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Hankel Forms and the Fock Space

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Abstract

We consider Hankel forms on the Hilbert space of analytic functions square integrable with respect to a given measure on a domain in $\mathbb{C}^n$. Under rather restrictive hypotheses, essentially implying «homogeneity» of the set-up, we obtain necessary and sufficient conditions for boundedness, compactness and belonging to Schatten classes S_p, $p \geqslant 1$, for Hankel forms (analogues of the theorems of Nehari, Hartman and Peller). There are several conceivable notions of «symbol»; choosing the appropriate one, these conditions are expressed in terms of the symbol of the form belonging to certain weighted L^p-spaces.

Our theory applies in particular to the Fock spaces (defined by a Gaussian measure in $\mathbb{C}^n$). For the corresponding L^p-spaces we obtain also a lot of other results: interpolation (pointwise, abstract), approximation, decomposition etc. We also briefly treat Bergman spaces.

A specific feature of our theory is that it is «gauge invariant». (A gauge transformation is the simultaneous replacement of functions f by $f\phi$ and $d\mu$ by $|\phi|^{-2}\,d\mu$, where ϕ is a given (non-vanishing) function). For instance, in the Fock case, an interesting alternative interpretation of the results is obtained if we pass to the measure $\exp(-y^2)\,dx\,dy$. In this context we introduce some new function spaces E_p, which are Fourier, and even Mehler invariant.

0. Introduction

0.1. Background. By a *Hankel form* we will in this paper informally refer to any (continuous) bilinear form H defined on a Hilbert space $\mathcal{H}$ of analytic functions (usually consisting of (all) functions square integrable with respect to a given measure μ; cf. *infra* §0.2) such that its value $H(f, g)$ for any $f, g \in \mathcal{H}$ only depends on the product $f \cdot g$. In particular, one then has the functional equation

$$H(\phi f, g) = H(f, \phi g)$$

where ϕ is any (analytic) multiplier on $\mathcal{H}$.

Example 0.1. In the case of the usual Hardy class $\mathcal{H} = H^2(\mathbb{T})$ ($\mathbb{T}$ = unit circle) the Hankel for $H_b^{\mathbb{T}}$ with symbol b is defined by

$$H_b^{\mathbb{T}}(f, g) = \frac{1}{2\pi}\int_{\mathbb{T}} \bar{b} fg|dz| = \langle fg, b\rangle_{H^2(\mathbb{T})}$$

In the canonical basis $\{z^j\}_{j\geq 0}$ it is given by the *Hankel matrix* $(\hat{b}(i + j))_{i,j\geq 0}$.

For the (classical) theory of Hankel forms in this case, highlighted by a number of agenda such as the issue of

finite rank (Kronecker)
boundedness (Nehari)
compactness (Hartman)
belonging to Schatten-von Neumann class (Peller),

we refer to, Sarason (1978), Power (1980, 1982*a*), Nikol'skiĭ (1985, 1986), Nikol'skiĭ and Peller (1987?).

Usually, though, one formulates the results for *operators*, not forms. With the form H one can associate the Hankel operator $\tilde{H}$ defined by

$$H(f, g) = \langle f, \tilde{H}g\rangle. \tag{0.1}$$

Notice that $\tilde{H}$ is an anti-linear operator in $\mathcal{H}$. To get a linear operator one combines $\tilde{H}$ with a conjugation; e.g. on $\mathbb{T}$ one usually considers $f \mapsto \overline{\tilde{H}f}$ with the range $\overline{H^2(\mathbb{T})}$, or a variant with range $H^2_-(\mathbb{T})$.

For various reasons we prefer to work with bilinear forms instead. For instance, this «zwanglos» suggests the extension of our theory to the *multilinear* case (§5).

An easy extension of the $H^2(\mathbb{T})$-theory concerns the space $B_s^2(D)$ ($s < 0$; D unit disc, $\partial D = \mathbb{T}$) defined by the condition

$$\frac{1}{\pi}\int_D |f(z)|^2(1 - |z|^2)^a\, dm(z) < \infty \qquad (a = -1 - 2s > -1),$$

where the letter B may at will be read as Bergman on Besov (see Peller (1982), Peetre (1983, 1984, 1985) and, for an extension to the case of the unit ball in several complex variables, Ahlmann (1984) and Burbea (1986)); one also writes $A^{a,2}(D)$ for the same spaces. Actually, already on this level the theory bifurcates according to (speaking of (linear) operators) whether one wants the range to be $\overline{B_s^2(D)}$ or $B_s^2(D)^{\perp}$. Here we will only be concerned with the first alternative. (The study of Hankel operators of the second species —which do *not* correspond to Hankel forms in our sense —was initiated only recently by Axler (1985) and then further pursued in Arazy, Fisher and Peetre (1986)).

See also the works of Luecking (1985) and Zhu (1985) for *Toeplitz* operators in Bergman space. (Some remarks in the case of general (homogeneous) domains are further made in Arazy and Upmeier (1985)).

As a formal limiting case ($a \to -1$) of the spaces $B_s^2(D)$ one recaptures the previous Hardy class $H^2(\mathbb{T})$ (the normalized 2-dimensional measure $(a+1)\cdot(1-|z|^2)^a\,dm(z)/\pi$ over D tends to the 1-dimensional measure $|dz|/2\pi$ concentrated on $\mathbb{T}$).

Another limiting case ($a \to \infty$) deals with the Fock space $F_\alpha^2(\mathbb{C})$ ($\alpha > 0$) defined by the condition

$$\frac{\alpha}{\pi}\int_{\mathbb{C}} |f(z)|^2 e^{-\alpha|z|^2}\,dm(z) < \infty.$$

(If one writes the definition for the B-spaces for a concentric disc of radius R then the weight factor becomes $[1-|z|^2/R^2]^a$. If now $a = \alpha R^2$ and $R \to \infty$ we formally get the weight $e^{-\alpha|z|^2}$).

The number α plays a role similar to Planck's constant in physics.

Remark 0.1. Besides Fock, other names occasionally are attached to this spaces, viz. Bargmann-Segal, Fisher and possibly others. The same is true for Bergman spaces (see e.g. Dzhrbashyan (1983)), so perhaps a more appropriate appelation, without digging too deeply into the history of the subject, would have been spaces of Bargmann-Besov-Bergman-Dzhrbashyan-Fisher-Fock-Segal type.

Toeplitz operators in Fock space are considered in Berger and Coburn (1985), (1986?) and Berger, Coburn and Zhu (1985).

0.2. Main Results (General Theory, §§1-6, 14). The aim of the present work is to develop a theory of Hankel forms over quite general (in practise «homogeneous») domains, which comprises both the Bergman and the Fock case (the other limiting case of the Hardy class being *excluded*) and this in any number of dimensions (a few results for the Fock space being formally valid also in the physically most interesting case of dimension ∞, see §7). As there is in general no boundary (and no Besov spaces) one has to proceed differently then before. Note that *potentially* our theory is applicable to a much broader range (including arbitrary symmetric domains and various limiting cases).

