

# Variants of the Calderón-Zygmund Theory for $L^p$ -spaces

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## 1. Introduction

Familiar tools in Fourier Analysis such as the Littlewood-Paley theory and the Cotlar-Stein lemma lead us to believe, very roughly, that if we take an operator  $T$ , and decompose it somehow into its «lacunary pieces»  $T_j$ , then to a large extent the different pieces  $T_j$  and  $T_k$ ,  $j \neq k$ , act «independently» of each other. For example we may be dealing with the Hilbert transform on  $L^2(\mathbb{R})$  with convolution kernel  $1/x$ . If now  $T_j$  is the operator with convolution kernel  $1/x \chi_{|x| \sim 2^j}$ , the operators  $T_j$  are uniformly bounded on  $L^2$  (because their kernels are uniformly in  $L^1$ ) and an application of the Cotlar-Stein lemma (see for example [8]) allows us to conclude that indeed the Hilbert transform is bounded on  $L^2(\mathbb{R})$ . One drawback of the Cotlar-Stein lemma is that it is only valid in the setting of Hilbert space. On the other hand, we may be dealing with a Fourier multiplier  $m$  on  $L^p(\mathbb{R}^n)$ . Crudely, the content of the Hörmander multiplier theorem (see [12]) is that if each «piece»  $m_j = m \chi_{|\xi| \sim 2^j}$  has enough regularity to guarantee via the Sobolev embedding theorem that the  $m_j$ 's are uniformly bounded  $L^1$ -multipliers, then we may conclude that  $m$  is a multiplier of  $L^p(\mathbb{R}^n)$ ,  $1 < p < \infty$ . However, examples abound to show that we cannot weaken the Hörmander theorem so as to totally avoid the regularity condition: Littman, McCarthy and Rivière [10] produce a very nice example where the  $m_j$  are uniformly bounded  $L^1$ -multipliers but  $m$  fails to be a multiplier of any  $L^p$  other than  $L^2$ .

The last decade or so has seen a flourishing of the study of certain operators in Fourier Analysis which are closely related to the singular integrals of

Calderón-Zygmund theory, but somehow seem to fall just outside the scope of that theory—for example singular integrals along curves, Stein’s spherical maximal function and the Bochner-Riesz operators. Fairly recently it has been observed by Duoandikoetxea and Rubio de Francia [7] and Christ and Stein [4] that if one is a little more flexible in one’s point of view as to what constitutes «classical» Calderón-Zygmund theory, many results on these operators hitherto regarded as «exotic» may be very easily obtained.

The purposes of this paper may be described as follows:

- (i) to provide a useful substitute for the Cotlar-Stein lemma for  $L^p$ -spaces (the orthogonality conditions are replaced by certain fairly weak smoothness assumptions);
- (ii) to investigate the «gap» between the Hörmander multiplier theorem and the Littman-McCarthy-Rivière example—just how little regularity is really needed?
- (iii) to simplify and extend the work of Duoandikoetxea and Rubio de Francia and Christ and Stein, which sometimes has unnecessarily strong assumptions, and to introduce a sensitivity to different  $L^p$ -spaces which does not appear in their work.

In §2 we deal with the decomposition of an integral operator into lacunary pieces on the kernel side, introducing some smoothness conditions which are  $L^p$ -analogues of Hörmander’s condition. In §3 we look at pseudo-differential and multiplier operators: theorem 2 may be regarded as a sharper version of the Hörmander multiplier theorem as applied to pseudo-differential operators, and theorem 3 is a multiplier theorem which is sensitive to different  $L^p$ ’s. Roughly, theorem 2 may be paraphrased as: let  $\sigma \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$  be the symbol of an  $L^2$ -bounded pseudo-differential operator. If for some  $\epsilon > 0$ ,

$$\left(\frac{\partial}{\partial \xi}\right)^\epsilon \sigma(2^i x, 2^{-i} \xi) \chi_{|\xi| \sim 1}$$

have uniformly bounded  $L^1$ -operator norms, then  $\sigma$  is of weak-type 1. In §4 we look at some results for the maximal operators which fall under the scope of the theory, but, for simplicity, we do not present them in the sharpest possible form. Although all our statements are in terms of the usual isotropic dilations of  $\mathbb{R}^n$ , analogous statements hold for parabolic dilations.

We proceed to set up some notation. Let  $\phi$  be a non-negative radial  $C^\infty$  bump function of compact support in  $\{|x| \leq 1\}$ , with  $\int \phi = 1$ ; convolution with  $\phi_{2^j} = 2^{-jn} \phi(2^{-j} \cdot)$  is denoted by  $P_j$ . Let  $\psi$  be a non-negative radial  $C^\infty$  bump function of compact support in  $\{1 \leq |\xi| \leq 4\}$ , such that  $\sum_j \psi(2^j \xi) \equiv 1$  on  $\mathbb{R}^n - \{0\}$ ; let  $Q_j f(\xi) = \psi(2^j \xi) \hat{f}(\xi)$ . Let  $\tilde{\psi}$  denote a generic radial non-negative member of the Schwartz class  $\mathcal{S}$  with  $\tilde{\psi}(0) = 0$  and correspondingly

define  $\tilde{Q}_j$ . If  $K(x, y) = K$  is an integral kernel,  $K_j$  denotes  $K\psi^j$  where  $\psi^j \in \mathcal{S}$  is defined by

