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Weak Type Endpoint Bounds for Bochner-Riesz Multipliers

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The Bochner-Riesz multipliers are defined for testing functions f on $\mathbb{R}^n$ by

$$(T_\lambda f)^\wedge(\xi) = (1 - |\xi|^2)_+^\lambda \hat{f}(\xi).$$

Questions concerning the convergence or multiple Fourier series have led to the study of their L^p boundedness. It is conjectured that for $n > 1$, for all exponents $p \in (1, 2(n-1)/n)$, T_λ is bounded on L^p for all

$$\lambda > \lambda(p) = n(p^{-1} - 2^{-1}) - 2^{-1} > 0.$$

What is known is that the conjecture holds for the full range of exponents in dimension two [1], and for the smaller range $1 < p \leqslant 2(n+1)/(n+3)$ for all $n \geqslant 3$. Moreover it is very easy to see that T_λ is unbounded for all $\lambda \leqslant \lambda(p)$; it suffices to compute the associated convolution kernel and to examine its action on the characteristic function of the unit ball. Nevertheless there is a positive result at the critical value $\lambda(p)$, at least for a certain range of exponents:

Theorem. *For all $n \geqslant 2$ and $1 < p < 2(n+1)/(n+3)$, $T_{\lambda(p)}$ is of weak type (p, p).*

Temporarily define

$$(T_\lambda^r f)^\wedge(\xi) = (1 - |r^{-1}\xi|^2)_+^\lambda \hat{f}(\xi).$$

Corollary. *For all $n \geqslant 2$, $1 < p < 2(n+1)/(n+3)$ and $f \in L^p(R^n)$, $T_\lambda^r f \to f$ in measure as $r \to \infty$.*

The result for $p = 1$ was recently proved in [4]. Our proof involves an application of the method of [4], a slight refinement of estimates already known on L^{p_0}, where $p_0 \equiv 2(n+1)/(n+3)$, and an interpolation between L^1 and L^{p_0}.

To begin fix $p \in (1, p_0)$. Write $p^{-1} = \theta \cdot 1 + (1-\theta)p_0^{-1}$, where $0 < \theta < 1$. Fix $\lambda = \lambda(p) = n(p^{-1} - 2^{-1}) - 2^{-1}$, and set $m(\xi) = (1 - |\xi|^2)_+^\lambda$. Let $f \in L^p$ and $\alpha > 0$ be arbitrary. In order to estimate the measure of the set where $|T_\lambda f| > \alpha$, apply the Calderón-Zygmund decomposition to f^p at height α^p to obtain $f = g + b$ where $\|g\|_p \leqslant C\|f\|_p$, $\|g\|_\infty \leqslant C\alpha$, and $b = \Sigma_Q b_Q$ where each b_Q is supported on a dyadic cube Q,

$$\int |b_Q|^p \leqslant \alpha^p |Q|,$$

the cubes Q have pairwise disjoint interiors, and

$$\Sigma_Q |Q| \leqslant C\alpha^{-p} \|f\|_p^p.$$

Since T_λ is bounded on L^2,

$$|\{x: |T_\lambda g(x)| > \alpha/2\}| \leqslant C\alpha^{-2}\|g\|_2^2 \leqslant C\alpha^{-p}\|f\|_p^p.$$

Let E be the union of the doubles of the cubes Q. Then

$$|E| \leqslant C\alpha^{-p}\|f\|_p^p,$$

so it suffices to show that

$$|\{x \notin E: |T_\lambda b(x)| > \alpha/2\}| \leqslant C\alpha^{-p}\|b\|_p^p.$$

This will follow by Chebychev's inequality from

$$\|T_\lambda b\|_{L^2(\mathbb{R}^n \setminus E)}^2 \leqslant C\alpha^{2-p}\|b\|_p^p. \tag{1}$$

Fix $\varphi_0 \in C_0^\infty(\mathbb{R}^n)$, radial and supported in $\{|x| \leqslant 1\}$ and satisfying $\varphi_0(x) \equiv 1$ for $|x| \leqslant 1/4$ and

$$\int (\partial^k \hat{\varphi}_0 / \partial \xi_1^k)(\xi) \cdot (\xi_1)_+^\lambda \, d\xi = 0 \tag{2}$$

for $k = 0, 1$. Let

$$\varphi_j(x) = \varphi_0(2^{-j}x) \quad \text{and} \quad \psi_j = \varphi_j - \varphi_{j-1}.$$

For $j > 0$ let

$$K_j(x) = \psi_j(x)\check{m}(x),$$

and let $K_0 = \varphi_0 \cdot \check{m}$, so that $\check{m} = \Sigma K_j$.

