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Non-Negative Solutions of Generalized Porous Medium Equations

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Dedicated to Professor A. P. Calderón
on the occasion of his 65th birthday

Introduction

The purpose of this paper is to study nonnegative solutions u of the nonlinear evolution equations

$$\frac{\partial u}{\partial t} = \Delta\varphi(u), \qquad x \in \mathbb{R}^n, \qquad 0 < t < T \leqslant +\infty. \tag{1.1}$$

Here the nonlinearity φ is assumed to be continuous, increasing, with $\varphi(0) = 0$.

The equation (1.1) arises in various physical problems, and specializing φ leads to models for nonlinear filtrations, or for the gas flow in a porous medium. For a recent survey on these equations see [9].

The main object of this work is to study the initial value problem for (1.1). Before stating our results, we will recall the known results in the linear case, i.e., the equation of heat conduction

$$\frac{\partial u}{\partial t} = \Delta u \quad \text{in} \quad \mathbb{R}^n \times (0, T]. \tag{1.2}$$

In this case, the Widder theory ([14]) gives a complete description of all the non-negative solutions of (1.2). To each non-negative solution u of (1.2), there corresponds a non-negative Borel measure μ on $\mathbb{R}^n$ such that

$$\lim_{t \downarrow 0} \int_{\mathbb{R}^n} u(x, t)\theta(x)\, dx = \int_{\mathbb{R}^n} \theta\, d\mu, \tag{1.3}$$

for all continuous functions θ in $\mathbb{R}^n$, with compact support. We will call the measure μ the trace of u. Furthermore, the trace μ satisfies the growth condition

$$\int_{\mathbb{R}^n} e^{-C|x|^2/T}\, d\mu(x) < \infty, \tag{1.4}$$

where C is an absolute constant.

The trace μ determines the solution uniquely, i.e., if u, v are two nonnegative solutions of (1.2) in $\mathbb{R}^n \times (0, T]$, with equal traces, then u is identicallly equal to v. Finally, for each non-negative Borel measure μ on $\mathbb{R}^n$, satisfying the growth condition (1.4) there is a non-negative solution u of (1.2) in $\mathbb{R}^n \times (0, T]$ with trace μ. Of course, by the uniqueness result mentioned before, this solution is unique.

Our aim in this work is to find the analogues of the results of the linear theory mentioned above, for a wide class of non-linearities φ.

In the case of pure powers, i.e., $\varphi(u) = u^m$, $m > 1$, (1.1) becomes the well known porous medium equation, which has been studied extensively. In fact, a combination of the results in [1], [3] and [5] gives the complete analogue of the Widder theory for this case. To simplify the description of these results, we will do it for the case when $T = +\infty$. Aronson and Caffarelli ([1]) showed that for every non-negative solution of $\partial u/\partial t = \Delta u^m$, $m > 1$ in $\mathbb{R}^n \times (0, \infty)$, the initial trace μ exists and satisfies the growth estimate

$$\frac{1}{R^n} \int_{|x| \le R} d\mu(x) = o(R^{2/m-1}) \quad \text{as} \quad R \to \infty. \tag{1.5}$$

Furthermore, Bénilan, Crandall and Pierre ([3]) showed that for every measure μ on $\mathbb{R}^n$ satisfying (1.5) there is a non-negative solution u of $\partial u/\partial t = \Delta u^m$, $m > 1$ in $\mathbb{R}^n \times (0, \infty)$. Finally, Dahlberg and Kenig ([5]) established uniqueness, i.e., two non-negative solutions of $\partial u/\partial t = \Delta u^m$ with equal traces are identical.

The main estimate established in Dahlberg and Kenig [5], was that for all non-negative solutions of the porous medium equation in $\mathbb{R}^n \times (0, \infty)$ we have the pointwise growth

$$u(x, t) \leqslant C_t(u)(1 + |x|^2)^{2/m-1}, \tag{1.6}$$

where

$$C_t(u) = O(t^{-\lambda}), \qquad \lambda = \frac{n}{2 + n(m-1)}$$

as $t \downarrow 0$.

These results constitute a very complete generalization of the Widder theory. However, from the point of view of the general equation (1.1), the proofs of the results mentioned above are not very satisfactory. This is due to several facts. The first one is the use of the scaling properties of the porous medium equation, i.e., the fact that if $\partial u/\partial t = \Delta u^m$ in $\mathbb{R}^n \times (0, \infty)$, then $v(x, t) = (\beta/\alpha)^{1/m-1} u(\alpha x, \beta t)$ also solves the same equation for any $\alpha, \beta > 0$. Another important fact used in these proofs is the existence of an explicit formula for the solution whose initial trace is the δ mass at the origin. This is the Barenblatt solution

$$B(x, t) = t^{[-(m-1)n+2]/n} \left[\left(a - \frac{c|x|^2}{t^{2+1/n(m-1)}} \right)_+ \right]^{1/m-1},$$

where a, c depend only on m, n.

In this work we are able to extend the above results for the following class of non-linearities φ. We work with continuous, strictly increasing φ on $0 \leqslant u < +\infty$, that are positive on $0 < u < +\infty$, with $\varphi(0) = 0$. We also impose the growth conditions

$$\text{(1.7)} \qquad \begin{cases} 0 < a \leqslant \dfrac{u\varphi'(u)}{\varphi(u)} \leqslant \dfrac{1}{a}, & 0 < u < \infty \\ 1 + a \leqslant \dfrac{u\varphi'(u)}{\varphi(u)} & \text{for} \quad u \geqslant u_0, \end{cases}$$

for some constant a, $0 < a < 1$, and some $u_0 > 0$. We will denote by Γ_a the cass of φ's that satisfy (1.7) and the normalization conditions $u_0 = 1$, $\varphi(1) = 1$.

For a φ as in (1.7) we say that u is a solution of (1.1) in a region $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ if u is continuous, and non-negative in Ω, and u solves the equation (1.1) in the distribution sense.

We will base our study of solutions of (1.1) for φ's verifying (1.7), in finding the analogues of (1.6) for them. Our key estimate is that, in this case, we have

$$\text{(1.8)} \qquad \frac{\varphi(u)}{1 + u} \leqslant C_t(u)(1 + |x|^2),$$

where $C_t(u) = O(t^{-1+\delta})$ for some $\delta = \delta(\varphi) > 0$. (In fact, this is a corollary of a more precise growth estimate, Theorem 5.5.) Using (1.8) we establish the existence of a trace μ for all non-negative solutions of (1.1) in $\mathbb{R}^n \times (0, \infty)$. We also prove that the solution is uniquely determined by its trace. The main new feature of our approach is that we treat all φ's satisfying (1.7) simultaneously. We thus recover the important scaling properties of the porous medium equation and we dispense with the use of the explicit form of special solutions of the equation.

Finally, in Section 7 we obtain several results about solutions of the initial value problem. First, the trace μ of a positive solution of (1.1) in $\mathbb{R}^n \times (0, \infty)$, with φ verifying (1.7), satisfies the growth estimate

$$\frac{1}{R^n}\int_{|x|<R} d\mu(x) = o(\Lambda(R^2)) \quad \text{as} \quad R \to \infty, \tag{1.9}$$

where Λ is the inverse function of $\varphi(u)/u$, $u \geqslant u_0$. Also, if μ is a non-negative measure on $\mathbb{R}^n$, which satisfies (1.9), there is a unique solution $u \geqslant 0$ of (1.1), with trace μ. We also establish corresponding results for finite strips $\mathbb{R}^n \times (0, T]$, $T < \infty$, together with «blow up» results at T. We refer the reader to the body of the paper for these results.

We would like to conclude these remarks by pointing out that using the estimates in this paper, one can carry out the program of [4] to obtain pointwise limit theorems as $t \downarrow 0$ (Fatou type theorems) for non-negative solutions of (1.1), when φ verifies (1.7).

We have also obtained a complete description of the non-negative solutions of (1.1) in $\Omega \times (0, T]$, which vanish on $\partial\Omega$, where Ω is a bounded Lipschitz domain $\subset \mathbb{R}$, and φ verifies (1.7). We will return to these questions in future publications.

In the rest of this work we will restrict ourselves to φ's in the class Γ_a. The general case of a φ which verifies (1.7) easily follows from this by dilation and division.

2. Preliminary Results

In this section we will establish a preliminary version of the maximum principle. This will be used to give an approximation procedure that justifies our use of the a priori inequalities that we will establish in Section 4. Set $B = \{x \in \mathbb{R}^n: |x - x_0| < r\}$, $\tau_1 < \tau_2$, and let $Q = B \times (\tau_1, \tau_2)$. Denote by $\partial_p Q$ the parabolic boundary of Q, i.e., $\partial_p Q = \bar{Q} \setminus (B \times \{\tau_2\})$.

Let $\varphi \in \Gamma_a$, $0 < a < 1$, and let $g \in C(\partial_p Q)$ be a given non-negative function. Consider the boundary value problem

$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta\varphi(v) & \text{in} \quad Q \\ v = g & \text{on} \quad \partial_p Q \end{cases} \tag{2.1}$$

A function $v(x, t)$ is said to be a weak solution of (2.1) in Q, if $v \in C([\tau_1, \tau_2], L^1(B)) \cap L^\infty(Q)$ and $v \geqslant 0$ satisfies the integral identity

$$
(2.2)\quad \begin{cases} \displaystyle\iint_Q \left[\varphi(v)\Delta\eta + v\frac{\partial \eta}{\partial t}\right] dx\,dt = \int_{\tau_1}^{\tau_2}\int_{\partial B} \varphi(g)\frac{\partial \eta}{\partial N}\, d\sigma\, dt \\ \displaystyle\qquad\qquad + \int_B v(x,\tau_2)\eta(x,\tau_2)\,dx - \int_B g(x,\tau_1)\eta(x,\tau_1)\,dx \end{cases}
$$

for all smooth functions η on $\bar{Q}$, which vanish on $\partial B \times [\tau_1, \tau_2]$. Here $\partial/\partial N$ denotes the exterior normal derivative on ∂B, and σ denotes the surface measure on ∂B.

We have the following comparison principle for weak solutions of (2.1). The corresponding result for the case of the porous medium equation was established in [2].

Lemma 2.3. *Let $g_1, g_2 \in C(\partial_p Q)$, and let v_1, v_2 be weak solutions of* (2.1), *with boundary values g_1 and g_2 respectively. If $0 \leqslant g_1 \leqslant g_2$, then $v_1 \leqslant v_2$ in Q.*

PROOF. The main difference between this case and the porous medium case is in the treatment of the situation when the values of the solutions are close to zero.

Fix $s \in (\tau_1, \tau_2]$, and let $A = (\varphi(v_1) - \varphi(v_2))/(v_1 - v_2)$ whenever $v_1 \neq v_2$ and $A = \varphi'(v_1)$ elsewhere. Putting $b = v_1 - v_2$, we see from (2.2) that whenever η is a non-negative test function, with $\eta \equiv 0$ and $\partial\eta/\partial N \leqslant 0$ on $\partial B \times [\tau_1, s]$, then

$$
(2.4)\qquad \int_B b(x,s)\eta(x,s)\,dx \leqslant \iint_{Q(s)} b\left[\frac{\partial \eta}{\partial t} + A\,\Delta\eta\right] dx\,dt,
$$

where $Q(s) = B \times (\tau_1, s]$.

Let now $h \in C_0^\infty(B)$, $h \geqslant 0$. For $E \in C^\infty(\bar{Q})$, $E > 0$ in $\bar{Q}$, let $\eta = S(E)$ solve the equation

$$
(2.5)\qquad \begin{cases} E\,\Delta\eta + \dfrac{\partial \eta}{\partial t} = 0 \quad \text{in} \quad Q(s) \\ \eta = 0 \quad \text{on} \quad \partial B \times [\tau_1, s] \\ \eta(x,s) = h(x), \qquad x \in B \end{cases}
$$

By the maximum principle, $\eta \geqslant 0$ in $Q(s)$ and $\partial\eta/\partial N \leqslant 0$ on $\partial B \times [\tau_1, s]$, and so (2.4) holds. We also have that

$$
(2.6)\qquad \iint_{Q(s)} E(\Delta\eta)^2\,dx\,dt \leqslant \frac{1}{2}\int_B |\nabla h|^2\,dx,
$$

since

$$\iint_{Q(s)} E(\Delta\eta)^2\,dx\,dt = -\iint_{Q(s)} \Delta\eta \frac{\partial\eta}{\partial t}\,dx\,dt$$

$$= -\int_B h\Delta h\,dx + \int_B \eta(x,\tau_1)\Delta\eta(x,\tau_1) + \iint_{Q(s)} \eta \frac{\partial\Delta\eta}{\partial t}$$

$$\leqslant \int_B |\nabla h|^2 - \iint_{Q(s)} E(\Delta\eta)^2\,dx\,dt.$$

Because of (2.6), an approximation argument shows that if we only assume that $0 < c \leqslant E \leqslant C$ in $Q(s)$, then there is an $\eta = S(E)$ with $\partial\eta/\partial t$, $\Delta\eta \in L^2(Q(s))$ such that (2.6) holds, $\partial\eta/\partial t + E\Delta\eta = 0$ in $Q(s)$ and

$$\int_B b(x,s)h(x)\,dx \leqslant \iint_{Q(s)} b\left[\frac{\partial\eta}{\partial t} + A\Delta\eta\right]dx\,dt.$$

Choose now a sequence $\epsilon_k \downarrow 0$, and put

$$\alpha_k = \frac{|\varphi(v_1) - \varphi(v_2)|}{\epsilon_k + |v_1 - v_2|}, \qquad A_k = \alpha_k + \epsilon_k.$$

Letting $\eta_k = S(A_k)$ we have that

$$I = \int_B h(x)b(x,s)\,dx \leqslant \iint_{Q(s)} b(A - A_k)\Delta\eta_k\,dx\,dt \tag{2.7}$$

From (2.7), it follows that

$$\operatorname{Max}(I,0)^2 \leqslant \left(\iint_{Q(s)} A_k(\Delta\eta_k)^2\,dx\,dt\right)\left(\iint_{Q(s)} \frac{(A - A_k)^2}{A_k} b^2\,dx\,dt\right)$$

$$\leqslant \left(\frac{1}{2}\int_B |\nabla h|^2\right)\left(\iint_{Q(s)} \frac{(A - A_k)^2}{A_k} b^2\,dx\,dt\right).$$

Observe that

$$\frac{1}{2}(A - A_k)^2 \leqslant (A - \alpha_k)^2 + \epsilon_k^2, \quad \text{and} \quad A - \alpha_k = \frac{\epsilon_k A}{|v_1 - v_2| + \epsilon_k}.$$

Hence

$$\frac{(A - \alpha_k)^2}{A_k} b^2 \leqslant \frac{\epsilon_k^2 A^2 b^2}{A_k(|v_1 - v_2| + \epsilon_k)^2} \leqslant \frac{\epsilon_k^2 A|b|}{|v_1 - v_2| + \epsilon_k} \tag{2.8}$$

$$\leqslant \epsilon_k|\varphi(v_1) - \varphi(v_2)| \leqslant C\epsilon_k,$$

since $v_1, v_2 \in L^\infty(Q)$. We also have

$$\frac{\epsilon_k^2 b^2}{A_k} \leqslant \epsilon_k b^2 \leqslant C\epsilon_k, \tag{2.9}$$

and so, letting $k \to \infty$, it follows that

$$\int_B h(x)b(x,s)\,dx \leqslant 0.$$

The lemma now follows since h was an arbitrary non-negative function in $C_0^\infty(B)$.

