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A Weak Regularity Theorem for Real Analytic Optimal Control Problems

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Abstract

We consider real analytic finite-dimensional control problems with a scalar input that enters linearly in the evolution equations. We prove that, whenever it is possible to steer a state x to another state y by means of a measurable control, then it is possible to steer x to y by means of a control that has an extra regularity property, namely, that of being analytic on an open dense subset of its interval of definition. Since open dense sets can have very small measure, this is a very weak property. However, it is absolutely general, depending on no assumptions other than real analyticity. This shows that real analyticity alone suffices to imply some regularity, and leaves open the question of how much more regularity can be proved in general. To show that real analyticity is essential, we prove, by constructing a class of examples, that no such theorem is true in the C^∞ case. The regularity result implies a similar regularity theorem for time-optimal controls.

1. Introduction

Existence theorems for optimal controls usually give existence of such a control in some large function space (e.g. the space of all bounded measurable functions on an interval). However, in problems where one can actually compute the solutions explicitly, they often turn out to be much more regular than might have been expected from the existence theory. This raises the question whether optimal controls really can be as pathological as the general theory appears to allow, or perhaps there are general regularity theorems that limit the possible pathologies that can occur.

The purpose of this note is to argue that there exists a fundamental distinction between systems of class C^∞ and real-analytic systems. In the former case, we show by means of a simple example that no *a priori* restriction on the pathology of optimal controls is possible, in the sense that, given any measurable control η, then one can construct an optimal control problem of which η is the only solution. For real-analytic systems, on the other hand, general regularity theorems exist. We establish this by proving one such theorem. The precise regularity property of optimal controls obtained in the theorem is rather weak: if a control $u(\cdot)$ is defined on an interval $[0, T]$, let us say that $u(\cdot)$ has property (R) if there is a relatively open dense subset Ω of $[0, T]$ such that $u(\cdot)$ is real-analytic on Ω. Our regularity theorem asserts that, whenever it is possible to steer a point x_1 to a point x_2 with cost c, then this can also be done by means of a control that has property (R). This statement can obviously be applied if c is the optimal cost, in which case we get the conclusion that, if there is an optimal control that steers x_1 to x_2, then there is one that has property (R).

We believe that our formulation of the regularity problem («find a class $\mathcal{R}$ such that, whenever x_1 can be optimally steered to x_2, then this can also be done by means of a control in $\mathcal{R}$») is more natural than the simpler one of asking for a class $\mathcal{R}$ such that every optimal control is in $\mathcal{R}$. The reason is that there are problems that have very pathological solutions because they are very degenerate. (The most extreme case is that of problems where every control is optimal. This can happen, e.g., if the control does not appear at all in the dynamical equations and the cost functional.) For these problems, the appropriate question is not whether *all* solutions are «nice», but whether there is at least one «nice» solution. This is precisely captured by our formulation. Naturally, when the optimal controls are unique, both formulations agree.

The particular class $\mathcal{R}$ given by our theorem is still very large since, for instance, the complement of the set Ω could have Lebesgue measure arbitrarily close to T. However, the main point we wish to make here is that real analyticity alone, in the absence of any other hypothesis, already has some nontrivial regularity implications. At the moment, it is an open question whether stronger regularity properties can be proved under the same hypothesis. The gap between the worse pathology that has been found in examples (e.g. Fuller's problem, cf. Marchal [9]) and our positive result is quite large. On the other hand, some partial knowledge is available which, for some cases, gives better regularity. (E.g. bang-bang theorems for linear systems, cf. Lee and Markus [8], nonlinear bang-bang theorems, cf. Krener [7], Sussmann [14], and a general piecewise analytic regularity theorem for time-optimal control of real-analytic systems $\dot{x} = f(x) + ug(x)$, $|u| \leqslant 1$, in the plane, cf. Sussmann [15], [17], [18], [19].)

For simplicity, we will confine our discussion to time-optimal control of systems of the form

$$\dot{x} = f(x) + ug(x), \qquad |u| \leqslant 1, \tag{1}$$

where the state x evolves in a finite-dimensional manifold M of class C^∞, and f, g are C^∞ vector fields on M. We call such a system *real-analytic* if M is a real-analytic manifold and f, g are real-analytic vector fields. Our theorem is actually true for more general systems $\dot{x} = f(x, u)$, with more general cost functionals, but the proof is much more delicate and requires the use of difficult stratification theorems for subanalytic sets. The case considered here is much simpler, but sufficient to show the main ideas. The general proof will appear elsewhere.

