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Total Curvature of Non-Differentiable Curves

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Introduction

This paper deals with the total curvature of curves γ in euclidean space. It is defined as the supremum of the expressions $\sum \alpha_i$ where α_i = angle formed by succesive chords C_i, C_{i+1} determined by a partition of the parameter interval, and we denote it by $T(\gamma)$. Notice that when γ has a curvature k then $T(\gamma) = \int k\, ds$, where ds is the element of arc-length (see comments at the end of section 2).

Our aim is to study curves for which $T(\gamma) < +\infty$ without a priori conditions regarding smoothness of γ: in this sense, the paper is more «real variables» than «differential geometry». This approach has been used by Borsuk [2], Fáry [3] and Milnor [7] among others in their study of knots, and some results below are extensions or improvements of their findings. In particular, Proposition 4.5 below (whose proof was communicated to us by A. P. Calderón) generalizes a statement by Fáry (third paragraph on *p.* 130 of [3]; see also [1]).

Furthermore, the hypothesis $T < +\infty$ in conjunction with an interior cone condition was used by McGowan and Porta (see [6]) as a substitute for convexity to extend Paul Levy's integral representation to distances in the plane which are not norms. This notion also appears in Finsler spaces (see Rund [8]), at least in the general form given in section 6 below, and in isoperimetric problems (see Bandle [0]); however differentiability or rectifiability is usually required. Finally we mention the following result of Gleason (see [4]): if γ_n

is the longest polygon with n vertices all on a curve γ, then the correspondir
lengths L_n and L satisfy

$$\lim n^2(L - L_n) = (1/24)\left(\int k^{2/3}\,ds\right)^3.$$

Since

$$\lim T(\gamma_n) = \int k\,ds$$

we may ask for other relations involving T and length and also for tl
geometric significance of other moments of the curvature (for the second, s
Weinèr [12]).

The main results obtained are the following: under the hypothesis $T(\gamma) < +$
we pove that γ has one-sided tangents everywhere (Theorem 2.3) which coi
cide at all but countable many points (Corollary 3.8). Furthermore, γ can l
decomposed into finitely many graphs of Lipschitz functions (Propositic
3.9) and, if τ denotes the Gauss map of γ defined by $\tau(t)$ = right unit tange
vector at $\gamma(t)$, then $T(\gamma)$ = length of τ considered as a curve in the unit sphe
S under the geodesic distance (the distance is relevant because τ is disco
tinuous in general).

The last two sections are devoted to the non-Hilbert case and to the rel
tions among total curvature, rectifiability, bounded variation and the like

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ments.

1. Preliminary Remarks

1.1. In the sequel we often consider angles formed by elements of a Hilbe
space H. If U, V are non-zero elements of H we define ang(U, V) by

$$\cos \operatorname{ang}(U, V) = \langle U, V\rangle / \|U\|\,\|V\|,$$

where we require that $0 \leqslant \operatorname{ang}(U, V) \leqslant \pi$. It is clear that ang is a continuo
function from $(H - \{0\}) \times (H - \{0\})$ into $[0, \pi]$, and that it verifies

$$\operatorname{ang}(U, V) + \operatorname{ang}(V, W) \geqslant \operatorname{ang}(U, W) \tag{1.1a}$$

whenever U, V, W are non-zero. This *angle triangle inequality* has the usu
consequences (like its iterated form $\Sigma \operatorname{ang}(U_i, U_{i+1}) \geqslant \operatorname{ang}(U_1, U_n)$, f
example).

When restricted to the unit sphere S of H, ang is a distance. Furthermor

$$\operatorname{ang}(U, V) \geqslant \|U - V\| \geqslant (1 - d^2/6)\operatorname{ang}(U, V), \tag{1.1b}$$

when $\| U \| = \| V \| = 1$ and $d = \| U - V \|$. Occasionally UV is used as an abbreviation of $\operatorname{ang}(U, V)$.

1.2. Let $G(u, v)$ be an interval function, i.e., a real valued function defined for $a \leqslant u \leqslant v \leqslant b$. We denote by $G'(c) = \lim G(u, v)$ taken when $u < c < v$ and $u, v \to c$. Admitedly, this limit may not exist; if G is monotonic, G' exists almost everywhere (see [5], page 94).

1.3. Suppose now that X is a metric space, with distance d and let σ be a (not necessarily continuous) function $\sigma\colon [a, b] \to X$. The total variation $\operatorname{Sup} \Sigma d(\sigma(t_i), \sigma(t_{i+1}))$ is called the *length* of σ. When the length is finite, we say that σ is *rectifiable*. This notion appears below in two different settings: when X is a Hilbert space H with the norm distance (in which case the length of σ is denoted by $l(\sigma)$) and when X is the unit sphere S of H and $d = \operatorname{ang}$ (and then we use $l_S(\sigma)$ for the length of σ).

We remark without proof that $l(\sigma) \leqslant l_S(\sigma)$ for all $\sigma\colon [a, b] \to S$ with equality when σ is continuous (the proof uses 1.1*b*).

By a «curve» in H we mean a continuous simple curve defined by a parametrization $\gamma\colon [a, b] \to H$; therefore γ is a homeomorphism from $[a, b]$ onto its image, and $l(\gamma)$ is the length of the curve.

The following notation will be used throughout: if U, V are distinct vectors, $C(U, V) = (V - U)/\| V - U \|$. If $\gamma(t)$, $a \leqslant t \leqslant b$ is a curve, and $a \leqslant u \leqslant v \leqslant b$, then $C(u, v) = C(\gamma(u), \gamma(v))$, so that $C(u, v)$ is the normalized chord from $\gamma(u)$ to $\gamma(v)$. Also, the curve $\bar{\gamma}$ is a *shortcut* of the curve γ if $\bar{\gamma}(t) = \gamma(t)$ for $a \leqslant t \leqslant u$ and $v \leqslant t \leqslant a$ while $\bar{\gamma}(t)$, $u \leqslant t \leqslant v$, coincides with the straight line segment joining $\gamma(u)$ and $\gamma(v)$.

2. Total Curvature of Curves in Hilbert Space

Suppose that $\gamma(t)$, $a \leqslant t \leqslant b$ is a curve in H, Hilbert space.

Let $\Pi = \{c_0, c_1, c_2, \dots, c_{n+1}\}$ satisfy $a \leqslant c_0 < c_1 < c_2 < \cdots < c_{n+1} \leqslant b$ (a «partition» in $[a, b]$). We set $T(\Pi) = T\{c_0, c_1, \dots, c_{n+1}\} = \Sigma \alpha_i$ where $\alpha_i = \operatorname{ang}(C(c_{i-1}, c_i), C(c_i, c_{i+1}))$. If the particular curve under consideration has to be identified, we write $T(\gamma; \Pi)$, etc.

Suppose now that I is a interval (of any kind) contained in $[a, b]$, with end-points $u < v$. We set $T(I) = T(u, v) = \operatorname{Sup} T(\Pi)$, where Π ranges over all partitions satisfying $u \leqslant c_0 < c_1 < \cdots < c_{n+1} \leqslant v$. We repeat that this definition does not distinguish between $I = [u, v]$, $I = [u, v)$, etc. Just as above we write $T(\gamma; u, v)$ when necessary.

2.1. Definition. *The curve γ has finite total curvature when $T(a, b) < +\infty$. In this case $T(a, b)$ is called the total curvature of γ.*

This terminology is justified at the end of this section.

We list below a few properties of the interval function T and indicat of the proofs.

2.2*a*. *Positivity*: $T \geqslant 0$.

2.2*b*. *Monotonicity with respect to shortcuts*: if $\bar{\gamma}$ is a shortcut of γ $T(\bar{\gamma}) \leqslant T(\gamma)$. (cf. [7], Cor. 1.2).

PROOF. Consider the family of partitions Π_0 having u and v as adjace tition points, where u, v have the same meaning as in (1.3). $\operatorname{Sup} T(\gamma; \Pi_0) \leqslant \operatorname{Sup} T(\gamma; \Pi) = T(\gamma)$. But for $\bar{\gamma}$ the partitions Π_0 are j good as all partitions since adding new points between u and v do change the value of $T(\bar{\gamma}; \Pi_0)$. Hence $\operatorname{Sup} T(\bar{\gamma}; \Pi_0) = T(\bar{\gamma})$ and 2.2*b* fo

Observe that this implies the following «bang-bang» principle. If th γ is a polygonal line with vertices $P_0, P_1, \ldots, P_{n+1}$, then the total cur of γ is the smallest among the curves passing throught $P_0, P_1, \ldots, P$ that order. In other words: «least twisted = shortest».