$$\psi^j(x, y) = \psi\left(\frac{x - y}{2^j}\right).$$

If  $T$  is the singular integral operator corresponding to  $K$ ,  $T_j$  is the operator corresponding to  $K_j$ . If  $\sigma(x, \xi)$  is a symbol (possibly independent of  $x$ ),

$$\sigma_j(x, \xi) = \sigma(x, \xi)\psi(2^j\xi),$$

and

$$\sigma * \rho(x, \xi) = \int \sigma(x, \eta)\rho(\xi - \eta) d\eta$$

for  $\rho \in \mathcal{S}(\mathbb{R}^n)$ .  $\Lambda_\epsilon$  is the usual Lipschitz space as defined for example in [12]. Finally,  $L^p$ -operator norms are denoted by  $|K|_p$ ,  $|T|_p$ ,  $|\sigma|_p$  or  $\|m\|_{\mathcal{S}T_p}$  as appropriate.

It is a pleasure to thank the Analysis Group at Yale University for their kind hospitality during March and April 1986, when the final stages of this research was being carried out, and particular thanks are due to José Luis Rubio de Francia who pointed out that commuting two steps in the proof of theorem 1 led to much nicer statements all round. Thanks are also due to the referee who made several useful suggestions which have improved the exposition of the paper considerably. Some of the results presented here have also been independently obtained by A. Seeger, [11].

## 2. Singular Integrals

In this section we study singular integral operators of the form

$$Tf(x) = \int K(x, y)f(y) dy.$$

**Theorem 1.** *Suppose that  $T$  is bounded on  $L^2$ , and that for some  $1 < p < 2$  satisfies*

$$\sum_{k \in \mathbb{Z}} \sup_{j \in \mathbb{Z}} \left| \sum_{l \geq 0} Q_{j+k} T_{j+l}(I - P_j) \right|_p < \infty. \tag{1}$$

*Then  $T$  is of weak-type  $p - p$ .*

**Corollary 1.** *Suppose that  $T$  is bounded on  $L^2$ , and that for some  $r$ ,  $1 \leq r < 2$ , satisfies*

$$\sup_j |T_j|_p < \infty, \quad r < p \leq 2.$$

If moreover

$$\sup_{j \in \mathbb{Z}} |Q_{j+k} T_{j+l}(I - P_j)|_2 \leq C 2^{-\epsilon(|k|+l)}$$

for some  $\epsilon > 0$  ( $k \in \mathbb{Z}, l \geq 0$ ), then  $T$  is bounded on  $L^p$ ,  $r < p \leq 2$ .

*Remarks.* (i) Theorem 1 should be compared with the classical Calderón-Zygmund theorem, which, upon replacing (1) by the Hörmander condition

$$\sup_{j \in \mathbb{Z}} \left| \sum_{l \geq 0} T_{j+l}(I - P_j) \right|_1 < \infty \tag{H}$$

gives the weak-type 1 of  $T$ . It would be very interesting to know whether  $L^2$  boundedness of  $T$  together with

$$\sup_j \left| \sum_{l \geq 0} T_{j+l}(I - P_j) \right|_p < \infty$$

is sufficient to imply the weak-type  $p$  of  $T$ .

(ii) If  $Q_{j+k}$  and  $T_{j+l}$  commute for all  $j, k, l$ , (in particular if  $T$  is a convolution operator), then we may replace (1) by

$$\sum_{k \leq 0} \sup_j \left| \sum_{l \geq 0} T_{j+l} Q_{j+k} \right|_p + \sup_j \left| \sum_{l \geq 0} T_{j+l} \tilde{Q}_j \right|_p < \infty, \tag{2}$$

since

$$I - P_j - \sum_{k \leq 0} Q_{j+k} = \tilde{Q}_j.$$

If now

$$\alpha(l) = \sup_j |T_{j+l}(I - P_j)|_p, \quad |T_{j+l} Q_{j+k}|_p \leq C \alpha(l - k) \quad \text{if } l \geq 0, k \leq 0,$$

and so  $T$  is of weak type  $p$  in the convolution case if

$$\sum_{l \geq 0} l \alpha(l) < \infty.$$

(iii) The corollary follows immediately from the theorem, because under the hypotheses of the corollary, the hypotheses of the theorem are satisfied for each  $p > r$ , by interpolation. A very similar result to this in the case  $r = 1$  had previously been obtained by M. Christ and E. M. Stein, [4]. Their result is that if  $T$  is bounded on  $L^2$ ,

$$\sup_j |T_j|_1 < \infty, \quad |T_j^* T_k|_2 \leq C 2^{-\epsilon|j-k|} \quad \text{and} \quad |T_{j+l}(I - P_j)|_2 \leq C 2^{-\epsilon l},$$

then  $T$  is bounded on  $L^p$ ,  $1 < p \leq 2$ .

(iv) Suppose that  $T$  is a convolution operator and that

$$\sup_j |T_j|_2 \leq C.$$

Then the smoothness hypothesis

$$|Q_{j+k}T_{j+l}(I - P_j)|_2 \leq C2^{-\epsilon(|k|+l)}, \quad l \geq 0, \quad k \in \mathbb{Z}, \quad (\text{some } \epsilon > 0)$$

is equivalent to

$$|T_{j+l}(I - P_j)|_2 \leq C2^{-\epsilon l}, \quad l \geq 0 \quad (\text{some } \epsilon > 0)$$

and to

$$|T_{j+l}Q_j|_2 \leq C2^{-\epsilon l}, \quad l \geq 0 \quad (\text{some } \epsilon > 0).$$

Each of these conditions is also clearly equivalent to the decay condition  $|\widehat{K_j}(\xi)| \leq C(2^j|\xi|)^{-\epsilon}$ ,  $(2^j|\xi|) \geq 1$ , all  $j$ , (some  $\epsilon > 0$ ) which has previously appeared in [2] and [7].

