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Multipliers for Hermite Expansions

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1. Introduction

The aim of this paper is to prove certain multiplier theorems for the Hermite series. Given a function μ defined on the set of positive integers we can define, at least formally, the operator T_μ by the prescription

$$T_\mu f(x) = \sum_{\alpha \geq 0} \mu(2|\alpha| + n) f^\wedge(\alpha)\Phi_\alpha(x) \tag{1.1}$$

whenever f has the Hermite expansion

$$f(x) = \sum_{\alpha \geq 0} f^\wedge(\alpha)\Phi_\alpha(x) \tag{1.2}$$

We want to find conditions on the function μ so that the operator T_μ is bounded on L^p, for all p, $1 < p < \infty$. Clearly the boundedness of the function μ is a necessary condition which is also sufficient when $p = 2$. But for p different from 2 some more conditions are needed to ensure the boundedness. The classical Marcinkiewicz multiplier theorem for the Fourier series asserts the following. If f has the Fourier series expansion $f(\theta) = \sum a_k e^{ik\theta}$ and if (μ_k) is a bounded sequence of complex numbers satisfying the condition

$$\sup_j \sum_{2^j \leq k \leq 2^{j+1}} |\mu_k - \mu_{k+1}| \leqslant C \tag{1.3}$$

then the following inequality holds for $1 < p < \infty$.

$$\|\sum \mu_k a_k e^{ik\theta}\|_p \leqslant C \|\sum a_k e^{ik\theta}\|_p \tag{1.4}$$

Our aim in this paper is to generalize this result to the Hermite expansions.

A version of the Marcinkiewicz multiplier theorem for the Spherical Harmonic expansions was proved by Bonami-Clerc [1] and Strichartz [17]. Bonami-Clerc used the arguments of Muckenhoupt-Stein [9] together with the Cesaro summability results. On the other hand Strichartz used the method employed by Stein [15] in his proof of Hormander-Mihlin multiplier theorem for Fourier integrals. To state their results let us introduce the following finite difference operators. These operators are defined inductively as follows:

$$\Delta\mu(N) = \mu(N+1) - \mu(N)$$

and for $k \geqslant 1$, they are defined by

$$\Delta^{k+1}\mu(N) = \Delta^k\mu(N+1) - \Delta^k\mu(N).$$

The following is the Marcinkiewicz multiplier theorem for the Spherical Harmonic expansions.

Let be (μ_k) a bounded sequence of complex numbers satisfying the condition

$$\sup_j 2^{j(N-1)} \sum_{2^j \leq k \leq 2^{j+1}} |\Delta^N \mu_k| \leqslant C \tag{1.5}$$

where N is the smallest integer greater than $n/2$. Then we have the inequality for $1 < p < \infty$,

$$\|\sum \mu_k H_k f\|_p \leqslant C\|\sum H_k f\|_p \tag{1.6}$$

where $H_k f$ is the orthogonal projection of f into the k-th eigenspace.

In [8] G. Mauceri studied Marcinkiewicz multiplier theorm for the Hermite expansions. His conditions on μ involve finite difference operators of order $(n+1)$. If we use the summability results proved in [19] and [20] we can greatly improve Mauceri's result. The arguments of Bonami and Clerc can be used together with the summability results to prove the following result.

Assume that the function μ satisfies the condition

$$\sup_j 2^{j(k-1)} \sum_{2^j \leq N \leq 2^{j+1}} |\Delta^k \mu(N)| \leqslant C$$

where $k = [(3n-2)/6] + 2$. Then the operator T_μ is bounded on L^p, $1 < p < \infty$.

This result is already an improvement over Mauceri's result when $n > 1$. As we noted before, in the case of Spherical Harmonics and Fourier series the number k of finite differences entering the conditions is the smallest integer bigger than $n/2$. We will show that this is true in the case of Hermite series also. The following is our version of the Marcinkiewicz multiplier theorem for Hermite series.

Theorem 1. *Assume that the function μ satisfies the conditions*

$$|\Delta^r \mu(N)| \leqslant CN^{-r} \quad \text{for} \quad r = 0, 1, \ldots, k,$$

where $k > n/2$. Then the operator T_μ is bounded on L^p, $1 < p < \infty$.

Observe that our conditions on μ are just like the Hormander-Mihlin conditions. The proof depends on some boundedness properties of g and g^* functions. We introduce and study these functions in section 2. So much for the Marcinkiewicz multiplier theorem. Another multiplier theorem we are interested in is given by the function

$$\mu(|\nu|) = (2|\nu| + n)^{-\alpha} e^{(2|\nu| + n)it}. \tag{1.7}$$

This defines the operator $T_t(\alpha)$ given by

$$T_t(\alpha)f(x) = \sum_{\nu \geq 0} (2|\nu| + n)^{-\alpha} e^{(2|\nu| + n)it} f^\wedge(\nu)\Phi_\nu(x) \tag{1.8}$$

This function μ does not satisfy the conditions of Theorem 1 unless $\alpha > n$ and so we cannot apply the Marcinkiewicz multiplier theorm. Fortunately, the kernel of this operator can be calculated explicitly and studied by other means. This operator behaves more or less like the operator given by convolution with the oscillating kernel $|x|^{-\alpha} e^{ix.x}$. Such operators have been studied by many authors, see e.g. [12], and [13]. For these operators we prove the following theorem.

Theorem 2. *When $\alpha = n|1/p\text{-}1/2|$, $1 < p < \infty$, the operators $T_t(\alpha)$ are bounded on L^p, i.e.,*

$$\|T_t(\alpha)f\|_p \leqslant C\|f\|_p. \tag{1.8}$$

When $p = 1$ and $\alpha = n/2$, $T_t(\alpha)$ is bounded from H^1 into L^1, where H^1 is the Hardy space.

$$\|T_t(\alpha)f\|_1 \leqslant C\|f\|_{H^1}. \tag{1.9}$$

This theorem extends the classical Hardy-Littlewood theorem for the Fourier transform. Recall that Hardy-Littlewood inequalities state the following.

$$\int |f^\wedge(x)|^p |x|^{n(p-2)} dx \leqslant C \int |f(x)|^p dx, \quad \text{for} \quad 1 < p \leqslant 2 \tag{1.10}$$

$$\int |f^\wedge(x)|^p dx \leqslant C \int |f(x)|^p |x|^{n(p-2)} dx, \quad \text{for} \quad p \geqslant 2 \tag{1.11}$$

$$\int |f^\wedge(x)| |x|^{-n} dx \leqslant C\|f\|_{H^1}. \tag{1.12}$$

These inequalities were first proved by Hardy and Littlewood [5] in 1926. For an easy proof see Sadosky [11]. The first two inequalities follow easily if we apply Marcinkiewicz interpolation theorem to the operators f going to $f^\wedge(x)|x|^n$ in the space $L^p(|x|^{-2n}\,dx)$. The third inequality can be proved using the atomic theory of H^1 spaces.