2. An example

Let $T > 0$, and take an arbitrary measurable function η: $[0, T] \to [-1, 1]$. We will exhibit a system of the form (1), an initial state x_1 and a terminal state x_2, such that η steers x_1 to x_2 in time T, and no other control does.

Define a function θ: $[0, T] \to \mathbb{R}$ by

$$\theta(t) = \int_0^t \eta(s)\, ds, \qquad 0 \leqslant t \leqslant T. \tag{2}$$

Let $K = \{(t, \theta(t)) : 0 \leqslant t \leqslant T\}$. Then K is a compact subset of $\mathbb{R}^2$. Therefore there exists a C^∞ function ϕ: $\mathbb{R}^2 \to \mathbb{R}$ such that $\phi = 0$ on K and $\phi > 0$ on $\mathbb{R}^2 - K$. We then take our system to be

$$\dot{x} = 1, \tag{3.i}$$

$$\dot{y} = u, \tag{3.ii}$$

$$\dot{z} = \phi(x, y), \tag{3.iii}$$

where x, y, z are real variables, and the control u satisfies $|u| \leqslant 1$.

Let $x_1 = (0, 0, 0)$, $x_2 = (T, \theta(T), 0)$. Then η steers x_1 to x_2 and, if η': $[0, T'] \to$ $\to [-1, 1]$ is any other control that steers x_1 to x_2 in time T', then necesarily $T' = T$ and $\eta' = \eta$ almost everywhere.

3. The theorem

We now consider a control system of the form (1), where the variable x takes values in a finite-dimensional, real-analytic manifold M, and f and g are real-analytic vector fields on M. An *admissible control* is a measurable function, defined on some interval of the form $[0, T]$ for some $T \geqslant 0$, which takes values in $[-1, 1]$. If $u(\cdot): [0, T] \to [-1, 1]$ is an admissible control, and $\gamma(\cdot): [0, T] \to M$ is a trajectory for $u(\cdot)$, such that $\gamma(0) = x_1$, $\gamma(T) = x_2$, then we say that $u(\cdot)$ *steers* x_1 *to* x_2 *in time* T. If there exists a $u(\cdot)$ that steers x_1 to x_2 in time T, we say that x is *reachable from* x_1 in time T. The set of all such x is the *time* T *reachable set from* x_1, and will be denoted $\text{Reach}_T(x_1)$. We will prove:

Theorem. *Let f, g be C^ω vector fields on the C^ω manifold M. Suppose that, for the system* (1), *x_1, x_2, T are such that $x_2 \in \text{Reach}_T(x_1)$. Then there exists a control $u(\cdot)$ that steers x_1 to x_2 in time T and is such that $u(\cdot)$ is real-analytic on some open dense subset of the interval* $[0, T]$.

PROOF. We first show that, without loss of generality, we can make some extra assumptions and simplifications. Let L be the Lie algebra of vector fields on M generated by f and g. For $x \in M$, let $L(x) = \{X(x): X \in L\}$. Since L is a Lie algebra of real-analytic vector fields, M can be partitioned into *maximal integral manifolds* of L, i.e. connected real-analytic submanifolds S such that (i) whenever $x \in S$, then the tangent space of S at x is $L(x)$, and (ii) S is not properly contained in any connected submanifold S' that satisfies (i) (cf. [10], [11], [16]). If $S(x_1)$ is the maximal integral manifold through x_1, then the vector fields f, g have well defined restrictions $\tilde{f}, \tilde{g}$, that are tangent to $S(x_1)$. Moreover, every trajectory that goes through x_1 is entirely contained in $S(x_1)$. Hence it suffices to prove the theorem with $S(x_1), \tilde{f}, \tilde{g}$ insteady of M, f, g. Equivalently, we may assume that

(I) *$L(x)$ is the tangent space to M at x for every $x \in M$.*

This assumption will be made from now on.

