.d.

We now come to our main task: The description of the symbol chain of the cylinder algebra $\mathcal{C}$. First let us look at the ideal quotient $\mathcal{E}/\mathcal{K}(\mathcal{H})$. In that respect we observe that the algebra $\mathcal{SQ}$ is a subalgebra of the algebra $\mathcal{SJ}$ of singular integral operators on $\mathbb{R}^{n'}$ generated by $S_j = s_j(D_t) = (\mu_j *)$, $j = 1, \ldots, n'$, and the multiplications with functions in $C(\mathbb{B}^{n'})$, with the «ball compactification» $\mathbb{B}^{n'}$ of $\mathbb{R}^{n'}$, having one infinite point in each direction $\infty \cdot t_0$, $|t_0| = 1$ (cf. [C1], IV, 1, problems). The special comparison algebra $\mathcal{SJ} \subset \mathcal{L}(\mathfrak{K}_t)$ has symbol space

$$\mathbb{M}_J = \{(t, \tau) \in C(\mathbb{B}^{n'} \times \mathbb{B}^{n'}) : |t| + |\tau| = \infty\}, \tag{2.1}$$

(cf. ex. (A) of [C2], V, 4, or [CHe1]). The subalgebra $\mathcal{SQ}$ of $\mathcal{SJ}$ consists precisely of all operators in $\mathcal{SJ}$ with symbol vanishing at $|\tau| = \infty$, as follows from the Stone-Weierstrass theorem, looking at the symbols of the generators. (Note that the generators are written as multiplications by functions of the variable τ and convolutions by functions of τ as well, since we consider the F_t-conjugated ideal $\mathcal{E}^\wedge$. Accordingly we have t and τ reversed, compared to the normal notation for space and momentum coordinates.) It follows that $\mathcal{SQ}/\mathfrak{K}_t$ is isometrically isomorphic to the function algebra $CO(\mathbb{E})$ with the locally compact space compact space

$$\mathbb{E} = \mathbb{M}_Q = \{(t, \tau) \in \mathbb{M}_J : |t| = \infty, |\tau| < \infty\} = \partial\mathbb{B}^{n'} \times \mathbb{R}^{n'}. \tag{2.2}$$

In case of $n' = 1$ this space is a disjoint union of the two sets $\{\infty\} \times \mathbb{R} = \mathbb{E}^-$ and $\{-\infty\} \times \mathbb{R} = \mathbb{E}^+$. Both $\mathbb{E}^\pm$ are copies of $\mathbb{R}$, with the variable τ running over $\mathbb{R}$. In the general case $n' > 1$ the space $\mathbb{E}$ is connected, and is a product of the infinite sphere $\partial B^{n'} = \mathbb{B}^{n'} \backslash \mathbb{R}^{n'}$ with $\mathbb{R}^n$.

Clearly this also is just the wave front space $\mathbb{R}^{n'}$, but with t and τ interchanged. We have proven the following result:

Theorem 2.2. *The quotient algebra $\mathcal{E}/\mathcal{K}(\mathcal{H})$ is isometrically isomorphic to the function algebra $CO(\mathbb{E}, \mathfrak{K}_x)$, so that $\mathcal{E}$ is a C^*-algebra with (compact operator valued) symbol, with symbol space $\mathbb{E}$. In the special case $n' = 1$ we have*

$$CO(\mathbb{E}, \mathfrak{K}_x) = CO(\mathbb{E}^-, \mathfrak{K}_x) \oplus CO(\mathbb{E}^+, \mathfrak{K}_x). \tag{2.3}$$

The symbols of the generators of $\mathcal{E}$ are given as compact operator valued functions of (t, τ), for $t \in \partial\mathbb{B}^{n'}$ (i.e., $t = \infty \cdot t_0$, $t_0 \in \mathbb{R}^{n'}$, $|t_0| = 1$) and $\tau \in \mathbb{R}^{n'}$, as follows:

Let A_j, $j = 1, \ldots, 5$, be the generators (1.2) of $\mathcal{C}$, in the order listed (so that G_j are their Fourier transforms). Then $[A_1, A_2] = 0 = [A_3, A_4]$. The symbols $\gamma_{[A_3, A_5]}$, $\gamma_{[A_4, A_5]}$ are independent of t, as $t \in \partial\mathbb{B}^{n'}$, the value given by the terms of (1.10), respectively, where $\Lambda^\sim(\tau) = \Lambda_x T(\tau)$, while the commutator $[\Lambda^\sim, D_x]$ is obtained from the resolvent integral from (1.19). Similarly $\gamma_{[A_1, A_j]}$, $j = 3, 4, 5$, are independent of t, as $|t| = \infty$, and, for $\tau \in \mathbb{R}^{n'}$, their values are given by (1.11) and the resolvent integral (1.16).

Also, for $j \neq 2$, the symbol of a product $A_j[A_k, A_l]$ or $[A_k, A_l]A_j$ is obtained by multiplying $\gamma_{[A_k,A_l]}$ with the corresponding function G_j of (1.6), from the left or right, respectively. Also, for $j = 2$, the symbols of these products equal the product of $\gamma_{[A_k,A_l]}(\tau)$ with the value of the function $s(t)$ (extended continuously to $\partial\mathbb{B}^{n'}$) at t. More generally, for every function $b \in C(\mathbb{B}^{n'})$ the operator $b(t)[A_k, A_l] \in \mathcal{E}$ has the symbol

$$\gamma = b(t)\gamma_{[A_k,A_l]}(\tau), \qquad k, l \neq 2. \tag{2.4}$$

Next we turn to the discussion of the quotient $\mathcal{C}/\mathcal{E}$, i.e., of the symbol and symbol space of $\mathcal{C}$.

Theorem 2.3. *The C^*-algebra $\mathcal{C}/\mathcal{E}$ is isometrically isomorphic to the algebra $C(\mathbb{M})$ of continuous complex-valued functions over a compact space $\mathbb{M}$, called the symbol space of $\mathcal{C}$. Here $\mathbb{M}$ is (homeomorphic to) the bundle of cospheres with infinite radius of the compactified poly-cylinder $\mathbb{B}^{n'} \times B$ considered as a compact C^∞-manifold with boundary (i.e., the product $B^{n'} \times B$ of the unit ball $B^{n'} = \{t \in \mathbb{R}^{n'}: |t| = 1\}$ in $\mathbb{R}^{n'}$ with the compact manifold B).*

Let $A_1, \ldots, A_5$ be the generators (1.2) again. Then the $\mathcal{C}$-symbols $\sigma_{A_j} = \sigma_{A_j}(t, x, \tau, \xi)$, i.e., the functions in $C(\mathbb{M})$, associated to A_j by the above isomorphism, are given as explicit functions of (t, x, τ, ξ) as follows:

$$\begin{gathered} \sigma_{A_1} = a(x), \qquad \sigma_{A_2} = t/\langle t\rangle, \qquad \sigma_{A_3} = 0. \\ \sigma_{A_4} = i\tau/(\tau^2 + |\xi|^2)^{1/2}, \qquad \sigma_{A_5} = (b^j\xi_j)/(\tau^2 + |\xi|^2)^{1/2}, \end{gathered} \tag{2.5}$$

with

$$|\xi| = (g^{jk}(x)\xi_j\xi_k)^{1/2}, \qquad Dx = b^j(x)\partial_{x^j} + p(x), \tag{2.6}$$

in local coordinates of B, where $t \in \mathbb{B}^{n'}$, $x \in B$, while $(\tau, \xi) \in S^\infty_{t,x}$, the cosphere at (t, x) with infinite radius. (Actually the last two symbols are the limits of the full symbol quotients

$$\tau/(1 + \tau^2 + |\xi|^2)^{1/2}, \qquad (b^j(x)\xi_j + p(x))/(1 + \tau^2 + |\xi|^2)^{1/2}, \tag{2.7}$$

as (τ, ξ) is replaced by $(\rho\tau, \rho\xi)$, and $\rho \to \infty$.)