More precisely, we consider the following set-up. Let Ω be a domain in $\mathbb{C}^n$ and as in the beginning of §0.1, let $\mathcal{H}$ be a Hilbert space of analytic functions now defined on Ω. If ξ is a positive measure on Ω, which we, for simplicity, assume to be absolutely continuous with respect to the Lebesgue measure m on Ω, we say that a Hankel form H defined on $\mathcal{H}$ has *symbol b with respect to ξ* if (with a convenient interpretation of the integral, if the latter is not absolutely convergent; cf. §6)

$$H(f, g) = \int_\Omega \bar{b} f g \, d\xi \qquad (f, g \in \mathcal{H}),$$

notation:

$$H = H_b^\xi.$$

The point is that a form may have several (interesting) symbols with respect to different measures and to some extent our theory is about the interplay between various symbols.

In most of the discussion we *fix* once and for all one such measure (fulfilling the assumption V0 stated in Section 1) and take $\mathcal{H} = A^2(\mu)$, the subspace of $L^2(\mu)$ consisting of all square integrable (with respect to μ), analytic functions on Ω. Clearly $A^2(\mu)$ is a Hilbert space with a reproducing kernel denoted by $K(z, w)$ or $K_w(z)$. We let P denote the orthogonal («Bergman») projection of $L^2(\mu)$ onto $A^2(\mu)$ and we further set

$$\omega(z) = \frac{1}{K(z, z)}.$$

It then turns out to be advantageous to take symbols not with respect to μ, but with the *associated* measure ν defined as

$$d\nu = \omega(z) \, d\mu = \frac{d\mu}{K(z, z)}.$$

We will in the sequel use the notations

$$\Gamma_b = H_b^\nu, \qquad H_b = H_b^\mu.$$

We further let L and Q denote the reproducing kernel in the Hilbert space $A^2(\nu)$ and the corresponding projection, respectively.

Remark 0.2. For a general measure we similarly have the Hilbert space $A^2(\xi)$ with a reproducing kernel K^ξ and the projection P^ξ. We will use these concepts only for $\xi = \mu$ or ν, where μ and ν are as above. We summarize the special notations used for these cases in the form of a table. (The notation A^2_ω will be explained in (0.3) below).

The Hankel forms always act in $A^2(\mu)$.

Measure	*Hilbert space*	*Kernel*	*Projection*	*Hankel form*
ξ (general)	$A^2(\xi)$	K^ξ	P^ξ	H_b^ξ
μ (fixed)	$A^2(\mu)$	K	P	H_b
ν (associated)	A_ω^2	L	Q	Γ_b

We occasionally write $\Gamma(b)$ for Γ_b, etc.

To get a reasonable theory one has to introduce some supplementary assumptions V1, V2 and V3 (see §3). The most severe of these is V2 which amonts to requiring that

$$L(z, w) = \varkappa K(z, w)^2 \tag{0.2}$$

where $\varkappa$ is a constant.

Before stating our main result (*infra*) we need one more concept, the natural scale of *weighted* L^p and A^p-spaces pertinent to our situation. We say that f is in L_ω^p iff

$$\int_\Omega |f|^p \omega^{p-2} \, d\nu < \infty \tag{0.3}$$

(f is in L_ω^∞ if and only if ωf is essentially bounded on Ω), and let A_ω^p be the subspace of L_ω^p consisting of analytic functions in L_ω^p. Let a_ω^∞ be the closure of A_ω^2 in the A_ω^∞-metric.

We can now announce:

Scholium 0.1. *Under the assumptions* V0 – V3 *the following is true.*

(*a*) Γ_b *is bounded (in* $A^2(\mu)$*) if and only if* $Qb \in A_\omega^\infty$.
(*b*) Γ_b *is compact if and only if* $Qb \in a_\omega^\infty$.
(*c*) Γ_b *is in* S_p, *where* $1 < p < \infty$, *if and only if* $Qb \in A_\omega^p$.

The Schatten-von Neumann classes of bilinear forms S_p (where in general $0 < p \leqslant \infty$) are discussed in Sub-Section 0.3. Some other comments are in order.

Comment 0.1. From this it is in principle easy to get results for general symbols, because H_b^ξ has the symbol $b\, d\xi/d\nu$ with respect to ν, $H_b^\xi = \Gamma_{b d\xi/d\nu}$. This is discussed in §6. Notice also that $H_b^\xi = H_{P^\xi b}^\xi$, so that in many cases it is natural to confine oneself to analytic symbols.

Comment 0.2. We expect part (*c*) of the Scholium to be true also in the range $0 < p < 1$ but this we have not been able to show.

Comment 0.3. The proofs can be found in §4, where also other results can be found, especially pertaining to the «Hankel projection». A crucial step is however taken already in §3, where the boundedness of the projection Q in L^p_ω, $1 \leqslant p \leqslant \infty$, is proved.

Our assumptions, in particular the crucial hypothesis V2, are fulfilled in all cases when the situation admits sufficiently many «automorphisms». This can in principle be found in the literature, but of course not in the Hankel context. We refer especially to Selberg (1957), Stoll (1977) and Inoue (1982). In particular, our theory applies in the B-case (the group is the Möbius group PSU (1,1)), see §§12-13, and in the F-case (the group is now its «contraction», the Heisenberg group).

We do not know of any other cases than homogeneous domains with highly symmetrical measures when the assumption V2 *is fulfilled.*

However, there is a deeper reason for the appearance of the strange looking hypothesis as condition V2 relating the square of the kernel K to the kernel L: Namely, that the whole set-up admits certain «supersymmetries», here termed *gauge transformations*. Let us briefly indicate what this is about.

Consider, quite generally, a closed subspace $\mathcal{H}$ of $L^2(\Omega, \mu)$, where Ω is some space equipped with a positive measure μ. We argue that we get an essentially equivalent theory if we simultaneously replace f by ϕf and μ by $|\phi|^{-2}\mu$, where ϕ is any non-vanishing (measurable) function. This is gauge transformation or change of gauge. The point is that one should work only with gauge invariant quantities. (A related point of view can be found e.g. in the works of Berezin (see e.g. Berezin (1975) for a start), but also elsewhere). Especially in our case (confining ourselves to analytic ϕ's), the («given») kernel K transforms according to the rule $K(z, w) \to \phi(z)\overline{\phi(w)}K(z, w)$, where as the («associated») kernel L experiences the change $L(z, w) \to \phi(z)^2\overline{\phi(w)^2}L(z, w)$ (see §§1 and 3). Thus V2 is a gauge invariant condition. Similarly, our preference for the Hankel operator Γ_b with symbol taken with respect to the measure ν (and not μ, as would seem natural at the first glance) is explained by the fact that Γ_b is gauge invariant (with the symbol transforming $b \to \phi^2 b$).

Note also that the measure λ defined as

$$d\lambda(z) = K(z, z)\, d\mu(z)$$

has a gauge invariant meaning; in all group theoretic cases it reduces to the usual invariant measure, in the very special case of the unit disc thus to a constant multiple of the Poincaré measure $(1 - |z|^2)^{-2}\, dm(z)$.