PROOF OF THEOREM 1. Fix  $f \in L^p$  and  $\alpha > 0$ , and apply the standard Calderón-Zygmund decomposition. (See [12].) Thus, we decompose  $f$  as  $f = g + \sum b_i$ , with  $\|g\|_\infty < \alpha$ ,  $b_i$  supported in a ball  $B_i$  of radius  $2^{j(i)}$ , the supports of the  $b_i$  pairwise disjoint,

$$\sum |B_i| \leq C\alpha^{-p}\|f\|_p^p, \quad |B_i|^{-1} \int_{B_i} |b_i|^p \leq C\alpha^p,$$

and the doubles  $B_i^*$  having bounded overlap. Let

$$G = g + \sum_i b_i * \phi_{2^{j(i)}}$$

be the good part; exactly as in the standard theory,  $L^2$ -boundedness of  $T$  gives that  $|\{x \mid |TG(x)| > \alpha\}| \leq C\|f\|_p^p/\alpha^p$ . So it suffices to treat

$$T\left(\sum_i [b_i - b_i * \phi_{2^{j(i)}}]\right) = \sum_i \sum_{l>0} T_{j(i)+l}(I - P_{j(i)})b_i + \sum_i \sum_{l \leq 0} T_{j(i)+l}(I - P_{j(i)})b_i.$$

The second term is supported in  $\cup B_i^{**}$ , whose measure is at most

$$\sum |B_i^{**}| \leq C \sum |B_i| \leq C\|f\|_p^p/\alpha^p.$$

We shall show that

$$\left\| \sum_j \sum_{l>0} T_{j+l}(I - P_j)f_j \right\|_p \leq C\|f_j\|_{L^p(Q^1)}. \tag{3}$$

If we apply (3) with

$$f_j = \sum_{i:j(i)=j} b_i,$$

we see that

$$\begin{aligned} \left\| \sum_i \sum_{l>0} T_{j(i)+l}(I - P_{j(i)})b_i \right\|_p &= \left\| \sum_j \sum_i T_{j+l}(I - P_j)f_j \right\|_p \\ &\leq C \left\| \sum_j \left| \sum_{i:j(i)=j} b_i \right| \right\|_p \leq C \left\| \sum |b_i| \right\|_p \\ &\leq C \left( \sum_i \int |b_i|^p \right)^{1/p} \leq C\alpha \left( \sum |B_i| \right)^{1/p} \leq C \|f\|_p, \end{aligned}$$

which completes the proof.

By duality, (3) is equivalent to

$$\left\| \sup_j \left| \sum_{l>0} (I - P_j)T_{j+l}^* g \right| \right\|_{p'} \leq C \|g\|_{p'}. \tag{4}$$

We decompose

$$\sum_{l>0} (I - P_j)T_{j+l}^* = \sum_{k \in \mathbb{Z}} \left[ \sum_{l>0} (I - P_j)T_{j+l}^* Q_{j+k} \right]$$

and treat each term in the  $k$ -sum using Littlewood-Paley theory. Thus,

$$\begin{aligned} \left\| \sup_j \left| \sum_{l \geq 0} (I - P_j)T_{j+l}^* Q_{j+k} g \right| \right\|_{p'} &\leq \left( \sum_j \int \left| \sum_{l \geq 0} (I - P_j)T_{j+l}^* Q_{j+k} g \right|^{p'} \right)^{1/p'} \\ &\leq C \sup_j \left| \sum_{l \geq 0} (I - P_j)T_{j+l}^* Q_{j+k} \right|_{p'} \left( \sum_j \int |\tilde{Q}_{j+k} g|^{p'} \right)^{1/p'} \\ &\leq C \sup_j \left| \sum_{l \geq 0} Q_{j+k} T_{j+l}(I - P_j) \right|_p \|g\|_{p'}. \end{aligned}$$

Hence (4) and (3) hold for a given  $p$  if (1) does.

*Remark.* The above proof is nothing but a minor variation on the classical Calderón-Zygmund theme. Indeed, the Calderón-Zygmund theorem follows by observing that (4) in the case  $p' = \infty$  is equivalent to the Hörmander condition (H), and the result of Christ and Stein [4], follows by observing that (4) holds for all  $p' < \infty$  provided that  $|T_{j+l}(I - P_j)|_1 \leq C$  and that

$$\left\| \left( \sum_j |(I - P_j)T_{j+l}^* g|^2 \right)^{1/2} \right\|_2 \leq C 2^{-\epsilon l} \|g\|_2.$$

By the Cotlar-Stein lemma this latter inequality will hold if

$$|(I - P_j)T_{j+l}^* T_{k+l}(I - P_k)|_2 \leq C 2^{-\epsilon(l+|j-k|)},$$

(or, in the case of convolution operators, if  $|T_{j+l}(I - P_j)|_2 \leq C 2^{-\epsilon l}$ ).

### 3. Multipliers and Pseudo-Differential Operators

In this section we study pseudo-differential operators of the form

$$Tf(x) = \int \sigma(x, \xi) \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$$

and multiplier operators when  $\sigma(x, \xi) = m(\xi)$  is independent of  $x$ .