Let us call a control $u(\cdot): [0, T] \to [-1, 1]$ *nice* if $u(\cdot)$ is real analytic on an open dense subset of the interval $[0, T]$. Call a trajectory nice if it corresponds to a nice control. We are trying to prove that, if x_2 can be reached from x_1 in time T, then x_2 can be reached from x_1 in the same time by means of a nice trajectory. We claim that it is sufficient to prove the weaker statement:

(II) *whenever* $x_1 \in M$, $x_2 \in M$ *are such that* x_2 *can be reached from* x_1, *then* x_2 *can be reached from* x_1 *by means of a nice trajectory.*

Indeed, if (II) holds, then we can consider, for given M, f, g, the «system obtained by adding time as a new variable». That is, we let $\tilde{M} = M \times \mathbb{R}$, and we let $(x, \xi) \in \tilde{M}$ evolve according to

$$\dot{x} = f(x) + ug(x), \tag{4.i}$$

$$\dot{\xi} = 1. \tag{4.ii}$$

Then x_2 is reachable from x_1 in time T for the old system if and only if (x_2, T) is reachable from $(x_1, 0)$ for the new system. Therefore, if we apply (II) to the new system, we obtain the desired conclusion.

We now show

(III) *if* x_2 *is an interior point of the set reachable from* x_1, *then* x_2 *can be reached from* x_1 *by means of a bang-bang control with a finite number of switchings.*

To see this, we first apply Chow's Theorem (cf. [7], [16]) to the system

$$\dot{x} = -f(x) - ug(x), \tag{5}$$

whose trajectories are those of the original system traversed backwards. We take $\epsilon > 0$ so small that every point reachable from x_2 by means of a trajectory of (5) in time $\leqslant \epsilon$ is in the reachable set from x_1 for the system (1). By the positive form of Chow's Theorem, there exists a nonempty open set Ω such that every $x_2' \in \Omega$ is reachable from x_2 in time $\leqslant \epsilon$ by means of a trajectory of (5) which is bang-bang with finitely many switchings. Then Ω is contained in the set reachable from x_1 for (1). Let $x_2 \in \Omega$ be reachable from x_1 by a trajectory $\gamma(\cdot)$ of (1) that corresponds to a control $u(\cdot)$, defined on $[0, T]$. Let $\{u_n(\cdot)\}$ be a sequence of bang-bang controls, defined on $[0, T]$, which converge weakly to $u(\cdot)$. Let $\{\gamma_n(\cdot)\}$ be the corresponding trajectories of (1), with initial condition $\gamma_n(0) = x_1$. Then $\gamma_n(T) \in \Omega$ if n is large enough. Pick n such that $\gamma_n(T) \in \Omega$, and let $x_2'' = \gamma_n(T)$. Then x_2'' is reachable from x_1, and x_2 is reachable from x_2'', by means of bang-bang trajectories of (1). Therefore, x_2 is bang-bang reachable from x_1, and (III) is proved.

In view of the preceding observations, it suffices to prove that, if (I) holds, then every point x_2 that can be reached from x_1 and belongs to the boundary of the reachable set from x_1 is reachable by a nice trajectory.

Let such x_1, x_2 be given. We can then apply the Pontryagin Maximum Principle, which we first state in the form that will be needed here. Let T^*M denote the cotangent bundle of M, and let $T^\# M$ be T^*M with the zero section removed, i.e.

$$T^{\#}M = \{(x,p): x \in M, p \in T_x^*M, p \neq 0\}, \tag{6}$$

where T_x^*M denotes the cotangent space of M at x. Let

$$X_+ = f + g, \qquad X_- = f - g. \tag{7}$$

Let

$$H_+(x,p) = \langle p, X_+(x)\rangle, \tag{8.i}$$

$$H_-(x,p) = \langle p, X_-(x)\rangle. \tag{8.ii}$$

Then H_+, H_- are real-analytic functions on the symplectic manifold $T^{\#}M$. Therefore, H_+, H_- give rise to Hamiltonian vector fields $X_+^{\#}, X_-^{\#}$ on $T^{\#}M$. The system (1) can also be written as

