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Vector Valued Inequalities for Strongly Singular Calderón-Zygmund Operators

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1. Introduction

Convolution operators with symbols of the form $e^{i|\xi|^a}|\xi|^{-b}$ have been extensively studied in the literature. Motivated by earlier work by C. Fefferman and E. M. Stein (cf. [8], [9]), we introduced in [1] a generalization of the classical Calderón-Zygmund operators. These operators, which include pseudodifferential operators with symbols in $S_{a,\delta}^{-b}$, where

$$\delta \leqslant a < 1, \qquad \delta \geqslant 0, \qquad (1-a)\frac{n}{2} \leqslant b < \frac{n}{2},$$

were termed strongly singular Calderón-Zygmund operators.

In this article we consider a theory of vector valued strongly singular operators. Our results include L^p, H^p and BMO continuity results. Moreover, as is well known, vector valued estimates are closely related to weighted norm inequalities. These results are developed in the first four sections of the paper. In §5 we use our vector valued singular integrals to estimate the corresponding maximal operators. Finally in §6 we discuss applications to weighted norm inequalities for pseudo-differential operators with symbols in the classes described above. Our results in this direction are related to some recent work by Chanillo and Torchinsky [3].

We refer to §2 for a review of the necessary definitions and notation to be followed in the paper.

2. Preliminaries

In [1] we considered the following class of singular integral operators.

(2.1) **Definition.** *Let $T: S \to S'$ be a bounded linear operator. T is called a strongly singular Calderón-Zygmund operator if the following conditions are satisfied.*

(S_1) *T extends to a bounded operator on* L^2.

(S_2) *T is associated with a standard kernel, in the sense that there exists a function $k(x,y)$ continuous away from the diagonal on $\mathbb{R}^{2n}$, such that*

$$\text{(i)} \qquad |k(x,y) - k(x,z)| + |k(y,x) - k(z,x)| \leqslant c \frac{|y-z|^{\delta}}{|x-z|^{n+\delta/\alpha}}$$

if $2|y-z|^{\alpha} \leqslant |x-z|$ *for some* $0 < \delta \leqslant 1$, $0 < \alpha \leqslant 1$.

$$\text{(ii)} \qquad (Tf, g) = \int k(x,y) f(y) g(x)\, dy\, dx,$$

for $f, g \in S$ *with disjoint supports.*

(S_3) *For some*

$$(1-\alpha)\frac{n}{2} \leqslant \beta < \frac{n}{2},$$

T and its adjoint T^* extend to continuous operators from L^q into L^2, where

$$\frac{1}{q} = \frac{1}{2} + \frac{\beta}{n}.$$

Let us observe that if T is associated with a standard kernel, then given $f \in C^{\infty} \cap L^{\infty}$, $T(f)$ can be defined as a continuous linear functional over the C_0^{∞} functions with vanishing mean.

In [1] we show that S.C.Z. Operators are bounded from L^{∞} into BMO and moreover, if T is a S.C.Z. Operator such that $T^*(1) = 0$,

$$\frac{1}{p_0} = \frac{1}{2} + \frac{\beta\left(\dfrac{n}{\alpha} + \dfrac{n}{2}\right)}{n\left(\dfrac{\delta}{\alpha} + \delta + \beta\right)},$$

then T defines a bounded operator

$$T: H^p(\mathbb{R}^n) \to H^p(\mathbb{R}^n), \qquad p_0 < p \leqslant 1.$$

In what follows we shall provide some background information in order to extend our theory to the frame of vector valued spaces. Moreover, we shall also be concerned with the issue of relaxing the smoothness condition (S_2) for more appropriate and sharper conditions in order to deal with the L^p theory of S.C.Z.O's.

The classical maximal function of Hardy Littlewood $M_p f$, and the sharp maximal operator $f^{\#}$, can be defined for vector valued functions simply replacing absolute values by norms. In fact, let X be a Banch space,

$$f \in L^1_{\text{loc}}(\mathbb{R}^n, X),$$

then we let

$$M_p f(x) = \sup_{B \ni x} \left(\frac{1}{|B|} \int_B \| f(y) \|^p \, dy \right)^{1/p},$$

$$f^{\#}(x) = \sup_{B \ni x} \inf_{c \in X} \left\{ \frac{1}{|B|} \int_B \| f(y) - c \| \, dy \right\},$$

where B denotes a ball in $\mathbb{R}^n$, $\| \quad \|$ stands for the norm of X, $1 \leqslant p < \infty$. In what follows we shall consider the usual vector valued $L^p(\mathbb{R}^n, X) = L^p(X)$ spaces (cf. [14]).

In a similar fashion we can define vector valued (p, q) atoms, and develop a theory of vector valued $H^p(X)$ spaces. Indeed, a (p, q) atom is a vector valued function supported in a ball, with moments zero up to order $\left[n\left(\frac{1}{p} - 1\right)\right]$ and such that

$$\| f \|_{L^q(X)} \leqslant |B|^{1/q - 1/p}.$$

Let $H^{p,q}(X) = \left\{ f: \mathbb{R}^n \to X \,\middle|\, f = \sum_{j=1}^{\infty} \lambda_j a_j \right.$ in the $S(\mathbb{R}^n, X')'$ sense, a_j is a (p, q) atom, $\left. j = 1, \ldots, \sum_{j=1}^{\infty} |\lambda_j|^p < \infty \right\}$. We let

$$\| f \|_{H^{p,q}(X)} = \inf \left\{ \left(\sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p} : f = \sum_{j=1}^{\infty} \lambda_j a_j \right\}.$$

As in the scalar valued case $H^{p,q}(X) = H^{p,r}(X)$, for $q \neq r$.

We also observe that if X' satisfies the Radon-Nikodym property (in particular if X is reflexive), then

$$(H^1(X))' = \mathrm{BMO}(X').$$

Moreover, if X satisfies the UMD property (cf. [14]), then $H^1(X)$ can be characterized using vector valued Riesz transforms (cf. [14]).

The $H^p(X)$ spaces can be interpolated in the expected way, in particular (cf. C. F. Folland and E. M. Stein [10]).

(2.2) **Theorem.** *Let T be an operator mapping $H^p(X)$ into $L^p(Y)$, $L^q(X)$ into $L^q(Y)$, where X, Y are Banach spaces and $0 < p \leqslant 1 < q \leqslant \infty$. Then, T maps $L^1(X)$ into $L(1, \infty)(Y)$, i.e., T is of weak type $(1, 1)$.*

We now formulate a vector valued version of S.C.Z.O's. Given X, Y Banach spaces, let $L(X, Y)$ denote the space of bounded linear operators from X into Y. Let T be an operator such that

$$T: S(\mathbb{R}^n, X) \to S(\mathbb{R}^n, Y')'.$$

We shall say that T is associated with a kernel if there exists a continuous function k: $\mathbb{R}^{2n}$/diagonal $\to L(X, Y)$, such that

$$(g, Tf) = \iint g(x) k(x, y) f(y)\, dy\, dx,$$

when f and g have disjoint supports.