Corollary 2.10. *Let $g \in C(\partial_p Q)$ be non-negative, and suppose that v is a weak solution of* (2.1). *Assume also that v is continuous in $\bar{Q}$, and that $G_k \in C^\infty(\mathbb{R}^n \times \mathbb{R})$ have been chosen so that $g_k = G_k|_{\partial_p Q}$ are strictly positive, $g \leqslant g_{k+1} \leqslant g_k$, and g_k converges to g uniformly. Let $\varphi_k \in C^\infty([0,\infty))$, $\varphi_k \in \Gamma_a$ and $\varphi_k \to \varphi$ uniformly on compact subsets of $[0,\infty)$. Let v_k solve $\partial v_k/\partial t = \Delta\varphi_k(v_k)$ in Q, with $v_k = g_k$ on $\partial_p Q$. Then, each $v_k \in C^\infty(\bar{Q})$, and v_k converges to v uniformly on compact subsets of Q.*

PROOF. The existence and smoothness of each v_k is well known, since $\inf g_k = m_k > 0$ (see e.g. [7]). From the maximum principle $0 < m_k \leqslant v_k \leqslant M = \max_{\partial_p Q} g_1$. By the results of Sacks ([11]) it follows that for each compact subset $K \subset Q$, the family $\{v_k\}$ is equicontinuous. If w is locally the uniform limit of a subsequence of $\{v_k\}$, then w is a weak solution of (2.1) with boundary values g. Hence, by Lemma 2.3, $w = v$, which yields the corollary.

Corollary 2.11. *Suppose that u is a non-negative continuous solution of $\partial u/\partial t = \Delta\varphi(u)$ in $\mathcal{D}'(\Omega)$, $\Omega \subset \mathbb{R}^n \times \mathbb{R}$, where $\varphi \in \Gamma_a$. Let $\varphi_k \in C^\infty([0,\infty)) \cap \Gamma_a$ and $\varphi_k \to \varphi$ uniformly on compact subsets of $[0,\infty)$. If*

$$Q = \{(x,t) \colon |x - x_0| < r, \tau_1 < t < \tau_2\}$$

and $\bar{Q} \subset \Omega$, there are non-negative solutions $v_k \in C^\infty(Q)$ of $\partial v_k/\partial t = \Delta\varphi_k(v_k)$ in Q, that converge uniformly to v on compact subsets of Q.

PROOF. It is easy to see that $v|_Q$ is a weak solution of (2.1) with boundary values $v|_{\partial_p Q}$ (see e.g. [1], Theorem 3.1). The corollary follows easily then from Corollary 2.10.

The next preliminary result that we need is an extension of M. Pierre's uniqueness theorem ([10]) to our class of non-linearities Γ_a. The proof by Pierre requires a modification to include non-linearities φ whose derivative is unbounded near 0.

Our general uniqueness result will ultimately remove all the extra conditions in the following lemma.

Lemma 2.12. *Let $\varphi \in \Gamma_a$, and suppose that u and v are continuous, non-negative weak solutions of $\partial u/\partial t = \Delta\varphi(u)$ in $\mathbb{R}^n \times (0, \infty)$, $n \geqslant 3$. Suppose that*

$$\sup_{t>0} \int_{\mathbb{R}^n} [u(x,t) + v(x,t)]\, dx < \infty,$$

and that $u, v \in L^\infty(\mathbb{R}^n \times [\tau, \infty))$ for each $\tau > 0$. Suppose also that

$$\lim_{t\to 0} \int_{\mathbb{R}^n} [u(x,t) - v(x,t)]\eta(x)\, dx = 0$$

for all $\eta \in C_0^\infty(\mathbb{R}^n)$. Then, $u \equiv v$ in $\mathbb{R}^n \times (0, \infty)$.

PROOF. As in Lemma 2.3, set $A = (\varphi(u) - \varphi(v))/(u - v)$ when $u \neq v$ and $A = \varphi'(u)$ elsewhere. Pick now ϵ_k in $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$, $\epsilon_k > 0$, and $\sup_x \epsilon_k(x) \underset{k}{\to} 0$, together with $\int_{\mathbb{R}^n} \epsilon_k(x)\, dx \underset{k}{\to} 0$. Define $A_k = \alpha_k + \epsilon_k$, where $\alpha_k = |\varphi(u) - \varphi(v)|/(\epsilon_k + |u - v|)$. Notice that A_k is continuous and strictly positive on each compact subset of $\mathbb{R}^n \times (0, \infty)$. Fix now $0 \leqslant \eta \in C_0^\infty(\mathbb{R}^n)$, and $R > 1$ so large that $\operatorname{supp} \eta \subset B_R = \{x \in \mathbb{R}^n : |x| < R\}$. Let θ_R be the Green's potential of η in B_R, i.e., θ_R solves the equation $\Delta\theta_R = -\eta$ in B_R, $\theta \equiv 0$ on ∂B_R. Fix $T > 0$, and let, for $\alpha > 0$ and smooth in $\mathbb{R}^{n+1}$, $\psi = S(\alpha, R)$ solve the equation

$$\text{(2.13)} \qquad \begin{cases} \dfrac{\partial \psi}{\partial t} + \alpha\Delta\psi = 0 & \text{in } B_R \times (0, T) \\ \psi(x, T) = \theta_R(x), & x \in B_R \\ \psi \equiv 0 & \text{on } \partial B_R \times [0, T]. \end{cases}$$

Letting $h = \Delta\psi$ we observe that

$$\text{(2.14)} \qquad \begin{cases} \dfrac{\partial h}{\partial t} + \Delta(\alpha h) = 0 & \text{in } B_R \times (0, T) \\ h(x, T) = -\eta(x) & x \in B_R \\ h \equiv 0 & \text{on } \partial B_R \times [0, T]. \end{cases}$$

From (2.13) it follows that

$$\text{(2.15)} \qquad \iint_{B_R \times [0,T]} \alpha(\Delta\psi)^2\, dx\, dt = \iint_{B_R \times [0,T]} \alpha h^2\, dx\, dt \leqslant \frac{1}{2} \int_{\mathbb{R}^n} |\nabla\eta|^2.$$

From the maximum principle it follows that

$$0 \leqslant \psi \leqslant \|\theta_R\|_\infty \quad \text{and} \quad h \leqslant 0. \tag{2.16}$$

Hence,

$$\frac{\partial}{\partial t}\int_{B_R} |h(x,t)|\,dx = -\int_{B_R} \frac{\partial h}{\partial t}\,dx = \int_{\partial B_R} \frac{\partial(\alpha h)}{\partial N}\,d\sigma \geqslant 0$$

so

$$\int_{B_R} |h(x,t)|\,dx \leqslant \int_{\mathbb{R}^n} \eta(x)\,dx \quad \text{for} \quad 0 < t < T. \tag{2.17}$$

Since $\partial\psi/\partial t = -\alpha\Delta\psi = -\alpha h$ we have that ψ is increasing in t, and so $0 \leqslant \psi(x,t) \leqslant \theta_R(x)$, $0 < t < T$. Hence

$$\left\|\frac{\partial\psi}{\partial N}\right\|_{L^\infty(B_R)} \leqslant C(\eta)R^{1-n}. \tag{2.18}$$

Our next aim is to solve (2.13) with $\alpha = A$. The problem is that A may be unbounded in $\mathbb{R}^n \times (0,\infty)$ at points where u and v are zero. From (2.18) and (2.2) it follows that if $b = u - v$, then

$$\begin{aligned} E &= \int_{B_R} b(x,T)\theta_R(x)\,dx - \int_{B_R} b(x,\tau)\psi(x,\tau)\,dx \\ &= \iint_{B_R\times(\tau,T)} b(A-\alpha)\Delta\psi\,dx\,dt + H, \end{aligned} \tag{2.19}$$

where

$$0 < \tau < T \quad \text{and} \quad |H| \leqslant C(\eta)R^{1-n}\iint_{\partial B_R\times[\tau,T]} [\varphi(u)+\varphi(v)]\,d\sigma\,dt,$$

where $C(\eta)$ is independent of α. Our assumptions on φ imply that there exists $1 > \delta > 0$ such that $\varphi(u) \leqslant C_\delta(u^\delta + u^{1/\delta})$. Using the fact that

$$u, v \in L^\infty(\mathbb{R}^n \times [\tau, T]),$$

we see that

$$\begin{aligned} R^{1-n}\iint_{\partial B_R\times[\tau,T]} [\varphi(u)+\varphi(v)]\,d\sigma\,dt \leqslant C_{\delta,\tau,T}\Big[R^{1-n}\int_{\partial B_R\times[\tau,T]} (u+v)\,d\sigma\,dt \\ + \Big(R^{1-n}\int_{\partial B_R\times[\tau,T]} (u+v)\,d\sigma\,dt\Big)^\delta\Big]. \end{aligned}$$

Let now $\alpha_\nu \in C^\infty(\mathbb{R}^n \times [0,\infty))$, α_ν be such that $\alpha_\nu \to A_k$ uniformly on each compact subset. Let ψ_ν, h_ν solve (2.13), (2.14) with $\alpha = \alpha_\nu$. For each sequence $\{\tau_j\}$, $T = \tau_0 > \tau_1 > \cdots > \tau_j \downarrow 0$, and each R, there is a sequence of measures $\{\lambda_j^{(k,R)}\}$, where each $\lambda_j^{(k,R)}$ is the weak limit in B_R of $-h_\nu(x,\tau_j)\,dx$, as $\nu \to \infty$.

Let $\psi_j^{(k,R)}$ be the Green's potential in B_R of the positive measure $\lambda_j^{(k,R)}$. By the weak convergence of $-h_\nu(x,\tau_j)\,dx$, we see that for a.e. x,

$$\psi_j^{(k,R)}(x) = \lim_{\nu\to\infty} \psi_\nu(x,\tau_j).$$

Also, since each $\psi_j^{(k,R)}(x)$ is a Green's potential, we have, for every x that

$$\psi_j^{(k,R)}(x) = \lim_{r\to 0} \frac{1}{|B_r(x)|}\int_{B_r(x)} \psi_j^{(k,R)}(y)\,dy,$$

where $B_r(x)$ is the ball of center x and radius r. Hence, the fact that $\psi_\nu(x,t)$ is increasing in t gives

$$0 \leqslant \psi_{j+1}^{(k,R)}(x) \leqslant \psi_j^{(k,R)}(x) \leqslant \theta_R \quad \text{for each } x \text{ in } \quad B_R. \tag{2.20}$$

Also, (2.17) yields

$$\int_{B_R} d\lambda_j^{(k,R)} \leqslant \int_{B_R} \eta(x)\,dx. \tag{2.21}$$

Furthermore,

$$\left|\int_{B_R} b(x,T)\theta_R(x)\,dx - \int_{B_R} b(x,\tau_j)\psi_j^{(k,R)}(x)\,dx\right| \leqslant E_j^{(k)},$$

where, by (2.15)

$$\begin{aligned} E_j^{(k)} \leqslant C_j(\eta)\Big[&\iint_{B_R\times[\tau_j,T]} A_k^{-1}b^2(A-A_k)^2\,dx\,dt \\ &+ R^{1-n}\iint_{\partial B_R\times[\tau_j,T]}(u+v)\,d\sigma\,dt \\ &+ \left(R^{1-n}\iint_{\partial B\times[\tau_j,T]}(u+v)\,d\sigma\,dt\right)^{\delta}\Big]. \end{aligned}$$

From (2.8) it follows that $A_k^{-1}b^2(A-A_k)^2 \leqslant C\epsilon_k$ on $B_R\times[\tau_j,T]$, and so

$$\iint_{B_R\times[\tau_j,T]} A_k^{-1}b^2(A-A_k)^2\,dx\,dt \to 0 \text{ as } k\to\infty.$$

Since

$$\sup_{t>0}\int_{\mathbb{R}^n}(u(x,t)+v(x,t))\,dx < \infty,$$

it follows that

$$R_l^{1-n}\int_{\partial B_{R_l}\times[0,T]}(u+v)\,d\sigma\,dt \to 0 \quad \text{when} \quad R_l\to\infty,$$

for a suitable sequence R_l.

Arguing as in the proof of (2.20) and (2.21), we see that by taking a suitable subsequence $k_s \to \infty$, the weak limits λ_j of $\lambda_j^{(k_s, R_s)}$ exist, and if ψ_j is the Newton potential of λ_j, then

$$\int_{\mathbb{R}^n} b(x, T)\theta(x)\, dx = \int_{\mathbb{R}^n} b(x, \tau_j)\psi_j(x)\, dx, \tag{2.22}$$

$$0 \leqslant \psi_{j+1}(x) \leqslant \psi_j(x) \leqslant \theta(x) = N\eta(x) \quad \text{for all} \quad x \in \mathbb{R}^n \tag{2.23}$$

and

$$\int_{\mathbb{R}^n} d\lambda_j \leqslant \int_{\mathbb{R}^n} \eta(x)\, dx. \tag{2.24}$$

Next, we remark that the non-negative measures λ_j have a weak limit λ_∞, as $j \to \infty$. For, if λ is a weak limit of a subsequence λ_{j_k}, then $\psi_\infty = \lim_{j\to\infty} \psi_j$ exists and $-\lambda = \Delta\psi_\infty$ in the distribution sense. This easily yields the existence of λ_∞.