PROOF. We may just apply the general results of [C2], VII, first verifying the assumptions. Let us shortly summarize these facts. First of all every symbol space of a comparison algebra contains the wave front space $\mathbb{W}$, normally identified with the bundle of unit spheres in the contangent space $T^*\Omega$ of the underlying manifold Ω (cf. [C2], VI, thm. 1.5). The space $\mathbb{W}$ is an open subset of $\mathbb{M}$. It precisely coincides with the set of points (x, t, τ, ξ), with $|t| < \infty$. Essentially this follows whenever $\mathcal{C}$ can be shown to contain all C_0^∞-functions and all operators $D\Lambda$, for a general first order differential expression with C^∞-coefficients and compact support.

In order to study the points at $|t| = \infty$ we require the compactification $\mathbb{P}^*\Omega$ of $T^*\Omega$ induced by the formal symbol quotients (2.7) together with the functions $a(x)$, $a \in \mathcal{A}_B^\#$, and $t/\langle t \rangle$, and $(1 + \tau^2 + |\xi|^2)^{-1/2}$ (in other words, by the formal symbols of the 5 types of generators (1.2)). (That is, $\mathbb{P}^*\Omega$ is defined as the smallest compactification of $T^*\Omega$ onto which all above functions can be continuously extended.) It is readily verified that $\mathbb{P}^*\Omega$ is given as the compactification of $T^*\Omega$ obtained by adding the infinite sphere $\{\infty(\tau, \xi): |\tau|^2 + |\xi|^2 = 1\}$ to each fiber $T^*_{t,x}$ of the cotangent bundle $T^*\Omega^c$, of the compactification $\Omega^c = \mathbb{B}^n \times B$ of Ω. In other words, $\mathbb{P}^*\Omega$ is the disjoint union of all balls $\{(t, x)\} \times \mathbb{B}^n$ of ininite radius, as $(t, x) \in \Omega^c$.

Moreover, Ω^c coincides with the compactification $\mathfrak{M}_{A^\#}$ of Ω defined by the functions $a(x)$, $t/\langle t \rangle \in C^\infty(\Omega)$, introduced in [C2], VI, and the algebra $\mathcal{C}$ satisfies conditions (m_1'), and (m_j), $j = 2, 3, 4, 5, 7$. (We noted before that (m_4) is implied by prop. 1.3. Cdn. (m_5) is trivial: We have $\Lambda \in \mathcal{C}$. Cdn. (m_1'), first used by McOwen, requires that the functions

$$|p(x)|^2 \quad \text{and} \quad g_{jk}(x)\bar{b}^j(x)b^k(x), \tag{2.8}$$

are in $\mathcal{A}^\#$, for every $D_x = b^j(x)\partial_{x^j} + p(x) \in \mathfrak{D}^\#$. This again is trivially true, since $\mathcal{A}^\#$ contains all of $C^\infty(B)$. Furthermore the conditions (m_j) involve some separation conditions which can be satisfied by enlarging $\mathcal{A}^\#$ and $\mathfrak{D}^\#$ in such a way that the generated algebra $\mathcal{C}$ remains the same. Details are left to the reader.

As a consequence we may apply [C2], VII, thm. 3.6. The conclusion is that the symbol space $\mathbb{M}$ of $\mathcal{C}$ is a compact subset of the boundary $\partial\mathbb{P}^*\Omega = \mathbb{P}^*\Omega \backslash T^*\Omega$ of our compactification $\mathbb{P}^*\Omega$, containing the wave front space $\mathbb{W}$, i.e., the bundle of cospheres of infinite radius over Ω.