$$\dot{x} = vX_-(x) + (1 - v)X_+(x), \qquad 0 \leqslant v \leqslant 1, \tag{9}$$

where the control v is related to u by

$$v = \frac{1}{2}(1 - u). \tag{10}$$

We can then consider the *Hamiltonian system associated to* (9), i.e. the system

$$\dot{z} = vX_-^{\#}(z) + (1 - v)X_+^{\#}(z), \tag{11}$$

where $z = (x,p) \in T^{\#}M$. If $\gamma(\bullet)$ is a trajectory of (9) for a control $v(\bullet)$, on an interval $[0, T]$, then any trajectory of (11) which is of the form $t \to (\gamma(t), p(t))$, $0 \leqslant t \leqslant T$, is called a *Hamiltonian lift* of γ. A trajectory $\zeta(\bullet) = (\gamma(\bullet), p(\bullet))$ of (11), and corresponding control $v(\bullet)$, are called *null-minimizing* if, for almost every $t \in [0, T]$, we have

$$\langle p(t), v(t)X_-(\gamma(t)) + (1 - v(t))X_+(\gamma(t))\rangle = \\ = \min_{0 \leq w \leq 1} \langle p(t), wX_-(\gamma(t)) + (1 - w)X_+(\gamma(t))\rangle. \tag{12}$$

The Maximum Principle says that if $v(\bullet)$ is a control on $[0, T]$, and $\gamma(\bullet)$ is a corresponding trajectory of (9), such that $\gamma(T)$ is on the boundary of the reachable set from $\gamma(0)$, then $(\gamma(\bullet), v(\bullet))$ has a Hamiltonian lift which is null-minimizing.

Now let x_1, x_2 be as above, and let $v(\bullet)$ steer x_1 to x_2 in time T. Let $\gamma(\bullet)$: $[0, T] \to M$ be the corresponding trajectory. Let $t \to p(t)$, $0 \leqslant t \leqslant T$, be such that $(\gamma(\bullet), p(\bullet))$ is a null-minimizing Hamiltonian lift of $\gamma(\bullet)$. Let

$\zeta(t) = (\gamma(t), p(t))$. We will show that $v(\bullet)$ is nice. In order to do this, we first have to construct a stratification of $T^{\#}M$ which has some special properties.

We partition $T^{\#}M$ into five sets S_1, S_2, S_3, S_4, S_5 by letting

$$
\begin{aligned}
&(x,p) \in S_1 \quad \text{if} \quad H_+(x,p) = H_-(x,p) = 0,\\
&(x,p) \in S_2 \quad \text{if} \quad H_+(x,p) = 0 \quad \text{but} \quad H_-(x,p) \neq 0,\\
&(x,p) \in S_3 \quad \text{if} \quad H_-(x,p) = 0 \quad \text{but} \quad H_+(x,p) \neq 0,\\
&(x,p) \in S_4 \quad \text{if} \quad H_+(x,p)H_-(x,p) > 0, \quad \text{and}\\
&(x,p) \in S_5 \quad \text{if} \quad H_+(x,p)H_-(x,p) < 0.
\end{aligned}
$$

Each of the sets S_i is semianalytic. Therefore there exists a real-analytic stratification Σ_0 of $T^{\#}M$ such that the strata of Σ_0 are semianalytic subsets of $T^{\#}M$, and each stratum of Σ_0 is entirely contained in one of the S_i. (cf. [3]).