Our operators will be associated with kernels satisfying certain conditions of which $(S_2)(i)$ is a limiting case. This will allow us also to refine the main results of our previous work [1].

(2.3) **Definition** (cf. [14]). *Let $1 \leqslant r \leqslant \infty$, $0 < \alpha \leqslant 1$. We shall say that a kernel k, k: $\mathbb{R}^{2n}$/diagonal $\to L(X, Y)$ satisfies the condition $D_{r,\alpha}(X, Y) \equiv D_{r,\alpha}$ if there exists a sequence $\{d_j\} \in l^1$ such that $\forall z \in \mathbb{R}^n$, $\forall \sigma > 0$, $\forall y \in \mathbb{R}^n$ such that $|y - z| < \sigma$,*

$$(2.4) \quad \left(\int_{C_j(z, \sigma^\alpha)} \|k(x, y) - k(x, z)\|^r\, dx\right)^{1/r} \leqslant d_j |C_j(z, \sigma^\alpha)|^{-1/r'}, \qquad j = 1, \ldots$$

where $C_j(z, \sigma^\alpha) = \{x: 2^j \sigma^\alpha < |x - z| \leqslant 2^{j+1}\sigma^\alpha\}$, and $\| \quad \|$ denotes the norm in $L(X, Y)$. For a kernel to be in $D_{r,\alpha}$ we also require that $\bar{k}(x, y) = k(y, x)$ satisfies (2.4). This assumption, although not always necessary, will simplify the formulation of our results.

It is readily verified that a $D_{p,\alpha}$ condition is stronger than a $D_{q,\alpha}$ condition if $p > q$. A limiting case of these conditions is the analogue of condition $(S_2)(i)$ obtained replacing absolute values by norms. We shall refer to this condition as D_α. Observe that a D_α condition implies all the $D_{r,\alpha}$ conditions, $r \geqslant 1$.

We shall now formulate vector valued versions of (S_1) and (S_3). Again, we shall refine our assumptions in order to be able to derive very precise results.

Let $1 < p < q \leqslant \infty$, $p/q \leqslant \alpha \leqslant 1$, we shall impose if $0 < \sigma < 1$

$$\left(\frac{1}{|B(z,\sigma)|}\int_{B(z,\sigma)} \|Tf\|_Y^q dx\right)^{1/q} \leqslant c\left(\frac{1}{|B(z,\sigma^\alpha)|}\int \|f\|_X^p dy\right)^{1/p} \tag{2.5}$$

and (if $\sigma > 1$),

$$\|Tf\|_p \leqslant c\|f\|_p \tag{2.5$'$}$$

for some absolute constant $c > 0$.

We shall refer to these conditions as $C_{p,q,\alpha}$ conditions. The strongly singular Calderón-Zygmund operators considered in this paper will satisfy conditions of the form $C_{p,q,\alpha}$ and $D_{r,\alpha}$.

(2.6) *Remark.* In [14] it is shown that if T is a sublinear operator on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, such that $\forall \epsilon > 0$, $(Tf)^{\#}(x) \leqslant C_\epsilon M_{1+\epsilon} f(x)$, T admits an extension to $L^p(\mathbb{R}^n, X)$ for all Banach spaces $X \in \text{UMD}$, with an unconditional basis. The extension is defined as follows. Let $\{b_j\}_{j\in\mathbb{N}}$ be an unconditional basis in X, then let $\tilde{T}(\sum_j f_j(x) b_j) = \sum_j Tf_j(x) b_j$.

In particular, let T_b be defined by

$$\widehat{T_b f(\xi)} = \theta(\xi) e^{i|\xi|^b} |\xi|^{-nb/2} \hat{f}(\xi), \qquad 0 < b < 1,$$

where θ is a smooth radial cutoff function, $\theta(\xi) = 1$ if $|\xi| \geqslant 1$ and $\theta(\xi) = 0$ if $|\xi| \leqslant 1/2$. Then, using (2.15) of Chanillo [4], we see that T_b can be extended to a bounded operator

$$\tilde{T}_b: L^p(\mathbb{R}^n, X) \to L^p(\mathbb{R}^n, X)$$

for all Banach spaces $X \in \text{UMD}$ with an unconditional basis.

3. H^p Theory

We consider first the L^∞, BMO continuity of S.C.Z.O's.

(3.1) **Theorem.** *Let T be an operator associated with a standard kernel satisfying the $C_{p,q,\alpha}$ and $D_{r,\alpha}$ conditions. Then, for $s = \max\{p, r'\}$, we have*

$$(Tf)^{\#}(x) \leqslant cM_s f(x), \qquad f \in L_c^\infty(X). \tag{3.2}$$

PROOF. Fix a ball $B(z,\sigma)$ and write $f = f_1 + f_2$, where

$$f_1 = f\chi_{B(z, 2\sigma^\alpha)}.$$

Let

$$c = \int k(z, y) f_2(y)\, dy,$$

then

$$\int_{B(z,\sigma)} \| Tf(x) - c\| \, dx \leqslant \int_{B(z,\sigma)} \| Tf_1(x)\| \, dx + \int_{B(z,\sigma)} \int_{\mathbb{R}^n/B(z, 2\sigma^\alpha)} \| k(x, y) - k(z, y)\| \, \| f(y)\| \, dy\, dx.$$

Dividing by $|B(z, \sigma)|$ we can estimate the two terms on the right hand side as follows. The first term is majorized using Hölder's inequality and the $C_{p,q,\alpha}$ condition. The second term is estimated by breaking up $\mathbb{R}^n/B(z, 2\sigma^\alpha)$ into disjoint annuli and estimating

$$\sup_{B(z,\sigma)} \sum_{j=1}^{\infty} \int_{C_j(z,\sigma^\alpha)} \| k(x, y) - k(z, y)\| \cdot \| f(y)\| \, dy$$

using Hölder's inequality and the $D_{r,\alpha}$ condition.