Because of our assumptions on u, v and φ, Pierre's argument ([10]) gives the existence of the initial trace of u and v. Let their common initial trace be μ. Let w be the Newton potential of u. We observe that

$$\frac{\partial w}{\partial t} = -\varphi(u) \leqslant 0. \tag{2.25}$$

In fact,

$$e(x, \tau, T) = w(x, T) - w(x, \tau) + \int_\tau^T \varphi(u(x, t))\, dt$$

is harmonic in x whenever $0 < \tau < T$. Since $w(x, t)$ is the Newton potential of an L^1 function, it follows that

$$\int_{|x|<R} w(x, t)\, dx \leqslant R^{n-2}, \ \tau < t.$$

Since $\varphi \in \Gamma_a$, it follows that

$$\int_\tau^T \varphi(u(x, t))\, dt \in L^p(\mathbb{R}^n)$$

for some $p \in (1, \infty)$. Therefore,

$$\int_{|x| \leq R} |e(x, \tau, T)|\, dx \leqslant CR^{n-\delta}$$

for some $\delta > 0$. The harmonicity of e now shows that $e \equiv 0$, which establishes (2.25). Since $w(\cdot, \tau_j)$ is increasing as $j \to \infty$, its limit F is superharmonic and $F = N\mu$. Hence, whenever $k \geqslant j$,

$$\int u(x, \tau_j)\psi_j(x)\, dx = \int w(x, \tau_j)\, d\lambda_j \geqslant \int w(x, \tau_k)\, d\lambda_j = \int u(x, \tau_k)\psi_j(x)\, dx.$$

By monotone convergence we then have

$$\liminf_{j\to\infty} \int u(x, \tau_j)\psi_j(x)\, dx \geqslant \int u(x, \tau_k)\psi_\infty(x)\, dx.$$

Also, $\psi_\infty \geqslant N\lambda_\infty$ since $N_\epsilon\lambda_j \to N_\epsilon\lambda_\infty$ as $j \to \infty$, pointwise, where $N_\epsilon\lambda = k^{(\epsilon)} * \lambda$, where $k(x) = c_n|x|^{2-n}$ is the kernel for the Newton potential, and

$$k^{(\epsilon)}(x) = \min(K(x), c_n\epsilon^{2-n}), \qquad 0 < \epsilon < 1.$$

Because $N_\epsilon\lambda_j \leqslant N\lambda_j$ and $N_\epsilon\lambda_\infty \downarrow N\lambda_\infty$ as $\epsilon \to 0$, the claim follows. Hence,

$$\liminf_{j\to\infty} \int u(x, \tau_j)\psi_j(x)\, dx \geqslant \int u(x, \tau_k)N\lambda_\infty(x)\, dx = \int w(x, \tau_k)\, d\lambda_\infty(x).$$

By monotone convergence we obtain

$$\liminf_{j\to\infty} \int u(x, \tau_j)\psi_j(x)\, dx \geqslant \int N(\mu)\, d\lambda_\infty.$$

The conclusion will now follow by an argument due to Pierre ([10]). For $s > 0$, let $v_s(x, t) = v(x, t+s)$, and put $b_s = u - v_s$. Let $W(\cdot, t) = Nv(\cdot, t)$ and notice that $0 \leqslant W_s \leqslant C_s$ in $\mathbb{R}^n \times [0, \infty)$.

$$\begin{aligned}\limsup_{j\to\infty} \int v_s(x, \tau_j)\psi_j(x)\, dx &= \limsup_{j\to\infty} \int w_s(x, \tau_j)\, d\lambda_j \leqslant \limsup_{j\to\infty} \int w_s(x, 0)\, d\lambda_j \\ &= \limsup_{j\to\infty} \int v(x, s)\psi_j(x)\, dx = \int v(x, s)\psi_\infty(x)\, dx \\ &= \int v(x, s)N\lambda_\infty(x)\, dx = \int w_s(x, 0)\, dt_\infty,\end{aligned}$$

since $\psi_\infty = N\lambda_\infty$ a.e. in $\mathbb{R}^n$. Thus,

$$\liminf_{j\to\infty} \int b_s(x, \tau_j)\psi_j(x)\, dx \geqslant \int [N\mu(x) - W_s(x, 0)]\, d\lambda_\infty(x) \geqslant 0$$

by (2.25). Letting $s \downarrow 0$ yields

$$\int b(x, T)\theta(x)\, dx \geqslant 0.$$

Reversing the roles of u and v yields

$$\int (u(x, T) - v(x, T))N\eta(x)\, dx = 0$$

for all non-negative test functions η. This easily shows that $u(x, T) = v(x, T)$ for all $T > 0$, which concludes the proof.

3. A Priori Estimates

We shall first deal with smooth, non-negative solutions of $\partial u/\partial t = \Delta\varphi(u)$, where $\varphi \in \Gamma_a \cap C^\infty([0, \infty))$, $0 < a < 1$. Our aim is to establish pointwise estimates of u in terms of spatial averages. This will be established by a variant of the Moser ([8]) iteration technique, along the lines used by the authors in [5].

Lemma 3.1. *Let u be a smooth, non-negative solution of the equation $\partial u/\partial t = \Delta\varphi(u)$ in $Q^* = \{(x,t) \in \mathbb{R}^{n+1}: |x| < 2, -4 < t < 0\}$, where $\varphi \in \Gamma_a \cap C^\infty([0, \infty))$. Let $Q = \{(x,t) \in Q^*: |x| \leqslant 1, -1 < t < 0\}$. Then,*

$$\|u\|_{L^\infty(Q)} \leqslant C\left\{1 + \iint_{Q^*} u^p\,dx\,dt\right\}^{\theta/p}, \tag{3.2}$$

where C, p, θ are positive constants which depend only on n and a.

To establish Lemma 3.1, we shall first deal with a slightly more general situation. Let $0 < r < \rho$, $0 < T < \tau$, $S = B_r \times (-T, 0]$, $R = B_\rho \times (-\tau, 0]$, where $B_r = \{x \in \mathbb{R}^n: |x| < r\}$.

Lemma 3.3. *Let $v \geqslant 0$ be a smooth non-negative solution of $\partial v/\partial t \leqslant \Delta\varphi(v)$ in R. For $\alpha \geqslant \epsilon$, there is a constant $C_\epsilon = C_\epsilon(a, n)$, such that*

$$\int_{Br} \alpha\phi_\alpha(v(x,0))\,dx + \iint_S |\nabla\varphi^\beta(v)|^2\,dx\,dt$$
$$\leqslant C_\epsilon\{(\rho - r)^{-2} + (\tau - T)^{-1}\} \iint_R [\varphi^{2\beta}(v) + \alpha\phi_\alpha(v)]\,dx\,dt\},$$

where

$$\phi_\alpha(v) = \int_0^v [\varphi(s)]^\alpha\,ds, \quad \text{and} \quad \beta = \frac{\alpha+1}{2}\cdot$$

PROOF. Choose $\psi \in C^\infty(\mathbb{R}^{n+1})$, $0 \leqslant \psi \leqslant 1$, such that $\psi(x,t) = 0$ whenever $|x| \geqslant \rho$ or $t \leqslant -\tau$, and $\psi(x,t) \equiv 1$ for $(x,t) \in S$. This ψ can also be chosen so that

$$\begin{cases} |\nabla\psi(x,t)| \leqslant C(\rho - r)^{-1} \\ \left|\dfrac{\partial\psi}{\partial t}(x,t)\right| \leqslant C(\tau - T)^{-1} \end{cases} \tag{3.4}$$

Whenever $\eta \geqslant 0$, $\eta \in C^\infty(\mathbb{R}^{n+1})$ is supported in $B_\rho \times \mathbb{R}$, we have

$$\iint_R \left[\eta\frac{\partial v}{\partial t} + \nabla\eta \cdot \nabla\varphi(v)\right] \leqslant 0.$$

Choosing $\eta = \psi^2\varphi^\alpha(v)$,

$$\iint_R \varphi^\alpha(v)\frac{\partial v}{\partial t}\psi^2\,dx\,dt + \alpha\iint_R \varphi^{(\alpha-1)}(v)|\nabla\varphi(v)|^2\psi^2\,dx\,dt \leqslant -2\iint_R \varphi^\alpha(v)\nabla\varphi(v)\nabla\psi\psi\,dx\,dt$$

Hence, for all $\delta > 0$, we have

$$\begin{aligned}\iint_R \varphi^\alpha(v)\frac{\partial v}{\partial t}\psi^2\,dx\,dt &+ \alpha\beta^2\iint_R |\nabla\varphi^\beta(v)|^2\psi^2\,dx\,dt\\ &\leqslant -2\beta^{-1}\iint_R \varphi^\beta(v)\nabla\varphi^\beta(v)\cdot\nabla\psi\cdot\psi\,dx\,dt\\ &\leqslant 2\delta\beta^{-1}\iint_R |\nabla\varphi^\beta(v)|^2\psi^2\,dx\,dt + 2(\delta\beta)^{-1}\iint_R \varphi^{2\beta}(v)|\nabla\psi|^2\,dx\,dt.\end{aligned}$$

Choosing $\delta = \alpha/4\beta$ and combining terms, we find that

$$\iint_R \varphi^\alpha(v)\frac{\partial v}{\partial t}\psi^2\,dx\,dt + \frac{\alpha\beta^{-2}}{2}\iint_R |\nabla\varphi^\beta(v)|^2\psi^2\,dx\,dt \leqslant 8\alpha^{-1}\iint_R \varphi^{2\beta}(v)|\nabla\psi|^2\,dx\,dt.$$

Integrating by parts in t now gives

$$\int_{Br} \phi_\alpha(v(x,0))\psi^2(x,0)\,dx + \frac{\alpha\beta^{-2}}{2}\iint_R |\nabla\varphi^\beta(v)|^2\psi^2\,dx\,dt \leqslant 8\alpha^{-1}\iint_R \varphi^{2\beta}(v)|\nabla\psi|^2\,dx\,dt + 2\iint_R \phi_\alpha(v)\psi\left|\frac{\partial\psi}{\partial t}\right|\,dx\,dt.$$

This easily yields the lemma.

Corollary 3.5. *Let v be as in Lemma 3.3, and $\alpha \geqslant \epsilon > 0$. Then*

$$\left[\sup_{t\in(-T,0)} \alpha\int_{B_r} \phi_\alpha(v(x,t))\,dx\right] + \iint_S |\nabla\varphi^\beta(v)|^2\,dx\,dt \leqslant C_\epsilon\{(\rho-r)^{-2} + (\tau-T)^{-1}\}\left\{\iint_R [\varphi^{2\beta}(v) + \alpha\phi_\alpha(v)]\,dx\,dt\right\}$$

PROOF. Pick $t_0 \in [0, T]$ such that

$$\int_{Br} \phi_\alpha(v(x,-t_0))\,dx \geqslant \frac{1}{2}\sup_{t\in(-T,0]}\int \phi_\alpha(v(x,t))\,dx.$$

Now, use Lemma 3.3 with R replaced by $R' = B\rho \times (-\tau, t_0]$, and S replaced by $S' = B_r \times (-T, \tau_0]$.

To complete the proof of Lemma 3.1, we need the following variant of Sobolev's inequality.

Lemma 3.6. *Let $w \geqslant 0$ be smooth in $R = B\rho \times (-\tau, 0)$. Let $q^* = q/(q-1)$, $q = n/2$ for $n \geqslant 3$, and $q = 2$ for $n = 1,2$. Then, for $1 < k < q^*$, we have*

$$\rho^{-n}\tau^{-1} \iint_R w^{2k}\,dx\,dt \leqslant C\Big\{\rho^{-n}\tau^{-1} \iint_R [w^2 + \rho^2|\nabla w|^2]\,dx\,dt\Big\} \cdot \cdot \sup_{t\in(-\tau,0)} \Big(\rho^{-n} \int_{B\rho} w(x,t)^{2(k-1)/q}\,dx\Big)^{1/q},$$

where C depends only on n.

PROOF. By rescaling, it is enough to treat the case $\rho = \tau = 1$. Letting $B = B_1 = \{x \in \mathbb{R}^n : |x| < 1\}$, we see that

$$\begin{aligned}\int_B w^{2k}(x,t)\,dx &= \int_B w^2(x,t) w^{2k-1}(x,t)\,dx \\ &\leqslant \Big(\int_B w^{2q^*}(x,t)\,dx\Big)^{1/q^*}\Big(\int_B w^{(2k-1)q}(x,t)\,dx\Big)^{1/q}.\end{aligned}$$

But, by the standard Sobolev inequality

$$\Big(\int_B w^{2q^*}(x,t)\Big)^{1/q^*} \leqslant C_n\Big[\int_B |\nabla w(x,t)|^2 + w^2(x,t)\,dx\Big].$$

We can now pass to the proof of Lemma 3.1.

PROOF OF LEMMA 3.1. For α large, let $k(\alpha)$ be determined by

$$k = \frac{1 + \dfrac{1}{q}}{1 + \dfrac{1}{\alpha}}.$$

If $\alpha \geqslant 2q$, then $1 < k < 1/(q+1) < q^*$. Assume now that $Q \subset S \subset R \subset Q^*$, where S and R are as in Lemma 3.3. Let $v = \max(u, 1)$, so that Lemma 3.3 applies to it. Combining Corollary 3.5 and Lemma 3.6 with $w = \varphi(v)^\beta$, $\beta = (\alpha+1)/2$ yields

$$\begin{aligned}(3.7)\quad &\iint_B \varphi(v)^{2\beta k}\,dx\,dt \\ &\leqslant C_n F\Big[\iint_R \varphi(v)^{2\beta} + \alpha\phi_\alpha(v)\,dx\,dt\Big] \sup_{t\in[-T,0]} \Big(\iint_{B_r} \varphi(v)^{2\beta(k-1)q}\,dx\Big)^{1/q},\end{aligned}$$

where $F = (\rho - r)^{-2} + (\tau - T)^{-1}$.