**Theorem 2.** *Suppose that  $T$  is a pseudo-differential operator which is bounded on  $L^2(\mathbb{R}^n)$  and satisfies  $|\sigma(x, \xi)| \leq C \forall x, \xi \in \mathbb{R}^n$ . Let  $\sigma_j(x, \xi) = \sigma(x, \xi)\psi(2^j\xi)$ , and suppose that  $|\sigma_i * (\hat{\psi})_{2^{-j}}|_1 \leq \alpha(i - j)$  with  $\sum_{k \leq 0} |k|\alpha(k) < \infty$ . Then  $T$  is of weak-type  $1 - 1$ , and bounded from  $H^1$  (the real Hardy space) to  $L^1$ .*

**Theorem 3.** *Suppose that  $T$  is a multiplier operator which is bounded on  $L^2(\mathbb{R}^n)$  and satisfies  $|m_i * (\hat{\psi})_{2^{-j}}|_p \leq \alpha(i - j)$  for some  $1 \leq p < 2$ . If*

$$\sum_{k \leq 0} |k|\alpha(k) < \infty,$$

*then  $T$  is of weak type  $p - p$ .*

**Corollary 3.** *If  $T$  is a multiplier operator which satisfies*

$$\|m_i(2^{-i}\cdot)\|_{\mathfrak{M}_r} + \|m_i(2^{-i}\cdot)\|_{\Lambda_\epsilon} \leq C_r$$

*for some  $\epsilon > 0$ , some  $1 \leq p < 2$ , all  $p < r \leq 2$ , then  $T$  is bounded on  $L^r$  for  $p < r < p'$ .*

*Remarks.* (i) The case  $p = 1$  of theorem 3 is of course contained in theorem 2; there is a variant of theorem 2 in which  $L^p$ -hypotheses produce a weak-type  $p$  operator, but because we have to use condition (1) instead of condition (H), some cancellation and minimal  $x$ -smoothness is also needed. This matter will be discussed in detail in a forthcoming paper of the author and A. Seeger.

(ii) Corollary 3 follows immediately from theorem 3, once we notice that  $\|m_i(2^{-i}\cdot)\|_{\Lambda_\epsilon} \leq C$  is equivalent to  $\|m_i * (\hat{\psi})_{2^{-j}}\|_{\mathfrak{M}_2} \leq C2^{\epsilon(i-j)}$ .

(iii) Hörmander's classical multiplier theorem (see [12]) and some of the things which are obtainable from it by interpolation (see for example [5]) are contained in theorem 3. To see this we merely observe that the hypothesis of the Hörmander theorem implies that  $\|m_i * (\hat{\psi})_{2^{-j}}\|_{L^\beta} \leq C2^{\epsilon(i-j)}$  for some  $\beta > n/2$  and  $\epsilon > 0$  and then the Sobolev embedding theorem allows us to apply theorem 3 with  $\alpha(k) = 2^{\epsilon k}$ . Recently, Baernstein and Sawyer have proved a sharp  $H^1$ -variant of the Hörmander theorem, (see [1]). Their result is that if  $\alpha(k)$  is as above, and  $\sum_{k \leq 0} \alpha(k)\omega_{-k} < \infty$  for some nondecreasing sequence  $\omega_k$

satisfying  $\sum_{k \geq 0} \omega_k^{-2} < \infty$ , then  $m$  is a multiplier of the real Hardy space  $H^1$ . Thus all the multipliers covered by the case  $p = 1$  of theorem 3 are also  $H^1$ -multipliers, since they satisfy the Hörmander condition  $(H)$ , and also the Baernstein-Sawyer condition.

(iv) As an application of Corollary 3 we obtain by simple methods a result previously obtained by Córdoba and López-Melero, [6], and Igari, [9] using heavily geometrical machinery. Let  $(T_j^\alpha f)^\wedge(\xi) = (1 - (2^j|\xi|)^2)_+^\alpha \hat{f}(\xi)$ . The theorem of Carleson and Sjölin, [3], states that for each  $\alpha > 0$ , there exists a  $p_0(\alpha) < 4/3$  such that  $\|T_j^\alpha f\|_{L^p(\mathbb{R}^2)} \leq C_{p,\alpha} \|f\|_{L^{p'}(\mathbb{R}^2)}$  for  $p_0 < p < p'_0$ . Write  $(1 - |\xi|^2)_+^\alpha = n(\xi) + m(\xi)$  with  $n \in C_c^\infty$  and  $m$  supported in  $1/2 \leq |\xi| \leq 1$ ,  $m \in \Lambda_\alpha$ . The term corresponding to  $n$  is controlled by the Hardy-Littlewood maximal function, and by Corollary 3,  $\sum_{j \in \mathbb{Z}} \pm m(2^j \xi)$  is a multiplier of  $L^p(\mathbb{R}^2)$ ,  $p_0 < p < p'_0$ , uniformly in the random choice of  $\pm$ . The usual argument with Rademacher functions (see Stein, [12]) now shows that

$$\left\| \sup_j |T_j^\alpha f| \right\|_p \leq C_{p,\alpha} \|f\|_{p'}, \quad p_0 < p < p'_0.$$

We return to these matters more systematically in the next section.