Finally, let us mention that we also prove a very general Kronecker theorem (concerning the structure of finite rank Hankel forms). This is basically an excercise in commutative algebra (sic!) and has little to do with the rest of the paper so it has been relegated to the end of the paper, more or less as an appendix (§14).

0.3. Schatten-von Neumann classes of bilinear forms. The Schatten-von Neumann classes (or trace ideals) S_p, $0 < p < \infty$, of (bounded) operators in Hilbert space have been studied extensively (see e.g. McCarthy (1967), Gohberg and Krein (1965), Simon (1979) and, as far as interpolation goes, Bergh and Löfström (1976)). To define the same classes for bilinear forms there are several (equivalent) avenues.

(*a*) Via operators (cf. Peetre (1985)). If H is a bilinear form on $\mathcal{H}_1 \times \mathcal{H}_2$, then $\tilde{H}$ defined by

$$\tilde{H}(g): f \to H(f, g) \tag{0.4}$$

is a linear operator from $\mathcal{H}_2$ into $\mathcal{H}_1^*$. (The natural, anti-linear, identification of $\mathcal{H}_1^*$ and $\mathcal{H}_1$ yields the anti-linear operator from $\mathcal{H}_2$ into $\mathcal{H}_1$ defined by (0.1)). We say that H is in S_p if and only if $\tilde{H}$ is in S_p, i.e. if and only if the positive operator $(\tilde{H}^*\tilde{H})^{p/2}$ has finite trace. We define S_∞ to be the space of all bounded bilinear forms (operators). (Some authors prefer to let S_∞ denote the compact operators).

One can also associate with H a linear operator $\tilde{\tilde{H}}$: $\mathcal{H}_2 \to \mathcal{H}_1$ doing the same job, but not in a canonical way. Indeed, if J: $\mathcal{H}_1 \to \mathcal{H}_1$ is any conjugation on $\mathcal{H}_1$ (J is antilinear with $J^2 = Id$), then J defines a *linear* isometry of $\mathcal{H}_1^*$ onto $\mathcal{H}_1$ (which we also denote by J) and we can take $\tilde{\tilde{H}} = J \circ \tilde{H}$. Notice in particular that $\tilde{\tilde{H}}^*\tilde{\tilde{H}} = \tilde{H}^*\tilde{H}$ independently of J. (If $\{h_{ij}\}$ is the matrix of H with respect to some orthonormal bases in $\mathcal{H}_1$ and $\mathcal{H}_2$, then this operator has the matrix $\{b_{ik}\}$ with $b_{ik} = \sum h_{ji}\overline{h_{jk}}$).

Remark 0.3. For some spaces there is a natural choice of J, e.g. if $\mathcal{H}_1 = \bar{\mathcal{H}}_1$ (say $\mathcal{H}_1 = L^2(\mu)$), $Jf = \bar{f}$ and if $\mathcal{H}_1$ is a suitable Hilbert space of analytic functions in the unit disc (or the complex plane) $Jf(z) = \overline{f(\bar{z})}$.

(*b*) Directly using s-numbers (Schmidt, approximation). Put

$$s_n(H) = \inf \|H - F\|_\infty \tag{0.5}$$

where $\|\bullet\|_\infty$ is the supremum norm and F runs through the set of all forms of finite rank $\leqslant n$. We say that H is in S_p if and only if $(s_n(H))_{n=0}^\infty \in l_p$, $0 < p \leqslant \infty$. (Note that H is compact if and only if $(s_n(H))_{n=0}^\infty \in c_0$).

0.4. Hankel forms of class S_2 (Hilbert-Schmidt). To give the reader at least a feel what it all is about we now briefly outline a direct treatment of the S_2 theory.

The Hankel form $\Gamma(L_z)$ with symbol L_z with respect to ν is

$$(f, g) \to \int \bar{L}_z fg\, d\nu = fg(z) = f(z)g(z) = \langle f, K_z\rangle\langle g, K_z\rangle \tag{0.6}$$

(This is a continuous form of rank 1, and belongs thus to every S_p). Thus

$$\langle \Gamma(L_z), \Gamma(L_w)\rangle_{S_2} = \langle K_w, K_z\rangle\langle K_w, K_z\rangle = K(z, w)^2.$$

If now

$$L(z, w) = \varkappa K(z, w)^2, \qquad \varkappa > 0,$$

it follows that

$$\varkappa\langle \Gamma(L_z), \Gamma(L_w)\rangle = L(z, w) = \langle L_w, L_z\rangle, \tag{0.7}$$

whence $b \to \varkappa^{1/2}\Gamma_b$ is an anti-linear isometry of $A^2(\nu)$ into S_2. Conversely, if $b \to \varkappa^{1/2}\Gamma_b$ is an isometry of $A^2(\nu)$ into S_2 for some $\varkappa$, then the argument above shows that

$$L(z, w) = \langle L_w, L_z\rangle = \varkappa K(z, w)^2.$$

This is closely related to the criterion by Aronszajn (1950), Theorem 8 II, p. 361, for $L = K^2$.

0.5. Contents. Results for Fock and Bergman spaces (§§7-13). Again for the benefit of the reader we pass to a more detailed description of the contents of the individual divisions, including an explicit mention of the main results in the Fock and Bergman cases.

Section 1 sets forth some basic material connected with Hilbert spaces with a reproducing kernel (for a more detailed treatment we refer to Aronszajn (1950)).

In the analytic case we also state the basic assumption V0 (p. 74).

In Section 2 we study the reproducing kernels when there are sufficiently many automorphisms. The main result is Theorem 2.1 proving the aforementioned condition V2 in such cases.

In Section 3 we introduce the assumptions V1-V4 (pp. 80-81) and we study the «Bergman» projection Q, especially establishing its boundedness in the full scale L^p_ω, $1 \leqslant p \leqslant \infty$ (Theorem 3.1). This result has a number of important corollaries (Cor. 3.1-3.8).

Section 4 is devoted to the study of Hankel forms in the general context of the assumption V0-V3 and we establish in particular all the results which above were summarized in Scholium 0.1.

In §5 the extension to the multilinear case is briefly treated. As far as we know, no theory is yet developed for S_p-classes of multilinear forms. Here we propose to define S_p, $1 < p < \infty$, using interpolation between S_1 and S_∞; in the two latter cases the definition is unambiguous.

§6 gives various complements to the previous discussion (§§1-5). In particular we discuss a more general definition of symbols (hitherto the defining integral was taken to be absolutely convergent) and consider also symbols with respect to a general measure ξ.

We also establish the minimality of A^1_ω in a certain sense, and prove a weak factorization result for A^1_ω.

In §7 we begin the study of Fock space proper. It then turns out to be natural to study the whole family of measures

$$d\mu_\alpha = (\alpha/\pi)^n e^{-\alpha|z|^2}\, dm(z) \qquad (\alpha > 0),$$

on $\mathbb{C}^n$, letting L^p_α $(1 \leqslant p \leqslant \infty)$ to the space of measurable functions f such that $f(z)e^{-\alpha|z|^2/2} \in L^p(m)$ and F^p_α be its analytic subspace, denoting the correspondingly projection by P_α. More precisely, we consider the action of Hankel forms on some fixed Hilbert space F^2_α but take symbols with respect to an arbitrary measure $d\mu_\beta$. The main result is Theorem 7.5 (= an almost immediate convergence of the results in §§1-6 in the «abstract» case; cf. Scholium 0.2 *infra*).