A very important special case of the above result obtains when T satisfies a $D_{1,\alpha}$ condition, in such case we have

(3.3) **Corollary.** *Let T satisfy, $C_{p,q,\alpha}$ and $D_{1,\alpha}$, then T can be defined as a continuous operator, $T\colon L^\infty(X) \to \mathrm{BMO}(Y)$.*

PROOF. Observe that $r = 1$ implies that $s = \infty$. Therefore, the estimate (3.2) can be written as

$$(Tf)^{\#}(x) \leqslant c\| f\|_\infty .$$

This gives the desired estimate when $f \in L_c^\infty(X)$. Moreover, it follows from the proof of (3.1) that Tf has a meaning for any $f \in L^\infty(X)$.

Our next result considers S.C.Z.O's as operators from $H^p(X)$ into $L^p(Y)$ spaces. The proof is the same as the one given in [1].

(3.4) **Theorem.** *Let T be an operator associated with a kernel k that satisfies a D_α condition and moreover suppose that T extends to a bounded operator, $T\colon L^2(X) \to L^2(Y)$, and from $L^q(X) \to L^2(Y)$,*

$$\frac{1}{q} = \frac{1}{2} + \frac{\beta}{n},$$

for some

$$(1-\alpha)\frac{n}{2} \leqslant \beta < \frac{n}{2}.$$

Let

$$\frac{1}{p_0} = \frac{1}{2} + \frac{\beta\left(\dfrac{\delta}{\alpha} + \dfrac{n}{2}\right)}{n\left(\dfrac{\delta}{\alpha} - \delta + \beta\right)},$$

and let $1 \geqslant p > p_0$, *then* T *maps* $H^p(X)$ *into* $L^p(Y)$.

In order to state the main result of this section we shall consider reflexive Banach spaces, X, Y. Then if T is associated with a kernel k, T^* can be defined as a linear continuous operator from $S(\mathbb{R}^n, Y')$ into $S(\mathbb{R}^n, X)'$ and T^* is associated with $k(y,x)^*$: $\mathbb{R}^{2n}/\text{diagonal} \to L(Y', X')$. Observe that if this kernel satisfies a $D_{1,\alpha}$ condition, then given $f \in L^\infty(Y') \cap C^\infty(Y')$, $T^*(f)$ has a meaning as a continuous linear functional over $D_0(X) = \left\{\varphi \in C_0^\infty(X): \int \varphi = 0\right\}$ (cf. [1]).

(3.5) **Theorem.** *Let* X, Y *be reflexive Banach spaces, let* T, T^* *satisfy the conditions of* (3.4). *Moreover, suppose that* $T^*(\bar{e}) = 0$, $\forall e \in Y'$, *and* $\bar{e}$: $\mathbb{R}^n \to Y'$ *given by* $\bar{e}(x) = e$. *Let*

$$\frac{1}{p_0} = \frac{1}{2} + \frac{\beta\left(\dfrac{\delta}{\alpha} + \dfrac{n}{2}\right)}{n\left(\dfrac{\delta}{\alpha} - \delta + \beta\right)},$$

then for $p_0 < p \leqslant 1$, T *is a continuous operator,* $T: H^p(\mathbb{R}^n, X) \to H^p(\mathbb{R}^n, Y)$.

PROOF. As in the scalar valued situation treated in detail in [1], one shows that Ta, the image of a $(p, 2)$ atom a, is a suitable molecule. The only new technical problem is to check that $\int Ta(x)\,dx = 0$. This can be done as follows. Let a be a $(p, 2)$ atom supported on a ball B, then $|B|^{-1+1/p}a$ is a (1,2) atom and given $e \in Y'$,

$$\begin{aligned}|B|^{-1+1/p}\left\langle e, \int Ta(x)\,dx\right\rangle &= \int \langle \bar{e}(x), T(|B|^{-1+1/p}a)(x)\rangle\,dx \\ &= \int \langle T^*(\bar{e}), |B|^{-1+1/p}a\rangle\,dx = 0\end{aligned}$$

where in the last line we have used the fact that $H^1(X)' = \text{BMO}(X')$ (cf. [14]) and that T^* maps $L^\infty(Y')$ into $\text{BMO}(X')$.

(3.6) *Remark.* If in the assumptions of (3.5) we have $\alpha = 1$, then we can let $\beta \to 0$ and we recover the well known fact that Calderón Zygmund operators are continuous from $H^p(X)$ into $H^p(Y)$ when $1 \geqslant p > n/(n+\delta)$ and $T^*(\bar{e}) = 0$, $e \in Y'$ (cf. [1]). Moreover, the cancellation condition can be dropped in the case of convolution operators, but in general is necessary. In fact, given a function φ slowly increasing with all derivatives, the operator $T_\varphi(f) = \varphi f$ is a linear continuous operator from $S(\mathbb{R}^n, \mathbb{C}) \to S(\mathbb{R}^n, \mathbb{C})'$, associated to the standard kernel identically zero. Moreover, $T_\varphi^*(1) = T_{\bar{\varphi}}(1) = \bar{\varphi}$. But $T_\varphi(f)$ will not have vanishing integral for any $f \in H^p(\mathbb{C})$, unless φ is a constant, i.e., $T_\varphi^*(1) = 0$ in the BMO sense.

Finally we consider continuity at the critical index p_0. The appropriate smoothness conditions on the kernel are given by

$$\int_{C_j(z,\sigma)} \| k(x,y) - k(x,z) \| \, dx \leqslant c 2^{-j\delta} \tag{3.7}$$

if $\sigma > 1$, $|y - z| < \sigma$.

$$\int_{C_j(z,\sigma^\gamma)} \| k(x,y) - k(x,z) \| \, dx \leqslant c 2^{-j\delta/\alpha} \sigma^{\delta(1-\gamma/\alpha)} \tag{3.8}$$

if $\sigma < 1$, $\gamma \leqslant \alpha$, $|y - z| < \sigma$.

Let us observe that if T is associated with a kernel satisfying the D_α condition then k satisfies (3.7) and (3.8).