We can rewrite the above inequalities in the following way. Consider the fractional powers of the Laplacian defined as follows.

$$\{(-\Delta)^{-\alpha}f\}^\wedge(x) = |x|^{-2\alpha}f^\wedge(x).$$

If we let $T(\alpha)f = (-\Delta)^{-\alpha}f$, then the above inequalities take the following form with $\alpha = n|1/p - 1/2|$.

$$\|\{T(\alpha)f\}^\wedge\|_p \leqslant C\|f\|_p, \quad \text{for} \quad 1 < p \leqslant 2 \tag{1.13}$$

$$\|f\|_p \leqslant C\|\{T(\alpha)f\}^\wedge\|_p, \quad \text{for} \quad p \geqslant 2 \tag{1.14}$$

$$\|\{T(\alpha)f\}^\wedge\|_1 \leqslant C\|f\|_{H^1} \tag{1.15}$$

Theorem 2 gives inequalities of this type for the operator $(-\Delta + |x|^2)^{-\alpha}$. Observe that

$$(-\Delta + |x|^2)^{-\alpha}f(x) = \sum_{\nu \geq 0}(2|\nu| + n)^{-\alpha}f^\wedge(\nu)\Phi_\nu(x) \tag{1.16}$$

Let F stand for the Fourier transform. Since F commutes with the operator $(-\Delta + |x|^2)^{-\alpha}$ and $F\{\Phi_\nu(x)\} = i^{|\nu|}\Phi_\nu(x)$ we have the following formula.

$$F(-\Delta + |x|^2)^{-\alpha}f(x) = e^{-in\pi/4}\sum_{\nu \geq 0}(2|\nu| + n)^{-\alpha}e^{(2|\nu| + n)i\pi/4}f^\wedge(\nu)\Phi_\nu(x) \tag{1.17}$$

Thus we get the Hardy-Littlewood inequalities for the operator $(-\Delta + |x|^2)^{-\alpha}$. The inequalities of Theorem 2 have another application to the solutions of the Schrodinger equation $-i\partial_t u(x, t) = (-\Delta + |x|^2)u(x, t)$. Let $u(x, t)$ denote the solution of the initial value problem

$$-i\partial_t u(x, t) = (-\Delta + |x|^2)u(x, t), \qquad u(x, 0) = f(x) \tag{1.18}$$

The solution of this problem has the following expansion in terms of the Hermite functions.

$$u(x, t) = \sum_{\nu \geq 0} e^{(2|\nu| + n)it}f^\wedge(\nu)\Phi_\nu(x) \tag{1.19}$$

We like to know if any inequality of the type $\|u(x, t)\|_p \leqslant C(t)\|f\|_p$ holds. But this is too much to ask for. Indeed, $u(x, t)$ is nothing but a fractional power of the Fourier transform of f and as we know, Fourier transform, and for that matter any fractional power of that, cannot map L^p into itself unless $p = 2$.

Therefore, following Sjostrand [14], we define the Riesz means

$$G_\tau(\alpha)f(x) = \alpha\tau^{-\alpha}\int_0^\tau (\tau - t)^{\alpha - 1}u(x, t)\,dt$$

and ask the question, «For what values of α the operators $G_\tau(\alpha)$ will be bounded on L^p?». As we will see, this boils down to the study of the operators $T_t(\alpha)$. Using the Hardy-Littlewood inequalities we can prove the following theorem.

Theorem 3. *If $\alpha \geqslant n|1/p - 1/2|$, then the operators $G_\tau(\alpha)$ are bounded on L^p.*

(1.20) $$\|G_\tau(\alpha)f\|_p \leqslant C(\tau)\|f\|_p, \qquad 1 < p < \infty.$$

When $p = 1$ and $\alpha = n/2$, $G_\tau(\alpha)$ is bounded from H^1 into L^1.

This paper is organised as follows. In the next section we study the Littlewood-Paley-Stein g functions. To fix the ideas we first consider the one dimensional case and then indicate how we prove the results in the general case. In section 3, we prove the Marcinkiewicz multiplier theorem. In section 4, we prove Hardy-Littlewood inqualities and study the Riesz means for the solutions of the Schrodinger equation in the last section.

This paper forms one part of my Princeton University thesis written under the guidance of Prof. E. M. Stein. The amount of help I got from him and the real interest he showed in the progress of this work cannot be exaggerated. I would like to thank him for everything. I am also grateful to Dr. Chris Sogge for turning my attention towards Marcinkiewicz multiplier theorem.

2. Littlewood-Paley-Stein Theory of g Functions

The g functions are defined in [16] in the more general context of semigroups of operators satisfying certain conditions. Here we are interested in the Hermite semigroup H^t. For $t > 0$ these operators are defined by

$$H^t f(x) = \sum e^{-Nt} f^\wedge(n)\varphi_n(x)$$

where $N = 2n + 1$ as usual and they have the kernel

$$K_t(x, y) = \sum e^{-Nt}\varphi_n(x)\varphi_n(y).$$

In view of the Mehler's formula K_t is given by

$$K_t(x, y) = (\sinh 2t)^{-1/2}e^{\varphi(t)}$$

where

$$\varphi(t) = -1/2(x^2 + y^2)\coth 2t + xy\,\text{cosech}\, 2t.$$

It is easy to see that H^t forms a semigroup of operators satisfying all the conditions except the last one listed in [16]. We can now define the g function by setting

$$g(f, x)^2 = \int_0^\infty t|\partial_t H^t f(x)|^2 \, dt.$$

Since the Hermite semigroup fails to satisfy the condition $H^t 1 = 1$, the general theory developed in [16] cannot be applied. But in view of the explicit form of the kernel $K_t(x, y)$ we can prove the following theorem without much difficulty.

Theorem 2.1. *With some constants C_1 and C_2 we have the following inequality*

$$C_1 \|f\|_p \leqslant \|g(f)\|_p \leqslant C_2 \|f\|_p, \qquad 1 < p < \infty.$$

PROOF. The L^2 boundedness of the g function is easy. Since

$$\partial_t H^t f(x) = -\sum e^{-Nt} N f^\wedge(n) \phi_n(x)$$

$$\int g(f, x)^2 \, dx = \int_0^\infty t \, dt \int |\partial_t H^t f(x)|^2 \, dx.$$

But it is immediate that

$$\int |\partial_t H^t f(x)|^2 \, dx = \sum e^{-2Nt} N^2 |f^\wedge(n)|^2$$

and hence we get

$$\|g(f)\|_2^2 = \sum_{n=0}^{\infty} \int_0^\infty t e^{-2Nt} N^2 \, dt |f^\wedge(n)|^2$$

which is equal to $1/4 \sum |f^\wedge(n)|^2 = 1/4 \|f\|^2$. This proves the L^2 boundedness. We will now prove that $g(f)$ is weak type (1, 1). That will prove the inequality $\|g(f)\|_p \leqslant C_2 \|f\|_p$ and the deduction of the inequality in the other direction is routine.

In proving the weak type (1,1) inequality we closely follow Stein [15]. We consider g as a Hilbert space valued singular integral operator. To be precise, g is a singular integral operator whose kernel $\partial_t K_t(x, y)$ is taking values in the Hilbert space $L^2(\mathbb{R}^+, t\, dt)$. Since g is already known to be bounded on L^2 we need to check the following condition on K_t.

$$\int_{|x - y^*| \geq 2|y - y^*|} \|\partial_t K_t(x, y) - \partial_t K_t(x, y^*)\| \, dx \leqslant C \tag{2.1}$$

where $\|\bullet\|$ is the norm of the Hilbert space $L^2(\mathbb{R}^+, t\, dt)$. Once this condition is checked we can invoke Theorem 5.1 in [15] to get the weak type estimate. The condition (2.1) is checked using the following estimate on the kernel $\partial_t K_t(x, y)$.

Lemma 2.1. $|\partial_y \partial_t K_t(x,y)| \leqslant Ct^{-3/2}|x-y|^{-1}(1+t^{-1/2}|x-y|)^{-2}$.

PROOF. The function φ can be written as

$$\varphi(t) = -(x-y)^2/(2\sinh 2t) - \tanh t\,(x^2+y^2)/2.$$

Since $\partial_y \partial_t K_t(x,y)$ is going to have many terms we indicate how to estimate one typical term viz.

$$J = (\sinh 2t)^{-7/2}\cosh 2t\,(x-y)^3 e^{\varphi(t)}.$$

First let us assume that $0<t<1$ so that $\sinh 2t = O(t)$ and $\cosh 2t = O(1)$. Then it is clear that we have the estimates

$$|J| \leqslant Ct^{-3/2}|x-y|^{-1} \quad \text{and} \quad |J| \leqslant Ct^{-1/2}|x-y|^{-3}$$

hence

$$|J| \leqslant Ct^{-3/2}|x-y|^{-1}(1+t^{-1/2}|x-y|)^{-2}.$$

The other terms are estimated similarly. Getting the estimates when t is greater than 1 is similar. In fact, we can get better estimates since $\sinh 2t = O(e^t)$ and $\cosh 2t = O(e^t)$ when $t \geqslant 1$. The details are omitted. This completes the proof of the Lemma.