We observe that $\varphi(v) \geqslant v$ for $v \geqslant 1$ and $\phi_\alpha(v) = \int_0^v \varphi^\alpha(s)\,dx \leqslant v\varphi^\alpha(v)$ for all $v > 0$. Hence,

$$\phi_\alpha(v) \leqslant \varphi^{\alpha+1}(v), \qquad v \geqslant 1. \tag{3.8}$$

We next observe that we can choose α_0 large enough so that there is a constant M such that, for $\alpha \geqslant \alpha_0$ and $v \geqslant 1$, we have

$$\varphi(v)^{2\beta(k-1)q} \leqslant M\alpha\phi_\alpha(v). \tag{3.9}$$

In fact,

$$\phi_\alpha(1) = \int_0^1 \varphi^\alpha(s)\,ds \geqslant \int_0^1 s^{\alpha/a}\,ds \geqslant \frac{a}{\alpha + a},$$

and for $v \geqslant 1$,

$$v\frac{d}{dv}[M\alpha\phi_\alpha(v) - C\varphi(v)^{2\beta(k-1)q}]$$
$$\geqslant \alpha M v\varphi^\alpha(v) - C(\alpha+1)(k-1)q\varphi(v)^{(\alpha+1)(k-1)q},$$

since $v\varphi'(v) \leqslant C\varphi(v)$. Note that $k > 1$ and that for $\alpha \geqslant 2q$, we have

$$2\beta(k-1)q = (\alpha+1)q\left[\frac{\alpha(q+1) - q(\alpha+1)}{q(\alpha+1)}\right] \leqslant \alpha,$$

and so, for $v \geqslant 1$,

$$v\frac{d}{dv}[M\alpha\phi_\alpha(v) - C\varphi(v)^{2\beta(k-1)q}] \geqslant \varphi^\alpha(v)[\alpha M v - C(\alpha - q)] \geqslant 0$$

for M large. This yields (3.9), which together with (3.7) gives that

$$\begin{aligned}\iint_B \varphi(v)^{2\beta k}\,dx\,dt &\leqslant \alpha CF\Big(\iint_R \varphi(v)^{2\beta}\,dx\,dt\Big) \sup_{t\in[-T,0]}\Big(\int_{B_r} \alpha\phi_\alpha(v)\,dx\Big)^{1/q} \\ &\leqslant C(\alpha F)^{1+1/q}\Big(\iint_R \varphi(v)^{2\beta}\,dx\,dt\Big)^{1+1/q}.\end{aligned} \tag{3.10}$$

Define sequences $\alpha_0, \alpha_1, \ldots$, and $\beta_0, \beta_1, \ldots$ inductively by letting α_0 be as in (3.9),

$$\beta_\nu = \frac{\alpha_{\nu+1}}{2}, \qquad \beta_{\nu+1} = k(\alpha_\nu)\beta_\nu.$$

Put

$$r_\nu = \frac{2(1+\nu)}{1+2\nu}, \qquad R_\nu = \{(x,t)\colon |x| < r_\nu,\ -r_\nu^2 < t < 0\}$$

and

$$M_\nu = \left(\int_{R_\nu} \varphi(v)^{2\beta_\nu}\right)^{1/2\beta_\nu}.$$

Observe that, since $\lim_{\alpha\to\infty} k(\alpha) = 1 + 1/q$ it follows that $E^\nu \leqslant \alpha_\nu \leqslant (E^*)^\nu$ for some numbers $1 < E < E^* < \infty$. (3.10) yields

$$M_{\nu+1} \leqslant e^{\gamma_\nu} M_\nu^{\theta_\nu}$$

where $\theta_\nu = (1 + 1/q)/k(\alpha_\nu)$, and $0 \leqslant \gamma_\nu \leqslant C(\nu + 1)E^{-\nu}$. Note that $1 < \theta_\nu < 1 + CE^{-\nu}$, and so, it easily follows that $\limsup_{\nu\to\infty} M_\nu \leqslant CM_0^{\theta_0}$, which yields Lemma 3.1.

We will also need to estimate the maximum of a solution in terms of spatial averages. In order to do so, we need some preliminary estimates.

Lemma 3.11. *Let u be a smooth, non-negative solution of the equation $\partial u/\partial t = \Delta\varphi(u)$ in $Q^* = \{(x, t): |x| < 2, -4 < t < 0\}$, where $\varphi \in \Gamma_a \cap C^\infty([0, \infty))$. Let $Q = \{(x, t) \in Q^*: |x| < 1, -1 < t < 0\}$. Then*

$$\|u\|_{L^\infty(Q)} \leqslant C\left\{1 + \sup_{-4<t<0} \int_{|x|<2} u(x, t)\, dx\right\}^\sigma, \tag{3.12}$$

where C and σ are positive constants which depend only on n and a.

PROOF. Let $S = B_r \times (-T, 0]$, $R = B_\rho \times (-\tau, 0]$ satisfy $Q \subset S \subset R \subset Q^*$, and set $v = \max(u, 1)$. Combining Corollary 3.5 with Lemma 3.6 gives, for $k \in (1, q^*)$, that

$$\iint_Q \varphi^{2\beta k}(v)\, dx\, dt \leqslant CF\left(\iint_R \varphi^{2\beta}(v)\, dx\, dt\right)\left(\sup_{t\in[-\tau,0]} \int_{B\rho} \varphi(v)^{2\beta(k-1)q}\, dx\right)^{1/q}.$$

Here $\beta = (\alpha + 1)/2$, and α is chosen so that $v^p \leqslant \varphi(v)^{2\beta k}$, for $v \geqslant 1$, where p is as in Lemma 3.1. As before, F equals $(\rho - r)^{-2} + (\tau - T)^{-1}$. We have that $\varphi(v) \leqslant v^M$ for some $M > 1$ and all $v \geqslant 1$, and so, if we now pick $k \in (1, q^*)$ so close to 1 that $2\beta(k - 1)qM \leqslant 1$, then

$$\iint_S \varphi^{2\beta k}(v)\, dx\, dt \leqslant CFI^{1/q} \iint_R \varphi^{2\beta}(v)\, dx\, dt, \tag{3.13}$$

where

$$I = \sup_{t\in[-4,0]} \int_{|x|<2} v(x, t)\, dx.$$

For $1/2 \leqslant r \leqslant 2$, let $L(r) = B_r \times [-r^2, 0]$, and for $s > 0$ let

$$m(r, s) = \iint_{L(r)} w^s\, dx\, dt,$$

where $w = \varphi(v)^{2\beta}$. (3.13) now shows that for $1/2 < r < \rho < 2$

$$m(r, k) \leqslant C(\rho - r)^{-2} I^{1/q} m(\rho, 1). \tag{3.14}$$

For $0 < s < 1$, $m(r, 1) \leqslant m(r, k)^{\theta/k} m(r, s)^{(1-\theta)/s}$, where $\theta = (1-s)k/(k-s) \in (0, 1)$. For $\gamma > 1$, $1/2 \leqslant r \leqslant 1$, (3.14) gives that

$$\log m(2r^\gamma, k) \leqslant \frac{1}{q} \log I + \log C + \log 4(r - r^\gamma)^2 + \frac{\theta}{k} \log m(2r, k) + \frac{1-\theta}{s} \log m(2, s), \tag{3.15}$$

since $m(r, s) \leqslant m(2, s)$, $1/2 < r \leqslant 1$. Integrating from $R = 3/4$ to 1, we see that

$$\gamma^{-1} \int_{R^{1/\gamma}}^{1} \log m(2r, k) \frac{dr}{r} \leqslant C_1 \log I + C_2 \log m(2, s) + C_3 + \frac{\theta}{k} \int_R^1 \log m(2\pi, k) \frac{dr}{r}. \tag{3.16}$$

Choose now s so small that $\varphi(v)^{2\beta s} \leqslant v$ for all $v \geqslant 1$, and γ so close to 1 that $\gamma^{-1} > \theta/k$. We now distinguish two cases. The first one is when $m(3/2, k) \leqslant 1$. In this case, Lemma 3.1 shows that $\|u\|_{L^\infty(Q)} \leqslant C$, and our Lemma follows. If, on the other hand, $m(3/2, k) > 1$, $\log m(2r, k) > 0$ for $r \in [R, 1]$, and so, since $R^{1/\gamma} > R$, it follows from (3.16) that

$$\left(\gamma^{-1} - \frac{\theta}{k}\right) \int_R^1 \log m(2\pi, k) \frac{dr}{r} \leqslant C_1 \log I + C_2 \log m(2, s) + C_3$$

which yields

$$\iint_{L(3/2)} \varphi(v)^{2\beta k}\, dx\, dt \leqslant C I^{\sigma_1} \left(\iint_{Q^*} \varphi(v)^{2\beta s}\, dx\, dt \right)^{\sigma_2},$$

where C, σ_1, σ_2 depend only on n, s and a. By our choice of s, it follows that

$$\iint_{L(3/2)} \varphi(v)^{2\beta k}\, dx\, dt \leqslant C I^{\sigma},$$

which together with Lemma 3.1 concludes the proof. The main result in this section is the next improvement of Lemma 3.11.

Theorem 3.17. *Let u be a smooth, non-negative solution of the equation $\partial u/\partial t = \Delta\varphi(u)$ in $Q^* = \{(x, t): |x| < 2, -4 < t < 0\}$, where $\varphi \in \Gamma_a \cap C^\infty([0, \infty))$. Then, there are positive constants C, γ and σ such that if $Q = \{(x, t): |x| < 1,$*

$-1 < t \leqslant 0\}$, *then*

$$\|u\|_{L^\infty(Q)} \leqslant C\{I^\sigma + I^\gamma\}, \tag{3.18}$$

where

$$I = \sup_{-4<t<0} \int_{|x|<1} u(x, t)\, dx.$$

PROOF. Lemma 3.11 implies that whenever $I \geqslant \epsilon_0 > 0$, then (3.18) holds with a constant C depending on ϵ_0. We also remark that if $0 \leqslant I \leqslant \epsilon_0 \leqslant 1$, then $0 \leqslant u \leqslant C_0 = C_0(r_0)$ in $Q_0 = \{(x, t): |x| < r_0, -r_0^2 < t < 0\}$, for $1 < r_0 < 2$. We now claim that there is an $\epsilon_0 \in (0, 1)$ such that if $I \leqslant \epsilon_0 = \epsilon_0(r_0, r_1)$, then $0 \leqslant u \leqslant 1$ in $Q_1 = \{(x, t): |x| < r_1, -r_1^2 < t < 0\}$, $1 < r_1 < r_0 < 2$. This follows from the results in [11], where it is shown that there is a modulus of continuity ω in Q_0, which depends only on C_0, a, n for all solutions of $\partial u/\partial t = \Delta\varphi(u)$ in Q^*.

Hence, it is enough to show that if $0 \leqslant u \leqslant 1$ in Q^* then

$$\|u\|_{L^\infty(Q)} \leqslant CI^\sigma, \quad \text{for some} \quad \sigma = \sigma(a, n), \tag{3.19}$$

Observe that there is an $\eta = \eta(a) > 0$ such that $v \geqslant \varphi^\eta(v)$ for $0 < v \leqslant 1$. For $\alpha > 0$, let $k = k(\alpha)$ be defined by $k = 1 + [(\alpha + \eta)/q(\alpha + 1)]$, where q is as in Lemma 3.6. Note that $\lim_{\alpha\to\infty} k(\alpha) = 1 + 1/q \in (1, q^*)$, where $q^* = q/(q-1)$. Choose $\alpha_0 > 0$ such that $1 < k < q^*$ for $\alpha > \alpha_0$. From Corollary 3.5 and Lemma 3.6, it follows that, for $\alpha \geqslant \alpha_0$

$$\iint_S \varphi^{2\beta k}(u)\, dx\, dt \leqslant CF \iint_R [\varphi(u)^{2\beta} + \alpha\phi_\alpha(u)]\, dx\, dt \cdot \cdot \sup_{t\in[-T,0]} \left(\int_{|x|<r} \varphi(u)^{2\beta(k-1)q}\, dx\right)^{1/q} \tag{3.20}$$

Here $Q \subset S \subset R \subset Q^*$,

$$S = \{(x, t): |x| < r, -T < t < 0\},$$
$$R = \{(x, t): |x| < \rho, -\tau < t < 0\},$$
$$F = (\rho - r)^{-2} + (\tau - T)^{-1},$$
$$\beta = (\alpha + 1)/2, \quad \text{and}$$
$$\phi_\alpha(u) = \int_0^u \varphi^\alpha(s)\, ds.$$

By our choice of k, it follows that, for $0 \leqslant u \leqslant 1$,

$$u\frac{d}{du}[\alpha M\phi_\alpha(u) - \varphi(u)^{2\beta(k-1)q}] \geqslant \alpha M u\varphi^\alpha(u) - c(\alpha+1)(k-1)q\varphi(u)^{\alpha+\eta} \geqslant 0$$

for $\alpha \geqslant \alpha_0$ if M is large enough. Since $\phi_\alpha(u) \leqslant u\varphi^\alpha(u) \leqslant \varphi^\alpha(u)$ for $0 \leqslant u \leqslant 1$, it follows from Corollary 3.6 and (3.20) that

$$\iint_S \varphi^{2\beta k}(u)\,dx\,dt \leqslant C(\alpha F)^{1+1/q}\left(\iint_R \varphi^\alpha(u)\,dx\,dt\right)^{1+1/q}. \tag{3.21}$$

(Here we used the fact that $\varphi^{2\beta}(u) \leqslant \varphi^\alpha(u)$ for $0 < u \leqslant 1$.) Arguing in a similar manner to the deduction of Lemma 3.1 from (3.10), it now follows from (3.21) that

$$\|u\|_{L^\infty(Q)} \leqslant C\left(\iint_{Q^*} \varphi^\alpha(u)\,dx\,dt\right)^{\sigma/\alpha}$$

for $\alpha \geqslant \alpha_0$ and some $\sigma = \sigma(a, n)$. Choosing now $\alpha > \alpha_0$ such that $\varphi^\alpha(u) \leqslant u$, $0 \leqslant u \leqslant 1$ finishes the proof.