(3.9) **Theorem.** *Let T be an operator associated with a kernel satisfying* (3.7) *and* (3.8). *Moreover, suppose that T extends to a bounded operator, $T: L^2(X) \to L^2(Y)$, and from $L^q(X)$ into $L^2(Y)$,*

$$\frac{1}{q} = \frac{1}{2} + \frac{\beta}{n},$$

for some

$$(1 - \alpha)\frac{n}{2} \leqslant \beta < \frac{n}{2}.$$

Then, $T: H^p(X) \to L^p(Y)$, $p_0 \leqslant p \leqslant 1$, where

$$\frac{1}{p_0} = \frac{1}{2} + \frac{\beta\left(\dfrac{\delta}{\alpha} + \dfrac{n}{2}\right)}{n\left(\dfrac{\delta}{\alpha} - \delta + \beta\right)}.$$

PROOF. Fix $p_0 \leqslant p \leqslant 1$ and let a be a (p, ∞) atom supported on $B(z, \sigma)$. Suppose first that $\sigma > 1$, then

$$\begin{aligned}\int \|Ta(x)\|^p \, dx &\leqslant \int_{B(z,2\sigma)} \|Ta(x)\|^p \, dx \\ &\quad + \sum_{j=1}^{\infty} \int_{C_j(z,\sigma)} \left(\int_{B(z,\sigma)} \|k(x,y) - k(x,z)\| \cdot \|a(y)\| \, dy \right)^p dx \\ &= I_1 + I_2 .\end{aligned}$$

Using Hölder's inequality with exponent $2/p$ and the L^2 continuity property, we can estimate I_1 by a constant. To take care of I_2 we use Hölder's inequality with exponent $1/p$ and (3.7), obtaining

$$I_2 \leqslant c \sum_j 2^{j(n-(n+\delta)p)} \leqslant c$$

since

$$1 \geqslant p \geqslant p_0 > \frac{n}{n+\delta} > \frac{n}{n + \dfrac{\delta}{\alpha}}.$$

Next we consider the case where $\sigma < 1$. Then,

$$\begin{aligned}\int \|Ta(x)\|^p \, dx &\leqslant \int_{B(z,2\sigma^{\gamma})} \|Ta(x)\|^p \, dx \\ &\quad + \sum_{j=1}^{\infty} \int_{C_j(z,\sigma^{\gamma})} \left(\int_{B(z,\sigma)} \|k(x,y) - k(x,z)\| \cdot \|a(y)\| \, dy \right)^p dx \\ &= I_1 + I_2\end{aligned}$$

with γ to be determined precisely later.

The first term can be majorized using Hölder's inequality with exponent $2/p$ and the L^q, L^2 continuity property,

$$I_1 \leqslant c\sigma^{n[\gamma(1-p/2)+p/q-1]}.$$

Thus, since $\sigma < 1$ and

$$\frac{1}{q} = \frac{1}{2} + \frac{\beta}{n},$$

I_1 will be bounded whenever γ satisfies

$$\gamma \geqslant \frac{1 - \dfrac{p}{2} - p\dfrac{\beta}{n}}{1 - \dfrac{p}{2}}. \tag{3.10}$$

On the other hand, using Hölder's inequality with exponent $1/p$ and condition (3.8), we get

$$I_2 \leqslant c\sigma^{-\gamma[p(n+\delta/\alpha)-n]+p(n+\delta)-n} \sum_{j=1}^{\infty} 2^{j[n-p(n+\delta/\alpha)]}.$$

Thus I_2 will be bounded whenever γ satisfies

$$\gamma \leqslant \frac{p(n+\delta)-n}{p\left(n+\dfrac{\delta}{\alpha}\right)-n}. \tag{3.11}$$

For the extreme value $p = p_0$ the right hand sides of (3.10) and (3.11) have the common value

$$\gamma_0 = \frac{\dfrac{n}{2}+\delta-\beta}{\dfrac{\delta}{\alpha}+\dfrac{n}{2}} \leqslant \alpha,$$

where the last inequality follows from our assumption $(1-\alpha)n/2 \leqslant \beta$. Observe that the right hand side of (3.10) decreases as $p \to 1$ and equals $1 - 2\beta/n$ when $p = 1$. Moreover, the right hand side of (3.11) increases as $p \to 1$ and equals α when $p = 1$. Consequently the choice $\gamma = \gamma_0$ will satisfy (3.10) and (3.11), $\forall 1 \geqslant p \geqslant p_0$.

(3.12) *Remark.* It would be interesting to obtain (3.9) imposing only a $D_{1,\alpha}$ condition on the kernel. Observe that when $\beta = (1-\alpha)n/2$ then $p_0 = 1$ and the (H^1, L^1) continuity can be obtained assuming a $D_{1,\alpha}$ condition (cf. [1], Theorem 1.2 *c*)).

4. Weak Type L^1 and L^p Inequalities

Let us observe that if T is an operator satisfying the conditions of (3.9) we can use intepolation (cf. (2.2) above) to obtain $T: L^1(X) \to L(1,\infty)(Y)$. In [14] the weak type (1,1) of Calderón-Zygmund operators (i.e., $\alpha = 1$, $\beta = 0$) was obtained for kernels satisfying only a $D_{1,1}$ condition. We now show how C. Fefferman's approach for convolution operators (cf. [8]) can be adapted to obtain the weak type (1, 1) estimates under sharp assumptions.

(4.1) **Theorem.** *Let T be an operator associated with a kernel satisfying the $D_{1,\alpha}$ condition. We also assume, moreover, that T extends to bounded* *operators,* $T: L^2(X) \to L^2(Y)$, *and* $T: L^q(X) \to L^2(Y)$,

$$\frac{1}{q} = \frac{1}{2} + \frac{\beta}{n}, \qquad (1-\alpha)\frac{n}{2} \leqslant \beta < \frac{n}{2}.$$

Then, $T: L^1(X) \to L(1, \infty)(Y)$.

PROOF. Let $f \in L^1(X)$, $\lambda > 0$. Consider a Calderón-Zygmund decomposition of f at level λ. Thus,

$$\Omega = \{Mf(x) > \lambda\} = \bigcup_{j=1}^{\infty} Q_j,$$

where the Q_j's are cubes and such that if f_{Q_j} denotes the mean value of f over Q_j, $f = f' + f''$, where

$$f' = f\chi_{\Omega^c} + \sum_{j=1}^{\infty} f_{Q_j}\chi_{Q_j}$$

$$f'' = \sum_{j=1}^{\infty} (f - f_{Q_j})\chi_{Q_j} = \sum_{j=1}^{\infty} f_j.$$

Moreover,

$$\|f'(x)\| \leqslant c\lambda \quad \text{a.e.}, \quad \|f'\|_1 \leqslant \|f\|_1, \tag{4.2}$$

where $\|\ \|_1$ denotes the norm in $L^1(X)$.

$$\int_{Q_j} \|f_j(x)\|\, dx \leqslant c|Q_j|\lambda, \qquad \int f_j(x)\, dx = 0 \tag{4.3}$$

$$|\Omega| \leqslant \frac{c}{\lambda}\|f\|_1. \tag{4.4}$$

From (4.2), $f' \in L^2(X)$ and $\|f'\|_2^2 \leqslant c\lambda\|f\|_1$. So,

$$\lambda^2|\{x/\|Tf'(x)\| > \lambda/2\}| \leqslant c\|Tf'\|_2^2 \leqslant c\lambda\|f\|_1.$$