Now it is easy to see how the condition follows from the Lemma. First we see that

$$\|\partial_t \partial_y K_t(x,y)\|^2 \leqslant C|x-y|^{-2}\int_0^\infty t^{-2}(1+t^{-1/2}|x-y|)^{-4}\,dt$$

which is less than or equal to $|x-y|^{-4}$. Now an application of the mean value theorem shows that

$$\begin{aligned}(2.2)\quad \int_{|x-y^*|\geq 2|y-y^*|} &\|\partial_t K_t(x,y) - \partial_t K_t(x,y^*)\|\,dx \\ &\leqslant C\int_{|x-y^*|\geq 2|y-y^*|} \|\partial_y \partial_t K_t(x,y_0)\|\,|y-y^*|\,dx \\ &\leqslant C\int_{|x-y^*|\geq 2|y-y^*|} |x-y_0|^{-2}|y-y^*|\,dx\end{aligned}$$

where y_0 lies between y and y^*. Since $|x-y_0| \geqslant 1/2|x-y^*|$ the condition (2.1) is verified.

To prove the Marcinkiewicz multiplier theorem, we have to introduce some more auxiliary functions. For any integer $k>1$, we can define the functions

$$g_k(f,x)^2 = \int_0^\infty t^{2k-1}|\partial_t^k H^t f(x)|^2\,dt.$$

Then it is an easy matter to prove that $g(f,x) \leqslant Cg_{k+1}(f,x)$. Indeed, by setting $u(x,t) = H^t f(x)$ we see that all t derivatives of $u(x,t)$ tend to zero as t goes to infinity. Therefore, writing

$$\partial_t^k u(x,t) = \int_t^\infty \partial_s^{k+1} u(x,s) s^k s^{-k}\, ds$$

we get the estimate

$$|\partial_t^k u(x,t)|^2 \leqslant \left\{ \int_t^\infty |\partial_s^{k+1} u(x,s)|^2 s^{2k}\, ds \right\} \left\{ \int_1^\infty s^{-2k}\, ds \right\}$$

Thus $g_k(f,x) \leqslant A_k g_{k+1}(f,x)$ and the claim is proved by induction. Another function we need is the g^* function which is defined by

$$g^*(f,x)^2 = \int_{-\infty}^{+\infty} \int_0^\infty t^{1/2}(1 + t^{-1/2}|x-y|)^{-2} |\partial_t H^t f(y)|^2\, dy\, dt.$$

The basic result about g^* which we are going to use is the following theorem. The proof is easy and for the sake of completeness we sketch it here.

Theorem 2.2. $\|g^*(f)\|_p \leqslant C\|f\|_p$, *for* $2 < p < \infty$.

PROOF. Let Ψ be a nonnegative function. We claim

$$\int g^*(f,x)^2 \Psi(x)\, dx \leqslant C \int g(f,x)^2 \Lambda\Psi(x)\, dx$$

where Λ is the Hardy-Littlewood maximal function. This is an easy consequence of the fact that

$$\sup_{t>0} \int t^{-1/2}(1 + t^{-1/2}|x-y|)^{-2} \Psi(y)\, dy \leqslant C\Lambda\Psi(x).$$

Since Λ is bounded on L^p, $1 < p < \infty$, an application of Hölder's inequality proves the theorem.

3. Marcinkiewicz Multiplier Theorem

We will prove that $g(F,x) \leqslant Cg^*(f,x)$ where $F(x) = T_\mu f(x)$. Then in view of theorems 2.1 and 2.2 it will follow that $\|F\|_p \leqslant C\|f\|_p$. Again, we need only to prove the inequality $g_2(F,x) \leqslant Cg^*(f,x)$. To prove this we introduce the function M.

(3.1) $$M(t,x,y) = \sum e^{-Nt} \mu(n) \varphi_n(x) \varphi_n(y)$$

If we let $u(x,t) = H^t f(x)$ and $U(x,t) = H^t F(x)$, then we can write

(3.2) $$U(x, t+s) = \int u(y,t) M(s,x,y)\, dy.$$

Differentiating the above expression with respect to s and t and then setting $t = s$ we get

$$\partial_t^2 U(x, 2t) = \int \partial_t u(y, t)\partial_t M(t, x, y)\, dy \tag{3.3}$$

Now we need to translate the hypothesis on the function μ into properties of M. This is done in the next lemma. For technical reasons we assume that $\mu(0) = 0$ without losing any generality.

Lemma 3.1. *Assume that μ satisfies the condition $k|\Delta\mu(k)| \leqslant C$. Then we have*

$$|\partial_t M(t, x, y)| \leqslant Ct^{-3/2} \tag{3.4}$$

$$\int |x - y|^2 |\partial_t M(t, x, y)|^2\, dy \leqslant Ct^{-3/2}. \tag{3.5}$$

Assuming the lemma for a moment we will first prove the inequality $g_2(F, x) \leqslant Cg^*(f, x)$.

$$\begin{aligned}\partial_t^2 U(x, 2t) &= \int_{|x-y| \leq t^{1/2}} \partial_t u(y, t)\partial_t M(t, x, y)\, dy \\ &\quad + \int_{|x-y| > t^{1/2}} \partial_t u(y, t)\partial_t M(t, x, y)\, dy \\ &= A_t(x) + B_t(x).\end{aligned} \tag{3.6}$$

Applying Schwarz inequality and using (3.4) we see that

$$\begin{aligned}|A_t(x)|^2 &\leqslant \int_{|x-y| \leq t^{1/2}} |\partial_t u(y, t)|^2\, dy \int_{|x-y| \leq t^{1/2}} |\partial_t M(t, x, y)|^2\, dy \\ &\leqslant Ct^{-5/2} \int (1 + t^{-1/2}|x - y|)^{-2} |\partial_t u(y, t)|^2\, dy.\end{aligned} \tag{3.7}$$

Another application of Schwarz inequality to $B_t(x)$ gives

$$\begin{aligned}|B_t(x)|^2 \leqslant &\int_{|x-y| > t^{1/2}} |x - y|^{-2} |\partial_t u(y, t)|^2\, dy \\ &\cdot \int_{|x-y| > t^{1/2}} |x - y|^2 |\partial_t M(t, x, y)|^2\, dy\end{aligned} \tag{3.8}$$

In view of the estimate (3.5) the above becomes

$$|B_t(x)|^2 \leqslant Ct^{-5/2} \int (1 + t^{-1/2}|x - y|)^{-2} |\partial_t u(y, t)|^2\, dy \tag{3.9}$$

Thus we have

$$|\partial_t^2 U(x, 2t)|^2 \leqslant Ct^{-5/2} \int (1 + t^{-1/2}|x - y|)^{-2} |\partial_t u(y, t)|^2\, dy$$

and hence

$$g_2(F,x)^2 \leqslant C \int_{t>0} \int t^{1/2}(1 + t^{-1/2}|x-y|)^{-2} |\partial_t u(y,t)|^2 \, dy \, dt \leqslant C g^*(f,x)^2. \tag{3.10}$$

Let us now prove the Lemma 3.1. (3.4) is a simple consequence of the boundedness of μ.

$$|\partial_t M(t,x,y)|^2 = |\sum e^{-Nt} N\mu(n)\varphi_n(x)\varphi_n(y)|^2 \leqslant \{\sum e^{-Nt} N\varphi_n(x)^2\}\{\sum e^{-Nt} N\varphi_n(y)^2\}. \tag{3.11}$$