For $0 \leqslant i \in \mathbb{Z}$ let $B_i = \Sigma b_Q$, the sum being taken over all Q with sidelength 2^i when $i > 0$ and sidelength less than or equal to one when $i = 0$. The contribution of B_0 turns out to be relatively easy to treat, so we shall ignore it until the end of the argument and concentrate instead on $\Sigma_{i>0} B_i$. Note that if Q has sidelength 2^i, then for all $j \leqslant i$, $b_Q * K_j$ is supported on the double of Q, hence on E. Consequently for all $x \notin E$,

$$T_\lambda(\Sigma_{i>0} B_i)(x) = \Sigma_{i>0} B_i * (\Sigma_{j>i} K_j)(x) = \Sigma_{s>0} \Sigma_{j>s} B_{j-s} * K_j(x).$$

Hence (1) is a consequence of

$$\|\Sigma_{j>s} B_{j-s} * K_j\|^2_{L^2(\mathbb{R}^n)} \leqslant C 2^{-\epsilon s} \alpha^{2-p} \|b\|^p_p \tag{3}$$

for all $s \in \mathbb{Z}^+$, for some $\epsilon > 0$.

Fix linear functions $l_1, l_2 : \mathbb{C} \to \mathbb{C}$ such that $Re(l_1(z)) \equiv p$ when $Re(z) = 1$, $\equiv p \cdot p_0^{-1}$ when $Re(z) = 0$, and $Re(l_2(z)) \equiv n(p^{-1} - 1)$ when $Re(z) = 1$ and $\equiv n(p^{-1} - p_1^{-1})$ when $Re(z) = 0$. Then $l_1(\theta) = 1$ and $l_2(\theta) = 0$. Define $B_{i,z}(x) = [B_i(x)]^{l_1(z)}$, interpreted as is customary in the standard proof of the Riesz-Thörin interpolation theorem. Define $K_{j,z}(x) = 2^{j l_2(z)} K_j(x)$. Then (3) follows by interpolation between the two endpoint estimates

$$\|\Sigma_{j>s} B_{j-s,z} * K_{j,z}\|^2_2 \leqslant C 2^{-\epsilon s} \alpha^p \|b\|^p_p \tag{4}$$

when $Re(z) = 1$ and

$$\|\Sigma_{j>s} B_{j-s,z} * K_{j,z}\|^2_2 \leqslant C \alpha^{p(2p_0^{-1} - 1)} \|b\|^p_p \tag{5}$$

when $Re(z) = 0$.

To justify (4) consider any collection $\{A_j : j > 0\}$ of functions satisfying

$$\int_Q |A_j| \leqslant C \alpha^p |Q|$$

for all cubes Q in R^n of sidelength 2^j. Consider further any collection of kernels

$$H_j(x) = \Phi(x) h_j(x)$$

where

$$\Phi(x) = \cos(2\pi|x| - \pi(n-1)/4)$$

and each h_j is supported in $\{2^{j-3} \leqslant |x| \leqslant c_2 2^j\}$ and satisfies

$$\|h_j\|_\infty + 2^j \|\nabla h_j\|_\infty \leqslant 2^{-nj}.$$

It is proved in [4] that

$$\|\Sigma_{j>s} A_{j-s} * H_j\|^2_2 \leqslant C 2^{-\epsilon s} \alpha^p \|\Sigma |A_j|\|_1 \tag{4'}$$