2.2*c*. *Superadditivity with respect to intervals*: if (u_j, v_j) are disjoint su vals of (u, v), then $\Sigma T(u_j, v_j) \leqslant T(u, v)$. In particular, T is *monotonic*

2.2*d*. *Invariance under parameter changes*: if γ and γ_1 are parametri of the same curve, then $T(\gamma) = T(\gamma_1)$.

The following theorem is the key result of this section.

2.3. Theorem. *Let $\gamma(t)$, $a \leqslant t \leqslant b$, be a curve with finite total cur Then for each $a \leqslant c < b$ the limit.*

2.3*a*. $$T^+(c) = \lim_{c \leq u < v,\, v \to c} C(u, v)$$

exists in the following sense: for each $\epsilon > 0$ there exists $\delta > 0$ suc $\|C(u, v) - T^+(c)\| < \epsilon$ whenever $c \leqslant u < v \leqslant c + \delta$. A similar statemen for

2.3*b*. $$T^-(c) = \lim_{u < v \leq c,\, u \to c} C(u, v)$$

when $a < c \leqslant b$. In particular, $T^+(c)$ and $T^-(c)$ are the right and le tangent vector to γ at $\gamma(c)$, respectively.

PROOF. First step: consider a sequence $t_1 > u_1 > t_2 > u_2 > \cdots > c$ with and form the series

$$s = \Sigma \operatorname{ang}(C(u_k, t_k), C(u_{k+1}, t_{k+1})).$$

If we abbreviate $\Pi_n = \{u_n, t_n, u_{n-1}, t_{n-1}, \ldots, u_1, t_1)$ then the partial sums s_n of s satisfy $s_n \leqslant T(\Pi_n)$, since the k^{th} term of s is majorated by

$$w_k = T\{u_{k+1}, t_{k+1}, u_k, t_k\} \quad \text{and} \quad T(\Pi_n) = \sum_{1 \leq k \leq n-1} w_k.$$

Thus $s_n \leqslant T(a, b)$ and s is a convergent series.

On the other hand, for $k > j$:

$$\operatorname{ang}(C(u_k, t_k), C(u_j, t_j)) \leqslant s_k - s_{j-1}$$

and therefore we have

$$\lim_{k,j \to \infty} \operatorname{ang}(C(u_k, t_k), C(u_j, t_j)) = 0.$$

This means that $\{C(u_n, t_n)\}$ is a Cauchy sequence for the ang distance whence, by $(1.1b)$, it is also a Cauchy sequence in the norm. Therefore there exists the limit $V = \lim C(u_n, t_n)$.

This limit is independent of the particular sequence (u_n, t_n): if (u'_n, t'_n) is a second such sequence with $V' = \lim C(u'_n, t'_n)$ we can thin out both of them to obtain subsequences (denoted by the same symbols) satisfying

$$t_1 > u_1 > t'_1 > u'_1 > t_2 > u_2 > t'_2 > u'_2 > \cdots$$

But this combined sequence is again convergent, which can only happen if $V = V'$.

Second step: Suppose only that $c < u_n < t_n$ with $t_n \to c$. Any subsequence of $C(u_n, t_n)$ has itself a subsequence with limit V for, discarding enough terms, we can obtain the alternation $t_1 > u_1 > t_2 > u_2 > \cdots$ and the argument of the first step applies. But then the whole sequence $C(u_n, t_n)$ converges to V.

Third step: Let now $c \leqslant u_n < t_n$ with $t_n \to c$. Choose $u_n < u'_n < t_n$ such that $\|C(u_n, t_n) - C(u'_n, t_n)\| < 1/n$. Then, $C(u'_n, t_n)$ being convergent to V by the second step, we also have $\lim C(u_n, t_n) = V$ as claimed.

We complete the definition of T^+ and T^- setting

$$T^+(b) = T^-(b),$$
$$T^-(a) = T^+(a).$$

Then:

2.4. Corollary. *The functions $T^+ : [a, b] \to S$, $T^- : [a, b] \to S$ have the following properties*:

2.4*a*. *T^+ is right continuous and T^- is left continuous.*

2.4*b*. $$T^+(c) \to T^-(c_0) \quad \textit{when} \quad c \to c_0, \qquad c < c_0.$$
$$T^-(c) \to T^+(c_0) \quad \textit{when} \quad c \to c_0, \qquad c > c_0.$$

2.4*c*. *T^+ and T^- are rectifiable for the* ang *distance on S, and*

$$l_S(T^+) \leqslant T(\gamma),$$
$$l_S(T^-) \leqslant T(\gamma).$$

Note. The inequalities in 2.4*c* are equalities (see 4.5*b*).

PROOF. Let $c_n \to c_0$, $c_n \geqslant c_0$. For $\epsilon > 0$ we have

$$\| C(c_n, c_n + 1/j) - T^+(c_0) \| \leqslant \epsilon,$$

for n, j large enough. Taking limits as $j \to \infty$ we get from 2.3

$$\| T^+(c_n) - T^+(c_0) \| \leqslant \epsilon$$

for the same values of n, so that 2.4*a* is proved for T^+. The proof for T^- is similar.

Assume now that $c_n \to c_0$, $c_n < c_0$ and let $\epsilon > 0$. Then by 2.3*b* there exists N_ϵ such that for $j > 1/(c_0 - c_n)$ and $n \geqslant N_\epsilon$ we have

$$\| C(c_n, c_n + 1/j) - T^-(c_0) \| \leqslant \epsilon.$$

Taking the limit as $j \to \infty$ we get $\| T^+(c_n) - T^-(c_0) \| \leqslant \epsilon$ which proves the first part of 2.4*b*. The second part is similar.

Finally, if $\epsilon > 0$ and $a = t_0 < t_1 < \cdots < t_n = b$, we can find $\Pi: t_0 < t'_0 < t_1 < t'_1 < \cdots$ with

$$\operatorname{ang}(T^+(t_i), C(t_i, t'_i)) \leqslant \epsilon/n.$$

Then

$$\sum \operatorname{ang}(T^+(t_i), T^+(t_{i+1})) \leqslant \operatorname{ang}(C(t_i, t'_i), C(t_{i+1}, t'_{i+1})) + 2\epsilon \leqslant T(\epsilon) + 2\epsilon$$

and therefore $l_S(T^+) \leqslant T(\gamma)$ as claimed. The proof for T^- is similar.

Remark. We close this section with a sketch of the proof that $T(a, b)$ is the «total curvature» of the curve γ when γ has a curvature $k = dT/dS$, where T = unit tangent vector = $d\gamma/ds$ and s is arclength. For a complete proof see [7], Theorem 2.2. For simplicity we assume that γ is a curve in $\mathbb{R}^3$ parametrized by arclength.

Setting cartesian coordinates in convenient way we have (see [10], Vol. 1, Chapter 1):

2.5. $$\gamma(s) = se_1 + (1/2)s^2ke_2 + \text{terms of higher order in } s$$

where $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$. Here k denotes the curvature of γ at $s = 0$.

Let now $\epsilon > 0$ and $K > 1$, and pick $\eta > 0$ so that $0 < \eta < \epsilon$, that 2.5 is valid in the interval $0 < s < \eta$, and that $\|e_1 - d\gamma/ds\| < \epsilon$ for $0 < s < \eta$. Let $0 = s_0 < s_1 < \cdots < s_n = \eta$ be chosen so that $\|\gamma(s_j) - \gamma(s_{j-1})\| = \delta$ is independent of j. It is easy to see that $\delta \leqslant s_j - s_{j-1} \leqslant (1 - \epsilon)\delta$. Also, if η is small enough, for $C_j = C(s_j, s_{j+1})$ we have

$$\|C_{j-1} - C_j\| \leqslant \text{ang}\,(C_{j-1}, C_j) \leqslant K\|C_{j-1} - C_j\|.$$

Then

$$T(\Pi) = \sum \text{ang}\,(C_{j-1}, C_j) \leqslant K \sum \|C_{j-1} - C_j\|.$$

On the other hand $C_{j-1} = (\gamma(s_j) - \gamma(s_{j-1}))/\delta$ and therefore

$$\|C_{j-1} - C_j\| = \|2\gamma(s_j) - \gamma(s_{j+1}) - \gamma(s_{j-1})\|/\delta.$$

Using 2.5 we obtain

$$2\gamma(s_j) - \gamma(s_{j+1}) - \gamma(s_{j-1}) = (2s_j - s_{j+1} - s_{j-1})e_1 + (k/2)(2s_j^2 - s_{j+1}^2 - s_{j-1}^2)e_2 + h \cdot o \cdot t.$$

($h \cdot o \cdot t$ = higher order terms) so that, from $s_j - s_{j-1} = \delta + h \cdot o \cdot t$ we conclude that

$$\|2\gamma(s_j) - \gamma(s_{j+1}) - \gamma(s_{j-1})\| = k\delta^2 + h \cdot o \cdot t.$$

Thus

$$T(\Pi) \leqslant K\eta\delta k + h \cdot o \cdot t = Kk\eta + h \cdot o \cdot t$$

because $n\delta = \eta + h \cdot o \cdot t$. Therefore $T(0, \eta) \leqslant Kk\eta$ and $K > 1$ being arbitrary we get $dT/ds \leqslant k$. It follows that $T(a, b) \leqslant \int k\,ds$.