We turn also the reader's attention to Theorem 7.8, which gives an exact result (not just a norm equivalence), and thus is potentially susceptible to an extension to infinitely many variables. This is however only for the special powers $p = 2$ and $p = 4$ and why this is so is a tantalizing question we do not quite understand.

In §8 we go on studying the spaces F^p_α and especially establish decomposition approximation and interpolation (pointwise, not abstract interpolation!) descriptions.

We interrupt at this junctune the exposition by the collecting the results for Hankel forms on the Fock space F^2_α as a Scholium (for those who like many equivalent conditions). For simplicity we state them in terms of the symbol taken with respect to the measure $d\mu_{2\alpha}$ which corresponds to the associated measure $d\nu$ in the general case (§4), when $d\mu = d\mu_\alpha$. We thus consider the Hankel form $H^{2\alpha}_b$ given by

$$H^{2\alpha}_b(f, g) = \int_{\mathbb{C}^n} \bar{b} f g \, d\mu_{2\alpha}$$

acting in the Hilbert space F^2_α. Let k^α_w be the normalized reproducing kernel in F^2_α, viz.

$$k^\alpha_w(z) = e^{\alpha\langle z, w\rangle - \alpha|w|^2/2},$$

and use the notation $k^{2\alpha}_w$ in the same sense.

Scholium 0.2. *The following are equivalent for* $1 \leqslant p \leqslant \infty$ *and any entire function* b.

(i) $H^{2\alpha}_b \in S_p$.

(ii) $H_b^{2\alpha}(f,g) = \sum \lambda_i \langle f, k_i^\alpha\rangle\langle g, k_i^\alpha\rangle$ *where* $\{\lambda_i\} \in l^p$ *(and* $k_i^\alpha = k_{w_i}^\alpha$ *for a suitable sequence* $\{w_i\}$, *separation condition etc.).*

(iii) $b = \sum \lambda_i k_i^{2\alpha}$ *with* $\{\lambda_i\} \in l^p$ *(same qualifications for* $k_i^{2\alpha}$*).*

(iv) $b \in P_{2\alpha}(L_{2\alpha}^p)$.

(v) $\{d_N\}_0^\infty \in l^p$ *where* $d_N = \inf\{\|b-g\|_{F_{2\alpha}^\infty} : g \in P_N\}$, P_N *being the set of linear combinations* $\sum_{j=1}^N a_j k_j^{2\alpha}$ *of length N.*

(vi) $b \in F_{2\alpha}^p$.

In §9 we first investigate for which values of the parameters involved the projection P_α is bounded as a map from L_β^p into F_γ^p. (Answer: The n and s condition is $\alpha^2/\gamma > 2\alpha - \beta$). This improves on an old result of Sjögren's (1976), who was interested when P_α maps $L^p(\mu_\alpha)$ into $L^q(\mu_\alpha)$. (Answer: $q < 4/p$, or $p = q = 2$). It is also connected with a duality result (Theorem 9.2):

$$(F_\beta^p)^* \cong F_{\alpha^2/\beta}^{p'},$$

in the duality induced by the inner product in F_α^2. We also study the (complex) interpolation of the 2-parameter family F_α^p (Theorem 9.3). It is somewhat surprising but at second thought quite understandable that the parameter α interpolates «logarithmically» (Theorem 9.4):

$$[F_{\alpha_0}^{p_0}, F_{\alpha_1}^{p_1}]_\theta = F_\alpha^p \quad \text{if} \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \alpha = \alpha_0^{1-\theta}\alpha_1^\theta.$$

It is an interesting (*open*) question to determine the spaces which arise by *real* interpolation from this scale. This can for *p fixed* be rephrased as a problem about spectral analysis for the dilation operator $D_\delta : f(z) \to f(\delta z)$ $(0 < \delta < 1)$ in the space F_α^p.

Example 0.2. Let

$$f(z) = \sum_{n=0}^\infty \frac{(az)^n}{n!(1 + t\delta^n)}$$

where a is complex, $t > 0$ and δ fixed, $0 < \delta < 1$. Is it true that

$$\iint_{\mathbf{C}} |f(z)| e^{-|z|^2/2}\, dm(z) = 0(e^{|a|^2/2}),$$

with a constant *independent* of t? If this were the case we could prove that D_δ is a «positive» operator so the usual Grisvard type machinery can be set at work (see e.g. Triebel (1977), Section 1.14).

§10 is likewise devoted to Fock space and treats various left-overs from the previous sections.

In §11 we treat Fock space in a different gauge (from the group representation point of view this is something half way in between the Bargman-Segal representatition thriving on $F^2_\alpha(\mathbb{C}^n)$ and the Heisenberg representation acting in $L^2(\mathbb{R}^n)$). In this connection we are led to introduce some new function (distribution) spaces E_p, whose definition formally reminds of the use of Besov spaces, only that the convolution parameter enters in an additive way ($f \in E_p \Leftrightarrow \phi_y * f(\cdot) \in L^p(L^p)$ where $\hat{\phi}_y(\xi) = \hat{\phi}(\xi + y)$ and ϕ is a «test» function) and have the conspicuous property of being *Fourier*, and even *Mehler invariant*. Indeed, it turns out that they are special cases of more general spaces known as *modulation spaces* and studied by Feichtinger (see, e.g. Feichtinger (1981*a*), (1981*b*) and the discussion in remark 11.3).

The following two sections (§§12 and 13) are devoted to B space theory. In §12 we spell out our results in the case of weighted Bergman spaces on the complex unit ball (the «Rudin» ball). In §13 again we make changes of variables and gauge and consider the case of the upper half plane, but only for $n = 1$. (This is really a pity, for the case $n > 1$ when one thus has a Siegel domain of the second kind (a generalized upper half plane) should be susceptible to potentially interesting considerations. Cf. Gindikin (1964)).

Finally, as already recorded at the end of §0.2, we give in §14 our general Kronecker result.

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Note. (added Jan. 1988). In two loose appendices (written in the spring of 1987), for which the middle author alone is responsible, we indicate some further developments after the main body of the paper was completed (June 1986).

1. Reproducing Kernels

In this section we collect some elementary, presumably well-known results which will be used later. We begin with a very general setting, see Aronszajn (1950).