We now define a sequence of stratifications $\Sigma_0, \Sigma_1, \ldots$, each of which is obtained from the preceding one by refining it in a suitable way. The procedure for refining a stratification is as follows. Let Σ be an arbitrary C^ω stratification of $T^{\#}M$, whose strata are subanalytic sets. Let $S \in \Sigma$. We partition S into five sets $B_1(S), B_2(S), B_3(S), B_4(S), B_5(S)$ as follows: if $(x,p) \in S$, we let $(x,p) \in B_1(S)$ if both vectors $X_+^{\#}(x,p), X_-^{\#}(x,p)$ are tangent to S; we let $(x,p) \in B_2(S)$ if $X_+^{\#}(x,p)$ is tangent to S but $X_-^{\#}(x,p)$ is not, $(x,p) \in B_3(S)$ if $X_-^{\#}(x,p)$ is tangent to S but $X_+^{\#}(x,p)$ is not, $(x,p) \in B_4(S)$ if $X_+^{\#}(x,p)$ and $X_-^{\#}(x,p)$ are not tangent to S but some convex combination of them is, and $(x,p) \in B_5(S)$ if no convex combination $vX_-^{\#}(x,p) + (1-v)X_+^{\#}(x,p)$, $v \in [0,1]$, is tangent to S. It is clear that the $B_i(S)$ are subanalytic subsets of $T^{\#}M$. Moreover, on the set $B_4(S)$ we can define a function $v_{4,S}\colon B_4(S) \to (0,1)$ by letting $v_{4,S}(x,p)$ be the unique v such that $0 < v < 1$ and $vX_-^{\#}(x,p) + (1-v)X_+^{\#}(x,p)$ is tangent to S. The function $v_{4,S}$ is obviously subanalytic (i.e. its graph is a subanalytic subset of $T^{\#}M \times \mathbb{R}$). Moreover, $v_{4,S}$ is real-analytic, in the sense that, if $(\bar{x},\bar{p}) \in B_4(S)$, then there is a neighborhood W of $(\bar{x},\bar{p})$ in S, and a real-analytic real-valued function w on W, such that $v_{4,S} = w$ on $W \cap B_4(S)$. (To see this, choose a C^ω coordinate chart $(z_1, \ldots, z_m)$ that maps a neighborhood $\tilde{W}$ of $(\bar{x},\bar{p})$ in $T^{\#}M$ diffeomorphically onto the unit cube in $\mathbb{R}^m$, in such a way that $(\bar{x},\bar{p})$ is mapped to 0, and the set $W = \{z\colon z_1 = \cdots = z_r = 0\}$ is, for some r, a neighborhood of $(\bar{x},\bar{p})$ in S. Relative to $(z_1, \ldots, z_m)$, the vector fields $X_+^{\#}, X_-^{\#}$ have components $\sigma_1^+, \ldots, \sigma_m^+$ and $\sigma_1^-, \ldots, \sigma_m^-$. Since $(\bar{x},\bar{p}) \in B_4(S)$, there exists a j such that $1 \leqslant j \leqslant r$ and $\sigma_j^+(\bar{x},\bar{p}) \neq \sigma_j(\bar{x},\bar{p})$. Then we can assume, after shrinking $\tilde{W}$ if necessary, that $\sigma_j^+(z) \neq \sigma_j^-(z)$ for all $z \in \tilde{W}$. For $z \in \tilde{W}$, let

$$\tilde{w}(z) = \frac{\sigma_j^+(z)}{\sigma_j^+(z) - \sigma_j^-(z)}. \tag{13}$$

Then $\tilde{w}$ is real-analytic on $\tilde{W}$, and

$$\tilde{w}\sigma_j^- + (1 - \tilde{w})\sigma_j^+ = 0. \tag{14}$$

If $(x,p) \in B_4(S) \cap W$, then $v_{4,S}(x,p)$ also satisfies

$$v_{4,S}(x,p)\sigma_j^-(x,p) + (1 - v_{4,S}(x,p))\sigma_j^+(x,p) = 0. \tag{15}$$

Therefore $v_{4,S}(x,p) = \tilde{w}(x,p)$. Hence we can take w to be the restriction of $\tilde{w}$ to W.) In particular, if $B_4(S) = S$, then $v_{4,S}$ is real-analytic on S.

We now describe the procedure for refining a stratification. If Σ is a real-analytic stratification of $T^{\#}M$ by subanalytic sets, we call Σ *good up to codimension* k if every stratum $S \in \Sigma$ of codimension $\leqslant k$ is such that $S = B_i(S)$ for some $i \in \{1,2,3,4,5\}$. We call Σ *good* if Σ is good up to codimension ν, where $\nu = \dim T^{\#}M = 2\dim M$. If Σ is good up to codimension k, we can refine Σ by letting $\mathcal{E}$ be the set of all sets $B_i(S)$, where S ranges over all the strata of Σ of codimension $>k$, and $i \in \{1,2,3,4,5\}$. Then $\mathcal{E}$ is a locally finite family of subanalytic subsets of $T^{\#}M$. Therefore there exists a C^ω stratification $\tilde{\Sigma}$ of $\bigcup\{E : E \in \mathcal{E}\}$ by subanalytic sets, such that every $E \in \mathcal{E}$ is a union of strata of $\tilde{\Sigma}$. We then define Σ to consist of all the strata of Σ of codimension $\leqslant k$, as well as all the strata of $\tilde{\Sigma}$. Then Σ is a C^ω stratification by subanalytic sets. Moreover, Σ is good up to codimension $k+1$. (Indeed, if $S \in \Sigma$ and $\operatorname{codim} S \leqslant k$, then $S \in \Sigma$ and so $S = B_i(S)$ for some i. If $\operatorname{codim} S = k+1$, then $S \subseteq E$ for some $E \in \mathcal{E}$, i.e. $S \subseteq B_i(F)$ for some i and some $F \in \Sigma$ such that $\operatorname{codim} F > k$. But then $\operatorname{codim} F = k+1$ and S is relatively open in F, so that $B_i(S) = B_i(F) \cap S$, and then $B_i(S) = S$.)