4. A Harnack Inequality

In order to show the existence of an initial trace for non-negative solutions of (1.2), we will establish a suitable Harnack inequality which controls the size of spatial averages in terms of the value of the solution at one point. For the porous medium equation $\partial u/\partial t = \Delta u^m$, $m > 1$ this was established by Aronson and Caffarelli ([1]). Their proof used, among othe things, the explicit formula for the solution of the porous medium equation with initial trace the Dirac measure. Ughi ([12]) extended this proof to a class of non-linearities that were asymptotically the pure power u^m, $m > 1$ both at $u = 0$ and at $u = \infty$.

The idea in our proof is to use scaling properties of solutions of

$$\frac{\partial u}{\partial t} = \Delta\varphi(u),$$

$\varphi \in \Gamma_a$, combined with compactness properties of solutions of these equations. The main point is that solutions of (1.1) with $\gamma \in \Gamma_a$, $0 < a < 1$, with initial trace the Dirac mass at the origin, have $u(0, T_0)$ uniformly bounded from below by a positive constant C_n, for a suitable choice of T_0.

To explain the scaling properties of solutions of (1.1), and our compactness result, we need to introduce some notation. $S(\varphi)$ will denote the class of continuous weak solutions of (1.1) in $\mathbb{R}^n \times (0, \infty)$.

$$P_\varphi(M) = \left\{u \in S(\varphi) \colon \sup_{t<0} \int_{\mathbb{R}^n} u(x, t)\,dx \leqslant M\right\}.$$

We remark that Pierre's proof ([10]) shows that all solutions $u \in P_\varphi(M)$ have an initial trace.

For $\varphi \in \Gamma_a$, $\varphi(u)/u$ is monotonically increasing on $[1, \infty)$, and

$$\lim_{u \to \infty} \frac{\varphi(u)}{u} = +\infty.$$

Let $\Lambda = \Lambda_\varphi$ be the inverse of $\varphi(u)/u$ on $[1, \infty)$. A computation shows that if $\partial u/\partial t = \Delta\varphi(u)$ in $\mathbb{R}^n \times (0, \infty)$ and $\alpha > 0$, $\beta > 0$, then

$$\frac{\partial v}{\partial t} = \Delta\psi(v) \quad \text{in} \quad \mathbb{R}^n \times (0, \infty) \tag{4.1}$$

where

$$v(x, t) = \frac{u(\alpha x, \beta t)}{\gamma}, \quad \text{and} \quad \psi(s) = \beta\varphi(\gamma s)\alpha^{-2}\gamma^{-1}.$$

If $\alpha^2 \geqslant \beta$ then the choice $\gamma = \Lambda(\alpha^2/\beta)$ is possible, and with this choice $\psi(1) = 1$. It is also easy to see that with this choice of γ, $\psi \in \Gamma_a$ whenever $\varphi \in \Gamma_a$.

Lemma 4.2. *Let $\varphi \in \Gamma_a$ and $u \in P_\varphi(M)$. There is a constant $C = C(a, M, n)$ such that $u(x, t) \leqslant C$ for $t \geqslant 1$, and, for $0 < t < 1$,*

$$u(x, t) \leqslant C\Lambda(\rho^2/t), \qquad x \in \mathbb{R}^n,$$

where $\rho = \rho(t) \in [\sqrt{t}, 1]$ is determined by $\rho^n\Lambda(\rho^2/t) = 1$.

PROOF. If $t \geqslant 1$ we have

$$\int_{|\xi - x| < 1} u(\xi, \tau)\, d\xi \leqslant M$$

for $t - 1 < \tau < t$, and so $u(x, t) \leqslant C = C(a, M, n)$ by Corollary 2.11, and Theorem 3.17. If $0 < t < 1$ we put $\gamma = \Lambda(\rho^2/t)$, and $v(\xi, \tau) = \gamma^{-1}u(x + \rho\xi, t\tau)$. By (4.1), $\partial v/\partial t = \Delta\Psi(v)$, where $\Psi \in \Gamma_a$, and for $0 < \tau < \infty$, $v \in P_\Psi(M)$ since

$$\int_{\mathbb{R}^n} v(\xi, \tau)\, d\xi = \gamma^{-1}\rho^{-n} \int_{\mathbb{R}^n} u(y, t\tau)\, dy \leqslant M.$$

By the argument above

$$v(0, 1) = \frac{u(x, t)}{\Lambda(\rho^2/t)} \leqslant C(a, M, n),$$

which completes the proof.

In what follows, the restriction to $n \geqslant 3$ is purely technical, and will be removed later, once Lemma 2.12 has been established for $n = 1,2$.

Corollary 4.3. *Let $n \geqslant 3$, $\varphi \in \Gamma_a$, $u, v \in P_\varphi(M)$ have the same trace. Then $u \equiv v$ in $\mathbb{R}^n \times (0, \infty)$.*

PROOF. If follows immediately from Lemma 4.1 and Lemma 2.12.

We can now give our compactness result.

Lemma 4.4. *Suppose that $\varphi_k \in \Gamma_a$, $0 < a < 1$, and that $u_k \in P_{\varphi_k}(M)$, $k = 1, 2, \ldots$ Suppose also that if μ_k is the trace of u_k, then μ_k converges weakly to a non-negative measure μ, and that φ_k converges uniformly on compact subsets of $[0, \infty)$ to a $\varphi \in \Gamma_a$. Then, there is a unique $u \in P_\varphi(M)$ such that u_k converges to u uniformly on compact subsets of $\mathbb{R}^n \times (0, \infty)$, and u has trace μ.*

PROOF. The uniqueness follows from Corollary 4.3. From Lemma 4.2 it follows that $\{u_k\}$ is locally bounded in $\mathbb{R}^n \times (0, \infty)$. It follows then from Sacks ([11]), that $\{u_k\}$ is locally equicontinuous. To conclude the proof of the lemma, it follows from the uniqueness in Corollary 4.3 that it is enough to show that whenever w is locally the uniform limit of a subsequence of $\{u_k\}$, then $w \in P_\varphi(M)$ and has trace μ.

To this end, first notice that if $\varphi \in \Gamma_a$, then $\Lambda_\varphi(s) \geqslant s^\gamma$ for all $s \geqslant 1$ and some $\gamma = \gamma(a)$. It now $0 < t < 1$, and ρ is as in Lemma 4.1, then

$$1 = \rho^n \Lambda(\rho^2/t) \geqslant \rho^{n+2\gamma} t^{-\gamma},$$

and so $\rho \leqslant t^{\gamma/(n+2\gamma)}$. Thus, for $0 < t < 1$, $u \in P_\varphi(M)$, $\varphi \in \Gamma_a$ we have the estimate

$$\varphi(u) \leqslant 1 + Cut^{-1+\alpha}, \tag{4.5}$$

where the positive constants C and α can be taken to depend only on n and a. From (4.5), it follows that whenever $0 < \tau < T < 1$, $\eta \in C_0^\infty(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} u_k(x, T)\eta(x)\, dx = \int_{\mathbb{R}^n} u_k(x, \tau)\eta(x)\, dx + l, \tag{4.6}$$

where

$$\begin{aligned} |l| &= \left| \int_\tau^T \int_{\mathbb{R}^n} \eta(x) \Delta\varphi_k(u_k(x, s))\, dx\, ds \right| \\ &\leqslant \int_\tau^T \int_{\mathbb{R}^n} |\Delta\eta(x)| \varphi_k(u_k(x, s))\, dx\, ds \\ &\leqslant C(T - \tau) \int_{\mathbb{R}^n} |\Delta\eta(x)|\, dx + CM(T^\alpha - \tau^\alpha) \|\Delta\eta\|_{L^\infty(\mathbb{R}^n)}. \end{aligned}$$

In the limit, we have that

$$\left| \int_{\mathbb{R}^n} w(x, T)\eta(x)\,dx - \int_{\mathbb{R}^n} \eta\,d\mu \right| = O(T^\alpha) \quad \text{as} \quad T \to 0,$$

for any $\eta \in C_0^\infty(\mathbb{R}^n)$, and hence w has trace μ.

As was mentioned before, our proof of the Harnack inequality will be based on the following estimate for the solution of (1.1) with initial trace δ, where δ is the point measure of unit mass at $0 \in \mathbb{R}^n$.

Lemma 4.7. *Let $\varphi \in \Gamma_a$, $0 < a < 1$, and let $n \geqslant 3$. Then, there is a unique $Q = Q_\varphi \in P_\varphi(1)$ with initial trace δ. Furthermore, there is a $T_0 > 0$ that can be taken to depend only on a such that* $\inf_{\varphi \in \Gamma_a} Q(0, T_0) > 0$.

PROOF. It is standard (see e.g. [7]) that if $f \geqslant 0$ is smooth with compact support, and if $\Psi \in C^\infty([0, \infty)) \cup \Gamma_a$, with $\Psi'(x) \geqslant \epsilon > 0$ for $0 < x < 1$, then a smooth solution u of $\partial u/\partial t = \Delta\Psi(u)$ with $u(x, 0) = f(x)$ will exist. Furthermore,

$$\int_{\mathbb{R}^n} u(x, t)\,dx = \int_{\mathbb{R}^n} f(x)\,dx,$$

and hence, the existence part of the lemma follows from Lemma 4.4 and an approximation argument. The uniqueness part follows from Corollary 4.3. Also, from the results of Vázquez ([13]) it follows that Q is radial and decreasing in $|x|$, for each $t > 0$. Let now $\eta \in C_0^\infty(\mathbb{R}^n)$ be non-negative, with

$$\eta(0) = \max_{x \in \mathbb{R}^n} \eta = 1 \quad \text{and} \quad \int_{\mathbb{R}^n} \eta(x)\,dx = 1.$$

From (4.6) it follows that

$$\left| \int Q_\varphi(x, t)\eta(x)\,dx - 1 \right| \leqslant Ct^\alpha,$$

where $\alpha = \alpha(a) > 0$. Hence

$$Q_\varphi(0, T) \geqslant \int \eta(x) Q_\varphi(x, t) \geqslant 1/2 \quad \text{if} \quad 0 < t < T_0,$$

and T_0 is chosen sufficiently small, which concludes the proof of the lemma.

We will also need a slight variant of Lemma 4.7.

Corollary 4.8. *Let $\varphi \in \Gamma_a$, $0 < a < 1$, and let $n \geqslant 3$. Then, for every $M > 0$ there is a unique solution $Q_{\varphi,M} \in P_\varphi(M)$, with trace $M\delta$. Furthermore, there is an $M = M(a, n)$ such that* $\inf\{Q_{\varphi,M}(0, 1) \colon \varphi \in \Gamma_a\} > 0$.

PROOF. Arguing as in the proof of Lemma 4.7, we see that we only need to establish the bound from below. This can be seen by observing that if T_0 is as in Lemma 4.7, $\beta = T_0^{-1}$ and $\alpha^n\Lambda(\alpha^2/\beta) = M$, then

$$v(x, t) = \gamma^{-1}Q_{\varphi, M}(\alpha x, \beta t) \in P_\psi(1),$$

where

$$\gamma = \Lambda_\varphi(\alpha^2/\beta) \quad \text{and} \quad \psi(v) = \varphi(\gamma v)/\varphi(\gamma),$$

belongs to Γ_a. Since $Q_{M,\varphi}(0, 1) = \gamma v(0, T_0)$, and the trace of v is δ, the corollary follows from Lemma 4.7.

We are now in a position to give a preliminary version of our Harnack inequality.

Lemma 4.9. *Suppose that $\varphi \in \Gamma_a$, $0 < a < 1$ and $n \geqslant 3$. Suppose that u is continuous, non-negative in $\mathbb{R}^n \times [0, 1]$, and solves $\partial u/\partial t = \Delta\varphi(u)$ in $\mathbb{R}^n \times (0, 1)$. Let $H_\varphi(x)$ be 1 for $0 < s < 1$ and $s[\varphi(s)/s]^{n/2}$ for $s \geqslant 1$. Then, there is a constant $C = C(a, n) > 0$ such that*

$$\int_{|x| \leq 1} u(x, 0)\, dx \leqslant CH_\varphi(u(0, 1)) \tag{4.10}$$

PROOF. We will first establish (4.10) under the additional asumptions that $u \in S_\varphi$ and

$$\begin{cases} \operatorname{supp} u(\cdot, 0) \subset \{x \in \mathbb{R}^n : |x| < 1\} \\ \sup\limits_{0 < t < \infty} \int u(x, t)\, dx < \infty . \end{cases} \tag{4.11}$$

We will proceed by a contradiction argument. If (4.10) does not hold, then for each $k = 1, 2, \ldots$ there is a $\varphi_k \in \Gamma_a$ and a continuous, non-negative u_k in $\mathbb{R}^n \times [0, \infty)$, $u_k \in S_{\varphi_k}$, u_k verifying (4.11) and such that

$$I_k = \int_{\mathbb{R}^n} u_k(x, 0)\, dx \geqslant kH_{\varphi_k}(u_k(0, 1)).$$

Notice that $I_k \geqslant k$. Let $\Lambda_k = \Lambda_{\varphi_k}$, and define $\alpha_k \geqslant 1$ as the solution of the equation

$$\alpha_k^n \Lambda_k(\alpha_k^2) = I_k/M$$

where M is as in Corollary 4.8. Set $\gamma_k = \Lambda_k(\alpha_k^2)$, and $v_k(x, t) = u_k(\alpha_k x, t)/\gamma_k$. Then, $\partial v_k/\partial t = \Delta\Psi_k(v_k)$ in $\mathbb{R}^n \times (0, \infty)$ where

$$\Psi_k(u) = \frac{\varphi_k(\gamma_k u)}{\gamma_k \alpha_k^2}.$$

Observe that $\Psi_k \in \Gamma_a$ and

$$\begin{cases} \int v_k(x,0)\,dx = I_k \alpha_k^{-n} \gamma_k^{-1} = M \\ \operatorname{supp} v_k(x,0) \subset \{x: \alpha_k |x| < 1\} \end{cases}$$

and that $v_k \in P_{\Psi_k}(M)$. Since $I_k \to \infty$, $\alpha_k \to \infty$, and so $v_k(x, 0)$ converges weakly to $M\delta$. By the compactness of Γ_a, we can extract a subsequence, which we again denote by Ψ_k, such that Ψ_k converges uniformly on compact subsets to a $\Psi \in \Gamma_a$.