Let $\mathcal{H}$ be a Hilbert space of functions on some set Ω such that the point evaluations $f \to f(z)$ are continuous linear functionals on $\mathcal{H}$ for all $z \in \Omega$. Then there exist unique functions $K_z \in \mathcal{H}$, $z \in \Omega$, such that

$$f(z) = \langle f, K_z \rangle, \qquad f \in \mathcal{H} \quad \text{and} \quad z \in \Omega, \tag{1.1}$$

and we define the reproducing kernel as the function

$$K(z, w) = K_w(z), \qquad (z, w) \in \Omega^2. \tag{1.2}$$

The definitions (1.1) and (1.2) yield (for $z, w \in \Omega$)

$$K(z, w) = K_w(z) = \langle K_w, K_z \rangle. \tag{1.3}$$

Consequently,

$$K(w, z) = \overline{K(z, w)} \tag{1.4}$$

$$K(z, z) = \|K_z\|^2 \geqslant 0 \tag{1.5}$$

$$|K(z, w)|^2 \leqslant K(z, z)K(w, w) \tag{1.6}$$

$$|f(z)| \leqslant \|f\| \, \|K_z\| = K(z, z)^{1/2} \|f\|. \tag{1.7}$$

Furthermore,

$$K(z, z) = 0 \Leftrightarrow K_z = 0 \Leftrightarrow f(z) = 0 \quad \text{for every} \quad f \in \mathcal{H}. \tag{1.8}$$

If $\{\phi_\alpha\}$ is an ON-basis in $\mathcal{H}$, then

$$\begin{aligned} K(z, w) = K_w(z) &= \sum_\alpha \langle K_w, \phi_\alpha \rangle \phi_\alpha(z) \\ &= \sum_\alpha \overline{\langle \phi_\alpha, K_w \rangle} \phi_\alpha(z) \\ &= \sum_\alpha \phi_\alpha(z) \overline{\phi_\alpha(w)}. \end{aligned}$$

(The sums converge absolutely).

Finally we note that the linear span of $\{K_z\}$ is dense in $\mathcal{H}$, because no non-zero function is orthogonal to every K_z.

We next impose additional structures on Ω.

Continuity. If Ω is a topological space and every function in $\mathcal{H}$ is continous, then $K(w, z)$ is separately continuous (because of $K_z \in \mathcal{H}$ and (1.4)), but not necessarily continuous. (Counterexamples are easily constructed, but we leave that to the reader).

Proposition 1.1. *If every function in $\mathcal{H}$ is continuous, then the following are equivalent.*

(i) *$(z, w) \to K(z, w)$ is continuous*;
(ii) *$z \to K(z, z)$ is continuous*;
(iii) *$z \to K_z$ is continuous (mapping Ω into $\mathcal{H}$).*

PROOF. (i) $\Rightarrow$ (ii). Trivial.

(ii) $\Rightarrow$ (iii). Fix z. If $w \to z$ then $K(w, w) \to K(z, z)$ by (ii) and $K(w, z) = K_z(w) \to K_z(z) = K(z, z)$ because $K_z \in \mathcal{H}$.

Hence, using (1.3),

$$\begin{aligned} \|K_w - K_z\|^2 &= \langle K_w, K_w \rangle + \langle K_z, K_z \rangle - 2\,\mathrm{Re}\,\langle K_z, K_w \rangle \\ &= K(w, w) + K(z, z) - 2\,\mathrm{Re}\,K(w, z) \\ &\to 0. \end{aligned} \tag{1.10}$$

(iii) $\Rightarrow$ (i). Immediate by (1.3). □

L^2-spaces. In the remainder of this section we assume that μ is a measure on Ω and that $\mathcal{H}$ is a closed subspace of $L^2(\mu)$ such that the point evaluations are continuous on $\mathcal{H}$. (Note that the functions in $\mathcal{H}$ thus are defined everywhere although functions in $L^2(\mu)$ are defined only a.e.).

Let P denote the orthogonal projection $L^2(\mu) \to \mathcal{H}$. Then, if $f \in L^2(\mu)$ and $z \in \Omega$, by $K_z \in \mathcal{H}$, (1.2) and (1.4),

$$\begin{aligned} Pf(z) = \langle Pf, K_z \rangle = \langle f, K_z \rangle = \int_\Omega f(w)\overline{K_z(w)}\, d\mu(w) = \\ = \int_\Omega K(z, w) f(w)\, d\mu(w). \end{aligned} \tag{1.11}$$

Change of gauge. Let ϕ be a non-zero measurable function on Ω and consider the map

$$f \to \phi f, \qquad \mu \to |\phi|^{-2}\mu, \tag{1.12}$$

which maps $L^2(\mu)$ isometrically onto $\phi L^2(\mu) = L^2(|\phi|^{-2}\mu)$ and $\mathcal{H}$ onto the subspace $\phi\mathcal{H} = \{f: \phi^{-1} f \in \mathcal{H}\}$ of $L^2(|\phi|^{-2}\mu)$.

This map, which we call a change of gauge, obviously gives an isomorphic theory. It will later be important to see how various entities transform.

Proposition 1.2. *The reproducing kernel for $\phi\mathcal{H}$ is $\phi(z)\overline{\phi(w)}K(z, w)$.*

PROOF. E.g. by (1.9), since $\{\phi\phi_\alpha\}$ is an ON-basis in $\phi\mathcal{H}$. □

Corollary 1.1. *The measure $K(z, z)\, d\mu(z)$ is invariant under all changes of gauge.*

PROOF. A change of gauge transforms $d\mu(z) \to |\phi(z)|^{-2}\, d\mu(z)$ by definition. □

Change of variables. Let Ψ be a bijection of Ω onto Ω'. Then Ψ maps μ onto $\mu \circ \Psi^{-1}$, and the map $f \to f \circ \Psi^{-1}$ maps $\mathcal{H}$ isometrically onto

$$\mathcal{H} \circ \Psi^{-1} \subset L^2(\mu) \circ \Psi^{-1} = L^2(\mu \circ \Psi^{-1}).$$

Proposition 1.3. *The reproducing kernel for* $\mathcal{H} \circ \Psi^{-1}$ *is* $K(\Psi^{-1}(z), \Psi^{-1}(w))$, $z, w \in \Omega'$. □

This triviality will be useful in conjunction with a simultaneous change of gauge in the next section.

Analytic functions. In the remainder of the paper (except in §14), we make the following assumptions, for future references denoted V0.

V0: *Ω is a connected open subset of $\mathbb{C}^n$ and μ is an absolutely continuous measure on Ω with continuous, strictly positive Radon-Nikodym derivative $d\mu/dm$ (m is the Lebesgue measure).*

Our basic Hilbert space is the space $A^2(\mu) = \mathcal{H}(\Omega) \cap L^2(\mu)$, i.e. the space of square integrable analytic functions. ($\mathcal{H}(\Omega)$ is the Frechet space of all analytic functions in Ω. It is easily seen that $A^2(\mu)$ is a closed subspace of $L^2(\mu)$ and that point evaluations are continuous; in fact, the embedding $A^2(\mu) \to \mathcal{H}(\Omega)$ is continuous). We let K denote the reproducing kernel in $A^2(\mu)$; all previous considerations of this section apply. (In the special case $\mu = m$, K is known as the Bergman kernel (in Ω)).

We will henceforth only consider analytic changes of gauge and analytic changes of variables, and note that they preserve our setting; e.g. if ϕ is analytic and non-zero, then $\phi A^2(\mu) = A^2(|\phi|^{-2}\mu)$.

