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Several Characterizations for the Special Atom Spaces with Applications

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Introduction

The theory of functions plays an important role in harmonic analysis. Because of this, it turns out that some spaces of analytic functions have been studied extensively, such as H^p-spaces, Bergman spaces, etc. One of the major insights that has developed in the study of H^p-spaces is what we call the real atomic characterization of these spaces.

The first author in [3] introduced the special atom space B^p for $p > 1/2$ and it was shown that $H^p \subsetneq B^p$ for $1/2 < p < 1$ and $B^p \subsetneq H^p$ for $1 \leqslant p < \infty$.

For example in case $p = 1$, we have the B^1-space which is a Banach space whose dual is the derivative of the Zygmund space Λ_*. In [4], it was shown that this space is the real atomic characterization of the boundary values of those analytic functions f in the disk $D = \{z \in C; |z| < 1\}$ for which

$$\|f\|_S = |f(0)| + \frac{1}{2\pi}\int_0^1\int_{-\pi}^{\pi} |f'(re^{i\theta})|\, d\theta\, dr < \infty.$$

This space denoted by S was introduced in [1] and among other things it was shown there that the dual of S is the space β of Bloch functions. We refer the reader to [4] for a complete discussion of these matters. It was shown in

[5] that

$$B^1 \subsetneq H^1 \subsetneq L^1.$$

Subsequently, in [9] we were able to show that for $p > 1$, the B^p-space is the real atomic characterization of the boundary values of those analytic functions in the disk for which

$$\int_0^1 \int_{-\pi}^{\pi} |f'(re^{i\theta})|(1-r)^{(1/p)-1}\, d\theta\, dr < \infty.$$

These analytic function spaces, which we denoted by J^p, were studied extensively in [14], [15] and [16], while for $1/2 < p < 1$, B^p is the smallest Banach space that contains H^p and also gives real characterization for the boundary values of those analytic functions in the disk for which

$$\int_0^1 \int_{-\pi}^{\pi} |f(re^{i\theta})|(1-r)^{(1/p)-2}\, d\theta\, dr < \infty.$$

These analytic function spaces were introduced in [13].

We would also like to point out that the duals of the B^p spaces are Lipschitz spaces see [5], [6] and [10] and this fact is exploited in the present paper.

It should now be clear why so much attention has been placed in analyzing these spaces. But lots of our efforts are concerned with using these spaces as intermediaries in order to interpolate linear operators between L^p-spaces. For example, it is well-known that (see Theorem D in [2])

$$T\colon H^1 \to L^1 \quad \text{and} \quad T\colon L^2 \to L^2 \quad \text{boundedly}.$$

Then

$$T\colon L^p \to L^p \quad \text{boundedly for} \quad 1 < p \leqslant 2. \tag{1}$$

And there is also an analytic version of this, namely Theorem E of [2]. And so of course, we wish to replace H^1 by B^1 and ask what else is needed in order to obtain (1). In many cases, T is a concrete operator and so what else is needed may in fact be easy to check. However, this program still is incomplete.

In this paper, we give several descriptions of B^p for $p > 1$ and obtain various consequences from them.

First description. Let f be a function which may be decomposed into a finite or countable linear combinations of characteristic functions of intervals in T, where T is the unit circle in the plane which we may identify in the usual way with $[0, 2\pi)$.

$$f(x) = \sum_{n=1}^{\infty} \alpha_n \chi_{I_n}(x)$$

For a given f there may be many such decompositions. Let

$$\|f\|_{B^p} = \operatorname{Inf} \sum_{n=1}^{\infty} \|\alpha_n \chi_{I_n}\|_p$$

where the infimum is taken over all possible decompositions of f, and by $\|\cdot\|_p$ we mean the L^p-norm on T, that is

$$\|g\|_p = \left(\int_T |g|^p\right)^{1/p}$$

even for $p < 1$. We get by Minkowski's inequality for $p > 1$ that $\|f\|_p \leqslant \|f\|_{B^p}$ and it follows that $\|\chi_I\|_{B^p} = \|\chi_I\|_p$.

We denote by B^p the set of all $f \in L^p$ for which $\|f\|_{B^p}$ is finite, for $1 < p < \infty$.

We point out that convergence of

$$\sum_{n=1}^{\infty} \alpha_n \chi_{I_n}$$

is taken in the sense of the Lebesgue space L^p, actually we could just as well use the Lorentz space $L(p, 1)$ and observe that $\|f\|_{L(p,1)} \leqslant \|f\|_{B^p}$.

Note. $f \in L(p, 1)$ if and only if

$$\|f\|_{L(p,1)} = \frac{1}{p} \int_0^{\infty} f^*(t) t^{1/p-1}\, dt < \infty,$$

where f^* is the decreasing rearrangement of f, $0 < p < \infty$.

It is straightforward to show the following result.

Theorem 1. *The space B^p, endowed with the norm $\|\cdot\|_{B^p}$ is a Banach space.*

Second description. By a special atom we mean a function b supported on an interval $I \subset T$ which is of the form

$$b(x) = \alpha \chi_L(x) - \alpha \chi_R(x)$$

where L is the left half of I, R is the right half and α is scalar.

For a function f defined on T and which is a finite or countable sum of special atoms plus a constant

$$f(x) = a + \sum_{n=1}^{\infty} b_n(x)$$

(again there may be many ways of so decomposing f) we let

$$\|f\|^*_{B^p} = \operatorname{Inf} \sum_{n=1}^{\infty} \|b_n\|_p.$$

We denote by B^p the set of all $f \in L^p$ for which $\|f\|_{B^p}^*$ is finite, for $1 < p < \infty$.

Observe that this description of B^p holds whenever $1/2 < p < \infty$, but that when $1/2 < p < 1$, f may exist not as a function but only as a distribution. Indeed if

$$f(x) = \sum_{n=1}^{\infty} b_n(x) \in B^p$$

then given a test function ϕ, that is, an infinitely differentiable function on T, we define

$$(f, \phi) = \sum_{n=1}^{\infty} (b_n, \phi) = \sum_{n=1}^{\infty} \int_{I_n} b_n(x)\phi(x)\,dx,$$

where I_n is the support of b_n as in the definition of special atoms. We wish to show that the series converges absolutely. Let a_n denote the height of $b_n(x)$ and h_n the length of I_n.