The converse inequality also holds since, using 2.4c, we get

$$\int k\,ds = \int \left\| \frac{dT}{ds} \right\| ds = l(T) = l_S(T) \leqslant T(a, b).$$

Therefore

$$T(a, b) = \int k\,ds, \tag{2.6}$$

and this justifies the terminology «total curvature» used for $T(a, b)$.

3. Further Properties of the Total Curvature

The following result is easy to obtain:

3.1. *For $U, V, W \in H - \{0\}$ we have*

$$\operatorname{ang}(U, V) + \operatorname{ang}(V, W) = \operatorname{ang}(U, W)$$

whenever $U = -W$ or $V = rU + sW$ for some $r, s \geqslant 0$.

It corresponds to the fact that in any triangle an exterior angle is the sum of the non-adjacent interior angles.

The next result is a corollary of 3.1. Consider distinct vectors $V_0, V_1, \ldots, V_{n+1}$ in H and let $V'_0 = V_0,\ V'_1 = V_1, \ldots, V'_{j-1} = V_{j-1},\ V'_j = V_{j+1}, \ldots, V'_n = V_{n+1}$. Denote $D_i = V_{i+1} - V_i,\ D'_i = V'_{i+1} - V'_i,\ \eta_i = \operatorname{ang}(D_i, D_{i-1})$, $\eta'_i = \operatorname{ang}(D'_i, D'_{i-1})$. Then

3.2. $$\sum \eta_i \geqslant \sum \eta'_i.$$

We leave the special cases $j = 1$ and $j = n$ to the reader and prove 3.2 under the assumption $1 < j < n$. After cancellation of like terms, we get that 3.2 amounts to

3.2*a*. $$\eta_{j-1} + \eta_j + \eta_{j+1} \geqslant \eta'_{j-1} + \eta'_j.$$

Using $D'_{j-2} = D_{j-2},\ D'_{j-1} = D_{j-1} + D_j,\ D'_j = D_{j+1}$ we obtain from 3.1

$$\eta_j = \operatorname{ang}(D_{j-1}, D'_{j-1}) + \operatorname{ang}(D'_{j-1}, D_j).$$

On the other hand, by the angle triangle inequality 1.1*a* we get

$$\eta_{j-1} + \operatorname{ang}(D_{j-1}, D'_{j-1}) \geqslant \eta'_{j-1}$$

$$\eta_{j+1} + \operatorname{ang}(D'_{j-1}, D_j) \geqslant \eta'_j$$

so that, adding up the last three relations, we get 3.2*a*.

In the sequel the following property, which sharpens 1.1*b*, is used several times:

3.3. *When restricted to the unit sphere $S = \{\|U\| = 1\}$ the function* ang *is a distance equivalent to the norm distance, since*

$$\|U - V\| \leqslant \operatorname{ang}(U, V) \leqslant (\pi/2)\|U - V\|.$$

With the aid of 3.1, 3.2 and 3.3 we can obtain the following additional properties of the function T for a curve γ in a Hilbert space H.

3.4*a*. *Monotonicity with respect to partitions*: if Π_2 is a refinement of Π_1, then $T(\Pi_2) \leqslant T(\Pi_1)$. In particular $T(\Pi) = \lim T(\Pi)$. (see [2], pp. 254-256 or [7], Lemma 1.1).

PROOF. If suffices to consider the case where $\Pi_2 = \{c_0, c_1, \ldots, c_{n+1}\}$ and $\Pi_1 = \{c_0, c_1, \ldots, c_{j-1}, c_{j+1}, \ldots, c_{n+1}\}$ and then apply 3.2 with $V_i = \gamma(c_i)$.

3.4*b*. *Total curvature of polygonal lines*: if γ is a polygonal line with vertices at $\gamma(c_0)$, $\gamma(c_1), \ldots$, then $T(\gamma) = T(\Pi_0)$ where $\Pi_0 = \{c_0, c_1, \ldots\}$.

PROOF. Using 3.4a we have $T(\gamma) = \lim T(\Pi)$; but $T(\Pi) = T(\Pi_0)$ for any Π finer than Π_0.

3.4*c*. *Lower continuity with respect to intervals*: if $I_1 \supset I_2 \supset \cdots$ are intervals contained in $[a, b]$ with $\cap I_n = \varnothing$ and γ has finite total curvature on $[a, b]$ then $T(I_n) \to 0$.

PROOF. It is clear from the hypothesis that $\cap \bar{I}_n$ consists of exactly one point, say r. Also denote a_n, b_n the left and right endpoints of I_n. Since $r \notin \cap I_n$ we have $r \notin I_n$ for n large. This implies that $r = a_n$ for all $n \geqslant N$ or $r = b_n$ for all $n \geqslant N$. Consider the first case, the other being similar, and assume by contradiction that $T(I_n) > k > 0$ for all $n \geqslant N$. Let $\Pi_1 = \{c_0 = r, c_1, \ldots, c_{n+1} = b_N\}$ be a partition such that $T(\Pi_1) > k$. Replacing $c_0 = r$ by $e_1 = r + \epsilon$ we get a new partition Π_1'; using the continuity of ang (see 3.3) we may assume that $T(\Pi_1') > k$ by choosing ϵ small enough. Then setting $d_1 = b_N$ we get $T(e_1, d_1) > k$.

Since $b_n \to r$ we have $r < b_m < e_1$ for some m and repeating the argument we conclude that $T(e_2, d_2) > k$ for appropiate $r < e_2 < d_2 = b_m < e_1$. Continuing in this way we obtain disjoint intervals $J_n = (e_n, d_n)$ with $T(J_n) > k$ which contradicts $T(a, b) < +\infty$ in view of 2.2c.

3.4*d*. *Upper continuity with respect to intervals*: If $I_1 \subset I_2 \subset \cdots$ are intervals contained in $[a, b]$ and $I = \cup I_n$, then $T(I_n) \to T(I)$.

PROOF. Denote by a_n, b_n the endpoints of I_n. For $\epsilon > 0$ let $\Pi = \{a, c_1, c_2, \ldots, c_k, b\}$ be a partition of $[a, b]$ with $T(\Pi) > T(a, b) - \epsilon$. Using again the continuity of ang we see that $T(\Pi_n) > T(a, b) - \epsilon$, where $\Pi_n = \{a_n, c_1, \ldots, c_k, b_n\}$ and n is large enough. But then

$$\lim T(I_n) \geqslant \lim T(\Pi_n) \geqslant T(a, b) - \epsilon$$

and the result follows since $\epsilon > 0$ is arbitrary.

3.4*e*. If $I_1 = (u_1, v_1)$, $I_2 = (u_2, v_2)$ are contained in (a, b) then

$$\operatorname{ang}(C(u_1, v_1), C(u_2, v_2)) \leqslant T(a, b).$$

The following property of T is a valuable tool for the sequel.

3.5. Proposition (the addition formula). *Let γ be a curve with finite total curvature and let $a \leqslant u < c < v \leqslant b$. Then*

3.5*a*. $T(u, c) + \operatorname{ang}(T^-(c), T^+(c)) + T(c, v) = T(u, v)$.

PROOF. Let Π_n, $n = 1, 2, \dots$ be a sequence of partitions in $[u, v]$ such that $T(\Pi_n) \to T(u, v)$. By 3.4*a* the convergence is preserved if we add partition points to Π_n so that we may assume that $c \in \Pi_n$ and that $\Pi'_n = \{c_j \in \Pi_n; c_j < c\}$ and $\Pi''_n = \{c_j \in \Pi_n; c_j < c\}$ satisfy $T(\Pi'_n) \to T(u, c)$ and $T(\Pi''_n) \to T(c, v)$. Abbreviate now α_j = angle formed by the chords $C(c_{j-1}, c_j)$ and $C(c_j, c_{j+1})$ and use β for the α_i corresponding to $c_i = c$. Then

$$T(\Pi_n) = \sum \alpha_j,$$

$$T(\Pi'_n) = \sum \{\alpha_j; c_j < c\}$$

and

$$T(\Pi''_n) = \sum \{\alpha_j; c_j > c\},$$

and therefore

$$T(\Pi_n) = T(\Pi'_n) + \beta + T(\Pi''_n).$$

Taking limits we get the desired formula as an application of 2.3.