Since $\mathbb{M}$ is compact, it must contain all points of the infinite cosphere bundle over $\partial\Omega^c$ as well. Thus it remains to be shown that no other point of $\partial\mathbb{P}^*\Omega$ is contained in $\mathbb{M}$. In particular none of the points $|t| = \infty$, $\tau^2 + |\xi|^2 < \infty$ can be contained in $\mathbb{M}$.

This, on the other hand, is a consequence of [C2], VII, thm. 4.2. To indicate at least the idea, we find that $\Lambda \in \mathcal{E}$, for our present algebra, while the formal symbol of Λ is $(1 + \tau^2 + |\xi|^2)^{-1/2}$, i.e., is $\neq 0$ for the latter type of points. Hence such points can not be in $\mathbb{M}$, since the symbol of $\Lambda \in \mathcal{E}$ must vanish, while thm. 3.6 implies that symbol and formal symbol coincide for the points of $\partial P^*\Omega$ which are in $\mathbb{M}$.

This completes the proof of thm. 2.3.

3. Extending the E-Symbol to the Algebra C; a Fredholm Result

Our results on the symbol chain of the algebra $\mathcal{C}$ are not yet practical for an application. Note that the C^*-algebra $\mathcal{C}^\vee = \mathcal{C}/\mathcal{K}(\mathcal{H})$ has the closed two-sided ideal $\mathcal{E}^\vee = \mathcal{E}/\mathcal{K}(\mathcal{H})$, and in thm. 2.2 we proved $\mathcal{E}^\vee$ isometrically isomorphic to the «function algebra» $CO(\mathbb{E}, \mathfrak{k}_x)$. Now it will prove useful to look for an extension of this isometry mapping the left regular representation of $\mathcal{C}^\vee$ on $\mathcal{E}^\vee$ into the function algebra $CB(\mathbb{E}, \mathcal{L}_x) \supset CO(\mathbb{E}, \mathfrak{k}_x)$.

To be more specific, every $A^\vee = \mathfrak{A} + \mathcal{K}(\mathcal{H}) \in \mathcal{C}^\vee$ induces a continuous operator $T_{A^\vee}: \mathcal{E} \to \mathcal{E}$, by left multiplication $T_{A^\vee} E^\vee = A^\vee E^\vee$, $E^\vee \in \mathcal{E}^\vee$. Clearly this defines a continuous algebra homomorphism $\mathcal{C}^\vee \to \mathcal{L}(\mathcal{E}^\vee)$, called the left regular representation of $\mathcal{C}^\vee$ on $\mathcal{E}^\vee$.

We claim that the linear operator $T_{A^\vee}$ of the B-space $\mathcal{E}^\vee$ has a natural isometric representation as a function in $CB(\mathbb{E}, \mathcal{L}_x)$, which coincides with the symbol γ_A whenever $A \in \mathcal{E}$.

Since it is clear that $\mathcal{E}^\vee = \mathcal{E}/\mathcal{K}$ is an ideal of $\mathcal{C}/\mathcal{K} = \mathcal{C}^\vee$ it suffices to observe the action of the generators of $\mathcal{C}^\vee$ on $\mathcal{E}$-symbols $\gamma_E \in CO(\mathbb{E}, \mathfrak{k}_x)$. Here we again look at the Fourier conjugated generators G_j of (1.7). In the order listed $G_1 = a(x)$ is given as multiplication by the L_x-valued function $a(x)$, constant in t, τ; $G_2 = s_1(D_t)$ acts on $CO(\mathbb{E}, k_x)$ as the multiplication by the scalar-valued function $s_1(\infty t_0)$. For G_j, $j = 3, 4, 5$, we find the multiplication by $\Lambda^-(\tau) \in CO(\mathbb{E}, \mathfrak{k}_x)$, $\tau_j \Lambda^-(\tau) \in CB(\mathbb{E}, \mathfrak{k}_x)$ and $D_x \Lambda^-(\tau) \in CB(\mathbb{E}, \mathcal{L}_x)$, respectively. All the above was discussed in thm. 2.2.