Observe that

$$(b_n, \phi) = \int_I b_n(x)\phi(x)\,dx = a_n\left[\Phi\left(\cdot + \frac{h_n}{2}\right) + \Phi\left(\cdot - \frac{h_n}{2}\right) - 2\Phi(\cdot)\right]$$

where Φ is the indefinite integral of ϕ, that is, $\Phi(x) = \int_0^x \phi(t)\,dt$. Thus

$$|(b_n, \phi)| \leqslant 4|a_n|\,\|\Phi\|_\infty,$$

where

$$\|\Phi\|_\infty = \operatorname*{ess\,sup}_{t\in T} |\Phi(t)|.$$

Then using the mean value theorem twice on the second difference above we also see that

$$|(b_n, \phi)| \leqslant |a_n| h_n^2 \|\Phi''\|_\infty.$$

But $\Phi'' = \phi'$ and $\|\Phi\|_\infty \leqslant \|\phi\|_\infty \cdot (\operatorname{supp}\phi)$, so if $h_n \leqslant 1$ then

$$|(b_n, \phi)| \leqslant |a_n| h_n^2 \|\Phi'\|_\infty \leqslant |a_n| h_n^{1/p} \|\phi'\|_\infty = \|b_n\|_p \|\phi'\|_\infty$$

since $1/2 < p < 1$, while if $h_n > 1$ then

$$|(b_n, \phi)| \leqslant 4|a_n|\,\|\phi\|_\infty \leqslant 4\|\phi\|_\infty(\operatorname{supp}\phi)|a_n| h_n^{1/p} = 4\|\phi\|_\infty(\operatorname{supp}\phi)\|b_n\|_p.$$

Therefore

$$\sum_{n=1}^{\infty} |(b_n, \phi)| \leqslant [4\|\phi\|_\infty(\operatorname{supp}\phi) + \|\phi'\|_\infty] \sum_{n=1}^{\infty} \|b_n\|_p.$$

Third description. A triangular function is a function supported on an interval $I \subset T$ whose graph restricted to I is the two equal sides of an isosceles triangule.

Observe that $t(x)$ is a triangular function if and only if it is the indefinite integral of a special atom (see second description).

For a sum of triangular functions

$$f(x) = \sum_{n=1}^{\infty} t_n(x),$$

we set

$$\|f\|_{B^p}^T = \operatorname{Inf} \sum_{n=1}^{\infty} \|t_n\|_p$$

where the infimum is taken over all possible representations of f. We denote by B^p the set of all $f \in L^p$ for which $\|f\|_{B^p}^T$ is finite, for $1 < p < \infty$.

Fourth description. Given a non-negative integer n partition $T = [0,2\pi)$ into 2^n subintervals of equal length and let I_{nk} be the k^{th} of these for $k = 1$ to $k = 2^n$. (I_{nk} is called a dyadic interval). The Haar function ϕ_{nk} is a special atom supported on I_{nk} and such that $\|\phi_{nk}\|_2 = 1$. That is,

$$\phi_{nk}(x) = \begin{cases} \left(\dfrac{2^n}{2\pi}\right)^{1/2} & \text{on } \left[\dfrac{k-1}{2^n}2\pi, \dfrac{k-1/2}{2^n}2\pi\right) \\ -\left(\dfrac{2^n}{2\pi}\right)^{1/2} & \text{on } \left[\dfrac{k-1/2}{2^n}2\pi, \dfrac{k}{2^n}2\pi\right] \\ 0 & \text{elsewhere.} \end{cases}$$

It is well known that if we supplement the Haar functions by the constant function whose value is everywhere the reciprocal of the square root of 2π, i.e. $\phi_{00} = 1/\sqrt{2\pi}$, we obtain a complete orthonormal system. Given an integrable function f, we may form its Haar series

$$f(x) = a + \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} c_{nk}\phi_{nk}(x)$$

where

$$c_{nk} = \int_0^{2\pi} f(x)\phi_{nk}(x)\,dx.$$

Let

$$\|f\|_{B^p}^H = \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} \|c_{nk}\phi_{nk}\|_p,$$

it is easily seen that the set of all those f for which $\|f\|_{B^p}^H$ is finite form a Banach space which is continuously embedded in L^p or even in $L(p, 1)$.

We denote this set by B^p for $1 < p < \infty$.

Fifth description. Similar to the first one, except that f must be written as

$$f(x) = \sum_{n=1}^{\infty} \alpha_n \chi_{I_n}(x),$$

where the I_n are dyadic intervals.

Sixth description. Similar to the third one except that

$$f(x) = \sum_{n=1}^{\infty} t_n(x)$$

with the supports of the triangular functions t_n's being dyadic intervals.

Note that with these norms B^p for $p > 1$ is a Banach space.

2. Equivalence of descriptions

We shall now argue that the various descriptions are equivalent.

Let $I = (a, b)$ and $h = |I| = b - a$. We wish to express χ_I as a sum of triangular functions. In fact, let $v(x)$ be the triangular function whose height is 1 and whose support is I and for each positive integer n let $s_n(x)$ and $t_n(x)$ be triangular functions of height $1/2$ whose supports are

$$\left(a, a + \frac{h}{2^n}\right) \quad \text{and} \quad \left(b - \frac{h}{2^n}, b\right)$$

respectively. It is easily seen that

$$\chi_I(x) = v(x) + \sum_{n=1}^{\infty} [s_n(x) + t_n(x)], \tag{2.1}$$

in fact, let $x \in I$, then

$$x \in \left(a, \frac{a+b}{2}\right] \cup \left[\frac{a+b}{2}, b\right), \quad \text{say,} \quad x \in \left(a, \frac{a+b}{2}\right),$$

then there is an integer N

$$a + \frac{h}{2^{N+1}} \leqslant x < a + \frac{h}{2^N},$$

therefore the right hand side of (2.1) becomes

$$v(x) + \sum_{n=1}^{N} s_n(x)$$

and so, we need to prove this sum is 1. In fact, if we explicitly write down $v(x)$ and $s_n(x)$ we see that

$$v(x) + \sum_{n=1}^{N} s_n(x) = \frac{2}{h}(x-a) + \frac{2}{h}(x-a) + \frac{2^2}{h}(x-a) + \cdots$$
$$+ \frac{2^{N-1}}{h}(x-a) - \frac{2^N}{h}\left(x - a - \frac{h}{2^N}\right)$$

Therefore we have

$$v(x) + \sum_{n=1}^{N} s_n(x) = \frac{2^N}{h}(x-a) - \frac{2^N}{h}\left(x - a\frac{h}{2^N}\right) = 1.$$

Similar situations occur when x is in the other half of I, that is,

$$x \in \left[\frac{a+b}{2}, b\right),$$

so that the assertion is proved.