Using the notation

$$T'(c) = \lim T(a_n, b_n), \qquad a_n < c < b_n \qquad b_n - a_n \to 0$$

introduced in 1.2, we have

3.6. Corollary. *For any $c \in (a, b)$,*

3.6*a*. $T'(c) = \operatorname{ang}(T^-(c), T^+(c))$.

PROOF. Write

$$T(a_n, b_n) = T(a_n, c) + \operatorname{ang}(T^-(c), T^+(c)) + T(c, b_n).$$

Taking limits and using 3.4*c* we get 3.6*a*.

Of course the addition formula 3.5*a* holds more generally in the form

3.7. $$T(a_0, a_{n+1}) = \sum_{i=0}^{n} T(a_i, a_{i+1}) + \sum_{i=1}^{n} T'(a_i)$$

for $a \leqslant a_0 < a_1 < \cdots < a_{n+1} \leqslant b$.

3.8. Corollary. *If γ has finite total curvature, then*

(*a*) *The inequality $T'(c) \neq 0$ can happen only for countably many values $s_1, s_2, \ldots$ of c and the series $\sum T'(s_k)$ is convergent.*

(*b*) *The curve γ has a unique tangent at all but countably many points.*

PROOF. From 3.7 we obtain

$$\sum T'(a_i) \leqslant T(a, b) < +\infty$$

for any choice of $a_0 < a_1 < \cdots < a_{n+1}$ and this suffices to obtain 3.8*a*. Using 3.6*a* we see that 3.8*b* follows from 3.8*a*.

Property 3.8*b* can be sharpened in the following way:

3.9. Proposition. *If γ is a curve with finite total curvature, its graph splits in a finite number of graphs of Lipschitz functions. In particular the curve is rectifiable.*

PROOF. Observe first that given any $m > 0$ there is a partition $a = c_0 < c_1 < \cdots < c_{n+1} = b$ such that $T(c_i, c_{i+1}) < m$ for all i. In fact, there exist only finitely many t with $T'(t) \geqslant m$. Label them $t_1, t_2, \ldots, t_k$. For each u interior to an interval $J = [t_i, t_{i+1}]$ we have $T'(u) < m$ and therefore there is a neighborhood $(u - \epsilon, u + \epsilon)$ with $T(u - \epsilon, u + \epsilon) < m$. Also there exist $t_i < t_i'$ and $t_{i+1}' < t_{i+1}$ with $T(t_i, t_i') < m$ and $T(t_{i+1}', t_{i+1}) < m$, by 3.4*c*. Hence by compactness we obtain a partition

$$t_i = u_0 < u_1 < \cdots < u_l = t_{i+1} \quad \text{with} \quad T(u_i, u_{i+1}) < m.$$

This can be repeated for all $J = [t_i, t_{i+1}]$ to obtain the desired partition $c_0, \ldots, c_{n+1}$.

Suppose that $0 < m \leqslant \pi$. Then on each $I = [c_i, c_{i+1}]$ we have $T(I) < \pi$ and therefore $T'(c) < \pi$ for c interior to I. This means that $W(c) = T^+(c) + T^-(c)$ satisfies $W(c) \neq 0$ at all interior points.

Let now T^+ and T^- be arbitrary unit vectors with $W = T^+ + T^- \neq 0$. Denote by $L = \{\alpha W; \alpha \text{ real}\}$ the line generated by W and by $L^\perp$ the orthogonal complement of L, $L^\perp = \{U \in H; \langle U, W \rangle = 0\}$. Suppose that $X = \|X\| V$ with V a unit vector satisfying $\|T^+ - V\| \leqslant \|W\|/4$, and write the decomposition $X = hW + Y$ (with h real, $Y \in L^\perp$) induced by $H = L \oplus L^\perp$.

Under these conditions we claim that

3.9*a*. $$\|Y\| \leqslant 2h.$$

In fact,

3.9*b*. $$\langle X, W\rangle = \langle hW, W\rangle = h\|W\|^2$$

and

$$\begin{aligned}\langle X, W\rangle &= \|X\|\langle V, W\rangle \\ &= \|X\|(\langle T^+, W\rangle - \langle T^+ - V, W\rangle) \\ &\geqslant \|X\|(\langle T^+, W\rangle - \|W\|^2/4).\end{aligned}$$

But

$$\langle T^+, W\rangle = 1 + \langle T^+, T^-\rangle = (1/2)\|T^+ + T^-\|^2 = (1/2)\|W\|$$

so that

3.9*c*. $$\langle X, W\rangle \geqslant \|X\|\ \|W\|^2/2.$$

Combining 3.9*b* and 3.9*c* we get $\|X\| \leqslant 2h$; then a fortiori $\|Y\|$ ⋮ claimed.

Fix now c interior to $J = [c_i, c_{i+1}]$ and apply this to the case T^+ = $T^- = T^-(c)$, $X = \gamma(t) - \gamma(s)$, $V = C(s, t)$. Certainly the hy $\|T^+ - V\| \leqslant \|W\|/4$ holds if $c \leqslant s < t \leqslant c + \epsilon$ and ϵ is small (by 2.⁚ write $\gamma(u) = h(u)W + Y(u)$, then $X = (h(t) - h(s))W + Y(t) - Y$ therefore from 3.9*a* we get, for $c \leqslant s < t \leqslant c + \epsilon$:

3.9*d*. $$\|Y(t) - Y(s)\| \leqslant 2(h(t) - h(s)).$$

In particular $h(s) \leqslant h(t)$. However $h(s) = h(t)$ implies $Y(s) = Y(t)$ frc and then $\gamma(s) = \gamma(t)$, impossible. Thus $u \to h(u)$ is a strictly monotoı tion for $c \leqslant u \leqslant c + \epsilon$. This allows us to use $x = h(u)$ as a new variablε from $x = h(x)$ to $x_0 + \eta = h(c + \epsilon)$. Set $Z(x) = Y(u)$ when $x = h(u)$. ˜ $x = h(s)$, $y = h(t)$ we have $\|Z(x) - Z(y)\| \leqslant 2\|x - y\|$ and therefore . is a Lipschitz function from $[x_0, x_0 + \eta]$ into $L^\perp$. The equality

$$xW + Z(x) = h(u)W + Y(u) = \gamma(u)$$

shows that the curve $\gamma(t)$, $c \leqslant t \leqslant c + \epsilon$ is the graph of Z.

A similar reasoning yields the definition of Z on the interval $[x_0 - \eta$ glueing both halves together we conclude that the curve is the grε Lipschitz function on an interval with $\gamma(c)$ corresponding to an interiι

The special case when c is an end point of J is handled in the same way taking T^+ and T^- both equal to the one-sided tangent available.

Finally, a compactness argument yields a desired decomposition.

3.10 Corollary. *A curve with finite total curvature can be parametrized by $\gamma(t)$, $a \leqslant t \leqslant b$ in such a way that the right and left derivatives of γ exist at all t and neither of them vanishes. Further these derivatives coincide except at countable many values of t.*

4. Associated Functions

From this section on, all curves (unless specified) will be assumed to have finite total curvature.

4.1. Definition. *For $a \leqslant u < v \leqslant b$ set*

4.1*a*. $E(u, v) = \text{ang}\,(C(u, v), T^+(u)) + \text{ang}\,(C(u, v), T^-(v))$.

4.1*b*. $\Xi(u, v) = T(u, v) - E(u, v)$.

In order to study these functions we begin with a lemma about partitions.

4.2 Lemma. *For any partition $\Pi = \{u, u_1, \dots, u_n, v\}$ of $[u, v]$ we have*

$$T(u, v) \geqslant \text{ang}\,(T^+(u), C(u, u_1)) + T(\Pi) + \text{ang}\,(T^-(v), C(u_n, v)).$$

PROOF. Pick $u < u' < u_1$ and $u_n < v' < v$. Then

$$T(u, v) \geqslant T\{u, u', u_1, u_2, \dots, u_n, v', v\}.$$

But this last number is the sum of

$$x = \text{ang}\,(C(u, u'), C(u', u_1)) + \text{ang}\,(C(u_n, v'), C(v', v))$$

and $z = T(u', u_1, u_2, \dots, u_n, v)$, so that taking limits as $u' \to u$ and $v' \to v$ we get that x and z approach, respectively,

$$\text{ang}\,(T^+(u), C(u, u_1)) + \text{ang}\,(C(u_n, v), T^-(v))$$

and $T(\Pi)$, which proves the lemma.

4.3. Proposition. *The function Ξ has the following properties:*

4.3*b*. $$\sum \Xi(u_j, v_j) \leqslant \Xi(u, v)$$

for any system of disjoint intervals (u_j, v_j) *contained in* (u, v). *In part is monotonic.*

4.3*c*. $\Xi'(c) = 0$ *for all* c.