SKETCH OF PROOF OF THEOREM 2. We wish to apply the Calderón-Zygmund theorem, and so we need to study the operators  $T_{j+l}(I - P_j)$  in order to be able to apply the Hörmander condition  $(H)$ . Recall that  $T_{j+l}$  has kernel  $K(x, y)\psi((x - y)/2^{j+l})$  where  $K$  is the integral kernel associated to  $T$ , and so  $T_{j+l}$  has symbol

$$\int \sigma(x, \eta)(\hat{\psi})_{2^{-j-l}}(\xi - \eta) d\eta = \sum_{i \in \mathbb{Z}} \int \sigma_{i+j}(x, \eta)(\hat{\psi})_{2^{-j-l}}(\xi - \eta) d\eta.$$

If  $\hat{\psi}$  had compact support, the  $i$ 'th term in this sum would be supported in  $\{|\xi| \sim 2^{-i-j}\}$  if  $l \geq i$  and in  $\{|\xi| \leq 2^{-j-l}\}$  if  $l \leq i$ . If  $\hat{\psi}$  were identically one on a neighbourhood of 0 in  $\mathbb{R}^n$ , only those terms in the sum with  $i \leq 0$  would contribute, then, when computing  $T_{j+l}(I - P_j)$  for  $l \geq 0$ . Now

$$\left| \sum_{i \leq 0} \sigma_{i+j} * (\hat{\psi})_{2^{-j-l}} [1 - \hat{\phi}(2^j \cdot)] \right|_1 \leq \sum_{i \leq 0} \alpha(i - l),$$

and so

$$\begin{aligned} \sum_{l \geq 0} |T_{j+l}(I - P_j)|_1 &\leq \sum_{l \geq 0} \sum_{i \leq 0} \alpha(i - l) \\ &= \sum_{k \leq 0} \alpha(k) \# \{(i, l): i - l = k, i \leq 0, l \geq 0\} \\ &\leq \sum_{k \leq 0} (|k| + 1)\alpha(k) < \infty, \end{aligned}$$



by hypothesis. But of course  $\hat{\phi}$  cannot be constant near 0, nor can  $\hat{\psi}$  have compact support, without violating the compact support of  $\phi$  and  $\psi$  respectively. The remainder of the proof is thus devoted to this technical matter, and is accordingly postponed to §5.  $\square$

SKETCH OF PROOF OF THEOREM 3. The proof of theorem 3 is essentially the same as that of theorem 2, except that we wish to verify (2) instead of (H). The second term that appears in (2)

$$\sup_j \left| \sum_{l \geq 0} T_{j+l} \tilde{Q}_j \right|_p,$$

may be treated in exactly the same manner as

$$\sup_j \left| \sum_{l \geq 0} T_{j+l} (I - P_j) \right|_p$$

was in the case  $p = 1$  in theorem 2, all the  $L^1$  estimates being equally valid for  $L^p$  under the hypothesis of the present theorem. To study the first term appearing in (2), fix  $k \leq 0$ ,  $l \geq 0$  and  $j \in \mathbb{Z}$  and write the symbol of  $T_{j+l} Q_{j+k}$  as

$$\sum_{i \in \mathbb{Z}} m_{i+j} * (\hat{\psi})_{2^{-j-l}}(\xi) \psi(2^{j+k} \xi) = \sum_{i < k-1} + \sum_{i=k-1}^{k+1} + \sum_{i=k+2}^{\infty} = I + II + III.$$

The main term is II; I and III are the error terms. To estimate  $|II|_p$ , we just use the hypothesis that  $|m_i * (\hat{\psi})_{2^{-j}}|_p \leq \alpha(i-j)$  to obtain

$$|II|_p \leq \alpha(k-1-l) + \alpha(k-l) + \alpha(k+1-l),$$

and so

$$\sum_{k \leq 0} \sum_{l \geq 0} \sup_j |II|_p \leq C \sum_{k \leq 0} |k| \alpha(k) < \infty;$$

I and III are dealt with exactly as are the trivial error terms in theorem 2, and we refer to §5 for the details.  $\square$

### 4. Maximal Functions

Let  $K^i$ ,  $i \in \mathbb{Z}$ , be a sequence of convolution kernels with corresponding multipliers  $m^i$ . We examine under what conditions the maximal operator  $\sup_i |K^i * f|$  is bounded on some  $L^p(\mathbb{R}^n)$ .

**Theorem 4.** *Suppose there exists a sequence  $(\alpha^i) \in l^\infty$  such that*

$$|m^i(\xi) - \alpha^i| \leq C(2^i |\xi|)^\epsilon \tag{5}$$

and

$$(6) \quad |m^i(\xi)| \leq C(2^i|\xi|)^{-\epsilon}$$

for some  $\epsilon > 0$ . Suppose also that for some  $1 \leq p \leq \infty$  we have

$$\sup_i \|m^i\|_{\mathfrak{M}_p} < \infty. \tag{7}$$

a) If  $p \geq 2$ , then

$$\left\| \sup_i |K^i * f| \right\|_r \leq C \|f\|_r, \quad 2 \leq r < p.$$

b) If  $p < 2$  and either

(i)  $K^i \geq 0$  for all  $i \in \mathbb{Z}$ , or

$$(ii) \|m^i(2^{-l}\cdot)\psi(\cdot)\|_{\Lambda_{\epsilon'}} \leq C, \quad i, l \in \mathbb{Z} \text{ some } \epsilon' > 0, \tag{8}$$