We apply this refining procedure successively, starting with Σ_0, and construct C^ω subanalytic stratifications $\Sigma_1, \Sigma_2, \ldots$ such that Σ_k is good up to codimension k for each k. If $\nu = \dim T^{\#}M$, we let $\Sigma = \Sigma_\nu$, so that Σ is good. Hence Σ is a C^ω stratification of $T^{\#}M$ by subanalytic sets, such that every stratum $S \in \Sigma$ is subanalytic and satisfies $S \subseteq S_i$ for some i, and $S = B_j(S)$ for some j.

We now show that Σ contains no stratum S such that $S \subseteq S_1$ and $S = B_1(S)$. Indeed, suppose $S \subseteq S_1$, $S = B_1(S)$, $S \in \Sigma$. For each C^ω vector field X on M, we can consider the lifted Hamiltonian vector field $X^{\#}$, i.e. the Hamiltonian vector field that arises from the Hamiltonian function $(x,p) \to \langle p, X(t)\rangle$. Then $X \to X^{\#}$ is a Lie algebra homomorphism. Since $X_-^{\#}$ and $X_+^{\#}$ are tangent to S (because $S = B_1(S)$), it follows that $X^{\#}$ is tangent to S for all $X \in L$. (Recall that L is the Lie algebra generated by f and g.) If $X \in L$, then $X^{\#}$ is tangent to S. Therefore the derivative of H_- in the direction of $X^{\#}$ vanishes throughout S (because $H_- \equiv 0$ on S). So $\langle p, [X_-, X](x)\rangle = 0$ for all $(x,p) \in S$. Similarly, $\langle p, [X_+, X](x)\rangle = 0$ for $(x,p) \in S$, $X \in L$. Finally, we know that $\langle p, X_-(x)\rangle = \langle p, X_+(x)\rangle = 0$ if $(x,p) \in S$, and so $\langle p, X(x)\rangle = 0$

for all $(x, p) \in S$, $X \in L$. But then, if we pick $(x, p) \in S$, we have $\langle p, X(x) \rangle = 0$ for all $X \in L$, i.e. $\langle p, \theta \rangle = 0$ for all $\theta \in L(x)$. Since $L(x) = T_x M$, we have $\langle p, \theta \rangle = 0$ for all $\theta \in T_x M$, and so $p = 0$, contradicting the fact that $(x, p) \in T^\# M$.

Let us now return to our null-minimizing Hamiltonian trajectory $\zeta(\cdot)$, that corresponds to a control $v(\cdot)$. Let $\zeta(\cdot)$ and $v(\cdot)$ be defined on $[0, T]$. For each t, $\zeta(t)$ belongs to a stratum $S(t)$ of Σ. Let $k(t) = \dim S(t)$. Since Σ is a stratification, every t has a neighborhood I such that $k(\tau) \geqslant k(t)$ for $\tau \in I$. (Otherwise, there would exist a sequence $\{t_n\}$ such that $t_n \to t$ and $k(t_n) < k(t)$. Since Σ is locally finite, we may pass to a subsequence and assume that all the $S(t_n)$ are one on the same stratum S. Then $\dim S < \dim S(t)$, but $\zeta(t) \in S(t)$ and $\zeta(t) \in \operatorname{Clos} S$. So $S(t) \subseteq \operatorname{Clos} S$, and then $\dim S(t) < \dim S$, which is a contradiction.)