Observe now that

$$\|v\|_p = \frac{h^{1/p}}{(1+p)^{1/p}} \quad \text{and} \quad \|s_n\|_p = \|t_n\|_p = \frac{h^{1/p}}{2^{(n/p)+1}(1+p)^{1/p}}$$

so that

$$\|\chi_I\|_{B^p}^T \leqslant \|v\|_p + \sum_{n=1}^{\infty} \|s_n\|_p + \sum_{n=1}^{\infty} \|t_n(x)\|_p = A_p h^{1/p} = A_p |I|^{1/p} = A_p \|\chi_I\|_{B^p}$$

where

$$A_p \doteq \frac{2^{1/p}}{(2^{1/p}-1)(1+p)^{1/p}},$$

consequently for any

$$f(x) = \sum_{n=1}^{\infty} \alpha_n \chi_{I_n}(x) \tag{2.2}$$

we have $\|f\|_{B^p}^T \leqslant A_p \|f\|_{B^p}$.

Next suppose $t(x)$ is a triangular function of height 1 whose support is some subinterval $I \subset [0, 2\pi)$ and $h = |I|$. Let us expand $t(x)$ in a Haar series; that is

$$t(x) = \frac{h}{2\pi} + \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} c_{nk}\phi_{nk}(x)$$

where ϕ_{nk} are the Haar functions (see description four).

Choose an integer N such that

$$\frac{2\pi}{2^{N+1}} < h \leqslant \frac{2\pi}{2^N}. \tag{2.3}$$

For $n \leqslant N$ there are at most two dyadic intervals I_{nk} which intersect I. We now estimate $|c_{nk}|$ by the elementary way

$$|c_{nk}| = \left| \int_{I_{nk}} t(x)\phi_{nk}(x)\,dx \right| \leqslant |I| \, \|\phi_{nk}\|_\infty = \frac{h2^{n/2}}{(2\pi)^{1/2}}. \tag{2.4}$$

Again we estimate $|c_{nk}|$ by using a different approach, in fact

$$c_{nk} = \int_{I_{nk}} t(x)\phi_{nk}(x)\,dx = \int_{I_{nk}} (t(x) - t(x_0))\phi_{nk}(x)\,dx.$$

where x_0 is any point of I_{nk}. Therefore we have

$$\begin{aligned} |c_{nk}| &\leqslant \|\phi_{nk}\|_1 \cdot \max\{|t(x) - t(x_0)|; x \in I_{nk}\} \\ &\leqslant (2\pi)^{1/2}2^{-(n/2)}4\pi h^{-1}2^{-n} = C2^{-3n/2}h^{-1} \end{aligned} \tag{2.5}$$

where $C = (2\pi)^{1/2}4\pi$.

We now estimate $\|t\|^H_{BP}$, in fact, by definition we have

$$\|t\|^H_{BP} = h(2\pi)^{(1/p)-1} + \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} |c_{nk}| \, \|\phi_{nk}\|_p.$$

If we set

$$\sum_{n=0}^{\infty} = \sum_{n=0}^{N} + \sum_{n=N+1}^{\infty} = S + T.$$

Recall that for $n \leqslant N$ there are at most two dyadic intervals I_{nk}which intersect I, therefore using (2.4) we get

$$\begin{aligned} S &= \sum_{n=0}^{N} \sum_{k=1}^{2^n} |c_{nk}| \, \|\phi_{nk}\|_p \leqslant \sum_{n=0}^{N} 2h2^{n/2}(2\pi)^{-1/2}2^{n/2}(2\pi)^{-1/2}\left(\frac{2\pi}{2^n}\right)^{1/p} \\ &= 2h(2\pi)^{1/p-1} \sum_{n=0}^{N} 2^{n(1-1/p)} \leqslant B_p h(2\pi)^{1/p-1}2^{(N+1)(1-1/p)} \\ &\leqslant B_p h^{1/p} \end{aligned}$$

The inequality on the last step follows from (2.3) where B_p is a constant which depends upon p and observe that it tends to infinity as p approaches 1.

The estimate for T follows by the observation that for $n > N$, there are at most $2^{n-N} + 2 \leqslant 2^{n-N+1}$ which intersects I and by (2.5). In fact,

$$\begin{aligned} T &= \sum_{n=N+1}^{\infty} \sum_{k=1}^{2^N} |c_{nk}| \, \|\phi_{nk}\|_p \\ &\leqslant \sum_{n=N+1}^{\infty} 2^{n-N+1} C 2^{-3n/2} h^{-1} 2^{n/2} (2\pi)^{-1/2} \left(\frac{2\pi}{2^n}\right)^{1/p} \\ &= C_p 2^{-N} h^{-1} \sum_{n=N+1}^{\infty} 2^{-n/p} \leqslant C_p 2^{-N} h^{-1} 2^{-(N+1)/p} \\ &\leqslant C_p h^{1/p}. \end{aligned}$$

The constants C_p are not the same at every occurence.

Thus there is a constant G_p which blows up as p approaches 1 such that

$$\|t\|_{B^p}^H \leqslant G_p h^{1/p}.$$

Therefore if

$$f(x) = \sum_{n=0}^{\infty} t_n(x)$$

where $t_n(x)$ is a triangular function we have

$$\|f\|_{B^p}^H \leqslant G_p \|f\|_{B^p}^T. \tag{2.6}$$

It is obvious from the definition that a function in the fourth description is also in the second description and that

$$\|f\|_{B^p}^* \leqslant \|f\|_{B^p}^H. \tag{2.7}$$

Likewise a function in the second description is also in the first description and that

$$\|f\|_{B^p} < 2^{1-(1/p)} \|f\|_{B^p}^*. \tag{2.8}$$

Now putting together (2.2), (2.6), (2.7) and (2.8) and the fact that the intervals of the two other descriptions are dyadic it makes the equivalence of the definitions of the B^p obvious.

We would like to point out that for $p = 1$, B^1 is not the same as B^1-dyadic, as was shown in [11].

3. Some Consequences

In this section we present some theorems whose proof was made possible due to the several descriptions of the special atom spaces.

