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On Kolchin's Theorem

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To Alberto Calderón, a wonderful friend and colleague,
and a superb mathematician

A well-known theorem due to Kolchin states that a semi-group, G, of unipotent matrices over a field F can be brought to triangular form over the field F [4, Theorem H]. Recall that a matrix A is called *unipotent* if its only eigenvalue is 1, or, equivalently, if the matrix I-A is nilpotent.

Many years ago I noticed that this result of Kolchin is an immediate consequence of a too-little known result due to Wedderburn [6]. This result of Wedderburn asserts that if B is a finite dimensional algebra, over a field F, which has a basis consisting of nilpotent elements then B itself must be nilpotent, that is, $B^k = (0)$ for some positive integer k.

To see how this result of Wedderburn implies that of Kolchin we proceed as follows. Let F_n be the algebra of $n \times n$ matrices over F and let $S \subset F_n$ be the linear span over F of the elements $I - g$ where $g \in G$. Since, for g and h in G we have that $(I - g)(I - h) = (I - g) + (I - h) - (I - gh)$, S is a subalgebra of F_n. Since S has a basis over F of the form $I - g_i$ for some appropriate g_i in G and each g_i is nilpotent by the hypothesis that $I - g$ is unipotent for every g in G, S has a basis consisting of nilpotent elements. Thus, by the theorem of Wedderburn, S is nilpotent, hence $S^k = (0)$ and $S^{k-1} \neq (0)$ for some integer $k \geqslant 1$. Therefore if $u \neq 0$ is in S^{k-1} then we have that $u(I - g) \in S^{k-1}S = S^k = (0)$ for all $g \in G$. This gives us that there is a common eigenvector for all the elements of G. So every g in G can be brought to the form $\begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}$ and as is easy to see, the g form a semi group of unipotent matrices lying in F_{n-1}. By induction on n we thus obtain Kolchin's theorem.

This theorem of Kolchin is capable of extensions to rings satisfying a polynomial identity.

Let R be a ring with unit satisfying a polynomial identity. Call an element u of R *unipotent* if $1 - u$ is nilpotent, that is, if $(1 - u)^n = 0$ for some n depending on u. Let G be a multiplicative sub-semigroup of R which consists of unipotent elements. Again, if we let S be the additive subgroup of R generated by all the $1 - g$ for all g in G, then, as above, S is a subring of R. Every element of S is a sum of nilpotent elements, and since $S \subset R$ satisfies a polynomial identity, by a theorem of Posner [5], every element of S is nilpotent. Again, since S satisfies a polynomial identity, S must be locally nilpotent [2, Corollary 1 to Lemma 5.4]. So, given any finite set of elements $1 - g_1$, $1 - g_2, \dots, 1 - g_n$ where the g_i are in G, then the subring they generate is nilpotent, hence there is an element $u \neq 0$ in S such that $u(1 - g_i) = 0$ for $1 \leqslant i \leqslant n$. Thus $ug_i = u$ for $1 \leqslant i \leqslant n$.

Let R be a finitely-generated ring with 1 satisfying a polynomial identity, and S a nil subring of R. If $J = J(R)$ is the Jacobson radical of R then, by a beautiful result of Braun [1], J is nilpotent. Let $R' = R/J$ and let S' be the image of S in R'. Since R' is semi-prime (i.e., has no nilpotent ideals), R' can be embedded in C_n, the $n \times n$ matrices over a commutative ring C which has no nilpotent elements [3, Theorem 6.3.2]. We may suppose that C is Noetherian, for if $u_1, \dots, u_n$ generate R then their images generate R'. Viewing each u_i' as a matrix over C then R' is contained in C', the subring of C generated by all the matrix entries of all the u_i' over the integers.

Moreover C' is Noetherian by the Hilbert Basis Theorem. Thus we may suppose that C is Noetherian without nilpotent elements.

Since C is a commutative Noetherian ring without nilpotent elements there exist prime ideals $P_1, \dots, P_m$ of C such that $P_1 \cap P_2 \cdots \cap P_m = (0)$. Thus we can embed R' in $(C/P_1)_n \oplus \cdots \oplus (C/P_m)_n$, the direct sum of $n \times n$ matrices over commutative integral domains. If $F^{(i)}$ is the field of quotients of C/P_i then R' is embedded in $(F^{(1)})_n \oplus \cdots \oplus (F^{(m)})_n$; since this latter ring is artinian and since S' is its subring, by a classical result of Hopkins and Levitzki, S' is nilpotent. Since $(S')^k = (0)$ for some k, we have that $S^k \subset J$, and since $J^r = (0)$ for some r, $S^{kr} = (0)$. Thus S is nilpotent. We have proved the

Theorem. *If R is a finitely generated ring satisfying a polynomial identity and S is a nil subring of R then S is nilpotent.*

As an immediate corollary to this theorem we obtain the

Theorem. *If R is finitely generated ring satisfying a polynomial identity and G is a multiplicative sub-semigroup of R consisting of unipotent elements, then there exists an element $u \neq 0$ in R such that $ug = u$ for every g in G.*

PROOF. If S the additive subgroup of R generated by all the $1 - g$ with g in G then, as we showed above, S is a nil subring of R hence, by the theorem above, S is nilpotent. As we showed earlier, this leads to an element $u \neq 0$ in S such that $ug = u$ for every g in G.

References

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