On the other hand, if $c\Omega$ denotes the union of the cubes cQ_j with the same center as Q_j and c times the side length, we have, according to (4.4)

$$|\{x \in c\Omega/\|Tf''(x)\| > \lambda/2\}| \leqslant |c\Omega| \leqslant \frac{c}{\lambda}\|f\|_1.$$

So, it remains to prove that

$$|\{x \in \mathbb{R}^n \backslash c\Omega/\|Tf''(x)\| > \lambda/2\}| \leqslant \frac{c}{\lambda}\|f\|_1.$$

Let us denote σ_j the diameter of the cube Q_j. Let

$$F = \sum_{\sigma_j < 1} f_j, \qquad G = \sum_{\sigma_j \geq 1} f_j.$$

If z_j is the center of the cube Q_j, for a suitable c we will have $|x - z_j| > 4\sigma_j$ if $x \in \mathbb{R}^n \backslash c\Omega$. According to (4.3), we can write

$$\lambda |\{x \in \mathbb{R}^n \backslash c\Omega / \| TG(x) \| > \lambda/4\}| \leqslant \int_{\mathbb{R}^n \backslash c\Omega} \| TG(x) \| \, dx$$
$$\leqslant \sum_j \int_{|x - z_j| > 2\sigma_j^\alpha} \int_{Q_j} \| k(x, y) - k(x, z_j) \| \cdot \| f_j(y) \| \, dy \, dx.$$

Using condition $D_{1,\alpha}$, this can be majorized by

$$c \sum_j \int_{Q_j} \| f_j(y) \| \, dy \leqslant c \| f \|_1.$$

Now, let $\varphi \in C_0^\infty(\mathbb{R}^n)$ a scalar function such that

$$\operatorname{supp} \varphi \subset \{|x| \leqslant 1/c\}, \qquad \int \varphi \, dx = 1, \qquad \varphi \geqslant 0.$$

Let

$$\varphi_j(x) = \frac{1}{\sigma_j^{n/\alpha}} \varphi(x/\sigma_j^{1/\alpha}).$$

We will write

$$F = \sum f_j * \varphi_j + \sum (f_j - f_j * \varphi_j) = F' + F''.$$

If $x \in \mathbb{R}^n \backslash c\Omega$,

$$T(f_j - f_j * \varphi_j)(x) = \int k(x, y) f_j(y) \, dy - \int k(x, w) \int \varphi_j(w - y) f_j(y) \, dy \, dw.$$

Since $k(x, w)$ is a $L(X, Y)$-valued continuous function away from the diagonal, we can change the order of integration in the second term above. Moreover, $\int \varphi_j(w - y) \, dw = 1$ for any y. So, we can write

$$\int \left[\int (k(x, y) - k(x, w)) \varphi_j(w - y) \, dw \right] f_j(y) \, dy.$$

Then,

$$\lambda |\{x \in \mathbb{R}^n \backslash c\Omega / \| TF''(x) \| > \lambda/8\}| \leqslant c \int_{\mathbb{R}^n \backslash c\Omega} \| TF''(x) \| \, dx$$
$$\leqslant \sum_j \int_{\mathbb{R}^n \backslash cQ_j} \int \left[\int \| k(x, y) - k(x, w) \| \varphi_j(w - y) \, dw \right] \| f_j(y) \| \, dy \, dx.$$

If $x \in \mathbb{R}^n \backslash cQ_j$, $|y-w| < \sigma_j^{1/\alpha}$, we have

$$|x-w| \geqslant |x-z_j| - |z_j - y| - |y-w| > 4\sigma_j - \sigma_j - \sigma_{j/c}^{1/\alpha} > 2\sigma_j,$$

since $\alpha < 1$, $\sigma_j < 1$.

So, we can majorize the above sum as

$$\sum_j \int_{Q_j} \int_{|y-w|<\sigma_\sigma^{1/\alpha}} \int_{|x-w|>2\sigma_j} \|k(x,y) - k(x,w)\| \, dx \, \varphi_j(w-y) \, dw \, \|f_j(y)\| \, dy.$$

Using again condition $D_{1,\alpha}$ and the fact that $\int \varphi_j(w-y)\,dw = 1$ for any y, j, we obtain

$$\sum_j \int_{Q_j} \|f_j(y)\| \, dy \leqslant c\|f\|_1.$$

Now, it remains to prove that

$$\lambda |\{x \in \mathbb{R}^n \backslash c\Omega \;\|TF'(x)\| > \lambda/8\}| \leqslant c\|f\|_1. \tag{4.5}$$

Let J^β denote the Bessel potential of order β. Observe that these operators are positive and therefore admit a vector valued extension. Thus, our program now will be to obtain the estimate

$$\|J^\beta F'\|_2^2 \leqslant A\lambda \|f\|_1. \tag{4.6}$$

In fact, from this estimate it follows according to the continuity assumptions on T that

$$\|TF'\|_2^2 = \|TJ^{-\beta}J^\beta F'\|_2^2 \leqslant \|J^\beta F'\|_2^2 \leqslant A\lambda\|f\|_1.$$

Thus, (4.5) follows using Chebyshev's inequality.

Now, let us recall that

$$F' = \sum_{\sigma_j<1} f\chi_{Q_j} * \varphi_j - \sum_{\sigma_j<1} f_{Q_j}\chi_{Q_j} * \varphi_j.$$

For x fixed, we will write $x \sim Q_j$ if x belongs to the closure of any cube Q_l which is adjacent or coincides with Q_j.

Now,

$$J^\beta \sum_{\sigma_j<1} f\chi_{Q_j} * \varphi_j(x) = \sum_{x\sim Q_j} J^\beta f\chi_{Q_j} * \varphi_j(x) + \sum_{x \nsim Q_j} J^\beta f\chi_{Q_j} * \varphi_j(x) = F_1(x) + F_2(x).$$

So that,

$$(J^\beta F')(x) = F_1(x) + F_2(x) - \sum_{\sigma_j<1} f_{Q_j} J^\beta(\chi_{Q_j} * \varphi_j)(x).$$

With the same proof as in Fefferman's proof, we can show that if $x \not\sim Q_j$,

$$\|J^\beta f\chi_{Q_j} * \varphi_j(x)\| \leqslant cJ^\beta \|f\|_{Q_j}\chi_{Q_j} * \varphi_j(x).$$

So, according to (4.3), we have

$$\begin{aligned}\Big\|F_2(x) - \sum_{\sigma_j<1} J^\beta f_{Q_k}\chi_{Q_j} * \varphi_j(x)\Big\| &\leqslant c\lambda \sum_{\sigma_j<1} J^\beta \chi_{Q_j} * \varphi_j(x)\\ &\leqslant c\lambda \|J^\beta\|_1 \Big\|\sum_{\sigma_j<1} \chi_{Q_j} * \varphi_j\Big\|_\infty\\ &\leqslant c\lambda\end{aligned}$$

since the supports of $\chi_{Q_j} * \varphi_j$ have finite intersection.