Theorem 3.1. *If $1/2 < p < 1 < q < \infty$ and $1/q = 1/p - 1$ then $f \in B^p$ if and only if $F \in B^q$, where F is the indefinite integral of f, that is, $F(x) = \int_0^x f(t)\,dt$.*

PROOF. Given a special atom $b_n(x)$ let

$$t_n(x) = \int_{x_n}^{x} b_n(t)\,dt$$

where x_n is the left hand endpoint of the interval supporting $b_n(x)$. Then $t_n(x)$ is a triangular function and there exists a constant $C_q \doteq 1/2(1+q)^{1/q}$ such that $\|t_n\|_q = C_q \|b_n\|_p$ for $1/q = 1/p - 1$. If we define $F(x)$ to be that function in L^q which is the sum of the $t_n(x)$, that is,

$$F(x) = \sum_{n=1}^{\infty} t_n(x),$$

where $t_n(x)$ is as above, then it is clear that

$$\|F\|_{B^q}^{T} \leqslant C_q \|f\|_{B^p}^{*}.$$

Conversely if $F \in B^q$ and we write

$$F(x) = \sum_{n=1}^{\infty} t_n(x)$$

and let $b_n(t) = (d/dx)t_n(x)$ (except at the corners) and let $f(x) = F'(x)$ be the distribution which is the sum of the $b_n(x)$, that is,

$$f(x) = \sum_{n=1}^{\infty} b_n(x),$$

then clearly

$$\|f\|_{B^p}^{*} \leqslant \frac{1}{C_q} \|f\|_{B^q}^{T}$$

so that

$$\|F\|_{B^q}^{T} = C_q \|f\|_{B^p}^{*}.$$

Observe that if f has two representations

$$f(x) = \sum_{n=1}^{\infty} b_n(x) = \sum_{n=1}^{\infty} c_n(x)$$

where $b_n(x)$ and $c_n(x)$ were special atoms and $s_n(x)$ and $t_n(x)$ their respective indefinite integrals then for every test function ϕ,

$$\sum_{n=1}^{\infty} (b_n, \phi) = \sum_{n=1}^{\infty} (c_n, \phi)$$

where both sums converge absolutely (recall that $(b_n, \phi) = \int_{I_n} b_n(x)\phi(x)\,dx$, where I_n is the support of b_n, likewise for (c_n, ϕ)), then by integration by parts we get

$$-\sum_{n=1}^{\infty} (s_n, \phi') = -\sum_{n=1}^{\infty} (t_n, \phi').$$

The dash means derivative.

Thus if

$$F(x) = \sum_{n=1}^{\infty} s_n(x) \quad \text{and} \quad G(x) = \sum_{n=1}^{\infty} t_n(x),$$

then F and G are in L^q and for every test function ϕ.

$$(F - G, \phi') = 0$$

so that F and G are functions in L^q which at most differ by a constant. Therefore Theorem 3.1 is proved.

Theorem 3.2 *If $f \in B^p$, $1/2 < p < \infty$ and $f_h(x) = f(x + h)$ then*

$$\lim_{h \to 0} \| f_h - f \|^*_{B^p} = 0.$$

PROOF. Suppose I is an interval of length m and consider

$$v(x) = \chi_I(x + h) - \chi_I(x).$$

$v(x)$ is supported on an interval of length $m + h$ and on this interval we expand $v(x)$ in a Haar series relative to that interval. For each $n \geqslant 1$ there are at most two ϕ_{nk} whose coefficients c_{nk} do not vanish and for these, we have

$$|c_{nk}| \leqslant \left(\frac{2^n}{m + h}\right)^{1/2} \cdot \left(h \Lambda \frac{m + h}{2^n}\right).$$

(Λ denotes the operation of taking the smaller of two numbers.)

Estimating the B^p norm of $v(x)$ we get

$$\|v\|_{B^p}^H = \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} \|c_{nk}\phi_{nk}\|_p$$

$$\leqslant \sum_{n=0}^{\infty} 2\left(\frac{2^n}{m+h}\right)^{1/2}\left(h\Lambda\frac{m+h}{2^n}\right)\left(\frac{2^n}{m+h}\right)^{1/2}\left(\frac{m+h}{2^n}\right)^{1/p}$$

$$< \frac{2}{m}\sum_{n=0}^{\infty} 2^{n(1-1/p)}\left(h\Lambda\frac{m+h}{2^n}\right).$$

This last series converges and individual terms tend monotonically to zero with h so that the sum tends to zero with h. Therefore

$$\lim_{h\to 0} \|v\|_{B^p}^* = 0.$$

Now since a special atom is a multiple of the difference of two characteristic functions it is clear that if $b(x)$ is a special atom and if $b_h(x) = b(x+h)$ then

$$\lim_{h\to 0} \|b_h - b\|_{B^p}^* = 0.$$

Given $\epsilon > 0$, any $f \in B^p$ can be written $f(x) = g(x) + K(x)$ where $g(x)$ is a finite sum of special atoms and $\|K\|_{B^p}^* < \epsilon$, therefore as $\|g_h - g\|_{B^p}^* < \epsilon$ for h small, we know that B^p is invariant under translation so that,

$$\|f_h - f\|_{B^p}^* \leqslant \|g_h - g\|_{B^p}^* + \|K_h\|_{B^p}^* + \|K\|_{B^p}^* < 3\epsilon$$

for h small enough. So Theorem 3.2 is proved.

By the modulus of continuity of $f \in B^p$ we mean for $\delta > 0$

$$w(\delta) = \sup\{\|f_h - f\|_{B^p}^*; -\delta < h < \delta\}.$$

(depending on the context we may replace $\| \quad \|_{B^p}^*$ by $\| \quad \|_{B^p}$ or $\| \quad \|_{B^p}^H$ etc.)

It is an immediate consequence of the definition of B^p and Theorem 3.2, that for $f \in B^p$, $w(\delta) \leqslant 2\|f\|_{B^p}^*$ and $w(\delta) \to 0$ as $\delta \to 0$.

Theorem 3.3. *If $1/2 < p < 1$ then L^1 of the boundary of the unit disc is continuously embedded in B^p.*

PROOF. Let I be a dyadic interval of length $h = 2\pi/2^m$. Let us expand χ_I

$$\chi_I(x) = c_{00}\phi_{00} + \sum_{n=0}^{\infty}\sum_{k=1}^{2^n} c_{nk}\phi_{nk}(x) \tag{3.4}$$

where ϕ_{nk} are the Haar functions and $\phi_{00}(x) = (2\pi)^{-1/2}$.