then

$$\left\| \sup_i |K^i * f| \right\|_r \leq C \|f\|_{r'}, \quad p < r \leq 2.$$

*Remark.* It is sometimes more convenient (for example when dealing with maximal functions along curves) to have a variant of (8) in which the kernel rather than the multiplier appears. An example of such a condition (easily verified in practical cases) is

$$|K_{i+l}^i|_2 \leq C2^{-\epsilon'l}, \quad l \geq 0 \text{ some } \epsilon' > 0 \tag{9}$$

which is equivalent to  $\|m^i(2^{-i}\cdot)\|_{\Lambda_{\epsilon'}} \leq C$ , which, together with (6) implies (8) (easy exercise). Condition (9) says that «most of the mass of  $K^i$  is concentrated around  $|x| \sim 2^i$ », while conditions (5) and (6) say that most of the mass of  $m^i$  is concentrated around  $|\xi| \sim 2^{-i}$ . It turns out to be useful to study this particular case first, which we single out as a lemma.

**Lemma.** *Suppose we are in the situation of theorem 4 with  $\alpha^i \equiv 0$  and  $m^i$  of compact support in  $|\xi| \sim 2^{-i}$ . Then*

$$\left\| \left( \sum_i |K^i * f|^2 \right)^{1/2} \right\|_r \leq C \|f\|_r, \quad p < r \leq 2.$$

**PROOF OF LEMMA.** The simplest approach is to observe that  $\sum \pm m^i(\xi)$  satisfies the hypothesis of corollary 3 uniformly in the choice of  $\pm$  as in §3 remark (iv) and apply Khintchine's inequality; alternatively one may observe that theorems 1 – 3 and their corollaries hold equally well in the context of Hilbert-

space-valued functions-all we have used is the Calderón-Zygmund decomposition and the Littlewood-Paley theory, which works equally well in the Hilbert-space-valued case. For this lemma we take the Hilbert spaces to be  $C$  and  $l^2(\mathbb{Z})$ .  $\square$

PROOF OF THEOREM 4. *a)* Let  $\tilde{K}^i = K^i - \alpha^i \phi_{2^i}$ ; by the Hardy-Littlewood maximal theorem it suffices to prove the result for  $\tilde{K}^i$ : this amounts to taking  $\alpha^i \equiv 0$  in (5). But now we are able to apply the Littlewood-Paley theory as in the proof of (4) in theorem 1 to obtain the required estimate.

*b)* (i) With  $\tilde{K}^i$  as above, we see that  $\tilde{K}^i$  is essentially positive in the sense of [2]; (5) and (6) show that the maximal operator associated to  $\{\tilde{K}^i\}$  is strongly bounded on  $L^2$  in the sense of [2]; inequality (7) allows us to apply theorem 2 of [2] to obtain the desired result.

*b)* (ii) Arguing as in part *a)*, we may first assume  $\alpha^i \equiv 0$  since clearly

$$\sup_i \|\hat{\phi}(2^i \cdot) \psi(\cdot)\|_{\Lambda_\epsilon} < \infty \quad \text{for all } \epsilon > 0.$$

Next, we write

$$m^i = \sum_{l \in \mathbb{Z}} m_{i+l}^i,$$

with  $m_{i+l}^i$  supported in  $\{|\xi| \sim 2^{-i-l}\}$ , as in §3. For each fixed  $l$ , we have that

$$\left( \sum_i |m_{i+l}^i(\xi)|^2 \right)^{1/2} \leq C 2^{-\epsilon|l|}$$

by (5) and (6);

$$\left( \sum_i |(m_{i+l}^i)_k * (\hat{\psi})_{2^{-j}}(\xi)|^2 \right)^{1/2} \leq C \|m_k^{k-l} * (\hat{\psi})_{2^{-j}}\|_\infty \leq C 2^{\epsilon(k-j)},$$

by (8); and

$$\left\| \left( \sum_i |[(m_{i+l}^i)_k * (\hat{\psi})_{2^{-j}}]^\vee * f|^2 \right)^{1/2} \right\|_p \leq C \|m_k^{k-l}\|_{\mathfrak{M}_p} \|f\|_p \leq C \|f\|_p,$$

by (7). Hence by the lemma, or the  $l^2$ -valued variant of corollary 3,

$$\left\| \sup_i |(m_{i+l}^i)_k^\vee * f| \right\|_r \leq C 2^{-\epsilon|l|} \|f\|_r, \quad p < r \leq 2.$$

Summing over  $l \in \mathbb{Z}$  finishes the proof.  $\square$

### 5. Proofs of Theorems 2 and 3. Technical Aspects

In §3 we postponed some of the technical aspects of the proofs of theorems

2 and 3. Here we give, for completeness, the full details; the significance of theorems 2 and 3 can be fully appreciated without reading this section.