On the other hand,

$$\begin{aligned}\Big\|F_2 - \sum_{\sigma_j<1} J^\beta f_{Q_j}\chi_{Q_j} * \varphi_j\Big\|_1 &\leqslant \sum_{\sigma_j<1} \|J^\beta \varphi_j * (f - f_{Q_j})\chi_{Q_j}\|_1\\ &\leqslant c\sum \int_{Q_j} \|f\|\, dy \leqslant c\|f\|_1.\end{aligned}$$

Thus, we get from the L^1 and L^∞ inequalities

$$\Big\|F_2 - \sum_{\sigma_j<1} J^\beta f_{Q_j}\chi_{Q_j} * \varphi_j\Big\|_2^2 \leqslant c\lambda\|f\|_1.$$

It only remains to show that $\|F_1\|_2^2 \leqslant c\lambda\|f\|_1$. This can be also done in the same way as in Fefferman's proof.

For x fixed, let

$$F_1^j(x) = \begin{cases} J^\beta f\chi_{Q_j} * \varphi_j(x) & \text{if } \ x \sim Q_j\\ 0 & \text{if not.}\end{cases}$$

Thus,

$$F_1(x) = \sum_{\sigma_j<1} F_1^j(x).$$

Moreover, for a fixed x, $F_1^j(x) \neq 0$ for at most N values of j. So,

$$\begin{aligned}\|F_1(x)\|_Y^2 &\leqslant \Big(\sum_{j=1}^N \|F_1^{j(x)}(x)\|_Y\Big)^2 = \sum_{j,h=1}^N \|F_1^{j(x)}(x)\|_Y \|F_1^{h(x)}(x)\|_Y\\ &\leqslant 2\Big(\sum_{j,h=1}^N \|F_1^{j(x)}(x)\|_Y^2 + \|F_1^{h(x)}(x)\|_Y^2\Big)\\ &\leqslant 4N \sum_{\sigma_j<1} \|F_1^j(x)\|_Y^2.\end{aligned}$$

So that,

$$\|F_1\|_2^2 \leqslant 4N \sum \|F_1^j\|_2^2 \leqslant 4N \sum_j \|J^\beta \varphi_j\|_2^2 \, \|f\chi_{Q_j}\|_1^2$$

$$\|J^\beta \varphi_j\|_2^2 = \int 1 + (|\xi|^2)^{-2\beta} |\hat{\varphi}(\sigma_j^{1/\alpha}\xi)|^2 \, d\xi \leqslant \frac{c}{|Q_j|},$$

since $|Q_j| \leqslant 1$.

Finally,

$$\|F_1\|_2^2 \leqslant c \sum_j \frac{1}{|Q_j|} \|f\chi_{Q_j}\|_1^2 \leqslant c\lambda \|f\|_1.$$

This completes the proof of the theorem.

Let us note in passing that we actually needed less than condition $D_{1,\alpha}$ on the kernel. In fact, what we really used is

$$\int_{|x-z|>c\sigma^\alpha} \|k(x,y) - k(x,z)\| \, dx \leqslant c, \qquad 0 < \sigma < 1$$

$$\int_{|x-z|>c\sigma} \|k(x,y) - k(x,z)\| \, dx \leqslant c, \qquad \sigma > 1$$

for $|y - z| < \sigma$.

Interpolating the results of §3 and (4.1), we can derive L^p continuity theorems and we shall leave details to the interested reader. Also, in the standard way, (3.2) gives weighted L^p inequalities. S. Chanillo proved in [4] sharp estimates for T_b.

Supposing the condition in Theorem 3.1 with $p = r' < \infty$, (3.2) reads

$$(Tf)^{\#}(x) \leqslant c_p M_p(f)(x).$$

It is well known that this inequality holds for any $p > 1$ when T is a Calderón-Zygmund operator. It is an open problem to know whether this is true or not for a general S.C.Z. Operator.

5. Maximal Inequalities

Let T be an operator associated with a standard kernel k. We assume that T satisfies (2.5) and (2.5)′ and moreover that the kernel k satisfies a D_∞ condition. Moreover, suppose that k satisfies the estimate

$$\|k(x,y)\| \leqslant \frac{c}{|x-y|^n}. \tag{5.1}$$

Let T_* denote the maximal operator defined by

$$T_*f(x) = \sup_{\epsilon > 0} \| T_\epsilon f(x) \|_Y$$

where

$$T_\epsilon f(x) = \int_{|x-y| > 3\epsilon^\alpha} k(x,y) f(y)\, dy.$$

Then, the following extension of Cotlar's estimates holds (cf. [5]).

(5.2) **Theorem.** *Let T be an operator satisfying the conditions described above, then $\forall f \in C_0^\infty(X)$,*

$$T_*f(x) \leqslant c(M(Tf)(x) + M_p f(x)) \tag{5.3}$$

where $1 < p < \infty$ is given by condition (2.5)′.

For the proof we require the following Lemma (cf. [5]).

(5.4) **Lemma.** *Let $f \in L^1(\mathbb{R}^n, X)$, $0 < \delta \leqslant 1$, $0 < \alpha \leqslant 1$. Then, there exists a constant $c > 0$ such that*

$$\text{(i)} \qquad \sup_{0<\epsilon<1} \epsilon^\delta \int_{|x-y|>3\epsilon^\alpha} \frac{\|f(y)\|}{|x-y|^{n+\delta/\alpha}}\, dy \leqslant cM(f)(x)$$

$$\text{(ii)} \qquad \sup_{|x-z|<\epsilon<1} \int_{|x-y|>3\epsilon^\alpha} \|k(x,y) - k(z,y)\| \, \|f(y)\|\, dy \leqslant cM(f)(x).$$

Proof. (i) can be readily derived using polar coordinates as in [5], Lemma 4. Alternatively,

$$\begin{aligned}
\epsilon^\delta \int_{|x-y|>3\epsilon^\alpha} \frac{\|f(y)\|}{|x-y|^{n+\delta/\alpha}}\, dy &= \sum_{j=0}^{\infty} \epsilon^\delta \int_{3\epsilon^\alpha 2^{j+1} > |x-y| > 3\epsilon^\alpha 2^j} \frac{\|f(y)\|}{|x-y|^{n+\delta/\alpha}}\, dy \\
&\leqslant c \sum_{j=0}^{\infty} \frac{\epsilon^\delta}{(2^j\epsilon^\alpha)^{n+\delta/\alpha}} \int_{B(x, 3\epsilon^\alpha 2^{j+1})} \|f(y)\|\, dy \\
&\leqslant c\left(\sum_{j=0}^{\infty} 2^{-j(\delta/\alpha)}\right) Mf(x).
\end{aligned}$$