Notice that for $n < m$ there is precisely one k such that c_{nk} does not vanish, therefore

$$|c_{nk}| = \left| \int_I \phi_{nk}(x)\, dx \right| = h\left(\frac{2^n}{2\pi}\right)^{1/2} \quad \text{and} \quad c_{00} = \frac{h}{(2\pi)^{1/2}}.$$

For $n \geqslant m$ all the Haar coefficients c_{nk} vanish, so that (3.4) becomes

$$\chi_I(x) = \frac{h}{2\pi} + \sum_{n=0}^{m-1} c_{nk}\phi_{nk}(x),$$

consequently using the definition of B^p norm

$$\|\chi_I\|_{B^p}^* \leqslant \sum_{n=0}^{m-1} h\left(\frac{2\pi}{2^n}\right)^{(1/p)-1} < C_p h = C_p \|\chi_I\|_1.$$

Thus if f is a step function whose steps are dyadic intervals we have

$$\|f\|_{B^p}^* \leqslant C_p \|f\|_1.$$

Since for any $f \in L^1$ there is a sequence of such step fuction f_n which approach f in L^1 and since B^p is a Banach space we get our result.

Corollary 3.5. *For $1/2 < p < 1$, L^2 of the boundary of the unit disc is continuously embedded in B^p. For $1 < p < \infty$, $(L^2)_1$ = the set of indefinite integrals of L^2 functions whose constant terms vanish, is continuously embedded in B^p, in particular every C^∞ function on the boundary of the unit disc is in every B^p.*

PROOF. The first part of the statement will follow from the fact that L^2 is continuously embedded in L^1 and Theorem 3.3. The last part follows from Theorem 3.1 and the first part of this corollary.

The use of lemmas of the following type in functional analysis is due to G. G. Lorentz.

Lemma 3.6. *Let H be a Hilbert space of measurable functions on T with inner product given by*

$$(f, g) = \int_T f(x)g(x)\, dx$$

and let F and G be real Banach spaces of measurable functions such that H is embedded continuously in F. Assume that

$$|(f, g)| \leqslant \|f\|_F \cdot \|g\|_G$$

and that

$$\|g\|_G = \sup\{|(f, g)|; f \in H, \|f\|_F \leqslant 1\}.$$

Then for every $f \in H$,

$$\|f\|_F = \sup \{ |(f, g)|; \|g\|_G \leqslant 1 \}.$$

PROOF. Given a subset C of H, let $C' = \{g \in H\colon |(f, g)| \leqslant 1 \text{ for all } f \in C\}$. It is a geometrical fact about Hilbert spaces that if C satisfies the three properties, 1. closed, 2. convex, 3. $f \in C$ implies $-f \in C$, then C' satisfies these three properties as well and moreover $C'' = C$.

Let us take $C = \{f \in H\colon \|f\|_F \leqslant 1\}$ then C satisfies the three properties, so it follows that $C' = \{g \in H\colon \|g\|_G \leqslant 1\}$. Therefore $C'' = C$ which implies the final statement of the lemma. So the lemma is proved.

Following Zygmund [17] for $0 < \alpha < 2$ we denote by Λ_α the set of all functions g on T, for which there is a constant A such that $|g(x + h) + g(x - h) - 2g(x)| \leqslant Ah^\alpha$, and $\|g\|_{\Lambda_\alpha}$ is meant to be the infimum of all such A, that is,

$$\|g\|_{\Lambda_\alpha} = \sup_{\substack{h>0 \\ x}} \frac{|g(x + h) + g(x - h) - 2g(x)|}{h^\alpha}.$$

For $0 < \alpha < 1$, $\Lambda_\alpha = \operatorname{Lip} \alpha = \{g; g(x + h) - g(x) = 0(h^\alpha)\}$ and for $1 < \alpha < 2$, $f \in \Lambda_\alpha$ if and only if its derivative $f' \in \operatorname{Lip}(\alpha - 1)$.

Lemma 3.7. *If* $f \in B^p$ *and* $g \in \operatorname{Lip} \alpha$ *for* $\alpha = 1/p$, $1/2 < p < 1$ *then for* $f \in L^2$ *we have*

$$\|f\|^*_{B^p} = \sup \left\{ \left| \int_T f(x)g(x)\,dx \right| : \left\| \int g \right\|_{\Lambda_{\alpha+1}} \leqslant 1 \right\}.$$

PROOF. By Corollary 3.5 and Lemma 3.6 with

$$H = L^2, \qquad F = B^p \quad \text{and} \quad G = \operatorname{Lip}(\alpha - 1)$$

we get the first part since for any special atom b we have

$$|(b, g)| \leqslant \|g\|_{\operatorname{Lip}(\alpha-1)} \cdot \|b\|_{B^p},$$

with the norm of $g \in \operatorname{Lip}(\alpha - 1)$ given by $\left\| \int g \right\|_{\Lambda_\alpha}$, where $\int g$ is the periodic indefinite integral of g.

For the second part, let $1/r = 1/p + 1$, then $\|f'\|^*_{B^r} = C\|f\|^T_{B^p}$ by Theorem 3.1, where the dash means the derivative of f. But integration by parts yields

$$\int_T f(x)g(x)\,dx = -\int_T f'(x)g_1(x)\,dx \quad \text{where} \quad g_1(x) = \int_0^x g(t)\,dt.$$

We now use the first part of this lemma to calculate $\|f\|^*_{B^r}$.

Lemma 3.8. *Suppose that given a family G of measurable functions on T or R^n, we define a norm on a space X for which this supremum is finite by*

$$\|f\|_X = \sup\left\{\left|\int f(x)g(x)\,dx\right|; g \in G\right\}.$$

Then a formula which recalls Minkowski's inequality for integrals holds, namely if

$$F(x) = \int f(x, y)\,dy$$

then

$$\|F\|_X \leqslant \int \|f(\cdot, y)\|_X\,dy.$$

If in addition G is translation invariant then for the convolution $f * g$ where $g \in L^1$ we have

$$\|f * g\|_X \leqslant \|f\|_X \cdot \|g\|_1$$

PROOF. From the definition of norm in X we have that

$$\|F\|_X = \operatorname{Sup}\left\{\left|\int g(x)\,dx \int f(x, y)\,dy\right|; g \in G\right\}.$$

But for each such g in G, the integral inside the brackets is bounded by

$$\int dy \left|\int g(x) f(x, y)\,dx\right|,$$

so that

$$\|F\|_X \leqslant \int \|f(\cdot, y)\|_X\,dy.$$

Now suppose G is translation invariant:

$$\|f * g\|_X = \sup\left\{\left|\int h(x)\,dx \int f(x - y)g(y)\,dy\right|; h \in G\right\}$$

For each h, the integral is bounded by

$$\int |g(y)|\,dy \left|\int h(x) f(x - y)\,dy\right| = \int |g(y)|\,dy \left|\int h(u + y) f(y)\,du\right|$$