We now let Ω be the set of those $t \in [0, T]$ such that, for some $\epsilon > 0$, the stratum $S(\tau)$ is the same for all $\tau \in (t - \epsilon, t + \epsilon) \cap [0, T]$. Then Ω is relatively open in $[0, T]$. We claim that Ω is dense. To see this, let $t \in [0, T]$, $\epsilon > 0$. $\tau' \in (t - \epsilon, t + \epsilon) \cap [0, T]$ be such that $k(\tau') = \max\{k(\tau): |\tau - t| < \epsilon, \tau \in [0, T]\}$. We know that $k(\tau) \geqslant k(\tau')$ for $\tau \in I$, where I is some interval that contains τ' in its interior relative to $[0, T]$, and $I \subseteq [0, T]$. We can also assume that $I \subseteq (t - \epsilon, t + \epsilon)$, and then $k(\tau) = k(\tau')$ for $\tau \in I$. It then follows that $S(\tau) = S(\tau')$ for $\tau \in I$. (Otherwise, there would exist a sequence $\{\tau_n\}$ such that $\tau_n \in I$, $\tau_n \to \tau$, $\tau \in I$, and $S(\tau_n) = S$ for some S such that $S \neq S(\tau)$. But then $\zeta(\tau) \in \operatorname{Clos} S$ but $\zeta(\tau) \notin S$, and so $S(\tau) \subseteq \operatorname{Clos} S$, $S(\tau) \neq S$. Therefore $\dim S(\tau) < \dim S$, contradicting the fact that $k(\tau) = k(\tau_n)$.) Then $\tau' \in \Omega$, and so Ω is dense.

We now prove that $v(\cdot)$ is real-analytic on Ω.

Let $t \in \Omega$, and let I be an interval such that $t \in I \subseteq [0, T]$, I is relatively open in $[0, T]$, and $S(\tau) = S(t)$ for all $\tau \in I$. Let $S = S(t)$. Let $\tilde{\zeta}$, $\tilde{v}$ be the restrictions of ζ, v to I.

We know that S is contained in one of the sets S_i. Since ζ is null-minimizing, at least one of the functionals H_+, H_- vanishes at $\zeta(t)$. Hence S is contained in S_1 or S_2 or S_3. If $S \subseteq S_2$ then $H_+ = 0$ on S and H_- never vanishes on S. Since $\tilde{\zeta}$ is contained in S and is null-minimizing, it follows that $\tilde{v} \equiv 0$. If $S \subseteq S_3$, it follows that $\tilde{v} \equiv 1$. Finally, we must consider the case when $S \subseteq S_1$. In this case, we know that $S = B_i(S)$ for some i. Also, we know that Σ contains no stratum S' such that $S' \subseteq S_1$ and $S' = B_1(S)$. Hence $S = B_i(S)$ for some $i \in \{2, 3, 4, 5\}$. Since $\tilde{\zeta}$ is contained in S, the tangent vector $\dot{\tilde{\zeta}}(\tau)$ (which exists for almost every $\tau \in I$, and is a convex combination of $X_+^\#(\tilde{\zeta}(\tau))$ and $X_-^\#(\tilde{\zeta}(\tau))$) is actually tangent to S. This excludes the possibility that $S \subseteq B_5(S)$. Hence $S = B_i(S)$, where $i = 2$ or $i = 3$ or $i = 4$. If $i = 2$ then $\tilde{v}(\tau)$ must be equal to 0 for all $\tau \in I$. Similarly, if $i = 3$ we have $\tilde{v}(\tau) = 1$ for $\tau \in I$. If $i = 4$, then $\tilde{v}(\tau) = v_{4,S}(\tilde{\zeta}(\tau))$. But then $\tilde{\zeta}$ is an integral curve of the vector field V given by

$$V(x, p) = v_{4,S}(x, p) X_-^\#(x, p) + (1 - v_{4,S}(x, p)) X_+^\#(x, p) \tag{16}$$

for $(x, p) \in S$. Since $v_{4,S}$ is real-analytic on S, the vector field V is real-analytic on S, as well as tangent to S. But then $\tau \to \tilde{\xi}(\tau)$ is real-analytic, and so $\tilde{v}$ is real-analytic.

This completes the proof.

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