Since

$$\sum e^{-Nt} N\varphi_n(x)^2 = -\partial_t\{\sum e^{-Nt}\varphi_n(x)^2\} = -\partial_t\{(\sinh 2t)^{-1/2} \exp(-x^2 \tanh t)\},$$

(3.4) follows immediately. To prove (3.5) we use the following recursion formula (see [18]).

$$2x\varphi_n(x) = \{2(n+1)\}^{1/2}\varphi_{n+1}(x) + (2n)^{1/2}\varphi_{n-1}(x). \tag{3.12}$$

Let us introduce the operators A and B defined by $A = -d/dx + x$ and $B = -d/dy + y$. These operators have the following effect on the Hermite functions:

$$A\varphi_n(x) = (2(n+1))^{1/2}\varphi_{n+1}(x) \quad \text{and} \quad B\varphi_n(y) = (2(n+1))^{1/2}\varphi_{n+1}(y).$$

We use the recursion formula to calculate $2(x-y)\partial_t M(t,x,y)$. An easy calculation using the recursion formula and the action of A and B show that

$$2(x-y)\partial_t M(t,x,y) = (B-A)\{\sum \Delta\Psi(n)\varphi_n(y)\varphi_n(x)\} \tag{3.13}$$

where $\Psi(n) = e^{-Nt}N\mu(n)$. Applying the Leibnitz rule for the finite differences, a typical term will be of the form $(B-A)\{\sum e^{-(N+2)t}(N+2)\,\Delta\mu(n)\varphi_n(x)\varphi_n(y)\}$. Since $(B-A)$ brings down a factor of $(2n+2)^{1/2}$ the square of the L^2 norm of this series is bounded by

$$\sum e^{-2(N+2)t}(N+2)^2|\Delta\mu(n)|^2(N+1)\varphi_n(x)^2.$$

By the hypothesis on μ the term $(N+2)^2|\Delta\mu(n)|^2$ is bounded independent of n and hence the above sum is dominated by a constant times $\sum e^{-Nt}N\phi_n(x)^2$ which is bounded by $Ct^{-3/2}$. Similar estimates hold for all other terms and this completes the proof of the Lemma.

Let us now consider the n dimensional case. The n dimensional Hermite semigroup H^t is defined by means of the kernel $K_t(x,y)$ which is given by

$$K_t(x,y) = \sum e^{-(2|\alpha|+n)t}\Phi_\alpha(x)\Phi_\alpha(y)$$

where Φ_α are the n dimensional Hermite functions. In view of the Mehler's formula we have

$$K_t(x,y) = (\sinh 2t)^{-n/2} \exp\{\Phi(t)\}$$

where

$$\Phi(t) = -1/2(|x|^2 + |y|^2)\coth 2t + x \cdot y \operatorname{cosech} 2t.$$

Denoting the differentiation with respect to y_j by ∂_j the following estimates can be obtained just like the one dimensional case.

$$|\partial_t \partial_j K_t(x,y)| \leqslant Ct^{-n/2-1}|x-y|^{-1}(1 + t^{-1/2}|x-y|)^{-n-1} \tag{3.14}$$

for $j = 1, 2, \ldots, n$. If we define the g and g_k functions as in the one dimensional case, then in view of the above estimate it is easily seen that Theorem 2.1 holds true in the n dimensional case also. We also have the relation

$$g(f,x) \leqslant A_k g_{k+1}(f,x)$$

between g and g_k. We need one more auxiliary function which is the n dimensional version of the g^* function. For $k > 0$ we define g_k^* by

$$(g_k^*(f,x))^2 = \int \int_0^\infty t^{(2-n)/2}(1 + t^{-1/2}|x-y|)^{-2k}|\partial_t H^t f(y)|^2 \, dy \, dt.$$

For $k > n/2$ the function $(1 + |x-y|)^{-2k}$ belongs to L^1 and hence it is easy to prove Theorem 2.2 for the g_k^* function i.e. we have the inequality

$$\|g_k^*(f)\|_p \leqslant C\|f\|_p,$$

provided $k > n/2$. As in the one dimensional case we set $F(x) = T_\mu f(x)$ and will prove that

$$g_{k+1}(F,x) \leqslant C g_k^*(f,x)$$

where $k > n/2$ is an integer. This will then prove the multiplier theorem. We start by defining

$$M(t,x,y) = \sum e^{-(2|\alpha|+n)t}\mu(2|\alpha| + n)\Phi_\alpha(x)\Phi_\alpha(y). \tag{3.15}$$

The following Lemma translates the hypothesis on μ into properties of $M(t,x,y)$.

Lemma 3.2. *Assume the function μ satisfies the hypothesis of Theorem 1. Then we have*

$$|\partial_t^k M(t,x,y)| \leqslant Ct^{-n/2-k} \tag{3.16}$$

$$\int |x-y|^{2k}|\partial_t^k M(t,x,y)|^2 \, dy \leqslant Ct^{-n/2-k}. \tag{3.17}$$

PROOF. The estimate (3.16) follows from the boundedness of μ as in the one dimensional case. The other estimate is a consequence of the following estimates:

$$(3.18)\quad \int |(x-y)^\beta \partial_t^k M(t,x,y)|^2\, dy \leqslant Ct^{-n/2-k}, \quad \text{for all} \quad \beta \quad \text{with} \quad |\beta| = k.$$

To prove these estimates we have to introduce some more notation. Consider the following operators A_j and B_j defined by

$$A_j = -d/dx_j + x_j \quad \text{and} \quad B_j = -d/dy_j + y_j.$$

These operators have the following effect on Φ_α:

$$A_j \Phi_\alpha(x) = \{2(\alpha_j + 1)\}^{1/2} \Phi_{\alpha+e^j}(x)$$

and

$$B_j \Phi_\alpha(y) = \{2(\alpha_j + 1)\}^{1/2} \Phi_{\alpha+e^j}(y)$$

where e^j is the j-th co-ordinate vector. Given a series

$$M(t,x,y) = \sum \Psi(|\alpha|)\Phi_\alpha(x)\Phi_\alpha(y)$$

we denote by $\Delta^r M(t,x,y)$ the series defined by

$$\Delta^r M(t,x,y) = \sum \Delta^r \Psi(|\alpha|)\Phi_\alpha(x)\Phi_\alpha(y)$$

where $\Delta^r \Psi$ is the finite difference of order r of Ψ. For technical convenience we assume that $\Psi(|\alpha|) = 0$ for all α with $|\alpha| \leqslant k$. Let us first calculate $2(x_j - y_j)M(t,x,y)$. Proceeding as in the one dimensional case we obtain

$$2(x_j - y_j)M(t,x,y) = (B_j - A_j)\,\Delta M(t,x,y).$$

Now it is clear how to proceed further. Iteration of the above procedure produces

$$(*)\qquad 2^m(x_j - y_j)^m M(t,x,y) = \sum C_{rs}(B_j - A_j)^r\, \Delta^s M(t,x,y)$$

where the sum is extended over all r and s satisfying the conditions $2s - r = m$, $s \leqslant m$.