PROOF. (*a*) Taking the trivial partition $\Pi = \{u, v\}$ we obtain from

$$T(u, v) \geqslant \operatorname{ang}(T^+(u), C(u, v) + \operatorname{ang}(T^-(v), C(u, v))$$

so that $T \geqslant E$ which means that $\Xi \geqslant 0$.

(*b*) It suffices to prove that

4.3*d*. $$\Xi(u, v) \geqslant \Xi(u, c) + \Xi(c, v), \quad \text{for} \quad u < c < v.$$

Denote

$U = T^+(u)$, $V = T^-(v)$, $C^- = T^-(c)$, $C^+ = T^+(c)$, $L = C(u, c)$, $R =$

and abbreviate $UL = \operatorname{ang}(U, L)$, $LC^- = \operatorname{ang}(L, C^-)$, etc.; by defini

$$\Xi(u, v) = T(u, v) - (WU + WV)$$
$$\Xi(u, c) = T(u, c) - (LU + LC^-)$$
$$\Xi(c, v) = T(c, v) - (RC^- - RV).$$

Using the addition formula 3.7 we get

$$\Xi(u, v) = \Xi(u, c) + \Xi(c, v) + \Delta$$

where

4.3*e*. $$\Delta = LU + LC' + RC^+ + RV + T'(c) - WU - WV$$

and therefore the desired inequality 4.3*d* is equivalent to $\Delta \geqslant 0$.

Observing that $T'(c) = C^- C^+$, from the triangle inequality 1.1 w

4.3*f* $$LC^- + RC^- + T'(c) \geqslant LR$$

and from the fact that $\gamma(u)$, $\gamma(c)$ and $\gamma(v)$ are the vertices of a triangle clude (see the figure 1 in the following page), that $LR = WL + WF$ from 4.3*e* and 4.3*f* we obtain (using again the triangle inequality):

$$\Delta \geqslant LU + RV + WL + WR - WU - WV'$$
$$= (LU + WL - WU) + (RV + WR - WV)$$

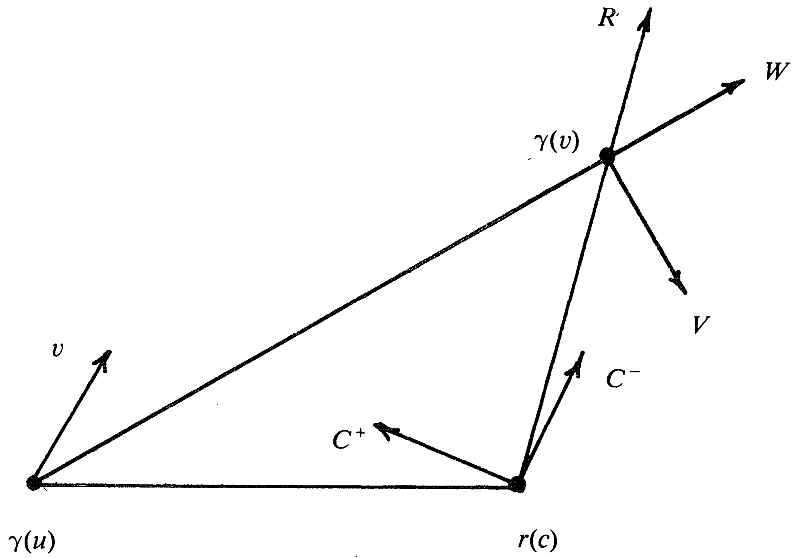

Figure 1

(c) First observe that if B is a limit of chords $C(c - \epsilon_n, c + \delta_n)$, then

$$B = pT^-(c) + qT^+(c)$$

for some $p, q \geqslant 0$. In fact, write

$$C(c - \epsilon_n, c + \delta_n) = p_n C(c - \epsilon_n, c) + q_n C(c, c + \delta_n)$$

with

$$p_n = \|\gamma(c) - \gamma(c - \epsilon_n)\| / \|\gamma(c + \delta_n) - \gamma(c - \epsilon_n)\|$$
$$q_n = \|\gamma(c + \delta_n) - \gamma(c)\| / \|\gamma(c + \delta_n) - \gamma(c - \epsilon_n)\|.$$

Taking limits we get $p = \lim p_n \geqslant 0$, $q = \lim q_n \geqslant 0$. This means that

$$BC^- + BC^+ = C^-C^+ = T'(c).$$

On the other hand, $T^+(c - \epsilon_n) \to C^-$ by 2.4a, whence

$$VC^- = \lim \text{ang}\,(C(c - \epsilon_n, c + \delta_n), T^+(c - \epsilon_n));$$

similarly

$$VC^+ = \lim \text{ang}\,(C(c - \epsilon_n, c + \delta_n), T^-(c + \delta_n)).$$

This of course gives $\lim F(c - \epsilon_n, c + \delta_n) = T'(c)$, or $T''(c) = 0$

Another function associated to a curve is the *Gauss map* τ: $[a, b] \to S$ (where S is the sphere $\|X\| = 1$) defined by $\tau(t) = T^+(t)$, $a \leqslant t < b$, and $\tau(b) = T^-(b)$. In general, τ is discontinuous.

Recall that the length $l_S(\tau)$ is defined by

$$l_S(\tau) = l_S(a, b) = \operatorname{Sup} \sum \operatorname{ang}(\tau(x_i), \tau(x_{i+1})),$$

the supremum taken over all partitions.

Observe that we have

4.4. $$l_S(a, b) = l_S(a, c) + T'(c) + l_S(c, b),$$

for $a < c < b$, so that «arc length» is additive only if the partition point is a point of continuity of τ. For the same reason, the length of τ, considered as a map in H, will be equal to $l_S(\tau)$ only if τ is continuous everywhere.

The main property of τ is given in 4.7, which requires the following proposition.

4.5. Proposition. *Let f: $[a, b] \to L$ (L a Hilbert space) be a Lipschitz function with right* (*resp. left*) *derivative $f'_+(t)$* (*resp. $f'_-(t)$*) *at all $a \leqslant t < b$* (*resp. $a < t \leqslant b$*), *and let γ: $[a, b] \to R \oplus L$ be the graph of f,* i.e., $\gamma(t) = (t, f(t))$. *Suppose that f'_+ is a function of bounded variation on $[a, b)$ with $l(f'_+) < 1$. Then*

4.5*a*. *γ has a finite total curvature*;

4.5*b*. $T(\gamma) = l_S(\tau)$.

PROOF. The hypotheses on f imply that γ is absolutely continuous with $\gamma'_+ = (1, f'_+)$ of bounded variation. Hence, for $a \leqslant u \leqslant v \leqslant b$ we have

$$\gamma(v) - \gamma(u) = \int_u^v \gamma'_+(t)\, dt = \int_u^v \tau(t) \|\gamma'_+(t)\|\, dt.$$

Now the integrand $\tau \|\gamma'_+\| = \gamma'_+$ is a function of bounded variation, hence Riemann integrable, and therefore $\gamma(v) - \gamma(u)$ is the limit of Riemann sums

$$R = \sum \tau(t_i) \|\gamma'_+(t_i)\|\, \Delta_i t$$

($\Delta_i t = t_i - t_{i-1}$). On the other hand, $R = h \sum a_i \tau(t_i)$ with $a_i = \|\gamma'_+(t_i)\|\, \Delta_i t/h$ and $h = \sum \|\gamma'_+(t_j)\|\, \Delta_j t$. Taking limits we get that $(\gamma(v) - \gamma(u))/l(\gamma)$ is the limit of convex combinations of points of the form $\tau(t)$, $u \leqslant t \leqslant v$. In particular, $C(u, v)$ belongs to the closed convex cone spanned by $\{\tau(t); u \leqslant t \leqslant v\}$.

We need the following lemma (whose proof is given below):

4.6. Lemma. *Let $N \subset S$ be a subset with diameter strictly less than $\pi/2$ for the* ang *distance, and let M denote the intersection of S with the closed convex cone spanned by N. Pick U, V in N and denote by g the function defined on S by $g(X) = \operatorname{ang}(U, X) + \operatorname{ang}(X, V)$. Then $\operatorname{Sup}_M g = \operatorname{Sup}_N g$.*

Let now $\Pi = \{t_0, t_1, \ldots, t_m\}$, $t_0 = a$, $t_m = b$ be a partition of $[a, b]$ and let $\epsilon > 0$.