Proposition 1.4. *$K(z, w)$ is continuous on $\Omega \times \Omega$, analytic in z and anti-analytic in w.*

PROOF. $K(z, w) = K_w(z)$ is analytic in z because $K_w \in A^2(\mu)$. By (1.4), $K(z, w)$ then is anti-analytic in w. Hence $K(z, \bar{w})$ is analytic in each variable on $\Omega \times \bar{\Omega}$ and thus, by Hartogs' theorem, analytic, in particular continuous. □

Corollary 1.2. *Proposition* 1.1 *yields that $z \to K_z$ is a continuous map of Ω into $A^2(\mu)$.* □

We next prove that K is determined by its restriction to the diagonal and the properties above.

Proposition 1.5. *Suppose that $J(z, w)$ is analytic in z and anti-analytic in w on $\Omega \times \Omega$ and that $J(z, z) = K(z, z)$, $z \in \Omega$. Then $J(z, w) = K(z, w)$.*

PROOF. We may assume that $0 \in \Omega$. The function $f(z, w) = J(z, \bar{w}) - K(z, \bar{w})$ is analytic, and $f(z, \bar{z}) = 0$ in a neighborhood of 0. Hence $f = 0$, see e.g. Bochner and Martin (1948), Chapter II, Theorem 7. □

2. Symmetries

Let $\Omega \subset \mathbb{C}^n$ and μ be as in V0 (see above). In §§3-6, we will impose further restrictions on Ω and μ. These restrictions seem very restrictive and we guess that the theory developed there only covers very special cases. The purposes of the present section is to show that at least highly symmetric cases, such as the Fock and Bergman spaces, are covered. Our main results extend results by Selberg (1957), Stoll (1977) and Inoue (1982). A general reference to the theory of automorphism groups is Narasimhan (1971), Chapters 5 and 9.

Let Aut (Ω) denote the group of analytic bijections of Ω onto itself. This group is too large for our purposes, while the subgroup of maps that leave μ invariant is too small (and has the further defect of not being gauge invariant). Instead, we study the subgroup of maps that leave μ invariant modulo an analytic change of gauge.

Definitions. $G(\mu)$ is the set of all $\gamma \in \mathrm{Aut}(\Omega)$ such that, for some analytic function ϕ on Ω,

$$\mu \circ \gamma^{-1} = |\phi|^2 \mu. \tag{2.1}$$

(Cf. (1.12). Since necessarily $\phi \neq 0$, we may here replace ϕ^{-1} by ϕ).

$$G^*(\mu) = \{(\gamma, \phi) \in \mathrm{Aut}(\Omega) \times \mathcal{H}(\Omega) \colon \mu \circ \gamma^{-1} = |\phi|^2 \mu\}.$$

$G^*(\mu)$ is a group with the natural group law

$$(\gamma, \phi) \circ (\delta, \psi) = (\gamma \circ \delta, \phi \cdot (\psi \circ \gamma^{-1}));$$

$G(\mu)$ is a subgroup of Aut (Ω) and a quotient group of $G^*(\mu)$.

Remark 2.1. ϕ is determined by (2.1) up to a unimodular constant. Hence $G^*(\mu)$ is an extension of $G(\mu)$ by $\mathbb{T}$. A unitary representation of $G^*(\mu)$ in $A^2(\mu)$ is defined by

$$R_{(\gamma,\phi)} f(z) = \phi(z) f(\gamma^{-1}(z)). \tag{2.2}$$

Remark 2.2. For the Fock spaces (§§7-11), $G(\mu)$ is strictly smaller than Aut (Ω), while $G(\mu) = \mathrm{Aut}(\Omega)$ for the Bergman spaces (§§12-13), and for any domain Ω when μ is the Lebesgue measure (let ϕ in (2.1) be the Jacobian of γ^{-1}).

Proposition 2.1. *If* $(\gamma, \phi) \in G^*(\mu)$, *then*

$$K(\gamma^{-1}(z), \gamma^{-1}(w)) = \phi(z)^{-1} \overline{\phi(w)}^{-1} K(z, w), \qquad z, w \in \Omega. \tag{2.3}$$

Proof. Immediate by propositions 1.2 and 1.3, since the change of gauge induced by ϕ^{-1} and the change of variables induced by γ map $A^2(\mu)$ onto the same space, and thus they transform K into the same kernel. □

Corollary 2.1. *The measure $K(z, z)\, d\mu(z)$ is invariant for all $\gamma \in G(\mu)$.* □

Corollary 2.2. *$|K(z, w)|^2/K(z, z)K(w, w)$ is a $G(\mu)$-invariant function of $(z, w) \in \Omega \times \Omega$.* □

Transitivity. We say that $G(\mu)$ is transitive if for every $z, w \in \Omega$ there exists $\gamma \in G(\mu)$ with $\gamma(z) = w$.

Lemma 2.1. *If $G(\mu)$ is transitive and $A^2(\mu) \neq \{0\}$, then $K(z, z) \neq 0$ for all $z \in \Omega$.*

Proof. Otherwise, by Proposition 2.1, $K(z, z) = 0$ for every $z \in \Omega$, which contradicts (1.7). □

Theorem 2.1. *Suppose that $G(\mu)$ is transitive and $A^2(\mu) \neq \{0\}$. Let r be an integer and let ν be the measure $K(z, z)^{-r}\, d\mu(z)$. Denote the reproducing kernel for $A^2(\nu)$ by L. Then, for some constant $c_r \geqslant 0$,*

$$L(z, w) = c_r K(z, w)^{r+1}. \tag{2.4}$$

In other words, if f is analytic and $f \in L^2(K(z, z)^{-r}\, d\mu)$,

$$f(z) = c_r \int_\Omega f(w) \frac{K(z, w)^{r+1}}{K(w, w)^r}\, d\mu(w). \tag{2.5}$$

Furthermore, $G(\nu) \supset G(\mu)$.

Proof. Let $\gamma \in G(\mu)$ and choose ϕ such that (2.1) holds. Then, using (2.3)

$$\frac{d\nu \circ \gamma^{-1}}{d\nu}(z) = \frac{K(\gamma^{-1}(z), \gamma^{-1}(z))^{-r}}{K(z, z)^{-r}} \frac{d\mu \circ \gamma^{-1}}{d\mu}(z)$$
$$= |\phi|^{2r}|\phi|^2 = |\phi^{r+1}|^2.$$

Since ϕ^{r+1} is analytic, γ belongs to $G(\nu)$. Corollary 2.1 now shows that γ preserves the measures $K(z, z)\, d\mu(z)$ and $L(z, z)\, d\nu(z)$ and thus the Radon-Nikodym derivative

$$\frac{L(z, z)\, d\nu(z)}{K(z, z)\, d\mu(z)} = L(z, z)K(z, z)^{-r-1}.$$

Hence this function is left invariant by every $\gamma \in G(\mu)$, and, since $G(\mu)$ is transitive, it has to be a constant, c_r say, i.e.

$$L(z, z) = c_r K(z, z)^{r+1}.$$

The proof is completed by Proposition 1.5. □

Remark 2.3. If Ω is simply connected, the theorem holds for any real number r. In particular, $K(z, w)^r$ is well-defined unless $c_r = 0$, i.e. unless $A^2(\nu) = \{0\}$. (The Lu Qi-keng conjecture states that $K(z, w) \neq 0$ for any simply connected domain (with the Lebesgue measure), cf. Lu Qi-keng (1966), Skwarczynski (1969)).†