Therefore

$$\|f * g\|_X \leqslant \sup\left\{\left|\int h(u + y) f(y)\,dy\right|; h \in G\right\} \cdot \int |g(y)|\,dy,$$

that is

$$\|f * g\|_X \leqslant \|g\|_1 \cdot \|f\|_X.$$

Observe now that if we take $X = B^p$ and $G = \{g \in \mathrm{Lip}(\alpha - 1); \|\int g\|_{\Lambda_\alpha} \leq 1\}$ for $\alpha = 1/p$, $1/2 < p < 1$ on T, and for $1 < p < \infty$ we set $G = \{g \in \mathrm{Lip}\,\alpha; \|\int g\|_{\Lambda_{\alpha+1}} \leq C\}$ then we have the following.

$$\|f*g\|^*_{B^p} \leq \|g\|_1 \cdot \|f\|^*_{B^p} \quad \text{for} \quad \tfrac{1}{2} < p < 1$$
$$\|f*g\|^T_{B^p} \leq \|g\|_1 \cdot \|f\|^T_{B^p} \quad \text{for} \quad 1 < p < \infty.$$

Theorem 3.9. *If $f \in B^p$ on the boundary of the unit disc, $1 < p < \infty$, and if f_r is its Poisson integral then $\|f_r\|^T_{B^p}$ is a non-decreasing function of r which tends to $\|f\|^T_{B^p}$ as $r \nearrow 1$, that is, $\|f\|^T_{B^p} \to \|f\|^T_{B^p}$ as $r \nearrow 1$. Moreover f_r tends to f in B^p.*

Conversely, if $\mu(r, \theta)$ is a harmonic function in the unit disc such that for each r, $0 < r < 1$

$$\|\mu(r, \cdot)\|_{B^p} \leq M < \infty.$$

then μ is the Poisson integral of $f \in B^p$. (A similar theorem is true for the real line.)

PROOF. We start with the converse. Notice that

$$\|\mu(r, \cdot)\|_p \leq \|\mu(r, \cdot)\|_{B^p} \leq M,$$

so that H^p-theory tells us that there is an $f \in H^p = L^p$ such that μ is the Poisson integral of f. We wish to prove that $f \in B^p$.

Expanding f into a Haar series we get

$$f(\theta) = a + \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} c_{nk} \phi_{nk}(\theta).$$

Let r_i be a sequence of $r_i \nearrow 1$. For each i expand $\mu(r_i, \theta)$ in a Haar series, namely

$$\mu(r_i, \theta) = c + \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} c^i_{nk} \phi_{nk}(\theta).$$

Notice that

$$c_{nk} = \int_T f(\theta)\phi \, d\theta = \lim_{i\to\infty} \int_T \mu(r_i, \theta)\phi_{nk}(\theta)\, d\theta = \lim_{i\to\infty} c^i_{nk}.$$

By Fatou's lemma (for sums rather than integrals) we get

$$\|f\|^H_{B^p} = \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} \|c_{nk}\phi_{nk}\|_p \leq \liminf_{i\to\infty} \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} \|c^i_{nk}\phi_{nk}\|_p$$
$$= \liminf_{i\to\infty} \|\mu(r_i, \cdot)\|^H_{B^p} \leq AM$$

where A is a constant. Therefore $f \in B^p$.

Next let $f \in B^p$. If we properly normalize the Poisson Kernel then $f_r = f * P_r$ where $\|P_r\|_1 = 1$. Let $\alpha = 1/p$, $p > 1$, since $f_r \in C^\infty$, we obtain from lemma 3.7 that

$$\|f_r\|_{B^p}^T = \sup\left\{\left|\int f_r(\theta)g(\theta)\,d\theta\right|; \left\|\int g\right\|_{\Lambda_{\alpha+1}} \leqslant 1\right\},$$

but

$$\left|\int_T f_r(\theta)g(\theta)\,d\theta\right| = \left|\int_T f(\theta)g_r(\theta)\,d\theta\right| \leqslant \|f\|_{B^p}^T \cdot \left\|\int g_r\right\|_{\Lambda_{\alpha+1}} \leqslant \|f\|_{B^p}^T.$$

Indeed if $r_1 < r < 1$ and if $\rho = r_1/r$, then it is well known that $f_{r_1} = f_r * P_\rho$, so that by Theorem 3.9 we get

$$\|f_{r_1}\|_{B^p}^T \leqslant \|f\|_{B^p}^T \cdot \|P_\rho\|_1 \leqslant \|f_r\|_{B^p}^T.$$

Thus $\|f_r\|_{B^p}^T$ is a non-decreasing function of r. Combining this with what was shown in the first part of the theorem we obtain as $r \nearrow 1$

$$\lim \|f_r\|_{B^p}^T \leqslant \|f\|_{B^p}^T$$

and

$$\liminf \|f_r\|_{B^p}^H \geqslant \|f\|_p^H.$$

Thus there exists a constant $C > 0$ such that as $r \nearrow 1$

$$\lim \|f_r\|_{B^p}^T \geqslant C\|f\|_{B^p}^H.$$

(We shall see later that $C = 1$.) Therefore given any $\epsilon > 0$ there is $r < 1$ such that $C\|f_r\|_{B^p}^T \geqslant C\|f\|_{B^p}^T - \epsilon$. Now as $f_r \in C^\infty$ there is a function g,

$$\left\|\int g\right\|_{\mathrm{Lip}\,\alpha} \leqslant 1$$

such that

$$C\|f\|_{B^p}^T - 2\epsilon \leqslant \left|\int_T f_r(\theta)g(\theta)\,d\theta\right| = \left|\int_T f(\theta)g_r(\theta)\,d\theta\right|.$$

But

$$\left\|\int g_r\right\|_{\mathrm{Lip}\,\alpha} \leqslant 1$$

so that if we set

$$\|f\|_{B^p}^{**} = \sup\left\{\left|\int_T f(\theta)g(\theta)d\theta\right|; \left\|\int g\right\|_{\Lambda_{\alpha+1}} \leqslant 1\right\}$$

Then

$$C\|f\|_{B^p}^T \leqslant \|f\|_{B^p}^{**} \leqslant \|f\|_{B^p}^T.$$

It is now easily seen that if f is a distribution not in B^p then $\|f_r\|_{B^p}^T \nearrow \infty$ and

$$\|f\|_{B^p}^{**} = \sup\left\{\left|\int_T f(\theta)g(\theta)\,d\theta\right|; g \in C^\infty, \left\|\int g\right\|_{\Lambda_{\alpha+1}} \leqslant 1\right\} = \infty.$$

For $f \in B^p$, $f_r(\theta) = \int_T f(\theta - \phi)P_r(\phi)\,d\phi$ where $\int_T P_r(\theta)\,d\theta = 1$ so that $f_r(\theta) - - f(\theta) = \int_T (f(\theta - \phi) - f(\theta))P_r(\phi)\,d\phi$. Therefore by a Minkowski type inequality (lemma 3.8) implies

$$\begin{aligned}\|f_r - f\|_{B^p}^{**} &\leqslant \int_T P_r(\phi)\|f(\theta - \phi) - f(\theta)\|_{B^p}^{**}\,d\phi \\ &\leqslant \int_T P_r(\phi)w(|\phi|)\,d\phi \leqslant 2\int_0^\pi P_r(\phi)w(\phi)\,d\phi,\end{aligned}$$

where w is the modulus of continuity defined right after Theorem 3.2.