Observe that, for $a \leqslant s, t \leqslant b$,

$$\begin{aligned}\operatorname{ang}(\tau(t), \tau(s)) &\leqslant (\pi/2)\|\tau(t) - \tau(s)\| \leqslant (\pi/2)\|\gamma'_+(t) - \gamma'_+(s)\| \\ &\leqslant (\pi/2)l(\gamma'_+) = (\pi/2)l(f'_+) < \pi/2\end{aligned}$$

so that the set $\{\pi(t); a \leqslant t \leqslant b\}$ has diameter less than $\pi/2$ for the ang distance. Thus, the lemma applies with $U = \tau(t_i)$, $V = \tau(t_{i+1})$ and $N = \{\tau(t), t_i \leqslant t \leqslant t_{i+1}\}$. Clearly there is $u_i \in [t_i, t_{i+1}]$ with $\operatorname{Sup}_N g \leqslant g(\tau(u_i)) + \epsilon/m$, and therefore (by the lemma),

$$\operatorname{ang}(\tau(t_i), X) + \operatorname{ang}(X, \tau(t_{i+1})) \leqslant \operatorname{ang}(\tau(t_i), \tau(u_i)) + \operatorname{ang}(\tau(u_i), \tau(t_{i+1})) + \epsilon/m$$

for all X in M. In particular, as proved above, this inequality holds for $X = C(t_i, t_{i+1})$.

Abbreviating $C(t_i, t_j) = C_{ij}$, $\tau(t_i) = \tau_i$ and $\operatorname{ang}(X, Y) = XY$, we get

$$\begin{aligned}T(\Pi) &= C_{01}C_{12} + C_{12}C_{23} + \cdots \\ &\leqslant \tau_0 C_{01} + C_{01}\tau_1 + \tau_1 C_{12} + C_{12}\tau_2 + \tau_2 C_{23} + \cdots \\ &\leqslant \tau_0\tau(u_0) + \tau(u_0)\tau_1 + \tau_1\tau(u_1) + \tau(u_1)\tau_2 + \cdots + \epsilon \\ &\leqslant l_S(\tau) + \epsilon,\end{aligned}$$

whence $T(\gamma) \leqslant l_S(\tau)$ which proves 4.5*a*. The converse inequality was proved in 2.4*c*, so that $T(\gamma) = l_S(\tau)$, as claimed in 4.5*b*.

PROOF OF 4.6. It suffices to prove that if F is a spherical polygon with vertices $P_1, P_2, \ldots, P_m$ in N (each side is a maximum circle segment) with diameter of F strictly less than $\pi/2$ (for the ang distance), then

4.6*a*. $$g(X) \leqslant \max\{g(P_1), g(P_2), \ldots, g(P_m)\}.$$

for X inside F.

Observe that g is the sum of functions of the form $g_1(X) = \operatorname{ang}(X, U)$. It is not hard to see that 4.6*a* follows if we prove that g_1 is convex, i.e.,

4.6*b*. $$g_1(X(\lambda\eta + (1 - \lambda)\xi)) \leqslant \lambda g_1(X(\eta)) + (1 - \lambda)g_1(X(\xi))$$

where $X(\theta)$ runs on a maximum circle. We will assume that coordinates ar set on a three-dimensional subspace containing U, $X(\theta)$, so that

$$X(\theta) = (\cos\theta, \operatorname{sen}\theta, 0), \qquad U = (v, 0, w), \qquad 0 \leqslant v \leqslant 1 \qquad 0 \leqslant w \leqslant 1.$$

Then, letting $\alpha(\theta) = g_1(X(\theta))$, we have $0 \leqslant \alpha < \pi/2$ and $\cos\alpha(\theta) = v\cos\theta$ Differentiating,

$$\frac{d^2\alpha}{d\theta^2}(\theta) = \operatorname{cotan}\alpha(1 - v^2\operatorname{sen}^2\theta/\operatorname{sen}^2\alpha)$$

$$= \operatorname{cotan}\alpha(1 - v^2)/\operatorname{sen}^2\alpha \geqslant 0$$

and this implies 4.6*b*, which completes the proof of the lemma.

Observe that the same proof applies to the case of

$$g(X) = \operatorname{ang}(X, U_1) + \cdots + \operatorname{ang}(X, U_n),$$

and that the U_i need not belong to N as long as $N \cup \{U_1, \ldots, U_n\}$ ha diameter less than $\pi/2$.

4.7. Corollary. *For a curve to have finite total curvature it is necessary an sufficient that it can be parametrized as* $\gamma(t)$, $a \leqslant t \leqslant b$, *in such a way tha*

4.7*a*. $\gamma(t)$ *is a Lipschitz function*;

4.7*b*. $\gamma'_+ = d^+\gamma/dt$ *exists and satisfies* $\|\gamma'_+(t)\| \geqslant 1$ *for all* t;

4.7*c*. γ'_+ *is rectifiable.*

4.8. Corollary. *Any curve* γ *with finite total curvature satisfies* $T(\gamma) = l_S(\tau$

For the proofs, combine 2.4*c*, 3.7, 3.9, 3.10 , 4.4 and 4.6.

We close this section by indicating a measure-theoretic interpretation of 7 Recall that any non-decreasing function $g\colon [a, b] \to \mathbb{R}$ determines a positiv regular Borel measure μ by means of the Lebesgue-Stieltjes integral, whic satisfies $\mu[u, v] = g(v^+) - g(u^-)$, $\mu[u, v) = g(v^-) - g(u^-)$, etc. Takin $g(t) = T(a, t)$, it follows from 3.4*c*, 3.4*d* and 3.5 that g is non-decreasing an left-continuous so that:

4.9. Proposition. *Let* $\gamma(t)$, $a \leqslant t \leqslant b$, *be a curve with finite total curvatur Then there is a unique positive real Borel measure* μ *on* $[a, b]$ *such that for an*

PROOF. Let $a < c < c + h < b$. Then from 3.5

$$T(a, c) + T'(x) + T(c, c + h) = T(a, c + h).$$

Letting $h \to 0^+$ and using 3.4*c* we obtain $T(a, c) + T'(x) = T(a, c^+)$ c $+ T'(c) = g(c^+)$. Since $g(c^-) = g(c)$ we get $T'(c) = g(c^+) - g(c^-)$ proves that $\mu(\{c\}) = T'(c)$. On the other hand, take $a < u < v < b$. T

$$T(u, v) = T(a, v) - T(a, u) - T'(u) = g(v) - g(u) - g(u^+) + g(u)$$
$$= g(v) - g(u^+) = \mu(u, v),$$

as claimed. The case $a = u$ is similar.

We have no answer for the following question: (*a*) which measures μ a in this way? (*b*) what is the measure-theoretic interpretation of Ξ and

5. Total Curvature of Plane Curves

In this section we assume that H is the plane $\mathbb{R}^2$.

5.1. Theorem. *Let $\gamma(t)$, $a \leqslant t \leqslant b$ be a plane curve. Then the followin ditions are equivalent*:

5.1.1. *γ has finite total curvature.*

5.1.2. *γ is the union of finitely many graphs of real functions f wi properties*:

(*a*) *f has a right derivative f'_+ everywhere,*
(*b*) *f'_+ is a function of bounded variation.*

5.1.3. *γ is the union of finitely many graphs of functions which a ferences of Lipschitz convex functions.*

PROOF. The equivalence between 5.1.1 and 5.1.2 is a special case c (2) implies (3): A theorem-of-the-mean-like argument shows that und hypothesis of 5.1.2 f satisfies: for $x < y$ there exists ξ and η between x with

$$f'_+(\eta) \leqslant (f(y) - f(x))/(y - x) \leqslant f'_+(\xi).$$

But then, f'_+ being (of bounded variation, hence) bounded we concluc f is a Lipschitz function. Therefore

$$f(z) = f(a) + \int_a^s f'_+ (t)\, dt.$$

Write now $f'_+ = h - g$ with $h \geqslant 0$, $g \geqslant 0$ and h, g non-decreasing. Then $f = H - G$ where G, H (the indefinite integrals of g and h) are convex Lipschitz functions.

(3) implies (1): If f is the difference of two Lipschitz convex functions then f'_+ exists at all points and, being the difference of two non-decreasing functions, it has bounded variation.

It is clear that in any Hilbert space $T \equiv 0$ characterizes straight lines. For plane curves the next result gives and interpretation of $\Xi \equiv 0$.

5.2. Proposition. *The plane curve $\gamma(t)$ with finite total curvature is convex on the interval $a \leqslant t \leqslant b$ if and only if $\Xi(a, b) = 0$.*

PROOF. Convex plane curves can be characterized by the following property: whenever $u < u' < v' < v$, the line segment $[\gamma(u), \gamma(v)]$ and $[\gamma(u'), \gamma(v')]$ do not cross each other, i.e., either they are disjoint or the first contains the second.

Suppose now that γ is not convex and that u, u', v and v' have been chosen so that the segments do cross (see the figure 2).