Remark 2.4. A similar argument shows that the group of μ-invariant automorphisms (i.e. those with $\phi \equiv 1$ in (2.1)) is transitive only in trivial cases. ($K(z, z)$ has to be constant, whence $K(z, w)$ is constant and $A^2(\mu) = \{0\}$ or $\mathbb{C}$. We do not know whether $A^2(\mu) = \mathbb{C}$ actually is possible). Note also the related fact (valid without any assumptions on $G(\mu)$) that $A^2(K(z, z)\, d\mu) = 0$ or $\mathbb{C}$, the latter case occuring if and only if $A^2(\mu)$ has finite dimension. (Sketch of proof. It follows from (1.9) that if $f \in A^2(K(z, z)\, d\mu)$, then $M_f : g \to fg$ defines a Hilbert-Schmidt operator in $A^2(\mu)$. Thus, the spectrum of M_f is discrete which implies that f is constant).

Isotropy. Define, for $z \in \Omega$, $G(\mu)_z = \{\gamma \in G(\mu) : \gamma(z) = z\}$. In this subsection we assume that $G(\mu)_z$ is large enough, more precisely:

> There exists a compact group H with $H \subset G(\mu)_z$ such that $(\gamma, z) \to \gamma(z)$ is continuous $H \times \Omega \to \Omega$ and that the only H-invariant analytic functions on Ω are the constant functions. (2.6)

We let $d\gamma$ denote the normalized Haar measure on H.

Lemma 2.2. *Assume that* $z \in \Omega$ *is such that* (2.6) *holds. Then, for any* $f \in \mathcal{H}(\Omega)$ *and* $w \in \Omega$,

$$\int_H f(\gamma(w))\, d\gamma = f(z).$$

PROOF. The integral defines an analytic H-invariant function of w, and is thus independent of w. Choosing $w = z$ we obtain

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† *Added May* 1987. After the above was written we have been told that the Lu-Gineng conjecture has been settled by Harold Boas.

$$\int f(\gamma(z))\,d\gamma = f(z). \qquad \square$$

The next lemma may be compared to the Lu Qi-keng conjecture in Remark 2.3.

Lemma 2.3. *Suppose that* $z \in \Omega$ *and that* $K(z, z) \neq 0$ *and that* (2.6) *holds. Then* $K(z, w) \neq 0$ *for all* $w \in \Omega$.

PROOF. Suppose on the contrary that $K(w, z) = 0$ for some w. By proposition 2.1,

$$K_z(\gamma(w)) = K(\gamma(w), z) = K(\gamma(w), \gamma(z)) = 0 \quad \text{for all} \quad \gamma \in H \subset \Gamma(\mu)_z.$$

Lemma 2.2 with $f = K_z$ yields

$$K(z, z) = K_z(z) = \int K_z(\gamma(w))\,d\gamma = 0,$$

a contradiction. $\square$

We may now extend the reproducing formula to functions outside $A^2(\mu)$.

Theorem 2.2. *Suppose that* $z \in \Omega$ *is such that* (2.6) *holds and* $K(z, z) \neq 0$. *If* f *is an analytic function such that*

$$\int_\Omega |K(z, w) f(w)|\,d\mu(w) < \infty,$$

then

$$\int_\Omega K(z, w) f(w)\,d\mu(w) = f(z).$$

PROOF. Corollary 2.2 implies that $|K(z, w)|^2/K(w, w)$ is a $G(\mu)_z$-invariant function of w. This and Corollary 2.1 imply that $|K(z, w)|^2\,d\mu(w)$ is a $G(\mu)_z$-invariant measure. Consequently, if $g = f/K_z$ (which is analytic by Lemma 2.3), then for any $\gamma \in H \subset G(\mu)_z$,

$$\int K(z, w) f(w)\,d\mu(w) = \int g(w) K(w, z) K(z, w)\,d\mu(w) = \int g(\gamma(w)) |K(z, w)|^2\,d\mu(w).$$

Integrating over H, we obtain by Fubini's theorem, Lemma 2.2 and (1.4)-(1.5),

$$\begin{aligned}
\int_\Omega K(z, w) f(w)\,d\mu(w) &= \int_H \int_\Omega g(\gamma(w)) |K(z, w)|^2\,d\mu(w)\,d\gamma \\
&= \int_\Omega \int_H g(\gamma(w))\,d\gamma |K(z, w)|^2\,d\mu(w) \\
&= \int g(z) |K_z(w)|^2\,d\mu(w) \\
&= g(z) \|K_z\|^2 \\
&= g(z) K(z, z) \\
&= f(z). \quad \square
\end{aligned}$$

Proper actions and invariant measures. We say that a topological group $G \subset \mathrm{Aut}(\Omega)$ acts properly on Ω if the action $\gamma(z)$ is continuous $G \times \Omega \to \Omega$ and the map $G \times \Omega \to \Omega \times \Omega$, $(\gamma, z) \to (\gamma(z), z)$ is proper. If G acts properly, then its topology coincides with the compact-open topology. $\mathrm{Aut}(\Omega)$ with the compact-open topology is a topological group, but is does not always act properly.

A related question concerns the existence of G-invariant metrics (defining the usual topology) on Ω. In fact, if such a metric exists and G is a closed subgroup of $\mathrm{Aut}(\Omega)$, then G acts properly, see van Dantzig and van der Waerden (1928) and Kaup (1967).

Now assume that

$$K(z, z) \neq 0 \quad \text{for every} \quad z \in \Omega.$$

Then the Bergman (pseudo)metric (with respect to μ) is defined as the Riemannian (pseudo) metric with the infinitesimal from

$$ds^2 = \sum_{ij} \frac{\partial^2 \log K(z, z)}{\partial z_i \, \partial \bar{z}_i} dz_i \, \overline{dz_j}, \tag{2.7}$$

cf. Bergman (1950), Chapter IX.3.

The form (2.7) is positive semidefinite (the proof of Kobayashi (1959), Theorem 3.1, holds verbatim in our situation too) and is positive definite if and only if

$$\{\mathrm{grad} f(z) : f \in A^2(\mu) \text{ and } f(z) = 0\} = \mathbb{C}^n. \tag{2.8}$$

For example, if $\int (1 + |z|^2)\, d\mu < \infty$, then all affine functions belong to $A^2(\mu)$, whence (2.8) is satisfied for every $z \in \Omega$ and the Bergman metric is a metric.

Furthermore the form (2.7) is invariant under (analytic) changes of coordinates and changes of gauge; hence (2.7) and the Bergman metric are $G(\mu)$-invariant.

We are now prepared to show that (1.7) can be improved to

$$f(z) = o(K(z, z)^{1/2})$$

in some cases, cf. Kobayashi (1959), Section 9.