By Lemma 4.4, v_k converges uniformly to $w = Q_{\Psi,M}$, on compact subsets of $\mathbb{R}^n \times (0, \infty)$. We notice that $v_k(0, 1) = \gamma_k^{-1} u_k(0, 1)$. Since $w(0, 1) > 0$, by Corollary 4.8, and $\gamma_k \to \infty$, since α_k does, it follows that $u_k(0, 1) \to \infty$ as $k \to \infty$.

Next, observe that if $\varphi \in \Gamma_a$, there are positive constans C_1, C_2, α and β that can be taken to depend only on a such that $A^\alpha \Lambda_\varphi(u) \leqslant \Lambda_\varphi(Au) \leqslant A^\beta \Lambda_\varphi(u)$, whenever $A, u \geqslant 1$. Let $F_k(u) = u^n \Lambda_k(u^2)$, and observe that, because of the remark above, $AF_k(u) \geqslant F_k(A^\sigma u)$ for some $\sigma = \sigma(a)$ and all $A, u \in [1, \infty)$. We have that

$$I_k = MF_k(\alpha_k) \geqslant kH_{\varphi_k}(u_k(0, 1)).$$

Since $u_k(0, 1) \to \infty$,

$$H_{\varphi_k}(u_k(0, 1)) = F_k(\sqrt{\varphi_k(u_k(0, 1))/u_k(0, 1)}),$$

it follows that

$$\alpha_k \geqslant (k/M)^\sigma [\varphi_k(u_k(0, 1))/u_k(0, 1)]^{1/2}$$

and hence

$$\lim_{k \to \infty} u_k(0, 1)\gamma_k^{-1} = 0 = w(0, 1),$$

a contradiction. To remove the assumptions that $u \in S_\varphi$, and (4.11), let $h \geqslant 0$ be continuous on $\mathbb{R}^n$, with $h(x) = 0$ for $|x| \geqslant 1$ and $0 \leqslant h \leqslant 1$. Let w_R solve the equation $\partial w/\partial t = \Delta\varphi(w)$ in $S_R = \{(x, t): |x| < R, 0 < t < R\}$ with $w_R(x, 0) = h(x)u(x, 0)$ for $|x| \leqslant R$, $w_R = 0$ on $\{(x, t): |x| = R, 0 < t < R\}$. (The existence of w_R follows easily by approximating φ with $\varphi_k \in C^\infty([0, \infty)) \cap \Lambda_a$, and with $\varphi_k'(x) \geqslant \epsilon$, using Lemma 2.3 and compactness arguments as in Lemma 4.4). By Lemma 2.3, in $\{|x| < R\} \times (0, 1)$, we have $w_R \leqslant w_\rho \leqslant u$ if $R < \rho$, and $w_R \leqslant w_\rho$ in S_R. Using compactness arguments as in Lemma 4.4, it follows that

$$\lim_{R \uparrow \infty} w_R \in P_\varphi(M) \quad \text{for some} \quad M > 0.$$

Hence the previous argument applies to w and

$$\int w(x,0)\,dx = \int h(x)u(x,0)\,dx \leqslant CH_\varphi(w(0,1)) \leqslant CH_\varphi(u(0,1)).$$

Since h is an arbitrary continuous function with supp $h \subset \{|x| \leqslant 1\}$ and $0 \leqslant h \leqslant 1$, the lemma follows.

We can now give our Harnack inequality. The proof will only be given for $n \geqslant 3$. However, once our uniqueness result, Lemma 2.12, is established for $n = 1, 2$, it will holds (as well as all the other results in this section) for all n.

Theorem 4.12. *Let $\varphi \in \Gamma_a$, $0 < a < 1$, and let $u \geqslant 0$ be continuous in $\bar{D}_T = \mathbb{R}^n \times [0, T]$, $T > 0$. Suppose that u solves $\partial u/\partial t = \Delta\varphi(u)$ in D_T, and $R > T^{1/2}$. Then, there is a constant $C = C(a, n)$ such that*

$$\int_{|x| \leq R} u(x,0)\,dx \leqslant C\{R^n\Lambda(R^2/T) + T^{n/2}H_\varphi(u(0,T))\}.$$

PROOF. Let $\gamma = \Lambda(R^2/T)$, $v(x,t) = u(Rx, Tt)/\gamma$ and $\psi(s) = \varphi(\gamma s)/\varphi(\gamma)$. $\psi \in \Gamma_a$, and by (4.1) $\partial v/\partial t = \Delta\psi(v)$. Now observe that if $v \geqslant 1$, or $u \geqslant \gamma$, then

$$\begin{aligned} H_\psi(v) = v[\psi(v)/v]^{n/2} &= \frac{u}{\gamma}\left[\frac{\varphi(u)}{u}\right]^{n/2}\left[\frac{\varphi(\gamma)}{\gamma}\right]^{-n/2} \\ &= [R^n\Lambda(R^2/T)]^{-1}T^{n/2}H_\varphi(u). \end{aligned}$$

Since $H_\psi(v) = 1$ otherwise, we have in all cases,

$$H_\psi(v) \leqslant 1 + [R^n\Lambda(R^2/T)]^{-1}T^{n/2}H_\varphi(u),$$

and so, by Lemma 4.9,

$$\int_{|x| \leq 1} v(x,0)\,dx = [R^n\Lambda(R^2/T)]^{-1}\int_{|x| < R} u(x,0)\,dx \leqslant CH_\psi(v(0,1)),$$

which yields Theorem 4.12.

5. Pointwise estimates

We will now combine the a priori inequalities of Section 3 with the Harnack inequality of Section 4, to give sharp upper bounds for the size of a solution $u(x,t)$ in $\mathbb{R}^n \times (0, T)$, as $t \to 0$, and as $x \to \infty$.

We need to introduce some notation. For μ a non-negative measure on $\mathbb{R}^n$, $\varphi \in \Gamma_a$, $0 < a < 1$, and $\rho \geqslant 1$ we define

$$|||\mu|||_\rho = \sup_{R \geq \rho} \frac{\mu\{|x| < R\}}{R^n \Lambda_\varphi(R^2)}. \tag{5.1}$$

Notice that for $1 \leqslant \rho < r$, $|||\mu|||_r \leqslant |||\mu|||_\rho$, and so

$$|||\mu|||_\infty = \lim_{\rho \to \infty} |||\mu|||_\rho \tag{5.2}$$

exists. We will set $|||\mu||| = |||\mu|||_1$, and for a non-negative function f on $\mathbb{R}^n$, $|||f||| = |||f dx|||$.

A consequence of our Harnack inequality (Theorem 4.12) is that if $\varphi \in \Gamma_a$ and $u \in S_\varphi(T)$ (the class of non-negative continuous solutions of (1.1) in $\mathbb{R}^n \times (0, T)$), then

$$\sup_{t \in (0, T-\delta)} |||u(\cdot, t)||| \leqslant C(u, T, \delta) < \infty \quad \text{for all} \quad \delta > 0. \tag{5.3}$$

For $s > 0$ and $0 < \tau < 1$, notice that the equation

$$R^n \Lambda_\varphi(R^2/\tau) = (1 + s)^n \Lambda_\varphi((1 + s)^2)$$

has a unique solution $R_\varphi(s, \tau)$.

Lemma 5.4. *For $\varphi \in \Gamma_a$, and $0 < \tau < 1$, $0 < s$, we have the estimate $R_\varphi(s, \tau) \leqslant (1 + s)\tau^\delta$, where $\delta > 0$ can be taken to depend only on n and a.*

PROOF. Let $F(x) = x^n \Lambda(x^2)$. Then, F is increasing on $[1, \infty)$ and if $A \geqslant 1$, $x \geqslant 1$, it is easy to see that $AF(x) \leqslant F(A^\sigma x)$, for some $\sigma = \sigma(a, n) > 0$. Since $F(R_\varphi/\sqrt{\tau}). = \tau^{n/2} F(1 + s)$, the lemma follows.

We are now ready to give our pointwise bound. As in Section 4, the proof will hold for $n \geqslant 3$, but will become valid for $n = 1,2$ once uniqueness is established in this case.

Theorem 5.5. *Let $\varphi \in \Gamma_a$, $u \in S_\varphi(T)$. Let $0 < \tau < \min(T, 1)$ and for $x \in \mathbb{R}^n$ set $R = R_\varphi(|x|, \tau)$. Then, for every $\delta > 0$, there is a constant $C = C(u, \delta, a)$ such that, for $\tau \leqslant t < T - \delta$, we have*

$$\varphi(u(x, t)) \leqslant 1 + Cu(x, t)R^2/\tau.$$

PROOF. The estimate is trivially satisfied if $u(x, t) \leqslant 1$, and so we will assume that $u(x, t) > 1$. Let

$$B = \sup_{0 < s < T - \delta} |||u(\cdot, s)|||.$$

For $(\xi, s) \in \mathbb{R}^n \times (0, T\tau^{-1})$, set $v(\xi, s) = u(x + R\xi, \tau s)/\gamma$, where $\gamma = \Lambda(R^2/\tau)$. Then, by (4.1), $\partial v/\partial s = \Delta\psi(v)$, where $\psi(v) = \varphi(\gamma v)/\varphi(\gamma)$.

For $0 < s < (T - \delta)/\tau$, we have that

$$\int_{|\xi| \leq 1} v(\xi, s)\, d\xi = R^{-n}\gamma^{-1} \int_{|x-\xi| < R} u(\xi, s\tau)\, d\xi.$$

By Lemma 5.4, we have that $R \leqslant C(1 + |x|)$, and so

$$\int_{|\xi| \leq 1} v(\xi, s)\, d\xi \leqslant R^{-n}\gamma^{-1} \int_{|\xi| < |x| + R} u(\xi, s\tau)\, d\xi$$

$$\leqslant CBR^{-n}\gamma^{-1}(1 + |x|)^n \Lambda((1 + |x|)^2) \leqslant C.$$

Hence, by the a priori estimate in Theorem 3.17, together with Corollary 2.10, we see that $v(0, 1) = u(x, t)/\gamma \leqslant C$, and so $u(x, t) \leqslant C\Lambda(R^2/\tau)$. Since Λ is the inverse of $\varphi(u)/u$ on $[1, \infty)$, the conclusion of the Theorem follows.

From Theorem 5.5, it follows that if $u \in S_\varphi(T)$, then

(5.6) $$\varphi(u(0, t)) \leqslant 1 + Cu(0, t)\rho^2/t,$$

where for $0 < t < 1$, ρ is determined by $\rho^n \Lambda(\rho^2/t) = 1$. We next want to show that the growth retriction (5.6) is sharp. (Note that Lemma 5.4 implies that $\rho \leqslant t^\delta$, while $\varphi(s) \geqslant s^{1+\eta}$, $s \geqslant 1$, $\eta > 0$, so that (5.6) is in fact a growth restriction.)

Indeed, let $M \geqslant 1$ be large, and let v be the solution of $\partial v/\partial t = \Delta\varphi(v)$ with initial trace $M\delta$, where δ is the Dirac measure (see Lemma 4.7). By Theorem 4.12,

$$M = \int M\, d\delta \leqslant C\{\rho^n \Lambda(\rho^2/t) + t^{n/2} H_\varphi(v(0, t))\},$$

where C is independent of M, as long as $\rho^2/t \geqslant 1$. Choosing $M = 1 + 2C$, we see that

(5.7) $$t^{-n/2} \leqslant H_\varphi(v(0, t)), \qquad 0 < t < 1.$$

This implies that $v(0, t) > 1$. Also, since $H_\varphi(s) = s[\varphi(s)/s]^{n/2}$ for $s \geqslant 1$,

$$H_\varphi(v(0, t)) = F(\sqrt{\varphi(v(0, t))/v(0, t)}),$$

where $F(s) = s^n \Lambda(s^2)$. Moreover, $t^{-n/2} = \rho^n t^{-n/2} \Lambda(\rho^2/t) = F(\rho/\sqrt{t})$, and so (5.7) can be written as

$$F(\rho/\sqrt{t}) \leqslant F(\sqrt{\varphi(v(0, t))/v(0, t)}) \quad \text{for} \quad 0 < t < 1.$$

Since F is increasing in $[1, \infty)$, this implies that $\varphi(v(0, t))/v(0, t) \geqslant \rho^2/t$, which shows that (5.6) is sharp.

6. Uniqueness

This section will be devoted to establishing the existence of a initial trace for any solution u in $S_\varphi(T)$, to showing that solutions are uniquely determined by their initial trace, and to proving a general compactness result. The proof of the general uniqueness result follows the same strategy as in the case of the porous medium equation (Dahlberg and Kenig [5]). The proof will be based on the preliminary uniqueness result Lemma 2.12, and on the following lemma.

Lemma 6.1. *Let $\varphi \in \Gamma_a$, $0 < a < 1$, and suppose that $u, v \in S_\varphi(T)$ for some $T > 0$. Assume that*

$$\lim_{t \downarrow 0} \int_{|x| \leq R} [v(x,t) - u(x,t)]^+ \, dx = 0 \tag{6.2}$$

for all $R > 0$, where $A^+ = \max(A, 0)$. Then, $v \leqslant u$ in $\mathbb{R}^n \times (0, T)$.

PROOF. We first remark that it is enough to prove the lemma when $n \geqslant 3$. In fact, let $\xi = (x, \eta) \in \mathbb{R}^{n+2}$ and notice that $u^*(\xi, t) = u(x, t)$ solves (1.1) in $\mathbb{R}^{n+2} \times (0, T)$, if u solves $\partial u/\partial t = \Delta\varphi(u)$ in $\mathbb{R}^n \times (0, T)$. If u, v verify (6.2) in $\mathbb{R}^n \times (0, T)$, then u^*, v^* verify (6.1) in $\mathbb{R}^{n+2} \times (0, T)$, and thus the remark is established. We therefore assume $n \geqslant 3$ from now on.