(ii) Observe that if $|x-z| < \epsilon < 1$ and $|x-y| > 3\epsilon^\alpha$, then

$$|y-z| > |x-y| - |x-z| > 3\epsilon^\alpha - \epsilon > 2\epsilon^\alpha > 2|x-z|^\alpha,$$

and moreover,

$$|y - z| \geqslant |x - y| - \epsilon^{\alpha} > \frac{2}{3}|x - y|.$$

Consequently, using the D_∞ condition we get

$$\begin{aligned}\int_{|x-y|>3\epsilon^{\alpha}} \|k(x,y) - k(z,y)\| \, \|f(y)\| \, dy &\leqslant c \int_{|x-y|>3\epsilon^{\alpha}} \frac{|x-z|^{\delta}}{|x-y|^{n+\delta/\alpha}} \|f(y)\| \, dy \\ &\leqslant c\epsilon^{\delta} \int_{|x-y|>3\epsilon^{\alpha}} \frac{\|f(y)\|}{|x-y|^{n+\delta/\alpha}} \, dy\end{aligned}$$

and the assertion follows from the first part of the lemma.

PROOF OF (5.2). Except for minor details, the proof is a modification of the classical proof of Cotlar (cf. [5], p. 97). We provide the details for the reader's convenience. Consequently, we shall consider only the case where $0 < \epsilon < 1$. Given $f \in C_0^{\infty}(\mathbb{R}^n, X)$, $0 < \epsilon < 1$, $x \in \mathbb{R}^n$, let $f = f_1 + f_2$, where $f_1 = f\chi_{B(x,2\epsilon^{\alpha})}$, $f_2 = (1 - \chi_{B(z,2\epsilon^{\alpha})})f$. Then, $(T_\epsilon f)(x) = Tf_2(x)$. Next, we estimate the values of Tf_2 at other points in $B(x, \epsilon)$, In fact, if $z \in B(x, \epsilon)$, then

$$|z - y| \geqslant |x - y| - |x - z| > 2\epsilon^{\alpha}.$$

So,

$$Tf_2(x) - Tf_2(z) = \int_{|x-y|>3\epsilon^{\alpha}} [k(x,y) - k(z,y)] f(y) \, dy.$$

Thus, by (5.4) (ii) this term can be majorized by $cMf(x)$.

Consequently, if $z \in B(x, \epsilon) = B$,

$$\begin{aligned}\|T_\epsilon f(x)\| = \|Tf_2(x)\| &\leqslant \|Tf_2(x) - Tf_2(z)\| + \|Tf_2(z)\| \\ &\leqslant cMf(x) + \|Tf_1(z)\| + \|Tf(z)\|\end{aligned}$$

since $Tf_2(z) = Tf(z) - Tf_1(z)$.

In order to prove (5.3) we may assume that $\|Tf_2(x)\| > 0$. Let $0 < \lambda < \|Tf_2(x)\|$ be fixed. Define the sets

$$E_1 = \left\{ z \in B \Big/ \|Tf(z)\| > \frac{\lambda}{3} \right\},$$

$$E_2 = \left\{ z \in B \Big/ \|Tf_1(z)\| > \frac{\lambda}{3} \right\},$$

$$E_3 = \phi \quad \text{if} \quad cMf(x) < \frac{\lambda}{3} \quad \text{or} \quad E_3 = B \quad \text{otherwise.}$$

Let $d\mu$ be the probability measure $|B|^{-1}\chi_B\,dx$, then

$$1 \leqslant \mu(B) \leqslant \mu(E_1) + \mu(E_2) + \mu(E_3).$$

In order to evaluate $\mu(E_i)$ we use Chebychev's inequality. Trivially we get

$$\mu(E_1) \leqslant c(MTf)(x), \qquad \mu(E_3) \leqslant cMf(x).$$

The second term can be estimated using (2.5) as follows

$$\begin{aligned}\mu(E_2) = \frac{|E_2|}{|B|} &\leqslant \frac{3}{\lambda}\frac{1}{|B|}\int_B \|Tf_1(z)\|\,dz \\ &\leqslant \frac{3}{\lambda}\left(\frac{1}{B}\int_B \|Tf_1(z)\|^q\,dz\right)^{1/q} \\ &\leqslant \frac{3c}{\lambda}|B|^{-1/q}\left(\int_{B(x,3\epsilon^\alpha)} \|f(z)\|^p\,dz\right)^{1/p} \\ &\leqslant \frac{3c}{\lambda}\epsilon^{-n/q+n\alpha/p}M_pf(x) \\ &\leqslant \frac{3c}{\lambda}M_pf(x)\end{aligned}$$

since $p/q \leqslant \alpha$. Consequently,

$$\lambda < c[M(Tf)(x) + M_pf(x)]$$

and taking the sup over all $0 < \lambda < \|Tf_2(x)\|$ in the above inequality, we obtain (5.3).

(5.6) *Remark.* We would like to point out that T_* can be also viewed as a vector valued strongly singular operator. In fact if we let

$$T_\epsilon f(x) = \int_{|y|>\epsilon} k(x,y)f(x-y)\,dy,$$

where

$$k\colon \mathbb{R}^n \times \mathbb{R}^n \sim \{0\} \to L(X,Y)$$

is continuous and satisfies:

(i) $k(x,w) = 0$ for $|x| \geqslant 1$,

(ii) $\|k(x,w)\| \leqslant \dfrac{c}{|w|^n}$

and

(iii) $\int_{|x|>2\sigma^\alpha} \|k(x+z, x-y) - k(x+z, x)\| \, dx < c$

for some $0 < \alpha < 1$, $z \in \mathbb{R}^n$, $|y| < \sigma$, then

$$T_* f(x) \leqslant c\left[Mf(x) + \left\| \int k_\epsilon(x,y) f(x-y)\, dy \right\|_{L^\infty(Y)}\right].$$

Here, k_ϵ is a smooth cutoff of the kernel k given by

$$k_\epsilon(x,y) = k(x,y)\varphi\left(\frac{y}{\epsilon}\right),$$

where $\varphi \in C^\infty$ and $\varphi \equiv 0$ near the origin and $\varphi \equiv 1$ for $|x| \geqslant 2$, say. Moreover, it can be shown using the methods of [14] that $k_\epsilon(x,y)$ satisfies a $D_{1,\alpha}$ condition.