Denote also the following angles (not all drawn) by the indicated letters

$$\alpha = \text{ang}\,(T^+(u), C(u, v))$$
$$\alpha' = \text{ang}\,(T^+(u), C(u, u'))$$
$$\delta = \text{ang}\,(C(u, u'), C(u, v))$$

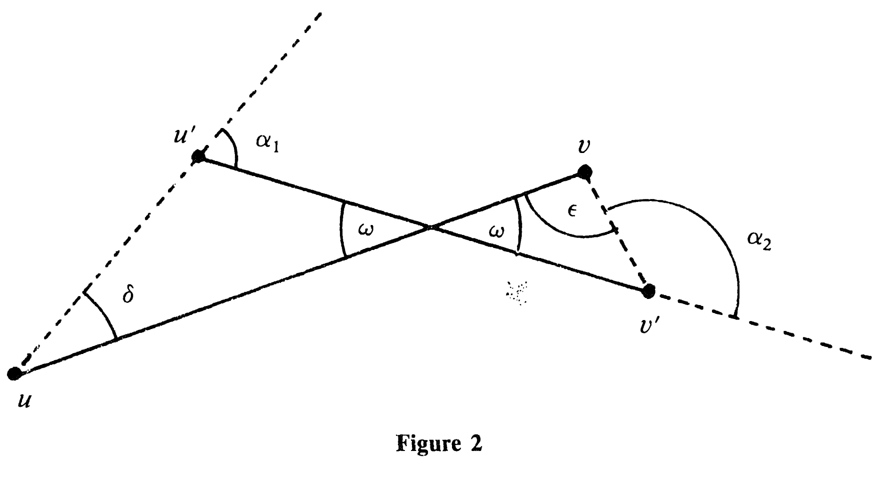

Figure 2

$$\beta = \text{ang}\,(T^-(v), C(u, v))$$
$$\beta' = \text{ang}\,(T^-(v), C(v', v))$$
$$\epsilon = \text{ang}\,(C(v', v), C(u, v)).$$

Using the angle triangle inequality 1.1.*a* we obtain $\alpha' \geqslant \alpha - \delta$ and $\beta' \geqslant \beta - \epsilon$. Now, according to 4.2

$$\begin{aligned} T(u, v) &\geqslant \alpha' + \alpha_1 + \alpha_2 + \beta' \\ &\geqslant \alpha - \delta + \alpha_1 + \alpha_2 + \beta - \epsilon = \alpha + \omega + \omega + \beta, \end{aligned}$$

so that $\Xi(u, v) \geqslant 2\omega > 0$. This shows that γ is convex when Ξ is zero.

The converse is easy since all polygonal lines inscribed in a convex curve satisfy $\alpha' + \beta' = \alpha_1 + \alpha_1 + \cdots + \alpha_n$,

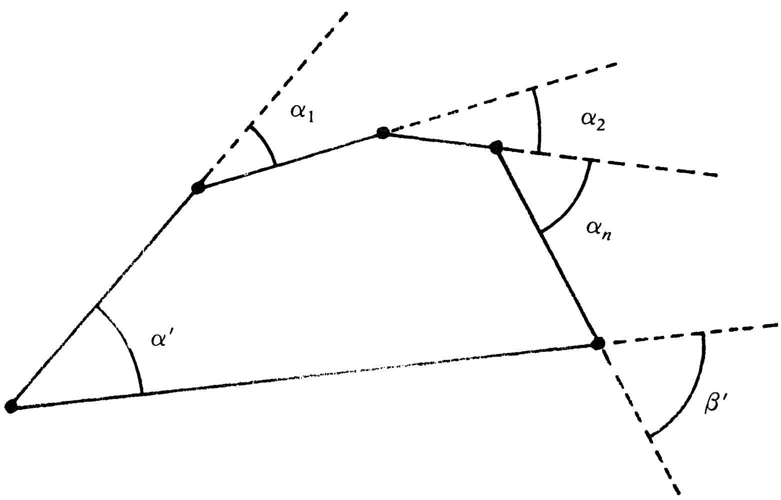

Figure 3

and taking limits we get $T = E$, or $\Xi = 0$.

In view of 5.1.3 and 5.2 it may be true that functions in general euclidean spaces whose graphs have finite total curvature can be written as differences of functions with $\Xi = 0$; we know no proof of this.

6. The non-Hilbert Case

Let X be a real Banach space, S the unit sphere $\|x\| = 1$ and δ the *geodesic distance* on S: $\delta(x, y) = \inf l_S(\sigma)$ where σ ranges over all continuous curves in

S joining x and y and l_S denotes the length of the curve σ,

$$l_S(\sigma) = \operatorname{Sup} \sum \|\sigma(t_{i+1}) - \sigma(t_i)\|.$$

It is well known that ([6]):

$$\|x - y\| \leqslant \delta(x, y) \leqslant 2\|x - y\|$$

for any pair $x, y \in S$.

Of course, in a Hilbert space $\delta(x, y) = \operatorname{ang}(x, y)$ and the inequalities are trivial consequences of 3.3. This suggests that we define, for a curv X and a partition $\Pi = \{t_0, t_1, \ldots\}$, the number $T(\Pi)$ by $T(\Pi) = \sum \delta(C_{i-}$ where $C_j = C(t_j, t_{j+1})$ and, as above,

$$C(u, v) = (\gamma(v) - \gamma(u))/\|\gamma(v) - \gamma(u)\| \in S.$$

In the same way, we set $T(\gamma) = \operatorname{Sup} T(\Pi)$, and define curves of *finite cur* by the property $T(\gamma) < +\infty$. With these definitions, all the results in Sec hold true without changing their proofs.

For the results in Section 3 the situation is different. In fact, the are on 3.1, 3.2 and 3.3. Now 3.3 is valid in any Banach space with $\pi/2$ re by 2, as observed above, which does not affect the use it is made of 3. on. Also, 3.2 is a corollary of 3.1. Thus, only 3.1 has to be checked f validity of all results in Section 3.

It turns out however that 3.1 holds for some Banach spaces and do hold for others (see below), so it may be thought of as too restrictive is not the case if the monotonicity of T with respect to partitions (3.4*a*) i sidered a natural condition. In fact we have:

6.1. Proposition. *Let X be a Banach space. Then the following pro are equivalent*:

6.1.1. *For any curve γ in X, $T(\Pi)$ increases when more partition poi added to Π.*

6.1.2. *The equality $\delta(U, V) + \delta(V, W) = \delta(U, W)$ holds for any U, V S with $U = -W$ or $V = pU + qW$ for some $p, q \geqslant 0$.*

6.1.3. *For any U, V in with $U + V \neq 0$, the function*

$$\sigma(t) = ((1 - t)U + tV)/\|(1 - t)U + tV\|$$

defines a curve with minimal length joining U and V, i.e., $l_S(\sigma) = \delta($

PROOF. First let us see that 6.1.1 is equivalent to

6.1.4. *For any U, W, V, R and Z in S satisfying $V = pR + qZ$ for son* $p, q \geqslant 0$, *the inequality*

6.1.5. $$\delta(U, V) + \delta(V, W) \leqslant \delta(U, R) + \delta(R, Z) + \delta(Z, W)$$

holds.

Assume 6.1.1 and consider a curve γ satisfying $\gamma(t_0) = 0$, $\gamma(t_1) = U$, $\gamma($ $= U + dpR$, $\gamma(t_3) = U + dV$, $\gamma(t_4) = U + dV + eW$ for a partition $\Pi = \{$ $t_1, t_2, t_3, t_4\}$ of its domain, where $0 < d \leqslant 1$ and $0 < e \leqslant 1$ are convenient chosen to avoid selfintersections (we are also assuming that $V \neq R$ and $V \neq$ since in either case 6.1.5 follows from the triangle inequality). If $\Pi_1 = \{t_0,$ $t_3, t_4\}$ then $T(\Pi_1)$ and $T(\Pi)$ are equal, respectively, to the left and right ha side terms of 6.1.5.

Conversely, if 6.1.4 holds it is easy to see that for any curve and any pa of partitions Π_1 and Π with Π having one more point than Π_1 we ha $T(\Pi_1) \leqslant T(\Pi)$. An induction argument finishes the proof.

Setting $R = U$ and $Z = W$ in 6.1.5 and using the triangle inequality we g 6.1.2; conversely, from 6.1.2 we obtain $\delta(R, Z) = \delta(R, V) + \delta(V, Z)$ a using twice the triangle inequality we get 6.1.5. This shows that 6.1.2 a 6.1.4 are equivalent.