The proof of $(*)$ is by induction. As we have seen the result is true for $m = 1$. Assuming the result for m, we will now consider

$$(3.19)\quad 2^{m+1}(x_j - y_j)^{m+1} M(t,x,y) = \sum C_{rs} 2(x_j - y_j)(B_j - A_j)^r\, \Delta^s M(t,x,y)$$

Let us write

$$(x_j - y_j)(B_j - A_j)^r = [(x_j - y_j), (B_j - A_j)^r] + (B_j - A_j)^r(x_j - y_j)$$

and calculate the commutator $[(x_j - y_j), (B_j - A_j)^r]$. It is easily seen that

$$[(x_j - y_j), (A_j - B_j)] = 2I,$$

where I is the identity operator. Now we claim that

$$[(x_j - y_j), (B_j - A_j)^r] = -2r(B_j - A_j)^{r-1}.$$

We prove the claim by induction. Suppose we have $[T, S] = 2I$ and $[T, S^r] = 2rS^{r-1}$.

$$\begin{aligned}(3.20)\qquad [T, S^{r+1}] &= (TS^r - S^rT + S^rT)S - S(S^rT - TS^r + TS^r)\\ &= 4rS^r + S^r(TS - ST + ST) - (ST - TS + TS)S^r\\ &= 4(r+1)S^r - [T, S^{r+1}]\end{aligned}$$

so that $[T, S^{r+1}] = 2(r+1)S^r$ and this proves the claim. Thus we have the equations

$$(3.21)\qquad 2(x_j - y_j)(B_j - A_j)^r = -2r(B_j - A_j)^{r-1} + (B_j - A_j)^r 2(x_j - y_j)$$

$$\begin{aligned}(3.22)\quad &2^{m+1}(x_j - y_j)^{m+1}M(t, x, y)\\ &= \sum C_{rs}\{-2r(B_j - A_j)^{r-1} + (B_j - A_j)^r 2(x_j - y_j)\}\,\Delta^s M(t, x, y)\end{aligned}$$

which equals to $\sum A_{rs}(B_j - A_j)^r\Delta^s M(t, x, y)$ with the conditions $2s - r = m + 1$, $s \leqslant m + 1$. This proves the equation (*).

Since the operator $(x_j - y_j)$ commutes with $(A_i - B_i)$ for i different from j, repeated application of (*) produces the following result

$$(3.23)\qquad (x - y)^\beta M(t, x, y) = \sum C_{\gamma\delta}(B - A)^\gamma \Delta^{|\delta|}M(t, x, y)$$

where $2\delta_j - \gamma_j = \beta_j$, $\delta_j \leqslant \beta_j$ and $(B - A)^\gamma$ stands for the product $\Pi(B_j - A_j)^{\gamma_j}$. Now we can complete the proof. Since

$$\partial_t^k M(t, x, y) = (-1)^k \sum e^{-(2|\alpha| + n)t}(2|\alpha| + n)^k \mu(2|\alpha| + n)\Phi_\alpha(x)\Phi_\alpha(y),$$

the above result (3.23) applied to $\partial_t^k M(t, x, y)$ gives

$$(3.24)\qquad (x - y)^\beta \partial_t^k M(t, x, y) = \sum C_{\gamma\delta}(B - A)^\gamma\,\Delta^{|\delta|}M_0(t, x, y)$$

where

$$M_0(t, x, y) = \sum \Psi(|\alpha|)\Phi_\alpha(x)\Phi_\alpha(y)$$

with

$$\Psi(|\alpha|) = (-1)^k e^{-(2|\alpha| + n)t}(2|\alpha| + n)^k \mu(2|\alpha| + n).$$

If we expand $(B - A)^{\gamma}$ and apply Leibnitz rule for finite differences, we see that a typical term in the sum (3.24) is of the form

$$\sum e^{-(2|\alpha|+n)t}(2|\alpha|+n)^k \Delta^{|\delta|}\mu(2|\alpha|+n) A^{\sigma}\Phi_{\alpha}(x) B^{\tau}\Phi_{\alpha}(y) \tag{3.25}$$

where $2|\delta| - |\tau| - |\sigma| = k$. Recalling the definition of the operators A and B we see that the square of the L^2 norm of the above sum is dominated by

$$\sum e^{-2(2|\alpha|+n)t}(2|\alpha|+n)^{2k+|\sigma|+|\tau|}|\Delta^{|\delta|}\mu(2|\alpha|+n)|^2|\Phi_{\alpha}(x)|^2. \tag{3.26}$$

Since

$$|\Delta^{|\delta|}\mu(2|\alpha|+n)|^2 \leqslant C(2|\alpha|+n)^{-2|\delta|} \quad \text{and} \quad 2|\delta| - |\tau| - |\sigma| = k,$$

the above sum is dominated by a constant times

$$\sum e^{-(2|\alpha|+n)t}(2|\alpha|+n)^k|\Phi_{\alpha}(x)|^2$$

which is bounded by $Ct^{-n/2-k}$. All other terms are similarly estimated. This completes the proof of Lemma 3.2.

Having proved the Lemma, Theorem 1 is proved just like the one dimensional version. We write

$$H^{t+s}F(x) = \int M(s, x, y) H^t f(y)\, dy.$$

Taking k derivatives with respect to s and one derivative with respect to t and then putting $t = s$, we get the expression

$$\partial_t^{k+1} H^{2t} F(x) = \int \partial_t^k M(t, x, y) \partial_t H^t f(y)\, dy. \tag{3.27}$$

In view of the Lemma we get $g_{k+1}(F, x) \leqslant C g_k^*(f, x)$ and this completes the proof.

4. Hardy-Littlewood Inequalities for $(-\Delta + |x|^2)$

We prove Theorem 2 when $n = 1$. There is absolutely no change in the proof for the general case. We first prove the inequality (1.9). The operators $T_t(\alpha)$ are all bounded on L^2. The other inequality (1.8) is then proved by interpolating between the L^2 result and the inequality (1.9). Then following interpolation theorem due Fefferman-Stein [3] is the one we are going to apply.

Theorem 4.1 (Fefferman-Stein). *Suppose S_z is an analytic family of operators satisfying*

(i) $\|S_{iy} f\|_1 \leqslant A_0(y) \|f\|_{H^1}$

(ii) $\|S_{1+iy} f\|_2 \leqslant A_1(y) \|f\|_2$

for all y, $-\infty < y < \infty$. *Assume that* $A_j(y)$ *satisfies the condition* $\log A_j(y) \leqslant c_j \exp\{d_j|y|\}$, $c_j > 0$ *and* $0 < d_j < \pi$. *If* $1/p = 1 - t/2$, $0 < t \leqslant 1$, *then* $\|S_t f\|_p \leqslant A\|f\|_p$.

The inequality (1.9) is proved using the atomic theory of H^p spaces. We say that a function φ is a p-atom if there is a ball B in $\mathbb{R}^n$ such that φ has the following properties.

$$\text{supp}(\varphi) \subset B, \quad \|\varphi\|_\infty \leqslant |B|^{-1/p} \tag{4.1}$$

$$\int \varphi(x)P(x)\,dx = 0 \tag{4.2}$$

for all polynomials of degree less than or equal to $k = n(1/p - 1)$. If f belong to $H^p(\mathbb{R}^n)$ it can be shown that there exists a sequence of p-atoms (φ_j) and a sequence of complex numbers (λ_j) such that $f = \sum \lambda_j \varphi_j$ in the sense of distributions and $\left(\sum|\lambda_j|^p\right)^{1/p} \leqslant C_p\|f\|_{H^p}$. Conversely, if f has the form $f = \sum \lambda_j \varphi_j$, then f belongs to $H^p(\mathbb{R}^n)$ and $\|f\|_{H^p} \leqslant C_p\left(\sum|\lambda_j|^p\right)^{1/p}$.

With these preliminaries consider the operator $T_t(\alpha)$ which is defined by

$$T_t(\alpha) = \sum (2n+1)^{-\alpha} e^{(2n+1)it} f^\wedge(n)\varphi_n(x) \tag{4.3}$$

The operator $T_t(\alpha)$ has the following kernel

$$K_t(x,y) = \sum (2n+1)^{-\alpha} e^{(2n+1)it}\varphi_n(y)\varphi_n(x). \tag{4.4}$$

We can write this kernel as

$$K_t(x,y) = 1/\Gamma(\alpha)\int_{t>0} \lambda^{\alpha-1} K_t^*(x,y,\lambda)\,d\lambda$$

where we have set

$$K_t^*(x,y,\lambda) = \sum e^{(2n+1)(-\lambda+it)}\varphi_n(y)\varphi_n(x). \tag{4.5}$$

In view of Mehler's formula we have with $r = e^{-2\lambda}$ the following expression

$$K_t^*(x,y,\lambda) = ce^{-(\lambda-it)}(1 - r^2e^{-4it})^{-1/2}\exp\{B_r(t,x,y)\} \tag{4.6}$$

where

$$B_r(t,x,y) = (1 - r^2e^{-4it})^{-1}\{-1/2(x^2+y^2)(1+r^2e^{-4it}) + 2xyre^{-2it}\}.$$

To prove that $T_t(\alpha)$ is bounded on L^p it is enough to show that

$$\int |K_t(x,y)|\,dx \leqslant C,$$

with a C independent of y.