PROOF OF THEOREM 2, CONTINUED. Having dealt with the  $i \leq 0$  terms previously, what remains is to show that

$$\begin{aligned} \sum_{l \geq 0} \left| \sum_{i > 0} \sigma_{i+j} * (\hat{\psi})_{2^{-j-l}} [1 - \hat{\phi}(2^j \cdot)] \right|_1 &= \\ &= \sum_{l \geq 0} \left| \sum_{i > 0} \sum_{k \in \mathbb{Z}} \sigma_{i+j} * (\hat{\psi})_{2^{-j-l}} \psi(2^{j+k} \cdot) [1 - \hat{\phi}(2^j \cdot)] \right|_1 < \infty. \end{aligned}$$

We write

$$\sum_{i > 0} \sum_{k \in \mathbb{Z}}$$

as

$$\begin{aligned} \sum_{k \leq 0} \sum_{i=0}^{\infty} + \sum_{k=0}^l \sum_{i=0}^k + \sum_{k=0}^l \sum_{i=k}^l + \sum_{k=0}^l \sum_{i=l}^{\infty} + \sum_{k=l}^{\infty} \sum_{i=0}^l + \sum_{k=l}^{\infty} \sum_{i=l}^{\infty} \\ = I + II + III + IV + V + VI. \end{aligned}$$

Claim 1. For  $k \leq 0$ ,

$$\left| \sum_{i > 0} \sigma_{i+j} * (\hat{\psi})_{2^{-j-l}}(\cdot) \psi(2^{j+k} \cdot) \right|_1 \leq C 2^{-\epsilon(l-k)},$$

some  $\epsilon > 0$ .

Claim 2.

$$\sum_{i=0}^k |\sigma_{i+j} * (\hat{\psi})_{2^{-j-l}}|_1 \leq \sum_{i=0}^k \alpha(i-l).$$

Claim 3. For  $k \leq i$ ,  $|\sigma_{i+j} * (\hat{\psi})_{2^{-j-l}}(\cdot) \psi(2^{j+k} \cdot)|_1 \leq C 2^{-\epsilon(l-k)}$ , so that

$$\sum_{i=k}^{\infty} | \quad |_1 \leq C(l-k) 2^{-\epsilon(l-k)} \leq C 2^{-\epsilon'(l-k)}.$$

Claim 4. For  $l \geq k$ ,

$$\left| \sum_{i=l}^{\infty} \sigma_{i+j} * (\hat{\psi})_{2^{-j-l}}(\cdot) \psi(2^{j+k} \cdot) \right|_1 \leq C 2^{-\epsilon(l-k)}.$$

Claim 5.

$$\sum_{i=0}^l |\sigma_{i+j} * (\hat{\psi})_{2^{-j-l}}|_1 \leq \sum_{i=0}^l \alpha(i-l).$$

Claim 6. For  $k \geq l$ ,

$$\left| \sum_{i=l}^{\infty} \sigma_{i+j} * (\hat{\psi})_{2^{-j-l}}(\cdot) \psi(2^{j+k} \cdot) \right|_1 \leq C.$$

If we accept claims 1 – 6 for the moment, the proof of the theorem is quickly concluded.

By claim 1,

$$\sum_{l \geq 0} |I|_1 \leq C \sum_{l \geq 0} \sum_{k \leq 0} 2^{-\epsilon(l-k)} < \infty;$$

by claim 2,

$$|II|_1 \leq C \sum_{k=0}^l \left\{ \sum_{i=0}^k \alpha(i-l) \right\} 2^{-\epsilon''k};$$

by claim 3,

$$|III|_1 \leq C \sum_{k=0}^l 2^{-\epsilon(l-k)} 2^{-\epsilon''k} \leq C 2^{-\epsilon l};$$

by claim 4,

$$|IV|_1 \leq C \sum_{k=0}^l 2^{-\epsilon(l-k)} 2^{-\epsilon''k} \leq C 2^{-\epsilon l};$$

by claim 5,

$$|V|_1 \leq C \sum_{k=l}^{\infty} \left\{ \sum_{i=0}^l \alpha(i-l) \right\} 2^{-\epsilon''k};$$

by claim 6,

$$|VI|_1 \leq C \sum_{k=l}^{\infty} 2^{-\epsilon''k} \leq C 2^{-\epsilon l}.$$

Hence

$$\sum_{i \geq 0} |III + IV + VI|_1 \leq C \sum_{l \geq 0} 2^{-\epsilon l} < \infty,$$

and

$$\begin{aligned} |II + V|_1 &\leq \sum_{k=0}^l \sum_{i=0}^k \alpha(i-l)2^{-\epsilon k} + \sum_{k=l}^{\infty} \sum_{i=0}^l \alpha(i-l)2^{-\epsilon k} \\ &= \sum_{i=0}^l \alpha(i-l) \sum_{k=i}^{\infty} 2^{-\epsilon k} \leq C \sum_{i=0}^l \alpha(i-l)2^{-\epsilon i}; \end{aligned}$$

thus

$$\begin{aligned} \sum_{l \geq 0} |II + V|_1 &\leq C \sum_{l=0}^{\infty} \sum_{i=0}^l 2^{-\epsilon i} \alpha(i-l) \\ &= C \sum_{j \leq 0} \alpha(j)2^{-\epsilon j} \sum_{l \geq -j} 2^{-\epsilon l} \leq C \sum_{j \leq 0} \alpha(j) < \infty. \end{aligned}$$