We prove that 6.1.3 implies 6.1.2: let

$$\sigma(t) = ((1 - t)U + tW)/\|(1 - t)U + tW\|$$

and denote by σ_1 and σ_2 the restrictions of σ that join U to V and V to respectively. We have, using 6.1.3,

$$\delta(U, W) = l_S(\sigma) = l_S(\sigma_1) + l_S(\sigma_2) = \delta(U, V) + \delta(V, W).$$

To prove the converse, pick a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such th (with σ as above):

$$l_S(\sigma) - \epsilon \leqslant \sum \|\sigma(t_{i+1}) - \sigma(t_i)\|.$$

From 6.1.2,

$$\sum \|\sigma(t_{i+1}) - \sigma(t_i)\| \leqslant \sum \delta(\sigma(t_i), \sigma(t_{i+1})) = \delta(\sigma(0), \sigma(1)) = \delta(U, V)$$

so that, ϵ being arbitrary, $l_S(\sigma) \leqslant \delta(U, V)$ whence $l_S(\sigma) = \delta(U, V)$ and 6.1 follows. Thus 6.1.1, 6.1.3 and 6.1.4 are all equivalent.

6.2. Theorem. *All properties of T stated in Sections 2 through 5 are va in Banach spaces having the equivalent properties of 6.1.*

6.3. *Remark.* Hilbert spaces and two-dimensional Banach spaces have tl equivalent properties of 6.1. It is possible that these are the only ones, but v know no proof of this fact. In support of this observe that such spaces ha\ the following property, not hard to obtain from 6.1.3: all curves of the for: $S \cap V$, where V is a two-dimensional subspace of X, have the *same* length. S(also [9] for related notions. Finally we observe that $\mathbb{R}^3$ with the nor: $(x^2 + y^2)^{1/2} + |z|$ is a Banach space where 6.1.1 (and then also 6.1.2 and 6.1.. fails.

7. Related Concept

Let γ be a plane curve given in polar coordinates by $\gamma(\theta) = r(\theta)(\cos\theta, \sin 6$ $0 \leqslant \theta \leqslant \alpha < 2\pi$, where $r(\theta) > 0$ is a continuous function.

7.1. Proposition. *Consider the following properties*:

- (a) *There exists $s > 0$ such that for each $0 \leqslant \theta \leqslant \alpha$ and each point z in t, plane satisfying $\|z\| \leqslant s$, the line segment joining z to $\gamma(\theta)$ meets t, curve only at $\gamma(\theta)$ («interior cone condition»).*
- (a') *$r(\theta)$ is a Lipschitz function.*
- (b) *$r(\theta)$ is a function of bounded variation.*
- (b') *γ is a rectifiable curve.*
- (c) *γ has finite total curvature.*

Then:

7.1.1. (a) *and* (a') *are equivalent.*

7.1.2. (b) *and* (b') *are equivalent.*

7.1.3. (a) *implies* (b) *and* (c) *implies* (b).

7.1.4. *All other implications fail in general.*

PROOF. (a') $\Rightarrow$ (a) is proved in [6], 7.1, and (a) $\Rightarrow$ (a') is proved in [11]; tl settles 7.1.1.

Assume now that γ is rectifiable. Then $r(\theta)\cos\theta$ and $r(\theta)\sin\theta$ are functio of bounded variation, which is equivalent to $r = (r^2 \cos^2\theta + r^2 \operatorname{sen}^2\theta)^{1/2}$ bei: of bounded variation (use $r \geqslant \min r > 0$ and the differentiability of squa root away from 0). The converse is just as easy, so that 7.1.2 is proved.

Next (a') $\Rightarrow$ (b) and (c) $\Rightarrow$ (b') (see 3.9) so that 7.1.3 follows.

We finish the proof with three examples: first, let γ_1 be the curve whos graph is the cusp $|x|^{1/2} + y^{1/2} = 1$, $-1 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$. Next, let γ_2 be th curve described by the figure 4

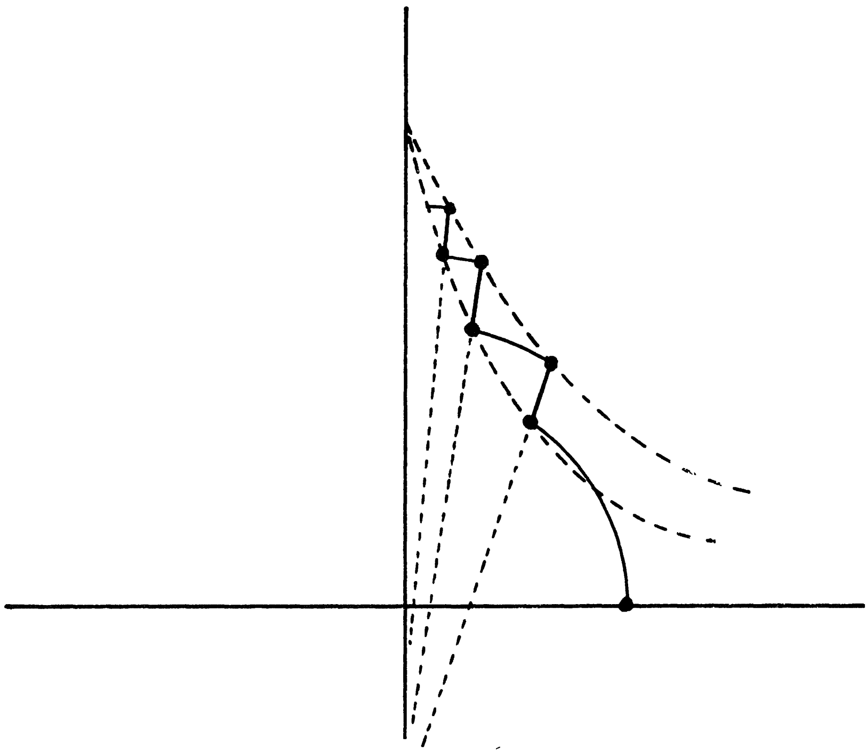

Figura 4

Here r increases from $\theta = 0$ to $\theta = \pi/2$. Finally, let $P_1, P_2, P_3, \ldots$ be sequence on $x^2 + y^2 = 1$ converging orderly to $(0, 1)$, and let γ_3 be the cur obtained by joining P_j to P_{j+1} with the broken line formed by tangents to th circle $x^2 + y^2 = 1/4$ (see the figure 5 in the next page).

It is not hard to see that (*b*) holds for γ_1, but (*a'*) and (*c*) fail; (*c*) holds f γ_2 but (*a*) fails, and (*a*) holds for γ_3 but (*c*) fails. This completes the proof 7.1.

Remark. Observe that 7.1 improves the statement (*a*) + (*c*) $\Rightarrow$ (*a'*) $\Rightarrow$ (proved in [6].

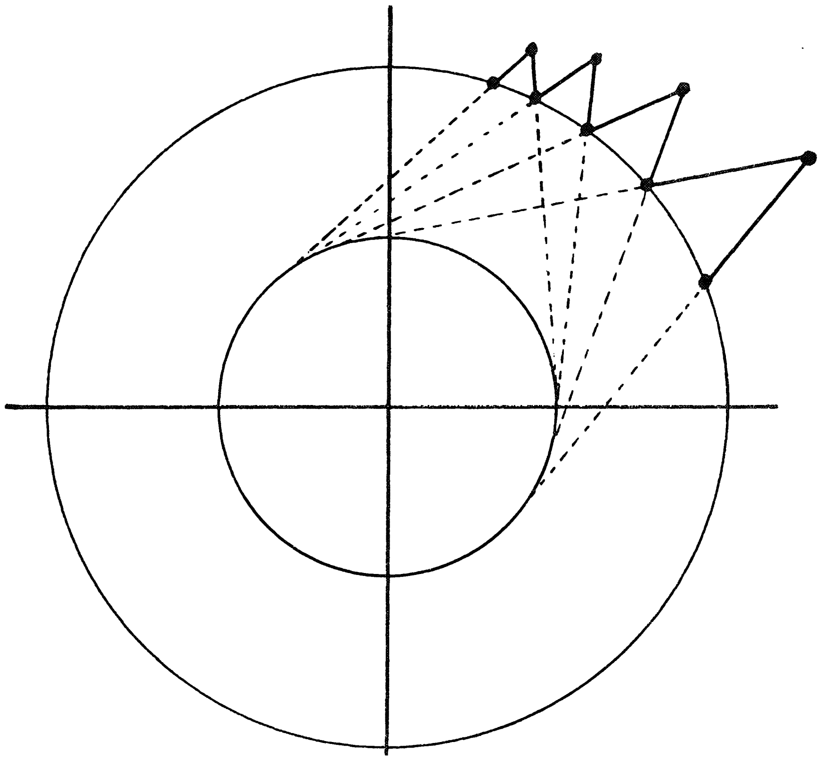

Figure 5

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