But $P_r(\phi)$ is monotone decreasing for $0 < \phi < \pi$ and w is monotone non-decreasing so that for any ξ, $0 < \xi < \pi$, the last integral is bounded by

$$2w(\xi)\int_0^\xi P_r(\phi)\,d\phi + 2P_r(\xi)\int_\xi^\pi w(\phi)\,d\phi < 2w(\xi) + 4\pi\|f\|_{B^p}^T P_r(\xi).$$

Choose ξ so as to make the first term arbitrarily small then for this ξ, choose r so as to make the second arbitrarily small. This shows that $f_r \to f$ in B^p which in turn implies $\|f_r\|_{B^p}^T \to \|f\|_{B^p}^T$ as $r \nearrow 1$.

Indeed in addition to proving our theorem we have also the following.

Theorem 3.10. *If $1 < p < \infty$ and $\alpha = 1/p$ then for any function $f \in B^p$*

$$\|f\|_{B^p}^T = \sup\left\{\left|\int_T f(\theta)g(\theta)\,d\theta\right|; \left\|\int g\right\|_{\Lambda_{\alpha+1}} \leqslant 1\right|,$$

and for any distribution f not in B^p

$$\sup\left\{\left|\int_T f(\theta)g(\theta)\,d\theta\right|; g \in C^\infty; \left\|\int g\right\|_{\Lambda_{\alpha+1}} \leqslant 1\right\} = \infty.$$

The interested reader is referred to [9], where Theorem 3.10 is related with the main result there.

For $1/2 < p < 1$ we have a result similar to Theorem 3.10, indeed we obtain the following.

Theorem 3.11. *If f_r is the Poisson integral of a distribution $f \in B^p$ of boundary of the unit disc, $1/2 < p < 1$, then*

$$\|f_r\|_{B^p}^* \nearrow \|f\|_{B^p}^* \quad \text{and} \quad \|f_r - f\|_{B^p}^* \to 0 \quad \text{as} \quad r \nearrow 1.$$

Conversely if $\mu(r, \theta)$ is a harmonic function in the unit disc such that for each r, $0 < r < 1$,

$$\|\mu(r, \cdot)\|_{B^p}^* \leqslant M < \infty$$

then there exists a distribution $f \in B^p$ whose Poisson integral is μ.

PROOF. Assume μ vanishes at the origin, so that it may be written in the form

$$\mu(r, \theta) = \sum_{n=1}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta)$$

and let

$$U(r, \theta) = \sum_{n=1}^{\infty} \frac{r^n(a_n \sin n\theta - b_n \cos n\theta)}{n}$$

so that

$$\frac{\partial U(r, \theta)}{\partial \theta} = \mu(r, \theta).$$

Let $1/q = 1/p - 1$, so $1 < q < \infty$ then for each r Theorem 3.1 tells us that $\|U(r, \cdot)\|_{B^q}^T = C\|\mu(r, \cdot)\|_{B^p}^*$ for some constant C. But there is a constant A such that

$$\|U(r, \cdot)\|_{B^q}^H \leqslant A\|U(r, \cdot)\|_{B^q}^T = AC\|\mu(r, \cdot)\|_{B^p}^* \leqslant ACM.$$

Thus there is a $F \in B^q$ such that U is its Poisson integral by Theorem 3.10. Let f be a distribution which is a derivative of F, that is,

$$F(x) = \sum_{n=1}^{\infty} t_n(x), \qquad f(x) = \sum_{n=1}^{\infty} b_n(x) \quad \text{where} \quad b_n(x) = \frac{dt_n(x)}{dx},$$

(recall t_n is a triangular function).

The Poisson Kernel and its derivatives are all C^∞ and so are test functions, therefore we have

$$\mu(r, \theta) = \frac{\partial U(r, \theta)}{\partial \theta} = F * \frac{\partial P_r}{\partial \theta} = \sum_{n=1}^{\infty} \int_T t_n(\theta - \phi) \frac{\partial P_r(\phi)}{\partial \phi} \, d\phi.$$

And so, integration by parts term by term yields

$$\mu(r, \theta) = \sum_{n=1}^{\infty} \int_T b_n(\theta - \phi) P_r(\phi) \, d\phi = \sum_{n=1}^{\infty} (b_n(\theta - \cdot), P_r) = (f(\theta - \cdot), P_r)$$

and this last is precisely the definition of the Poisson integral of a distribution.

The rest of the proof is the same as the proof of the previous theorem mutatis mutandis and is left to the reader.

The proof of the next theorem which is the equivalent to Theorem 3.10 and Theorem 3.11 for $p = 1$, seems to be more delicate and requires the use of fractional integration.

Following Herman Weyl we say that if $f(\theta)$ is a distribution whose Fourier series is $\Sigma' c_n e^{in\theta}$ (Σ' denotes summation for $n \in Z$, $n \neq 0$) then $f_\alpha(\theta)$ is the distribution defined by

$$f_\alpha(\theta) = \Sigma' \frac{c_n e^{in\theta}}{(in)^\alpha}.$$

Let I_α be the linear operator which maps f into f_α. It is a well known theorem of Hardy and Littlewood that $f \in \operatorname{Lip} \beta$ if and only if $f_\alpha \in \operatorname{Lip}(\alpha + \beta)$ provided $\alpha + \beta < 1$. This result was extended by Zygmund [17] who showed that $f \in \Lambda_\beta$ if and only if $f_\alpha \in \Lambda_{\alpha+\beta}$ provided $\alpha + \beta < 2$.

f_α is called fractional integration of f. Observe that the kernel for fractional integration is in L^2 if and only if $\alpha > 1/2$.