for a certain $\epsilon > 0$. This is done by first, for technical reasons, introducing a finite partition of unity $\{\eta_\beta\}$ on $\mathbb{R}^n \setminus \{0\}$ with each η_β homogeneous of degree zero and supported in some cone $\{x: \langle x, v_\beta\rangle > \delta|x|\}$ for some $\delta > 0$ and $v_\beta \in S^{n-1}$. (4′) follows from the variant of itself defined by replacing each H_j by $J_j = H_j \cdot \eta_\beta$, for then one may sum over β. This modified (4′) is an easy consequence of the estimates

$$|J_j * \tilde{J}_j(x)| \leqslant C2^{-nj}(1 + |x|)^{-\mu}$$

and

$$\|J_i * \tilde{J}_j\|_\infty \leqslant 2^{-nj}2^{-\mu i} \quad \text{for all} \quad 0 < i < j - 3,$$

where $\tilde{J}_j(x) \equiv J_j(-x)$ and $\mu = (n-1)/2$; these are not difficult to verify by direct computation using the stated properties of $\{H_j\}$.

When $Re(z) = 1$, $A_j = B_{j,z}$ and $H_j = K_{j,z}$ have all these properties (H_j does, by the known asymptotics for Bessel functions). Therefore we consider (4) to be proved and concentrate on (5). For a single term $B_{j-s,z} * K_{j,z}$, it turns out that the desired bound follows at once from the estimates in [7]; the technical manipulations which follow are designed to enable us to pass from bounds for these individual terms to a bound for the entire sum.

Let $m_j = \hat{K}_j = m * \hat{\psi}_j$ (for $j > 0$).

Lemma 1.

(6) $\|\partial^\alpha m_j/\partial\xi^\alpha\|_\infty \leqslant C_\alpha 2^{j|\alpha|}2^{-j\lambda}$ *for all multi-indices* α.

(7) $|m_j(\xi)| + 2^{-j}|\nabla m_j(\xi)| \leqslant C_M 2^{-jM}$ *for all* M *and all* $\xi \notin [1/2, 3/2]$.

(8) $|m_j(\xi)| + 2^{-j}|\nabla m_j(\xi)| \leqslant C_M 2^{-j\lambda}(1 + 2^j|1 - |\xi||)^{-M}$ *for all* $|\xi| \in [1/2, 3/2]$.

(9) *There exists* $\delta > 0$ *such that*

$$|m_j(\xi)| + 2^{-j}|\nabla m_j(\xi)| \leqslant C2^{-j\lambda}\max(2^j|1 - |\xi||, 2^{-j\delta})$$

for all $|\xi| \in [1 - 2^{-j}, 1 + 2^{-j}]$.

The conclusions are all totally routine bounds for $m_j = m * \hat{\psi}_j$ except for (9), which relies on the technical condition (2). To obtain the bound in (9) for $m_j(\xi)$, observe that since $|m_j(\xi)| \leqslant C2^{-j\lambda}$ when $|\xi| = 1 \pm 2^{-j}$ by (8), and since $\|\nabla m_j\|_\infty \leqslant C2^{j(1-\lambda)}$, it suffices by the fundamental theorem of calculus to prove (9) for $|\xi| = 1$. Both m and $\hat{\psi}_j$ are radial, so we may take $\xi = \xi_0 = (1, 0, \ldots, 0)$.

$$(m * \hat{\psi}_j)(\xi_0) = \int \hat{\psi}_j(\xi_0 - \zeta) \cdot [(1 - |\zeta|^2)_+^\lambda - |2(1 - \zeta_1)_+^\lambda|]\, d\zeta,$$

where $\zeta = (\zeta_1, \zeta_2, \ldots)$, since the term subtracted is actually zero by (2) (with $k = 0$). The function $\hat{\psi}_j(\xi_0 - \bullet)$ is essentially supported on a ball of radius 2^{-j}

centered at ξ_0. On this ball

$$|[1-|\zeta|^2]_+^\lambda - [2(1-\zeta_1)_+]^\lambda| \leqslant C2^{-2j\lambda};$$

the best way to see this is to introduce new coordinates centered at ξ_0 and rescaled by a factor of 2^j. In such coordinates the boundary of the unit ball becomes almost flat as $j \to \infty$, producing an extra factor of $2^{-j\lambda}$. Hence (9) holds for m_j; we omit the precise details. ∇m_j may be estimated in the same way, using (2) with $k = 1$.