Now claims 2 and 5 follow immediately from the definition of  $\alpha(j)$ ; claims 1, 3, 4 and 6 all follow from the hypothesis that  $\sigma \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$  and the facts that  $|\partial^\gamma \hat{\psi}(s-t)| \leq C_{k,\gamma}/|t|^k$ , all  $k \in \mathbb{N}$ , if  $|s| \leq 2|t|$ , and  $|\partial^\gamma \hat{\psi}| \leq C_\gamma$ . In fact, if  $\tau(x, \xi)$  is any symbol, then

$$|\tau(Rx, R^{-1}\xi)|_p = |\tau(x, \xi)|_p \quad \text{for all } R, p;$$

if now  $\tau$  is supported in  $\{|\xi| \sim 1\}$  and satisfies  $|\partial_\xi^\gamma \tau(x, \xi)| \leq C_\gamma A$  for all multi-indices  $\gamma$ , then  $|\tau(x, \xi)|_1 \leq CA$ , since the integral kernel  $L$  associated to  $\tau$  satisfies

$$|L(x, y)| \leq CA, \quad |L(x, y)| \leq CA/|x-y|^{n+1}.$$

So to establish claims 1, 3, 4 and 6 we examine

$$\partial_\xi^\gamma \int \sum_i \sigma_{i+j}(2^{j+k}x, \eta)(\hat{\psi})_{2^{-j-l}}(\xi 2^{-j-k} - \eta) d\eta$$

for  $|\xi| \sim 1$ , (where the sum is over a range of  $i$  depending on which claim we are proving). In any case, what we get is

$$\int \sum_i \sigma_{i+j}(2^{j+k}x, \eta) 2^{(j+l)n} 2^{(l-k)|\gamma|} \partial^\gamma \hat{\psi}(2^{l-k}\xi - 2^{j+l}\eta) d\eta.$$

For claim 6, we just use  $|\partial^\gamma \hat{\psi}| \leq C$  and the fact the integrand is supported in a set of measure  $\leq C2^{-(j+l)n}$  to estimate the integral by  $\|\sigma\|_\infty 2^{(l-k)|\gamma|}$ ; under the hypotheses of claims 1, 3 and 4 we have that (essentially)  $2^{l-k}|\xi| \geq 2 \cdot 2^{j+l}|\eta|$  if  $|\xi| \sim 1$ , and thus we may estimate the integral by  $\|\sigma\|_\infty 2^{(j+l)n} 2^{(l-k)(|\gamma|-l)}$  multiplied by the measure of the support of the integrand, for each  $l \in \mathbb{N}$ . These measures are  $2^{-jn}$ ,  $2^{-(i+j)n}$  and  $2^{-(j+l)n}$  respectively, and so in each case we can dominate the integral by  $2^{(l-k)(n+|\gamma|-l)}$  for all  $l \in \mathbb{N}$ .  $\square$

PROOF OF THEOREM 3, CONTINUED. We must deal with the terms I and III from the proof begun in §3; thus to control I we study

$$\begin{aligned} & \partial^\gamma \int \sum_{i < k-1} m_{i+j}(\eta) (\hat{\psi})_{2^{-j-l}}(\xi 2^{-j-k} - \eta) d\eta \\ &= \int_{|\eta| \geq 2^{-k-j+1}} \sum_{i < k-1} m_{i+j}(\eta) 2^{(j+l)n} 2^{(l-k)|\gamma|} \partial^\gamma \hat{\psi}(2^{l-k}\xi - 2^{j+l}\eta) d\eta \end{aligned}$$

for  $|\xi| \sim 1$ . Now  $2^{j+l}|\eta| \geq 2^{l-k+1} \geq 2 \cdot 2^{l-k}|\xi|$  if  $|\xi| \sim 1$ . Hence we may estimate the integral, for each  $t \in \mathbb{N}$ , by

$$\int_{|\eta| \geq 2^{-k-j+1}} \|m\|_\infty 2^{(j+l)n} 2^{(l-k)|\gamma|} \frac{C_t}{[2^{(j+l)}|\eta|]^t} d\eta.$$

If  $t$  is chosen sufficiently large, this is dominated by  $\|m\|_\infty 2^{(l-k)(n-t+|\gamma|)}$ , and so

$$\sum_{k \leq 0} \sum_{l \geq 0} \sup |I|_p < \infty.$$

For III we examine

$$\int_{|\eta| \leq 2^{-k-j-2}} \sum_{i \geq k+2} m_{i+j}(\eta) 2^{(j+l)n} 2^{(l-k)|\gamma|} \partial^\gamma \hat{\psi}(2^{l-k}\xi - 2^{j+l}\eta) d\eta$$

for  $|\xi| \sim 1$ . This time,  $2^{j+l}|\eta| \leq 2^{l-k-2} \leq 2^{-1} 2^{l-k}|\xi|$  if  $|\xi| \sim 1$ . Hence we may estimate the integral by  $\|m\|_\infty 2^{(j+l)n} 2^{(l-k)(|\gamma|-l)} 2^{-(k+j)n} \leq 2^{(l-k)(|\gamma|-l+n)}$ , which gives the same estimate for III as for I, concluding the proof of the theorem.  $\square$

### 6. Concluding Remarks

Applications of the theory to singular integrals and maximal functions along «well-curved» curves, classical singular integrals, and lacunary versions of Stein’s spherical maximal function are detailed in [4] and [7], although the application to the lacunary maximal Bochner-Riesz operator in remark (iv), §3 appears to be new. In view of the wide variety of applications of the theory, it would be of interest to obtain analogues of our theorems in other cases of «natural» decompositions of operators. (Some partial results in the product domain setting have recently been obtained by the author and A. Seeger.) It would also be interesting to extend theorem 4b) (ii) to cover the case of «full» maximal operators - parts a) and b) (i) appear in [2]. The stumbling block here is the use of the Littlewood-Paley theory. We can use theorem 2, which is valid

in the general Banach-space context, to obtain results for maximal functions valid for  $1 < p \leq \infty$ ; but it would seem to be harder to obtain results sensitive to different  $p$ 's in (1, 2), which might be useful in the study of the maximal Bochner-Riesz operator, for example.

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