Theorem 3.12. *If f_r is the Poisson integral of a function $f \in B (B = B^1)$ on the boundary of the unit disc then*

$$\|f_r\|_B^* \nearrow \|f\|_B^* \quad \text{and} \quad \|f_r - f\|_B^* \to 0 \quad \text{as} \quad r \nearrow 1.$$

Conversely if $\mu(r, \theta)$ is harmonic and for $0 < r < 1$,

$$\|\mu(r, \cdot)\|_B \leqslant M < \infty.$$

Then μ is the Poisson integral of f in B.

PROOF. We start with the converse. The main problem is to lay our hands on a special atomic decomposition of the boundary function. Assume μ vanishes at the origin. Then

$$\mu(r, \theta) = \sum_{n=-\infty}^{\infty}{}' c_n(r) e^{in\theta}$$

where $c_{-n} = \bar{c}_n$ (complex conjugate). Let

$$\mu_\alpha(r, \theta) = I_\alpha \mu(r, \theta) = \sum_{n=-\infty}^{\infty}{}' \frac{c_n(r) e^{in\theta}}{(in)^\alpha}.$$

Let $g \in \operatorname{Lip} \beta$ where $\alpha + \beta = 1$, then

$$\int_T \mu_\alpha(r, \theta) g(\theta)\, d\theta = \int_T \mu(r, \theta) h_\alpha(-\theta)\, d\theta \quad \text{where} \quad h_\alpha = I_\alpha h \quad \text{and} \quad h(\theta) = g(-\theta);$$

by Zygmund's Theorem $h_\alpha \in \Lambda_1$, the Zygmund class Λ_1, so that the last integral above is dominated by

$$\leqslant \|\mu(r, \cdot)\|_B \left\| \int h_\alpha \right\|_{\Lambda_1} \leqslant CM \left\| \int h \right\|_{\mathrm{Lip}(\beta+1)} = CM \left\| \int g \right\|_{\mathrm{Lip}(\beta+1)}.$$

This combined with Theorem 3.10 shows that if $q = 1/\beta$ then

$$\|\mu_\alpha(r, \cdot)\|_{B^q} \leqslant CM$$

so that $\mu_\alpha(r, \cdot)$ is the Poisson integral of $G \in B^q$. Let g be the «derivative» of G. (Recall we express G as a sum of triangular functions and define g as the term by term derivative.) Theorem 3.1 shows that $g \in B^p$ where $1/p = 1/q + 1$ and in the proof of Theorem 3.12 we have that the Poisson integral of g is precisely $\partial\mu_\alpha(r, \theta)/\partial\theta$. Observe that g has been defined as a sum of special atoms, also observe that if we differentiate a special atom we obtain a measure supported on three points which is fractionally integrated of order β, that is, convolved with the kernel $\sum_{n=-\infty}^{\prime\infty} e^{in\theta}/(in^\beta)$ is in L^2 provided $\beta > 1/2$. This in turn implies that $I_\beta b$ where $b(x)$ is a special atom is in $H = \int L^2 =$ the periodic indefinite integral of L^2-functions.

Now fix $\alpha < 1/2$, $\beta = 1 - \alpha$, if $f(\theta) = \sum_{n=0}^\infty b_n(\theta)$. For an individual $b_n(x)$ we have that $I_\beta b_n \in H$ so that for any g with $\left\| \int g \right\|_{\Lambda_1} = 1$

$$\int_T I_\beta b_n(\theta) g(\theta)\, d\theta = \int_T b_n(\theta) h_\beta(-\theta)\, d\theta$$

where $h_\beta = I_\beta h$ and $h(\theta) = g(-\theta)$ but $h_\beta \in \mathrm{Lip}\,\beta$, so that $I_\beta b_n \in B$ and given any $\epsilon > 0$ we may write $I_\beta b_n(\theta) = \sum_{j=1}^\infty c_{nj}(\theta)$ where $c_{nj}(\theta)$ are special atoms and

$$\sum_{j=1}^\infty \|c_{nj}\|_1 \leqslant C\|b_n\|_p + \frac{\epsilon}{2^n}.$$

Let

$$f(\theta) = \sum_{n=1}^\infty \sum_{j=1}^\infty c_{nj}(\theta),$$

clearly f is a function in B. Let us compute its Poisson integral

$$\begin{aligned} f_r(\theta) &= \frac{1}{2\pi} \int_T f(\theta - \phi) P_r(\phi)\, d\phi = \frac{1}{2\pi} \int_T \sum_{n=1}^\infty \sum_{j=1}^\infty c_{nj}(\theta - \phi) P_r(\phi)\, d\phi \\ &= \frac{1}{2\pi} \sum_{n=1}^\infty \int_T \sum_{j=1}^\infty c_{nj}(\theta - \phi) P_r^*(\phi)\, d\phi = \frac{1}{2\pi} \sum_{n=1}^\infty \int_T I_\beta b_n(\theta - \phi) P_r^*(\phi)\, d\phi \\ &= \frac{1}{2\pi} \sum_{n=1}^\infty \int_T b_n(\theta - \phi) I_\beta P_r^*(\phi)\, d\phi = \frac{1}{2\pi} \int_T f(\theta - \phi) I_\beta P_r^*(\phi)\, d\phi, \end{aligned}$$

where P_r^* is the Poisson integral minus its constant term. Notice that $I_\beta P_r^* \in C^\infty$ and so is a test function, therefore

$$(f(\phi - \bullet), I_\beta P_r^*) = \mu(r, \theta).$$

This last inequality is proved by expanding for each r both sides in a Fourier series. Since both sides are C^∞ functions pointwise equality follows.

The other part of the theorem follows by repeating the argument with the obvious modifications of Theorem 3.11 and Theorem 3.12 and is left to the reader.

Theorem 3.13. *If* $f \in B^p$, $1/2 < p < \infty$ *and* σ_n is the $(C, 1)$ *means of the partial sums* s_n *of the Fourier expansion of* f *then* σ_n *tends to* f *in* B^p.

PROOF. To fix ideas, let $1/2 < p < 1$, $\sigma_n = f * K_n$ where K_n is the normalized Fejer kernel, then

$$\sigma_n(\theta) - f(\theta) = \int_T [f(\theta - \phi) - f(\theta)] K_n(\phi)\, d\phi$$

which by the Minkowski type inequality (see Lemma 3.8) gives

$$\|\sigma_n - f\|_{B^p}^* \leqslant 2 \int_0^\pi K_n(\phi) w(|\phi|)\, d\phi$$

which must tend to zero as $n \to \infty$, since the $w(|\phi|)$ the modulus of continuity (see comments right after Theorem 3.2) of an f in B^p tends to zero as ϕ tends to zero.

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