It is easy to calculate the L^1 norm of $K_t(x,y)$. Let

$$C_r(t,x,y) = (1 - r^2e^{-4it})^{-1/2} \exp\{B_r(t,x,y)\}.$$

Then an easy calculation shows that

$$(4.7)\quad |C_r(t,x,y)| = a^{1/2} \exp\{-1/2a^2(1-r^4)(x^2+y^2) + 2rxya^2(1-r^2)\cos 2t\}$$

where

$$a^2 = \{(1-r^2)^2 + 4r^2\sin^2 2t\}^{-1}.$$

Letting $b^2 = a^2(1-r^4)$ and $c = 2r\cos 2t(1+r^2)^{-1}$ we have

$$(4.8)\quad |C_r(t,x,y)| = a^{1/2}\exp\{-1/2b^2(x-cy)^2\}\exp\{-1/2b^2(1-c^2)y^2\}.$$

It is easily seen that $b^2(1-c^2) = (1-r^2)/(1+r^2)$. Thus we have

$$(4.9)\quad \int |C_r(t,x,y)|\,dx = a^{1/2}\exp\{-1/2(1-r^2)/(1+r^2)y^2\}\int \exp\{-1/2b^2(x-cy)^2\}\,dx$$

which is equal to

$$Aa^{-1/2}(1-r^4)^{-1/2}\exp\{-1/2(1-r^2)/(1+r^2)y^2\}.$$

Therefore, we have

$$(4.10)\quad \int |K_t(x,y)|\,dx \leqslant A\int_{\lambda>0}\lambda^{\alpha-1}e^{-\lambda}(1-r^2)^{-1/2}\{(1-r^2)^2 + 4r^2\sin^2 2t\}^{1/4}\,d\lambda.$$

From this we see that when $\sin 2t = 0$, the kernel $K_t(x,y)$ is integrable for all $\alpha > 0$ but when $\sin 2t$ is not 0 the kernel is integrable only if $\alpha > 1/2$. Since we are interested in the case $\alpha = 1/2$ and $t = \pi/4$, we have to study the operators by other means. That is why we need the atomic theory of the Hardy spaces.

Suppressing α, let us considet the operator $T_t(1/2) = T_t$. The kernel $K_t(x,y)$ of this operator is given by

$$(4.11)\quad K_t(x,y) = \sum (2n+1)^{-1/2}e^{(2n+1)it}\varphi_n(y)\varphi_n(x)$$

Since

$$K_t^*(x,y,\lambda) = \sum e^{(2n+1)(-\lambda+it)}\varphi_n(y)\varphi_n(x)$$

we can write the kernel $K_t(x,y)$ as

$$K_t(x,y) = c\int_0^\infty \lambda^{-1/2} K_t^*(x,y,\lambda)\,d\lambda.$$

A simple calculation shows that

$$K_t^*(x,y,\lambda) = c\{\sinh 2(\lambda - it)\}^{-1/2} e^{-A_t(x,y,\lambda)} e^{iB_t(x,y,\lambda)} \tag{4.12}$$

where $A_t(x,y,\lambda)$ and $B_t(x,y,\lambda)$ are given by the following equations.

$$\begin{aligned} 2A_t(x,y,\lambda) = {} & (\sinh^2 2\lambda + \sin^2 2t)^{-1}(\sinh 2\lambda)\{\cos 2t(x-y)^2 \\ & + (\cosh 2\lambda - \cos 2t)(x^2+y^2)\} \end{aligned} \tag{4.13}$$

$$\begin{aligned} 2B_t(x,y,\lambda) = {} & -(\sinh^2 2\lambda + \sin^2 2t)^{-1}(\sin 2t)\{\cosh 2\lambda(x-y)^2 \\ & - (\cosh 2\lambda - \cos 2t)(x^2+y^2)\}. \end{aligned} \tag{4.14}$$

First consider the integral taken from 1 to infinity. Since $\sinh 2\lambda$ behaves like $e^{2\lambda}$ for $\lambda \geqslant 1$, it can be easily checked that the integral

$$\int_1^\infty \lambda^{-1/2}\{\sinh 2(\lambda - it)\}^{-1/2} e^{iB_t(x,y,\lambda)} e^{-A_t(x,y,\lambda)}\,d\lambda$$

defines a nice L^1 kernel and hence the operator corresponding to this kernel is bounded on L^p, for all p, $1 \leqslant p \leqslant \infty$. So we can very well assume that the kernel of the operator T_t is given by

$$K_t(x,y) = \int_0^1 \lambda^{-1/2}\{\sinh 2(\lambda - it)\}^{-1/2} e^{-A_t(x,y,\lambda)} e^{iB_t(x,y,\lambda)}\,d\lambda$$

In view of the atomic decomposition, T_t will be bounded from H^1 into L^1 once we prove the following proposition.

Proposition 4.1. $\int |T_t f(x)|\,dx \leqslant C$ *whenever f is an* 1-*atom.*

In proving this proposition we closely follow Phong and Stein. In [10], they studied the boundedness of the operator T whose kernel is of the form $K(x-y)e^{iB(x,y)}$ where K is a Calderón-Zygmund kernel and B is a non-degenerate bilinear form. To prove the proposition we need certain estimates for the kernel $K_t(x,y)$. These estimates are proved in the next lemma. Let us write $K_t(x,y,\lambda) = \{\sinh 2(\lambda - it)\}^{-1} e^{-A_t(x,y,\lambda)}$.

Lemma 4.1. *We have the following inequalities.*

$$\left|\int_0^1 \lambda^{-1/2}\partial_y K_t(x,y,\lambda) e^{iB_t(x,y,\lambda)}\,d\lambda\right| \leqslant C|x-y|^{-2}$$

$$\left|\int_0^1 \lambda^{-1/2} K_t(x,y,\lambda)\partial_y\{e^{iB_t(x,y,\lambda)}\}\,d\lambda\right| \leqslant C(\sin 2t)^{-3/2}$$

$$\left|\int_0^1 \lambda^{-1/2}\partial_\lambda e^{iB_t(x,y,\lambda)}\lambda K_t(x,y,\lambda)\,d\lambda\right| \leqslant C|x-y|^{-3}.$$

PROOF. The proof of this lemma is elementary. We prove it when $0 < t \leqslant \pi/8$. The proof of the lemma for other intervals of t is similar. First we calculate that

$$(4.15) \qquad |\sinh 2(\lambda - it)|^2 = c(\sinh^2 2\lambda + \sin^2 2t)$$

Observe that for $0 \leqslant \lambda \leqslant 1$, $(\sinh^2 2\lambda + \sin^2 2t)$ behaves like $(\lambda^2 + \sin^2 2t)$. Let us prove the first inequality of the lemma. $\partial_y K_t(x, y, \lambda)$ has two terms. We will estimate only the contribution of

$$(4.16) \qquad J = \{\sinh 2(\lambda - it)\}^{-1/2}(\sinh^2 2\lambda + \sin^2 2t)^{-1}$$
$$(\sinh 2\lambda)(\cos 2t)(x - y)e^{-A_t(x,y,\lambda)}$$