Thus we now extend the isometry $\gamma^\vee: \mathcal{E}^\vee \to CO(\mathbb{E}, \mathfrak{k}_x)$ induced by the $\mathcal{E}$-symbol γ to a homomorphism from the algebra $\mathcal{C}^{0\vee}$ generated by finite adjunction of the cosets of (1.2) by assigning to $A_j^\vee$ the function just designated for G_j. Let the extension be called $\gamma^\vee$ again, and let γ still denote the lifting to the corresponding dense subalgebra of $\mathcal{C}$.

Note that γ_{A_j} not only is in $CB(\mathbb{E}, \mathcal{L}_x)$ but even in $CB(\mathbb{E}, \mathcal{C}_x)$, with the C^*-algebra $\mathcal{C}_x \subset \mathcal{L}_x$ of singular integral operators over B, the unique Laplace comparison algebra of B (cf. [C2], VI, 3).

The map γ is well defined: The assignment $A_j \leftrightarrow \gamma_{A_j}$, $j \neq 2$, is directly given by F_t-conjugation and the isometry $CB(\mathbb{R}^{n'}, \mathcal{L}_x) \to \mathcal{L}(\mathcal{H})$ of (1.4). This trivially extends to an isometry $C^\blacktriangle \to CB(\mathbb{R}^{n'}, \mathcal{C}_x)$ of the C^*-algebra $C^\blacktriangle \subset \mathcal{C}$ generated by A_l, $l \neq 2$. Moreover the algebra $CB(\mathbb{R}^{n'}, \mathcal{C}_x)$ does not contain compact operators $\neq 0$: If $A(\tau) \neq 0$ near $\tau = \tau_0$, then $\{\phi(\tau) w(x): \phi \in C_0^\infty(\mathfrak{N}_{\tau_0})\}$, for $A(\tau_0) w \neq 0$ near $\mathfrak{N}_{\tau_0}$, defines an infinite dimensional subspace of $\mathcal{H}$ on which the operator $A(\tau)$ is bounded away from zero. Thus the isometry $\mathcal{C}^\blacktriangle \to CB$ induces a *-isomorphism $\mathcal{C}^{\blacktriangle\vee}$, CB, i.e. also an isometry $\mathcal{C}^{\blacktriangle\vee} \to CB(\mathbb{E}, \mathcal{C}_x)$, the functions in $CB(\mathbb{R}^{n'}, \mathcal{C}_x)$ to be considered as functions over $\mathbb{E}$ constant in

t. The general (F_t-conjugated) element of $\mathcal{C}^{0\vee}$ then is a finite sum $A^\vee = \sum a_j(D_\tau)A_j(\tau) + \mathcal{K}(\mathcal{H})$, $a_j \in CS(\mathbb{R}^{n'})$, $A_j(\tau) \in \mathcal{C}^{\blacktriangle\vee}$, since commutators $[s_1(D_t), G_j]$ were seen compact. Clearly, for an $E^\vee \simeq E(\infty t_0, \tau) \in CO(\mathbb{E}, k_x)$ we get

$$A^\vee E^\vee = \sum a_j(\infty t_0)A_j(\tau)E(\infty t_0, \tau). \tag{3.1}$$

Also, in view of the fact, that $CO(\mathbb{E}, \mathfrak{k}_x)$ contains all operators of the form $\phi(\tau)(v\rangle\langle w)$, $\phi \in CO(\mathbb{E})$, $v, w \in \mathfrak{k}_x$, it is clear that

$$\begin{aligned} \|T_{A^\vee}\| &= \sup\{\|A^\vee E^\vee\| : E^\vee \in \mathcal{E}^\vee \|E^\vee\| = 1\} = \\ &= \sup\left\{\left\|\sum a_j(\infty t_0)A_j(\tau)\right\| : \infty t_0 \in \partial\mathbb{B}^{n'}, \tau \in \mathbb{R}^{n'}\right\} \end{aligned} \tag{3.2}$$