Let $w = v - u$, and let q denote the characteristic function of the set where $u(x, t) < v(x, t)$. Suppose that $0 < \tau < t < T$, and that $\eta \in C_0^\infty(\mathbb{R}^n)$, $\eta \geqslant 0$. We claim that

$$\begin{aligned} \int \eta(x) w^+(x,t)\, dx \leqslant \int w^+(x,\tau)\eta(x)\, dx \\ + \iint_{0<\tau<s<t} \Delta\eta(x) q(x,s)[\varphi(v(x,s)) \\ - \varphi(u(x,s))]^+ \, dx\, ds \end{aligned} \tag{6.3}$$

By Corollary 2.11, it is enough to show (6.3) when u, and v are smooth. Then, by Kato's inequality ([6]),

$$\Delta[\varphi(v) - \varphi(u)]^+ \geqslant q\Delta[\varphi(v) - \varphi(u)]$$

in the distribution sense. Also, since $\partial w^+/\partial t = q\,\partial w/\partial t$ in the distribution sense, it follows that

$$\frac{\partial w^+}{\partial t} + \Delta[\varphi(v) - \varphi(u)]^+ \geqslant 0$$

in the distribution sense. Hence

$$\iint_{0<\tau<s<t} \eta(x)\left\{\Delta[\varphi(v)-\varphi(u)]^{+}+\frac{\partial w^{+}}{\partial t}\right\} dx\,dt \geqslant 0,$$

and so (6.3) follows by integration by parts. Let $A = (\varphi(v) - \varphi(u))/(v - u)$ if $v > u$ and zero elsewhere. We first observe that if $v \geqslant 2$, then for $0 < \delta < T$

$$A \leqslant C(1 + |x|^2)t^{\sigma-1}, \tag{6.4}$$

where $\sigma = \sigma(a, n) > 0$, and $C = C(u, v, \delta)$. To see (6.4), we observe first that if $0 \leqslant u \leqslant v/2$, and $v \geqslant 2$, then $A \leqslant 2\varphi(v)/v \leqslant C(1 + |x|^2)t^{\sigma-1}$ by Theorem 5.5 and Lemma 5.4. If $v \geqslant 2$ and $u > v/2$ then $A = \varphi'(\xi)$ for some $\xi \in (u, v)$. But $\varphi'(\xi) \leqslant C\varphi(\xi)/\xi$, and again Theorem 5.5 and Lemma 5.4 yield (6.4).

For $r \geqslant 1$, let $M_r(t) = |||w^+(\bullet, t)|||_r$, $0 < t < T$. By letting $\tau \downarrow 0$ in (6.3), we see that, from (6.2) we can conclude that, for $t \in (0, T)$,

$$\int w^+(x, t)\eta(x)\,dx \leqslant \iint_{0<s<t} A(x, s)w^+(x, s)|\Delta\eta(x)|\,dx\,ds. \tag{6.5}$$

Choose $R \geqslant r$ such that

$$\int_{|x|\leq R} w^+(x, t)\,dx \geqslant \frac{1}{2}M_r(t)R^n\Lambda(R^2),$$

and pick $0 \leqslant \eta \leqslant 1$, $\eta \in C_0^\infty(\mathbb{R}^n)$, $\eta \equiv 1$ on $\{|x| < R\}$, and $|\Delta\eta| \leqslant CR^{-2}$. Then,

$$\begin{aligned}\frac{1}{2}M_r(t)R^n\Lambda(R^2) &\leqslant \iint_{0<s<t} A(x, s)w^+(x, s)|\Delta\eta(x)|\,dx\,ds\\ &\leqslant \iint_{\{0<s<t,\, v\leq 2\}} \varphi(v)|\Delta\eta(x)|\,dx\,ds\\ &\quad + C\iint_{0<s<t}(1 + |x|^2)s^{\sigma-1}w^+(x, s)|\Delta\eta(x)|\,dx\,ds\end{aligned}$$

Since $1 \leqslant r \leqslant R$ it follows that

$$M_r(t) \leqslant C\left\{t(r^2\Lambda(r^2))^{-1} + \int_0^t s^{\sigma-1}M_r(s)\,ds\right\}, \tag{6.6}$$

for $0 < t < T - \delta$. Here $C = C(u, v, \delta)$ for every $\delta \in (0, T)$. (6.6) implies that if

$$F(t) = \int_0^t s^{\sigma-1}M_r(x)\,ds, \quad \text{then} \quad \frac{dF}{dt} \leqslant C\{\alpha t^\sigma + t^{\sigma-1}F(t)\},$$

where $\alpha = (r^2\Lambda(r^2))^{-1}$, for $0 < t < T - \delta$.

It is then easy to see that

$$F(t) \leqslant M\alpha, \quad \text{for} \quad 0 < t < T - \delta, \tag{6.7}$$

where $M = M(u, v, \delta, T)$. Hence, by (6.6),

$$M_r(t) \leqslant C\alpha, \quad \text{for} \quad 0 < t < T - \delta, \qquad C = C(u, v, \delta, T).$$

In particular,

$$\int_{|x|<r} w^+(x, t) \leqslant Cr^{n-2}, \qquad 0 < t < T - \delta. \tag{6.8}$$

Our next step will be to use (6.8) in (6.5). Since $\varphi \in \Gamma_a$, there is a $\theta = \theta(a) \in (0, 1]$ such that if $0 < u < v < 2$, then $\varphi(v) - \varphi(u) \leqslant C(v - u)^\theta$, $C = C(a)$. If now $R > r$ is chosen so that

$$\frac{1}{2} M_r(t) R^n \Lambda(R^2) \leqslant \int_{|x| \leq R} w^+(x, t)\, dx, \qquad 0 < t < T - \delta,$$

and η is chosen as before, then

$$\int_{|x| \leq R} w^+(x, t)\, dx \leqslant \iint_{0<s<t} \{\varphi(v) - \varphi(u)\}^+ |\Delta \eta|\, dx\, ds$$

$$\leqslant CR^n \Lambda(R^2) \int_0^t s^{\sigma - 1} M_r(s)\, ds + CR^{-2} \int_{0<s<t} \int_{|x|<2R} w^+(x, s)^\theta\, dx\, ds$$

Dividing through by $R^n \Lambda(R^2)$ and using Hölder's inequality and the fact that $R > r$, we see that

$$M_r(t) \leqslant C \int_0^t s^{\sigma-1} M_r(s)\, ds + CR^{-2} \Lambda(R^2)^{-1} R^{-n\theta} \int_0^t \left(\int_{|x|<2R} w^+(x, s)\, dx \right)^\theta ds$$

$$\leqslant C \int_0^t s^{\sigma-1} M_r(s)\, ds + Ct\Lambda(r^2)^{-1} r^{-2-2\theta}$$

Using (6.7) again, we find this time that

$$\int_{|x| \leq r} w^+(x, t)\, dx \leqslant Cr^{n-2-2\theta}, \qquad r > 1, \qquad 0 < t < T - \delta. \tag{6.9}$$

Let

$$h(x, t) = \int_0^t [\varphi(v(x, s)) - \varphi(u(x, s))]^+\, ds.$$

Then

$$r^{-n} \int_{|x| \leq r} h(x, t)\, dx \leqslant C \int_0^t r^{-n} \int_{|x| \leq r} w^+(x, s)^\theta\, dx\, ds$$

$$+ Cr^{2-n} \int_0^t s^{\sigma-1} \int_{|x| \leq r} w^+(x, s)\, dx\, ds$$

$$\leqslant Ctr^{-\theta(2+2\theta)} + Ct^\sigma r^{-2\theta}.$$

However, h is subharmonic in x for every $t \in (0, T)$, since, if $\eta \in C_0^\infty$, $\eta \geqslant 0$,

$$\int \eta(x)\Delta h(x,t)\,dx \geqslant \int_0^t \eta(x)\frac{\partial}{\partial s}w^+(x,s)\,dx\,ds = \int \eta(x)w^+(x,t)\,dx \geqslant 0.$$

Hence, as

$$h(x,t) \leqslant C_n r^{-n}\int_{|y-x|<r} h(y,t)\,dy \underset{r\to\infty}{\longrightarrow} 0,$$

h is identically 0, which easily yields the lemma.

We will now show that all solutions of (1.1), with $\varphi \in \Gamma_a$ have an initial trace.

Theorem 6.10. *Let $\varphi \in \Gamma_a$, $0 < a < 1$, and let $u \in S_\varphi(T)$ for some $T > 0$. Then, there is a non-negative measure μ on $\mathbb{R}^n$ such that*

$$\mu\{|x| < R\} = O(R^n\Lambda(R^2)) \quad \text{as} \quad R \to \infty, \tag{6.11}$$

and

$$\lim_{t\downarrow 0}\int_{\mathbb{R}^n} u(x,t)\eta(x)\,dx = \int_{\mathbb{R}^n}\eta\,d\mu,$$

for all $\eta \in C_0^\infty(\mathbb{R}^n)$.

PROOF. First notice that given $\delta > 0$ there is a constant $C = C(u,\delta,T)$ such that whenever $R \geqslant 1$, $0 < t < T - \delta$ we have that

$$\int_{|x|\le R} u(x,t)\,dx \leqslant CR^n\Lambda(R^2) \tag{6.12}$$

When $n \geqslant 3$, this was established in (5.3). For $n = 1,2$, notice that $u^*(x,\xi,t) = u(x,t)$, $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^2$ solves (1.1) in $\mathbb{R}^{n+2} \times (0,T)$, and so

$$R^2\int_{|x|\le R} u(x,t)\,dx \leqslant C\int_{\substack{|x|\le R\\ |\xi|\le R}} u^*(x,\xi,t)\,dx\,d\xi \leqslant CR^{n+2}\Lambda(R^2).$$

Thus, there exists a sequence $t_j \downarrow 0$ and a measure $\mu \geqslant 0$ on $\mathbb{R}^n$, verifying (6.11), and such that $u(x,t_j)\,dx$ converges weakly to $d\mu$ on $\mathbb{R}^n$. If $\eta \in C_0^\infty(\mathbb{R}^n)$, $0 < \tau < t < T$, then

$$\int [u(x,t) - u(x,\tau)]\eta(x)\,dx = \int_{0<\tau<s<t} \Delta\eta(x)\varphi(u(x,s))\,dx\,ds$$

From Theorem 5.5 and (6.12) it follows that

$$\left|\int [u(x,t) - u(x,\tau)]\eta(x)\,dx\right| \leqslant C[t^\sigma - \tau^\sigma],$$

where $\sigma = \sigma(a, n) \in (0, 1)$. Hence

$$\lim_{t \downarrow 0} \int u(x, t)\eta(x)\, dx = \int \eta\, d\mu,$$

which completes the proof of the theorem.

Our general uniqueness result will now be proved by the approximation technique of Dahlberg and Kenig [5].

Theorem 6.13. *Let* $\varphi \in \Gamma_a$, $0 < a < 1$, *and let* u, v *belong to* $S_\varphi(T)$ *for some* $T > 0$. *If* u *and* v *have the same trace, then* $u = v$ *in* $\mathbb{R}^n \times (0, T)$.

PROOF. Notice that if u, v have the same trace on $\mathbb{R}^n$, so do u^*, v^* in $\mathbb{R}^{n+2}$. We may therefore assume that $n \geqslant 3$.

Pick $h \in C_0^\infty(\mathbb{R}^n)$, with $0 \leqslant h \leqslant 1$ and let $w = w(x, t, h)$ be the unique solution in $\mathbb{R}^n \times (0, \infty)$ with initial trace h_μ and $\sup_{t>0} \int w(x, t)\, dx < \infty$, where μ is the common trace of u and v. (The existence of w follows from the estimates is Sections 4 and 5, while the uniqueness follows from Section 2.) We claim that $w \leqslant u$. To show this, let for $\epsilon \in (0, T/2)$, U_ϵ be the unique bounded solution in $\mathbb{R}^n \times (0, \infty)$ with initial data $h(x)u(x, \epsilon)$, satisfying

$$\sup_{t>0} \int U_\epsilon(x, t)\, dx \leqslant \sup_{\epsilon \in (0, T/2)} \int h(x)u(x, \epsilon)\, dx \leqslant C_h$$

where $C_h < \infty$ by the Harnack inequality. For any continuous function η we have that

$$\lim_{\epsilon \to 0} \int \eta(x)h(x)u(x, \epsilon)\, dx = \int \eta h\, d\mu,$$

and hence, our compactness result, Lemma 4.4 shows that

$$\lim_{\epsilon \to 0} U_\epsilon(x, t) = w(x, t) \quad \text{for all} \quad (x, t) \in \mathbb{R}^n \times (0, \infty).$$

From the maximum principle Lemma 6.1, $U_\epsilon(x, t) \leqslant u(x, t + \epsilon)$ for $0 < t < T - \epsilon$.

Hence, $w \leqslant u$ in $\mathbb{R}^n \times (0, T)$. Observe next that if $0 \leqslant h_j \leqslant h_{j+1} \leqslant 1$, $h_j \in C_0^\infty(\mathbb{R}^n)$ and $\lim_{j \to \infty} h_j(x) = 1$ for all $x \in \mathbb{R}^n$, then $w(x, t, h_j) \leqslant w(x, t, h_{j+1}) \leqslant u(x, t)$. Let $w_\infty = \lim_{j+\infty} w(x, t, h_j)$. Since $\{w_j\}$ is locally bounded in $\mathbb{R}^n \times (0, T)$, it follows that $\{w_j\}$ is locally equicontinuous and hence w_∞ solves (1.1) in $\mathbb{R}^n \times (0, T)$. We also notice that λ, the trace of w_∞ is between $h_j\mu$ and μ for all j, and hence $\lambda = \mu$. Since $w_\infty \leqslant u$, it follows that

$$\lim_{t \to 0} \int_{|x|<R} |u(x, t) - w_\infty(x, t)|\, dx = 0 \quad \text{for all} \quad R > 0.$$

Hence, $u = w_\infty$ by Lemma 6.1, and similarly $v = w_\infty$, which concludes the proof of the Theorem.