(5.7) *Remark.* Using (5.2), we can readily obtain weighted norm inequalities of the following type. If T is an operator satisfying the conditions of (5.2), then $T_* : L^r(w(x)\,dx, X) \to L^r(w(x)\,dx, Y)$, $\forall p < r < \infty$, $\forall w \in A_{r/p}$. In the special case of Calderón-Zygmund operators, the result remains valid for $1 < r < \infty$.

6. Applications

The H^p inequalities obtained in §3 are new also for vector valued Calderón-Zygmund operators, extending the results of [14]. Using the ideas of [14], we can also derive results of the following type («self improving»).

(6.1) **Theorem.** (i) *Let T be an operator associated with a standard kernel satisfying* (3.7) $\forall \sigma > 1$. *Suppose that T is continuous from $L^{r_0}(\mathbb{R}^n, X)$ into $L^{r_0}(\mathbb{R}^n, Y)$ for some $1 < r_0 \leqslant \infty$. Then,*

(6.2)
$$\left\| \left(\sum_{j=1}^{\infty} \|Tf_j\|^r \right)^{1/r} \right\|_p \leqslant c \| \{f_j\} \|_{H^p(\mathbb{R}^n, l^r(X))}$$

for $n/(n+\delta) < p \leqslant 1 < r \leqslant r_0 \leqslant \infty$.

(ii) *Assume, moreover, that T satisfies the cancellation conditions of* (3.6). *Then*

(6.3)
$$\| \{Tf_j\} \|_{H^p(\mathbb{R}^n, l^r(Y))} \leqslant c \| \{f_j\} \|_{H^p(\mathbb{R}^n, l^r(Y))}$$

for $n/(n+\delta) < p \leqslant 1 < r \leqslant r_0 < \infty$.

PROOF. We prove only (i), the proof of (ii) being similar. To do this observe that the operator $\tilde{T}\{f_j\} = \{Tf_j\}_{j=1}^{\infty}$, $\{f_j\} \in L_c^{\infty}(\mathbb{R}^n, l^r(X))$ is a Calderón-Zygmund operator associated with a kernel satisfying a $D_{1,1}$ condition with $d_j = 2^{-j\delta}$. Moreover, it is continuous from $L^r(\mathbb{R}^n, l^r(X))$ into $L^r(\mathbb{R}^n, l^r(Y))$. Thus, by (3.9) conveniently adapted for the case $\alpha = 1$, we obtain (6.2).

Suppose T is an operator associated to a standard kernel satisfying a $D_{1,\alpha}$ condition. Moreover, suppose that T maps continuously $L^2(X)$ into $L^2(Y)$ and $L^2(X)$ into $L^{q'}(Y)$, $1/q' = 1/2 - \beta/n$. Then, if $T(\bar{e}) = 0$, Theorem 3.5 shows that T^* maps continuously $H^1(Y')$ into $H^1(X')$.

That is to say, T maps continuously BMO(X) into BMO(Y). When $\alpha = 1$, $\beta = 0$, we get in particular the BMO-continuity of Calderón-Zygmund operators under the suitable cancellation condition. Once again, this cancellation condition can be dropped in the convolution case.

We shall consider some applications to pseudo-differential operators. Firstly, observe that pseudo-differential operators defined by amplitudes $a(x, y, \xi)$ in the class $S^0_{1,\theta}$, $0 \leqslant \theta < 1$, are Calderón-Zygmund operator satisfying conditions (S_2) of §2 with $\alpha = \delta = 1$ (i.e., a D_1 condition) (cf. [2]). Thus, our results apply for these operators. We obtain in particular the results of [13]. Moreover, pseudo-differential operators with amplitudes in $S^{-\beta}_{\alpha,\delta}$,

$$(1-\alpha)\frac{n}{2} \leqslant \beta < \frac{n}{2}, \qquad 0 < \delta \leqslant \alpha < 1,$$

are strongly singular Calderón-Zygmund operators satisfying a somewhat weaker $D_{1,\alpha}$ condition (cf. [1]).

Lemma 1.2 in [14] essentially proves that these operators also satisfy a $D_{2,\alpha}$ condition. Thus, using (3.2) with $p = r = 2$, gives

$$(Tf)^{\#}(x) \leqslant cM_2(f)(x), \tag{6.4}$$

an estimate derived by S. Chanillo and A. Torchinsky (cf. [3]). In [3] it is also asked whether the index 2 is the smallest possible one in the estimate (6.4). For example, the authors show that for a special class of symbols in $S^{-n(1-\alpha)/2}_{\alpha,\delta}$, M_2 can be replaced in (6.4) by M_p, for any $1 < p < \infty$.

Since one knows that operators T with symbols in $S_{\alpha,\delta}$, $(1-\alpha)n/2 \leqslant \beta < n/2$ are continuous in L^p for any $1 < p < \infty$, we can improve (6.4) each time we can assure that T maps continuously L^p into L^q, for some $p < 2$, $p/q \leqslant \alpha$.

For example, L. Hörmander has proved in [11] that an operator with symbol in $S^m_{\rho,\delta}$, is continuous from L^p into L^q, if

$$m < -n\left(\frac{1}{p} - \frac{1}{q}\right), \qquad 1 < p \leqslant 2 \leqslant q < \infty \tag{6.5}$$

$$(6.5)' \qquad m < -n\left[\frac{1}{p} - \frac{1}{q} + (1-\rho)\left(\frac{1}{q} - \frac{1}{2}\right)\right], \qquad 1 < p \leqslant q \leqslant 2.$$

In our case is $m = -\beta$. Let us fix any $1 < p < 2$, $q = p/\alpha$ and suppose $0 < \alpha < p/2$. Thus, (6.5) reads

$$\beta > \frac{n(1-\alpha)}{p}.$$

So, (6.4) holds with 2 replaced by p, $1 < p < 2$, if T is an operator with symbol in $S_{\alpha,\delta}^{-n(1-\alpha)/p-\epsilon}$, $\epsilon > 0$, provided $0 < \alpha < p/2$. *In particular, in* $S_{\alpha,\delta}^{-n(1-\alpha)}$, (6.4) holds with 2 replaced by any $p > 1$, provided $0 < \alpha < 1/2$.

When $1 > \alpha > p/2$, (6.5)′ yields to the condition

$$\beta > n(1-\alpha)\left(\frac{1+\alpha}{p} - \frac{1}{2}\right).$$

This time, (6.4) will hold for any $1 < p < 2$, if T has symbol in $S_{\alpha,\delta}^{-n(1-\alpha)(1/2+\alpha)}$ when $1/2 < \alpha < 1$.

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