Since $0 < t \leqslant \pi/8$, $\cos 2t \geqslant 2^{-1/2}$. We consider two cases. First assume that $t \leqslant \lambda$. In this case $(\sinh^2 2\lambda + \sin^2 2t)$ behaves like λ^2. Therefore,

$$(4.17) \qquad |J| \leqslant C(x - y)\lambda^{-3/2} \exp\{-c\lambda^{-1}(x - y)^2\}.$$

Integrating this agains $\lambda^{-1/2}$ we have

$$(x - y) \int_0^1 \lambda^{-2} e^{-c\lambda^{-1}(x-y)^2} d\lambda = (x - y) \int_1^\infty e^{-c\lambda(x-y)^2} d\lambda$$
$$\leqslant (x - y) \int_0^\infty \lambda^{1/2} e^{-c\lambda(x-y)^2} d\lambda.$$

This gives the estimate $C|x - y|^{-2}$. Next assume that $t > \lambda$. In this case $(\sinh^2 2\lambda + \sin^2 2t)$ behaves like t^2. Therefore,

$$|J| \leqslant C(x - y)\lambda t^{-5/2} \exp\{-c\lambda t^{-2}(x - y)^2\}.$$

This gives the integral

$$(x - y)t^{-5/2} \int_0^1 \lambda^{1/2} e^{-c\lambda t^{-2}(x-y)^2} d\lambda \leqslant (x - y)t^{-5/2} \int_0^\infty \lambda^{3/2-1} e^{-c\lambda t^{-2}(x-y)^2} d\lambda$$

which is bounded by $Ct^{1/2}|x - y|^{-2}$. This proves the first inequality when $0 < t \leqslant \pi/8$. If t is in the neighbourhood of $\pi/4$ we can use

$$(\sinh^2 2\lambda + \sin^2 2t)^{-1}(\sinh 2\lambda)(\cosh 2\lambda - \cos 2t)(x^2 + y^2)$$

in place of

$$(\sinh^2 2\lambda + \sin^2 2t)^{-1}(\sinh 2\lambda) \cos 2t(x - y)^2$$

since $(\cosh 2\lambda - \cos 2t) \geqslant 1 - \cos 2t \geqslant 2 \sin^2 t \geqslant c$. This completes the proof of the first inequality.

The proof of the second inequality is similar. Unfortunately the estimate we get is not uniform in t. We believe that a uniform estimate is possible though we are not able to prove it now. Again we will be having two terms. Consider the term

$$G = K_t(x, y, \lambda)(\sin 2t)(\sinh^2 2\lambda + \sin^2 2t)^{-1}(\cosh 2\lambda)(x - y).$$

Since $c \leqslant (\sinh^2 2\lambda + \sin^2 2t)^{-1} \leqslant (\sin^2 2t)^{-1}$, G in modulus is bounded by

$$|G| \leqslant C(x - y)(\sin 2t)^{-3/2} \exp\{-c\lambda(x - y)^2\}. \tag{4.18}$$

Integrating this against $\lambda^{-1/2}$ proves the desired inequality. The other term is estimated similarly. The proof of the third inequality follows along similar lines. Differentiation with respect to λ brings down a factor of λ and hence we get $|x - y|^{-3}$. Hence the lemma.

Having proved the required estimates, we can now prove Proposition 4.1. Assume that f is an 1-atom supported in $|x - y^*| \leqslant \delta$ i.e., f satisfies the following two conditions.

(i) $$\|f\|_\infty \leqslant \delta^{-1}$$

(ii) $$\int f(x)\, dx = 0.$$

Let Q_δ denote the ball of radius δ centered at y^* and let CQ_d stand for the complement of Q_δ. We write $F(x) = T_t f(x)$ as a sum of three functions, $F = F_1 + F_2 + F_3$ where $F_1(x) = F(x)$ on $Q_{2\delta}$, 0 elsewhere; $F_2(x) = F(x)$ on $CQ_{2\delta} \cap Q_{\delta^{-1}}$, 0 elsewhere and $F_3(x) = F(x)$ on $CQ_{2\delta} \cap CQ_{\delta^{-1}}$, 0 elsewhere. We note that $F_2 = 0$ when $\delta \geqslant 2^{-1/2}$.

To study F_1, we apply the L^2 theory of T_t. From the definition, it is clear that T_t is bounded on L^2. Therefore

$$\begin{aligned}\int |F_1(x)|\, dx &\leqslant |Q_{2\delta}|^{1/2}\left\{\int |F_1(x)|^2\, dx\right\}^{1/2} \\ &\leqslant \delta^{1/2}\left\{\int |T_t f(x)|^2\, dx\right\}^{1/2} \\ &\leqslant C\delta^{1/2}\left\{\int |f(x)|^2\, dx\right\}^{1/2}.\end{aligned}$$

Since

$$\left\{\int |f(x)|^2\, dx\right\}^{1/2} \leqslant C\delta^{-1/2}$$

we get $\int |F_1(x)|\, dx \leqslant C$.

To study F_2 we write the kernel $K_t(x, y)$ in the following way.

$$K_t(x, y) = E_t(x, y) + G_t(x, y) + H_t(x, y)$$

where

$$E_t(x, y) = \int_0^1 \lambda^{-1/2}\{K_t(x, y, \lambda) - K_t(x, y^*, \lambda)\} e^{iB_t(x, y, \lambda)}\, d\lambda$$

$$G_t(x, y) = \int_0^1 \lambda^{-1/2} K_t(x, y^*, \lambda)\{e^{iB_t(x, y, \lambda)} - e^{iB_t(x, y^*, \lambda)}\}\, d\lambda$$

Since f has mean value 0, the kernel $H_t(x,y)$ does not contribute anything. Observe that when x is in $\mathbf{C}Q_{2\delta}$ and y is in Q_δ we have $|x-y^*| \geqslant 2|y-y^*|$. In view of Lemma 4.1 we have the estimates

$$|E_t(x,y)| \leqslant C|y-y^*|\,|x-y^*|^{-2} \tag{4.19}$$

$$|G_t(x,y)| \leqslant C|y-y^*|. \tag{4.20}$$

Therefore,

$$|F_2(x)| \leqslant C\delta\{1+|x-y^*|^{-2}\}.$$

Since $F_2(x)$ is supported in $2\delta \leqslant |x-y^*| \leqslant \delta^{-1}$, we get

$$\int |F_2(x)|\,dx \leqslant C\delta\left\{\int_{|x-y^*|\leq 1/\delta} dx + \int_{|x-y^*|\geq 2\delta} |x-y^*|^{-2}\,dx\right\} \leqslant C.$$

Finally, we consider F_3. We write the kernel as the sum of the following three terms.

$$E_t(x,y) = \int_0^1 \lambda^{-1/2}\{K_t(x,y,\lambda) - K_t(x,y^*,\lambda)\}e^{iB_t(x,y,\lambda)}\,d\lambda$$
$$G_t(x,y) = \int_0^1 \lambda^{-1/2}K_t(x,y^*,\lambda)\{e^{iB_t(x,y,\lambda)} - e^{iB_t(x,y,0)}\}\,d\lambda$$
$$H_t(x,y) = \int_0^1 \lambda^{-1/2}K_t(x,y^*,\lambda)e^{iB_t(x,y,0)}\,d\lambda$$

Using Lemma 4.1 we get the following estimates.

$$|E_t(x,y)| \leqslant C|y-y^*|\,|x-y^*|^{-2} \tag{4.21}$$

$$|G_t(x,y)| \leqslant C|x-y^*|^{-3}. \tag{4.22}$$

These estimates will imply that

$$|E_tf(x)| \leqslant C\delta|x-y^*|^{-2} \quad \text{and} \quad |G_tf(x)| \leqslant C|x-y^*|^{-3}.$$

Therefore,

$$\begin{aligned}\int_{|x-y^*|\geq \delta^{-1}} |E_tf(x)|\,dx &\leqslant C\delta\left\{\int_{|x-y^*|\geq\delta^{-1}} |x-y^*|^{-2}\,dx\right\}\\ &\leqslant C\delta\left\{\int_{|x-y^*|\geq 2\delta} |x-y^*|^{-2}\,dx\right\}\\ &\leqslant C.\end{aligned}$$