which confirms that the map $\gamma^\vee$ is an isometry $T(\mathcal{C}^{0\vee}) \to CB(\mathbb{E}, \mathcal{C}_x)$. Taking continuous extension we then indeed get the required isometry, called $\gamma^\vee$ again. We also extend the map γ from $\mathcal{E}$ to $\mathcal{C}$, using the chain

$$\begin{array}{c} \mathcal{C} \to \mathcal{C}^\vee \to T(\mathcal{C}^\vee) \xrightarrow{\gamma^\vee} CB(\mathbb{E}, \mathcal{C}_x) \\ \downarrow \\ \mathcal{L}(\mathcal{E}^\vee) \end{array} \tag{3.3}$$

Theorem 3.1. *The extended map γ defines a continuous *-homomorphism $\mathcal{C} \to CB(\mathbb{E}, \mathcal{C}_x)$, with the unique Laplace comparison algebra $\mathcal{C}_x$ of the compact space B. All functions $A(\infty t_0, \tau) \in \operatorname{im}\gamma$ have their $\mathcal{C}_x$-symbol independent of τ. Moreover, there exists a continuous *-homomorphism $\iota: \operatorname{im}\gamma \to C(\mathbb{M}\backslash\mathbb{W})$ from $\operatorname{im}\gamma$ onto the space of continuous functions over the infinite points*

$$\begin{aligned} \mathbb{M}\backslash\mathbb{W} = \{(\infty t_0, x), \infty(\tau, \xi)) : t_0, \tau \in \mathbb{R}^{n'}, |t_0| = 1, \\ (\tau, \xi) \in T^*B, \tau^2 + |\xi|^2 = 1\} \end{aligned} \tag{3.4}$$

of the symbol space $\mathbb{M}$ of $\mathcal{C}$ (*with* $|\xi|^2 = g^{jk}\xi_j\xi_k$) *such that*

$$\sigma_A|\mathbb{M}\backslash\mathbb{W}) = \iota(\gamma_A), \quad \text{for all} \quad A \in \mathcal{C}. \tag{3.5}$$

In particular we have

$$\ker\gamma = \mathcal{J}_0, \tag{3.6}$$

where $\mathcal{J}_0$ denotes the minimum comparison algebra of $\Omega = \mathbb{R}^{n'} \times B$, i.e., the C^*-algebra generated by the multipliers of $C_0^\infty(\Omega)$ and the operators $D\Lambda$, with all first order differential expressions D of compact support (*and C^∞-coefficients*).

PROOF. The first statement was already discussed above, and one finds that γ_{A_j}, for the generators (1.2) have the $\mathcal{C}_x$-symbols

$$a(x), s(\infty t_0), 0, 0, b^j(x)/(g^{\nu\mu}(x)\xi_\nu\xi_\mu)^{1/2}, \tag{3.7}$$

in the order listed, with $D_x = b^j(x)D_{x^j} + p(x)$. All these functions are independent of τ, so that the general element in $\operatorname{im}\gamma$ must have the same property.

The minimal comparison algebra $\mathcal{J}_0$ has the generators

$$a(t,x), p(t,x)\Lambda, b(t,x)D_{t^j}\Lambda, D_x(t)\Lambda, \tag{3.8}$$

where $a, b, p \in C_0^\infty(\Omega)$, and $D_x(t)$ has compact support as well. All these operators clearly are in $\ker\gamma$, since they may be written in the form $A = \chi A$, with a suitable function $\chi(t) \in C_0^\infty(\mathbb{R}^{n'})$. Thus it follows that $\mathcal{J}_0 \subset \ker\gamma$. It is also known that $A \in \mathcal{C}$ is in $\mathcal{J}_0$ if and only if $\sigma_A = 0$ on $\mathbb{M}\backslash\mathbb{W}$ (cf. [C2] VII, 2). (This follows from the observation that for such A we must have $\sigma_{\chi_j A} = \chi_j(t)\sigma_A \to \sigma_A$ in $C(\mathbb{M})$, as $j \to \infty$, where $\chi_j(t) = \chi(t/j)$, $\chi(t) \in C_0^\infty(\mathbb{R}^{n'})$, $\chi = 1$ near $t = 0$, is a sequence of cut-off functions, equal to 1 on larger and larger subdomains of $\mathbb{M}$. Accordingly there exists a sequence $C_j \in \mathcal{K}(\mathcal{H})$ such that $\|A - (\chi_j A - C_j)\| \to 0$, while we get $\chi_j A + C_j \in \mathcal{J}_0$.)