Lemma 2. *There exist positive radial functions* $\{\eta_j: j > 0\}$ *such that* $\Sigma\eta_j^2 \in L^\infty$ *and the multipliers* $n_j = m_j/\eta_j$ *satisfy* (7) *and* (8).

Indeed, define $\eta_j(\xi) = 1$ if $|\xi| = 1 \pm 2^{-j}$, $=2^{-j\delta/2}$ if $|\xi| = 1$, where δ is the exponent in (9), and interpolate smoothly for intermediate values of $|\xi|$. Proceed similarly for $|\xi| \notin [1-2^{-j}, 1+2^{-j}]$.

We may now deduce (5). Suppose that $Re(z) = 0$.

$$\begin{aligned}\|\Sigma_{j>s}B_{j-s,z}*K_{j,z}\|_2^2 &= \int |\Sigma\hat{B}_{j-s,z}(\xi)\cdot 2^{jl_2(z)}n_j(\xi)\eta_j(\xi)|^2\,d\xi\\ &\leqslant \int (\Sigma\eta_j(\xi)^2)(\Sigma|\hat{B}_{j-s,z}(\xi)\cdot 2^{l_2(z)}n_j(\xi)|^2)\,d\xi\\ &\leqslant C\int \Sigma|\hat{B}_{j-s,z}\cdot 2^{jl_2(z)}n_j(\xi)|^2\,d\xi\\ &= \Sigma\|B_{j-s,z}*2^{jl_2(z)}\check{n}_j\|_2^2 = \Sigma\|B_{j-s,z}*2^{jn(p^{-1}-p_0^{-1})}\check{n}_j\|_2^2.\end{aligned}$$

Therefore it suffices to show that for all $F \in L^{p_0}(\mathbb{R}^n)$ satisfying

$$\int_Q |F|^{p_0} \leqslant \beta|Q| \tag{10}$$

for all cubes Q of sidelength 2^j, we have

$$\|F*2^{jn(p^{-1}-p_0^{-1})}n_j\|_2^2 \leqslant C\beta^{2p_0^{-1}-1}\|F\|_{p_0}^{p_0}, \tag{11}$$

for $B_{j-s,z}$ satisfies (10) uniformly for all $s \in \mathbb{Z}^-$, $z \in i\mathbb{R}$, with $\beta = \alpha^p$. Set $L_j = 2^{jn(p^{-1}-p_0^{-1})}\check{n}_j\varphi_j$, and for all $i > j$ set $L_i = 2^{jn(p^{-1}-p_0^{-1})}\check{n}_j\psi_i$. We will prove that there exists $\epsilon > 0$ such that for *all* $F \in L^{p_0}$ and all $i \geqslant j$,

$$\|F*L_i\|_2^2 \leqslant C2^{-\epsilon(i-j)}2^{-ni(2p_0^{-1}-1)}\|F\|_{p_0}^2. \tag{12}$$

Since L_i is supported on $\{|x| \leqslant 2^i\}$, it follows at once that

$$\|F*L_i\|_2^2 \leqslant C2^{-\epsilon(i-j)}\beta^{2p_0^{-1}-1}\|F\|_{p_0}^{p_0}$$

for all $F \in L^{p_0}$ satisfying (10). Summing over i gives (11).

Finally (12) is a straightforward consequence of the L^2 restriction theorem of Tomas and Stein, as in [7]. For if $I = [1/2, 3/2]$ and $B = \{|\xi| \notin I\}$, then

$$\|F * L_i\|_2^2 = \int_B |\hat{F}(\xi)|^2 |\hat{L}_i(\xi)|^2 \, d\xi + \int_I \left(\int_{S^{n-1}} |\hat{F}(r\theta)|^2 \, d\theta \right) \cdot |\hat{L}_i(r)|^2 r^{n-1} \, dr$$

where we have written $\hat{L}_i(r)$ for $\hat{L}_i(\xi)$ when $|\xi| = r$, recalling that $\hat{L}_i$ is radial. For $\xi \in B$,