When $\delta \leqslant 1$ we get

$$\int_{|x-y^*|\geq\delta^{-1}} |G_tf(x)|\,dx \leqslant C\left\{\int_{|x-y^*|\geq\delta^{-1}} |x-y^*|^{-3}\,dx\right\} \leqslant C\delta^2 \leqslant C$$

and when $\delta > 1$ we have

$$\int_{|x-y^*|\geq 2\delta} |G_tf(x)|\,dx \leqslant C\left\{\int_{|x-y^*|\geq 2\delta} |x-y^*|^{-3}\,dx\right\} \leqslant C\delta^{-2} \leqslant C.$$

This takes care of the terms $E_tf(x)$ and $G_tf(x)$. Finally $H_tf(x)$ is given by

$$H_tf(x) = F_tf(x)\left\{\int_0^1 \lambda^{-1/2}K_t(x, y^*, \lambda)\, d\lambda\right\} = F_tf(x)g_t(x)$$

where

$$F_tf(x) = \int e^{iB_t(x,y,0)}f(y)\, dy.$$

By Plancherel's theorem $\|F_tf\|_2 = \|f\|_2$ and we also have $|g_t(x)| \leqslant C|x - y^*|^{-1}$. Hence by Schwarz inequality we obtain

$$\int_{|x-y^*|\geq\delta^{-1}} |H_tf(x)|\, dx \leqslant C\left\{\int |F_tf(x)|^2\, dx\right\}^{1/2}\left\{\int_{|x-y^*|\geq\delta^{-1}} |x - y^*|^{-2}\, dx\right\}^{1/2}$$
$$\leqslant C$$

since $\|f\|_2 \leqslant \delta^{-1/2}$. This completes the proof of the Proposition 4.1 and hence proves that the operator T_t maps H^1 boundedly into L^1.

We can now prove Theorem 2. Consider the analytic family S_z of operators defined by $S_z = T_t\{(1 - z)/2\}$. Then clearly $S_{1+iy} = T_t(-iy/2)$ is bounded on L^2. Also $S_{iy} = T_t\{(1 - iy)/2\}$ and for this operator we can prove that it maps H^1 boundedly into L^1 by repeating the proof of Proposition 4.1. By applying the interpolation theorem of Fefferman and Stein we get the theorem for $1 < p \leqslant 2$. For $p \geqslant 2$, the theorem follows from duality.

5. An Application to the Solutions of the Schrodinger Equation

Consider the solution $u(x, t)$ of the initial value problem for the Schrodinger equation

$$(5.1) \qquad -i\partial_t u(x, t) = (-\Delta + |x|^2)u(x, t), \qquad u(x, 0) = f(x)$$

where f is a nice function, say f belongs to the Schwartz class $S(\mathbb{R}^n)$. The solution $u(x, t)$ has the Hermite expansion

$$u(x, t) = \sum e^{(2|\alpha| + n)it} f^\wedge(\alpha)\Phi_\alpha(x)$$

where $f^\wedge(\alpha)$ are the Hermite coefficients of the function f. $f \to u$ defines an operator $F(t)$ which, in view of the above expansion, is given by a kernel as follows:

$$(5.2) \qquad F(t)f(x) = \pi^{-n/2} \int (\sin 2t)^{-n/2} e^{i\varphi(t,x,y)} f(y)\, dy$$

where

$$\varphi(t, x, y) = -x \cdot y \operatorname{cosec} 2t + 1/2(|x|^2 + |y|^2) \cot 2t.$$

Clearly, $F(t)$ defines a unitary operator on $L^2(\mathbb{R}^n)$. It also has the group property $F(t)F(s) = F(t+s)$, $F(0) =$ identity. Furthermore, when t takes values in the set $\{\pi/k: k = 1, 2, 3, \ldots\}$ $F(t)$ is a fractional power of F, the Fourier transform. For example, $F(\pi/4) = F$ and $F(\pi/8) = F^{1/2}$. Because of this and the group property the operators $F(t)$ cannot be bounded on L^p. For otherwise the Fourier transform has to be bounded on L^p which, of course, is not true.

A similar situation occurs when we consider the Schrodinger equation without the potential. Let $H(t)$ be the solution operator for the equation $-i\partial_t u(x,t) = -\Delta u(x,t)$, $u(x,0) = f(x)$. Formally this operator can be defined by $(H(t)f)^\wedge(\xi) = \exp\{it|\xi|^2\} f^\wedge(\xi)$. This operator is unitary on $L^2(\mathbb{R}^n)$. But as shown in Hormander [6] and Littman et al. [7], it fails to be bounded on L^p when p is different from 2. In this connection Sjostrand [14] considered the Riesz means of the operator $H(t)$ defined by

$$G_\tau(\alpha)f(x) = \alpha\tau^{-\alpha} \int_0^\tau (\tau - t)^{\alpha - 1} H(t) f(x)\, dt$$

and proved that the Riesz means $G_\tau(\alpha)$ are bounded on L^p if and only if $\alpha > n|1/p - 1/2|$. So it is natural to ask the same question with regard to the Riesz means $G_\tau(\alpha)$ of the operators $F(t)$. It turns out that Theorem 3 is true. For the sake of simplicity we treat the one dimensional case.

We write down the Hermite expansion of $F(t)f(x)$ for a smooth function f.

$$F(t)f(x) = \sum e^{(2n+1)it} f^\wedge(n)\varphi_n(x) \tag{5.3}$$

When f is in the Schwartz class the series converges uniformly. Let g be the inverse Fourier transform of the function $h(t)$ defined to be $(1-t)^{\alpha-1}$ for $0 < t < 1$ and 0 elsewhere. Multiplying the above series by $(\alpha/\tau)h(t/\tau)$ and integrating with respect to t we obtain

$$G_\tau(\alpha)f(x) = \sum g\{(2n+1)\tau\} f^\wedge(n)\varphi_n(x). \tag{5.4}$$

The function $g(t)$ can be calculated explicitly. We can write

$$g(t) = \int_{-\infty}^{1} (1-s)^{\alpha-1} e^{its}\, ds - \int_{-\infty}^{0} (1-s)^{\alpha-1} e^{its}\, ds.$$

We have the following formula for the first integral as proved in Gelfand-Shilov [4], p. 171.

$$g_1(t) = c_\alpha t^{-\alpha} e^{it}, \qquad c_\alpha = \Gamma(\alpha) e^{-i\alpha\pi/2}. \tag{5.5}$$

An integration by parts will show that the second integral is equal to $d_\alpha t^{-1} + g_2(t)$ where the function $g_2(t) = O(t^{-2})$ at infinity.

Thus $G_\tau(\alpha)$ is a sum of three operators, $G_\tau(\alpha)f = T_\tau(\alpha)f + T_0(1)f + Tf$ where

$$Tf(x) = \sum g_2\{(2n+1)\tau\} f^\wedge(n)\varphi_n(x). \tag{5.6}$$

Since $|g_2(2n+1)| \leqslant C(2n+1)^{-2}$ and $\|\varphi_n\|_\infty \|\varphi_n\|_1 \leqslant C(2n+1)^{1/6}$, we see that T is given by an L^1 kernel and so T is bounded on L^p, for all p, $1 \leqslant p \leqslant \infty$. The operator $T_0(1)$ is clearly bounded on L^p, $1 \leqslant p \leqslant \infty$ for the same reason as we have noticed in the previous section. Finally we can apply Theorem 2 to the operators $T_\tau(\alpha)$. For $\alpha \geqslant |1/p - 1/2|$, these operators are bounded on L^p, for $1 < p < \infty$ and when $p = 1$ and $\alpha = 1/2$ it is bounded from H^1 into L^1. This proves Theorem 3.

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