In order to define the homomorphism ι we first note that $\gamma\mathcal{C}^0 = \gamma^\vee\mathcal{C}^{0\vee}$ is dense in $\operatorname{im}\gamma$, with the above finitely generated algebras $\mathcal{C}^0$ and $\mathcal{C}^{0\vee} = \mathcal{C}^0/\mathcal{K}$. Hence it suffices to define such a homomorphism in $\gamma^\vee C^{0\vee}$. For $A \in \mathcal{C}^0$ we may write γ_A as a finite sum

$$\gamma_A = \sum a_j(\infty t_0)A_j(\tau) + C(\infty t_0, \tau)C \in CO(\mathbb{R}^{n'}, \mathfrak{k}_x), \tag{3.9}$$

where the last term corresponds to an operator $E \in \mathcal{E}$ with $\sigma_E = 0$. On the other hand, the first term is γ-image of $A^\sim = \sum \alpha_j(t)A_j^\sim \in \mathcal{C}$ with $A_j^\sim \in \mathcal{C}^\blacktriangle$. One calculates that

$$\sigma_A(\infty t_0, x, \infty(\tau, \xi)) = \sum a_j(\infty t_0)\sigma_{A_j^\sim}(x, \infty(\tau, \xi)). \tag{3.10}$$

Also the restrictions $\sigma|\mathcal{C}^\blacktriangle$ and $\gamma|\mathcal{C}^\blacktriangle$ are related by a homomorphism. $\sigma|\mathcal{C}^\blacktriangle = \iota \circ (\gamma|\mathcal{C}^\blacktriangle)$, defined as follows: For a function $A_{jl}^\sim(\tau)$ take the F_t^{-1}-conjugation (an isometry), and then the symbol σ, a contraction map $\mathcal{C}^\blacktriangle \to C(\mathbb{M})$, where the $\sigma_{A_j^\sim}$ are independent of t, on

$$\mathbb{M} = \{(t, x, \infty(\tau, \xi)) : t \in \mathbb{B}^{n'}, x \in B, \tau^2 + g^{\nu\mu}(x)\xi_\nu\xi_\mu = 1\}.$$

It follows that (with $\sigma_X = \sigma(X)$, for a moment)

$$\begin{aligned} \|\sigma_A\|_{L^\infty(\mathbb{M}-\mathbb{W})} &= \sup_{|t_0|=1} \sup_{\mathbb{M}} \left|\sigma\left(\sum a_j(\infty t_0)A_j^\sim\right)\right| \\ &\leqslant \sup_{|t_0|=1} \sup\left\{\left\|\sum a_j(\infty t_0)A_j(\tau)\right\| : t \in \mathbb{R}^{n'}\right\}, \end{aligned} \tag{3.11}$$

where the right hand side is the norm of $\gamma_A \in C(\mathbb{E}, \mathcal{C}_x)$. This shows that the map $\iota: \gamma_A \to \sigma_A$ is continuous on $\gamma(\mathcal{C}^0)$. Also this map trivially defines a homomorphism on $\gamma(\mathcal{C}^0)$, and of course we have (3.5) satisfied for $A \in \mathcal{C}^0$.

Taking continuous extension we get the desired continuous *-homomorphism satisfying (3.5) on all of $\mathfrak{C}$.