$$\begin{aligned} |\hat{L}_i(\xi)| &= 2^{jn(p^{-1} - p_0^{-1})} \cdot |n_j * \hat{\psi}_i(\xi)| \quad (\text{or } \hat{\varphi}_j \text{ when } i = j) \\ &\leqslant C_M 2^{-(i-j)} 2^{-Mj} (1 + |\xi|)^{-M} \end{aligned}$$

for all $M < \infty$, by the bounds (7) and (8) for n_j and its gradient, and routine estimation. Hence the Hausdorff-Young inequality gives

$$\begin{aligned} \int_B |\hat{F}(\xi)|^2 |\hat{L}_i(\xi)|^2 \, d\xi &\leqslant C_M 2^{-2(i-j)} 2^{-Mj} \|F\|_{p_0}^2 \\ &= 2^{-\epsilon(i-j)} 2^{-ni(2p_0^{-1}-1)} 2^{j(-M + (2p_0^{-1}-1))} \|F\|_{p_0}^2 \end{aligned}$$

where $\epsilon = 2 - n(2p_0^{-1} - 1) = 2/(n+1) > 0$. Thus the desired bound follows as soon as $M \geqslant 2p_0^{-1} - 1$. On the other hand for $r \in I$ we have

$$\int_{S^{n-1}} |\hat{F}(r\theta)|^2 \, d\theta \leqslant C \|F\|_{p_0}^2$$

by the restriction theorem. Hence

$$\int_{\mathbb{R}^n \setminus B} |\hat{F}(\xi)|^2 |\hat{L}_i(\xi)|^2 \, d\xi \leqslant C \|F\|_{p_0}^2 \int_I |\hat{L}_i(r)|^2 \, dr.$$

It follows from (7), (8) and routine computation that for $r \in I$,

$$|\hat{L}_i(r)| \leqslant C_M 2^{jn(p^{-1} - p_0^{-1})} 2^{-j\lambda} (1 + 2^j |1 - |\xi||)^{-M} \cdot 2^{j-i}$$

for all $M < \infty$. Hence

$$\begin{aligned} \int_I |\hat{L}_i(r)|^2 \, dr &\leqslant 2^{2jn(p^{-1} - p_0^{-1})} 2^{-2j\lambda} \cdot 2^{-j} \cdot 2^{-2(i-j)} \\ &= 2^{-in(2p_0^{-1} - 1)} 2^{-\epsilon(i-j)} \end{aligned}$$

where again $\epsilon = 2/(n+1)$. This concludes the proof of (5).

Only the contribution of B_0 remains to be treated. Again form the analytic functions $B_{0,z}$ and $K_{j,z}$ as above. When $Re(z) = 0$ it follows from the L^2 restriction theorem that

$$\|B_{0,z} * K_{j,z}\|_2^2 \leqslant C\alpha^{p(2p_0^{-1} - 1)} \|B_0\|_p^p$$

as above; now it is not necessary to introduce the η_j and n_j, so the proof is straightforward. On the other hand it is shown in [4] that when $Re(z) = 1$,

$$\|B_{0,z} * K_{j,z}\|_2^2 \leqslant C2^{-j(n-1)/2}\alpha^p \|B_{0,z}\|_1.$$

Since the right-hand side is equal to $C2^{-j(n-1)/2}\alpha^p\|B_0\|_p^p$, interpolation gives $\|B_{0,z} * K_{j,z}\|_2^2 \leqslant C2^{-j\theta(n-1)/2}\alpha^{2-p}\|B_0\|_p^p$. So

$$\begin{aligned}\|B_0 * \Sigma K_j\|_2 &\leqslant \Sigma\|B_0 * K_j\|_2 \\ &\leqslant C\alpha^{(2-p)/2}\|B_0\|_p^{p/2}\Sigma 2^{-j\theta(n-1)/4} \\ &\leqslant C[\alpha^{2-p}\|B_0\|_p^p]^{1/2}.\end{aligned}$$

Remark. In dimension $n = 2$ T_λ is known [1] to be bounded on L^p for all $\lambda > \lambda(p)$, for all $p \leqslant 4/3$, but our proof applies only in the smaller range $p < 6/5$. It remains an open question whether weak type endpoint results hold in the full range of exponents, even in dimension two. In [2] this has been shown to be the case for radial functions.

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