Corollary 6.14. *Let $\varphi \in \Gamma_a$, $0 < a < 1$, and let $u, v \in S_\varphi(T)$ for some $T > 0$. Let u, v have traces μ, ν respectively. If $\mu \leqslant \nu$, then $u \leqslant v$ in $\mathbb{R}^n \times (0, T)$.*

PROOF. Let $h_j \in C_0^\infty(\mathbb{R}^n)$, $0 \leqslant h_j \leqslant h_{j+1} \leqslant 1$, and such that $\lim_{j\to\infty} h_j(x) = 1$ for all $x \in \mathbb{R}^n$. Let u_j, v_j be the solutions in $\mathbb{R}^n \times (0, \infty)$ with initial trace $h_j\mu$ and $h_j\nu$ respectively. Then $u_j \leqslant v_j$ since, by Lemma 4.4, $u_j = \lim_{\epsilon\to 0} u_{j,\epsilon}$, $v_j = \lim_{\epsilon\to 0} v_{j,\epsilon}$, where $u_{j,\epsilon}$, $v_{j,\epsilon}$ are the solutions with initial data $\eta_\epsilon * (h_j \cdot \mu)$, $\eta_\epsilon * (h_j \cdot \nu)$ respectively. Here $\eta_\epsilon(x) = \epsilon^{-n}\eta(x/\epsilon)$, where $\eta \geqslant 0$, $\int \eta = 1$, $\eta \in C_0^\infty(\mathbb{R}^n)$. By Lemma 6.1, $u_{j,\epsilon} \leqslant v_{j,\epsilon}$, and our claim follows. Since $u = \lim_{j\to\infty} u_j$, $v = \lim_{j\to\infty} v_j$, the Corollary follows.

Theorem 6.15. *Let $\varphi_k \in \Gamma_a$, $0 < a < 1$, and let $u_k \in S_{\varphi_k}(T)$ for some $T > 0$. Assume that $\sup_k u_k(0, T) < \infty$. Let μ_k denote the initial trace of u_k, and assume that μ_k converges weakly to a non-negative measure μ on $\mathbb{R}^n$. If $\varphi_k \to \varphi \in \Gamma_a$ uniformly on compact subsets of $[0, \infty)$, then there is a $u \in S_\varphi(T)$ such that u_k converges uniformly to u on compact subsets of $\mathbb{R}^n \times (0, T)$, and the initial trace of u equals μ.*

PROOF. From the Harnak inequality and the pointwise estimates, it follows that for each compact set $K \subset \mathbb{R}^n \times (0, T)$, we have

$$\sup_k \|u_k\|_{L^\infty(K)} < \infty.$$

From the continuity results of Sacks ([11]), it follows that $\{u_k\}$ is equicontinuous on each compact subset $K \subset \mathbb{R}^n \times (0, T)$. Let $w \in S_\varphi(T)$ be locally the uniform limit of a subsequence of $\{u_k\}$. Using our pointwise estimates as in the proof of Theorem 6.10, it follows that w has trace μ. By the uniqueness result, Theorem 6.13, the Theorem follows.

Remark. Since our general uniqueness result, Theorem 6.10 has been established for all dimensions, we can now remove the restriction $n \geqslant 3$ in Sections 4 and 5.

7. Existence and blow up

This section will be devoted to studying the solvability and maximum time interval of existence for the initial value problem for $\partial u/\partial t = \Delta\varphi(u)$. The analogous problem for the porous medium equation $\partial u/\partial t = \Delta u^m$, $m > 1$, was settled by Bénilan, Crandall and Pierre ([3]). We will use the notations introduced at the beginning of Section 5. Our existence results will be based on the following lemma.

Lemma 7.1. *Let $\varphi \in \Gamma_a$, $0 < a < 1$. There is a number $\delta = \delta(a, n) > 0$ such that if $|||\mu||| < \delta$, there exists a unique solution u of $\partial u/\partial t = \Delta\varphi(u)$ in $\mathbb{R}^n \times (0, 1)$, with initial trace μ.*

PROOF. Because of Theorem 5.5 and Theorem 6.15 it is enough to show that there exists $\delta = \delta(a, n) > 0$ such that if $f \geqslant 0$ is in $C_0^\infty(\mathbb{R}^n)$, $\varphi \in \Gamma_a \cap C^\infty([0, \infty))$, and u solves $\partial u/\partial t = \Delta\varphi(u)$, $u(x, 0) = f(x)$, then

$$\sup_{0<t<1} |||u(\cdot, t)||| \leqslant C = C(a, n), \tag{7.2}$$

whenever $|||f||| \leqslant \delta$.

Let $g(t) = |||u(\cdot, t)|||$, $\eta_R(x) = \eta(x/R)$, where $0 \leqslant \eta \leqslant 1$, $\eta \in C_0^\infty(\mathbb{R}^n)$, $\eta = 1$ for $|x| \leqslant 1$, and $\eta = 0$ for $|x| \geqslant 2$. Then,

$$\int u(x, t)\eta_R(x)\, dx = \int f(x)\eta_R(x)\, dx + \iint_{0<\tau<t} R^{-2}\varphi(u(x, \tau))\Delta\eta(x/R)\, dx\, d\tau. \tag{7.3}$$

For $0 < \tau < 1$, $R > 1$, let $v(x, s) = u(\rho x, s\tau)/\gamma$, where $\rho = 2R$, $\gamma = \Lambda(\rho^2/\tau)$. Then, $\partial v/\partial t = \Delta\psi(v)$ for a $\psi \in \Gamma_a$, and the pointwise estimate Theorem 3.17 shows that

$$\sup_{|x|<1} v(x, 1) = \sup_{|x|<2R} u(x, \tau)/\gamma \leqslant C\{A^\sigma + A^\gamma\}, \tag{7.4}$$

where $A = G(\tau)\Lambda(R^2)\gamma^{-1}$, and $G(\tau) = \sup\{g(s): \tau/2 < s < \tau\}$, $0 < \tau < 1$.

Recall now that $\Lambda(bs) \leqslant b^\nu\Lambda(s)$, $b \geqslant 1$, $s \geqslant 1$, for some $\nu = \nu(a) > 0$. Hence, $A \leqslant C\tau^\nu G(\tau)$, and therefore, it follows from (7.4) that

$$\sup_{|x|<2R} u(x, \tau) \leqslant C\gamma\{\tau^{\nu\sigma}G^\sigma(\tau) + \tau^{\nu\gamma}G^\gamma(\tau)\}, \tag{7.5}$$

for $\tau \in (0, 1)$.

Recall now that $\Lambda^{-1}(u) = \varphi(u)/u$ whenever $u \geqslant 1$ and that $\Lambda^{-1}(\alpha\gamma) \leqslant \alpha^\theta\Lambda^{-1}(\gamma)$ for some $\theta = \theta(a)$ whenever $\alpha\gamma \geqslant 1$, $\gamma = \Lambda(\rho^2/\tau) \geqslant 1$. Hence, it follows from (7.5) that

$$\sup\left\{\frac{\varphi(u(x, \tau))}{u(x, \tau)}: |x| < 2R, u(x, \tau) \geqslant 1\right\} \leqslant C\frac{R^2}{\tau}\{\tau^l G^r(\tau) + \tau^\alpha G^\beta(\tau)\},$$

for some positive constants l, r, α, β.

Hence, (7.3) shows that, if $0 < t < 1$, then

$$\int_{|x|<R} u(x,t)\,dx \leqslant \int_{|x|\leq 2R} f(x)\,dx$$

$$+ CR^{-2}\int_0^t \int_{|x|<2R} [1 + u(x,\tau)]\frac{R^2}{\tau}[\tau^l G^r(\tau) + \tau^\alpha G^\beta(t)\}\,dx\,d\tau.$$

Next, we divide by $R^n \Lambda(R^2)$ and take the supremum over $R > 1$. Hence,

$$G(t) \leqslant C\delta + C\int_0^t [\tau^{l-1}G^{r+1}(\tau) + \tau^{\alpha-1}G^{\beta+1}(\tau)]\,d\tau.$$

It is then easy to see that there are M_0 and $\delta_0 > 0$ such that if $0 < \delta < \delta_0$, then $\sup_{\tau\in(0,1)} G(\tau) \leqslant M_0 < \infty$, which finishes the proof of the Lemma.

Before studying the initial value problem for a general measure $\mu \geqslant 0$ in $\mathbb{R}^n$, we wish to make some remarks on blow up. Note that as a consequence of our Harnack inequality, if μ is the initial trace of a $u \in S_\varphi(T)$, $\varphi \in \Gamma_a$, then

$$|||\mu||| < \infty \tag{7.6}$$

(see (6.11)). In order to give an estimate of the largest possible time interval for a solution to exist, we introduce some notation. Let $\varphi \in \Gamma_a$, $0 < a < 1$. Set, for $r \geqslant 1$, and $0 < t < r^2$

$$\begin{cases} A_r(t) = \sup\limits_{x \geq r^2} [\Lambda_\varphi(x)/\Lambda_\varphi(x/t)] \\ B_r(t) = \inf\limits_{x \geq r^2} [\Lambda_\varphi(x)/\Lambda_\varphi(x/t)] \end{cases} \tag{7.7}$$

It is easy to see that the limits

$$A(t) = A_\varphi(t) = \lim_{r\to\infty} A_r(t), \quad \text{and} \quad B(t) = B_\varphi(t) = \lim_{r\to\infty} B_r(t)$$

exist. Furthermore. A_φ and B_φ are strictly increasing on $(0, \infty)$, with

$$A_\varphi(0) = B_\varphi(0) = 0, \qquad \lim_{t\to\infty} A(t) = \lim_{t\to\infty} B_\varphi(t) = +\infty.$$

Our blow up result is the following.

Theorem 7.7. *Let $\varphi \in \Gamma_a$, $0 < a < 1$, and let $u \in S_\varphi(T)$, for some $T > 0$. Let μ be the initial trace of u. Then,*

$$|||\mu|||_\infty \leqslant C/B_\varphi(T). \tag{7.8}$$

PROOF. Note that, by our Harnack inequality (Theorem 4.12), it follows that, for all $t \in (0, T)$, $R \geqslant \sqrt{t}$,

$$\mu\{|x| < R\} \leqslant CR^n \Lambda(R^2/t) + F(t),$$

where $F(t)$ is independent of R. Hence, for $r \geqslant 1$,

$$|||\mu|||_r \leqslant CB_r(t)^{-1} + F(t) r^{-n} \Lambda(r^2)^{-1}.$$

Letting $r \to \infty$, and $t \to T$, we obtain (7.8).

We will now give our general existence result.

Theorem 7.9. *Let $\varphi \in \Gamma_a$, $0 < a < 1$, and let μ be a measure with $|||\mu|||_\infty < \infty$. Then, there is a $T > 0$ such that μ is the initial trace of a $u \in S_\varphi(T)$. More precisely, there is a constant $C = C(a, n) > 0$ such that $A_\varphi(T_\varphi(\mu)) \geqslant C(|||\mu|||_\infty)^{-1}$, where $T_\varphi(\mu) = \sup\{T: \mu$ is a trace of a $u \in S_\varphi(T)\}$.*

PROOF. For $0 < \tau < \rho^2$, let $\psi(u) = \varphi(\gamma u)/\gamma(\gamma)$, where $\gamma = \Lambda(\rho^2/\tau)$. Then, $u \in S_\varphi(T)$ if and only if $v \in S_\psi(T\tau^{-1})$, where $v(x, t) = u(\rho x, \tau t)/\gamma$. Let the measure $\lambda \geqslant 0$ be defined by

$$\int f(x)\, d\lambda = \rho^{-n}\gamma^{-1} \int f(x/\rho)\, d\mu.$$

Then, λ is the initial trace of v if and only if μ is the initial trace of u. For $r \geqslant 1$, we have that

$$\lambda\{|x| < r\} = \rho^{-n}\gamma^{-1}\mu\{|x| < \rho r\} \leqslant |||\mu|||_{\varphi,\rho} r^n \Lambda_\varphi((\rho r)^2)\gamma^{-1},$$

where $|||\ \ |||_{\varphi,\rho}$ is as in (5.1), but emphasizing the dependence on φ.

Observe now that if $\xi \geqslant 1$ and $\eta = \Lambda_\Psi(\xi)$, then

$$\xi\varphi(\gamma)/\gamma = \psi(\eta)\varphi(\gamma)/\gamma\eta = \varphi(\gamma\eta)/\gamma\eta.$$

Since $\varphi(\gamma)/\gamma = \rho^2/\tau$, we have that

$$\gamma\eta = \gamma\Lambda_\Psi(\xi) = \Lambda_\varphi(\rho^2\xi/\tau).$$

Hence

$$|||\lambda|||_{\varphi,\rho} \leqslant \sup_{r \geq 1} \Lambda_\varphi(\rho^2((/\Lambda_\varphi((\rho r^2)/\tau) = |||\mu|||_{\varphi,\rho} A_\rho(\tau).$$

Suppose now that $|||\mu|||_{\varphi,\infty} < \infty$. Choose now $\tau > 0$ such that $|||\mu|||_\rho A(\tau) < \delta$, where $\delta = \delta(a, n)$ is as in Lemma 7.1. We can find now a $\rho > \max(\sqrt{\tau}, 1)$ such that $|||\mu|||_{\varphi,\rho} A_\rho(\tau) \leqslant \delta$. By Lemma 7.1, we can find a $v \in S_\psi(1)$, with initial trace λ. Since $u \in S_\varphi(\tau)$, where $u(x, t) = \gamma v(x/\rho, t/\tau)$, the Theorem follows.

Finally, we will give a necesary and sufficient condition for $T_\varphi(\mu) = +\infty$. Let S_φ denote the class of non-negative, continuous solutions of (1.1) in $\mathbb{R}^n \times (0, \infty)$.

Corollary 7.10. *Let $\varphi \in \Gamma_a$, $0 < a < 1$. A non-negative measure μ on $\mathbb{R}^n$ is the initial trace of a $u \in S_\varphi$ if and only if $|||\mu|||_\infty = 0$.*

PROOF. If $u \in S_\varphi$, (7.8) gives $|||\mu|||_\infty = 0$. Conversely, if $|||\mu|||_\infty = 0$, by Theorem 7.9 there is a solution $u_T \in S_\varphi(T)$, with initial trace μ for every $T > 0$. By our uniqueness result, $u_T = u_\tau$ if $\tau < T$, and this finishes the proof.

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