In particular (3.6) implies that $\sigma_A = 0$ on $\mathbb{M}\backslash\mathbb{W}$ (i.e., $A \in \mathfrak{J}_0$) whenever $A \in \ker\gamma$. Or, $\ker\gamma \subset \mathfrak{J}_0$, completing the proof of thm. 3.1.

Now, regarding the Fredholm property of an operator $A \in \mathfrak{C}$ we have the following result.

Theorem 3.2. *An operator* $A \in \mathfrak{C}$ *is Fredholm if and only if* (i) $\sigma_A(m) \neq 0$ *for all* $m = (t, x, \infty(\tau, \xi)) \in \mathbb{M}$, and (ii) $\gamma_A(e)$ *is invertible in* $\mathfrak{C}_x$ *(i.e. in* $\mathfrak{L}(\mathfrak{h}_x)$*) for every* $e = (\infty t_0, \tau) \in \mathbb{E}$, *and* $\gamma_A(e)^{-1}$ *is uniformly bounded on* $\mathbb{E}$.

PROOF. The conditions are clearly necessary, since a Fredholm inverse B of A (such that $1 - AB$, $1 - BA$ are of finite rank) also gives inverses σ_B and γ_B for σ_A and γ_A respectively, implying (i) and (ii) for A. Vice versa, let A satisfy (i) and (ii). Since $\operatorname{im}\gamma$ is a C^*-algebra it follows then that we have $\gamma_A(\ell)^{-1} = \gamma_P(\ell)$, for $\ell \in \mathbb{E}$ and some $P \in \mathfrak{C}$. Using the homomorphism ι we also get

$$(3.12) \qquad \sigma_{1-AP} = \iota(\gamma_{1-AP}) = 0, \qquad \sigma_{1-PA} = \iota(\gamma_{1-PA}) = 0, \qquad m \in \mathbb{M}\backslash\mathbb{W}.$$

In other words we get $\sigma_P = 1/\sigma_A$ on $\mathbb{M}\backslash\mathbb{W}$, and we then can find $Q \in \mathfrak{J}_0$ such that $\sigma_Q = 1/\sigma_A - \sigma_P$ on all of W, since σ maps onto $CO(\mathbb{W})$. Then let $B = P + Q$. Conclude that $\gamma_B = \gamma_P$, since $\gamma_Q = 0$, by (3.6).

In other words we get

$$(3.13) \qquad \gamma_{1-AB} = \gamma_{1-BA} = 0$$

and

$$(3.14) \qquad \sigma_{1-AB} = \sigma_{1-BA} = 0.$$

Relations (3.13) imply that $1 - AB$ and $1 - BA$ are in $\ker\gamma = \mathfrak{J}_0$. But J_0 is a compact commutator algebra, and the restriction of σ to $\mathfrak{J}_0$ is the symbol of $\mathfrak{J}_0$. Therefore (3.14) implies that $1 - AB$ and $1 - BA$ are compact, so that A has an inverse mod $\mathfrak{K}(\mathfrak{H})$ and must be Fredholm, q.e.d.

Remark. Thm. 3.2 has the trivial consequence that the Fredholm index of an operator $A \in \mathfrak{C}$ must be given by a group homomorphism

$$(3.15) \qquad \operatorname{ind}: (\langle\sigma_A\rangle, \langle\gamma_A\rangle) \to \mathbb{Z},$$

mapping the group of pairs of homotopy classes $(\langle\phi\rangle, \langle\phi\rangle)$ of maps

$$(3.16) \qquad \phi: \mathbb{M} \to \mathbb{C}^*, \qquad \phi: \mathbb{E} \to \mathfrak{C}_x, \qquad \phi \in C(M), \qquad \phi \in \operatorname{im}\gamma,$$

(with group operation induced by pointwise multiplication of functions) into the additive group of integers. Similarly for operators acting on crossections of vector bundles over Ω. For an explicit index formula, as in case of a compact manifold (cf. [ASj], $j = 1, 3, 4, 5$) one will have to obtain the homomorphism ind explicitely by calculating the Fredholm index of specific operators.

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