Theorem 2.3. *Assume that $G(\mu)$ is transitive and that* (2.8) *holds for some* (*and thus all*) $z \in \Omega$. *Then, for every $f \in A^2(\mu)$,*

$$f(z)/K(z, z)^{1/2} \in C_0(\Omega). \tag{2.9}$$

PROOF. It suffices to prove (2.9) when $f = K_w$, $w \in \Omega$, because of (1.7) and the fact that these functions span a dense subspace of $A^2(\mu)$. Assume thus, in order to achieve a contradiction, that $w \in \Omega$ and that (2.9) fails for $f = K_w$. Then there exists a sequence $\{z_n\} \subset \Omega$ that is not relatively compact such that $\inf_n K_w(z_n)/K(z_n, z_n)^{1/2} > 0$. Thus, for some $\delta > 0$ and every n,

$$|K(z_n, w)| > 2\delta K(w, w)^{1/2} K(z_n, z_n)^{1/2}. \tag{2.10}$$

Let $k_z(w) = K_z(w)/\|K_z\| = K(w, z)/K(z, z)^{1/2}$. Then $\|k_z\| = 1$ and

$$\langle k_z, k_w \rangle = \frac{\langle K_z, K_w \rangle}{\|K_z\| \, \|K_w\|} = \frac{K(w, z)}{(K(z, z)K(w, w))^{1/2}}.$$

Let $A = \{z \in \Omega : \|k_z - k_w\| < \delta\}$. Since $z \to k_z$ is continuous, A is open. Choose $\gamma_n \in G(\mu)$ such that $\gamma_n w = z_n$. Then, if $z \in A$, using Corollary 2.2,

$$|\langle k_w, k_{\gamma_n^{-1} z} \rangle| = |\langle k_{\gamma_n w}, k_z \rangle| \geqslant |\langle k_{z_n}, k_w \rangle| - \|k_z - k_w\| > 2\delta - \delta = \delta.$$

Consequently, by Corollary 2.1, for every n,

$$\int_{\gamma_n^{-1}A} |k_w(z)|^2 \, d\mu(z) = \int_{\gamma_n^{-1}A} |\langle k_w, k_z \rangle|^2 K(z, z) \, d\mu(z) > \int_A \delta^2 K(z, z) \, d\mu(z) > 0.$$

Now, let B be a compact subset of Ω such that

$$\int_{\Omega \setminus B} |k_w|^2 \, d\mu < \delta^2 \int_A K(z, z) \, d\mu(z).$$

Then $B \cap \gamma_n^{-1} A \neq \emptyset$ for every n. Since the Bergman metric is $G(\mu)$-invariant, and $G(\mu)$ is a closed subgroup of $\operatorname{Aut}(\Omega)$, $G(\mu)$ acts properly by the result referred to above. Hence $\{\gamma : \gamma A \cap B \neq 0\}$ is compact, whence $\{\gamma_n\}$ and $\{z_n\} = \{\gamma_n w\}$ are relatively compact, a contradiction. □

Remark 2.5. The assumptions of Theorem 2.3 imply also that $G(\mu)$ is a real Lie group, and Ω thus a homogenous space, cf. e.g. Kobayashi (1959).

3. The Bergman Projection

We assume that Ω and μ satisfy the basic condition V0 in Section 1 and furthermore:

V1: If $z \in \Omega$ then $f(z) \neq 0$ for some $f \in A^2(\mu)$.

Equivalently, $K(z, z) > 0$ for $z \in \Omega$.

We introduce, as in §0.2, additional notations and assumptions which will be used in the remainder of the paper (except Section 14).

Definitions. λ and ν are the measures given by

$$d\lambda(z) = K(z, z)\, d\mu(z), \tag{3.1}$$

$$d\nu(z) = K(z, z)^{-1}\, d\mu(z). \tag{3.2}$$

$L(z, w)$ is the reproducing kernel for $A^2(\nu)$. Q is the projection $L^2(\nu) \to A^2(\nu)$. (K and P denote as before the corresponding objects for $A^2(\mu)$). ω is the function

$$\omega(z) = K(z, z)^{-1}. \tag{3.3}$$

L^p_ω, $1 \leqslant p \leqslant \infty$, is the weighted L^p-space $\{f: \omega f \in L^p(\lambda)\}$ with the obvious norm, and A^p_ω is the subspace of analytic functions.

Note that λ is the invariant measure of Corollaries 1.1 and 2.1, and that $L^1_\omega = L^1(\mu)$ and $L^2_\omega = L^2(\nu)$, whence $A^2_\omega = A^2(\nu)$.

We wish to stress that the spaces L^p_ω are the natural L^p-spaces to consider in our setting, and not the differently weighted spaces $L^p(\nu)$. (For example, the results for L^p_ω in this section do not hold for $L^p(\nu)$, see Section 9).

It is easily seen that under the (analytic) change of gauge (1.12), $\nu \to |\phi|^{-4}\nu$, $L(z, w) \to \phi(z)^2\overline{\phi(w)}^2 L(z, w)$, $\omega \to |\phi|^{-2}\omega$ and $L^p_\omega \to \phi^2 L^p_\omega$, $A^p_\omega \to \phi^2 A^p_\omega$. Hence the transformation $f \to \phi^2 f$ («of weight 2») operates on L^p_ω and A^p_ω (in particular, on $A^2(\nu)$).

We make two additional basic assumptions. Presumably, the first is very restrictive while the second is more technical. Both assumptions are gauge invariant.

V2: $L(z, w) = \varkappa(K(z, w))^2$ for some constant $\varkappa \geqslant 0$.

V3: If f is analytic on Ω and

$$\int |L(z, w) f(w)|\, d\nu(w) < \infty$$

for every z, then

$$\int L(z, w) f(w)\, d\nu(w) = f(z), \qquad z \in \Omega. \tag{3.4}$$

At a few places we need a further assumption.

V4: If $f \in A^2(\mu)$, then $f(z)/K(z, z)^{1/2} \in C_0(\Omega)$.

(It suffices that this holds when $f = K_w$, $w \in \Omega$, because of (1.7) and density. Hence V4 is equivalent to $|K(z, w)|^2/K(z, z)K(w, w) \in C_0(\Omega)$ for every fixed w).

We will always let $\varkappa$ denote the constant in V2; it will appear in various norm estimates.

If V0 and V1 hold, then $K_z^2 \in A^2(\nu)$ because, by (1.6) and (1.5),

$$\begin{aligned}\int |K_z|^4\,d\nu &\leqslant \int |K_z(w)|^2 K(z,z)K(w,w)\,d\nu(w)\\ &= K(z,z)\int |K_z(w)|^2\,d\mu(w) \qquad (3.5)\\ &= K(z,z)^2 < \infty\end{aligned}$$

Hence (1.8) implies that $L(z,z) > 0$ and thus, if V2 too holds, $\varkappa > 0$. In fact, by (1.5) and (3.5), then

$$L(z,z) = \|L_z\|^2_{L^2(\nu)} = \varkappa^2 \int |K_z|^4\,d\nu \leqslant \varkappa^2 K(z,z)^2 = \varkappa L(z,z),$$

and thus $\varkappa \geqslant 1$ (with equality iff $A^2(\mu)$ is one-dimensional).

Remark 3.1. An inspection of the proofs below shows that in most places we could replace V2 by the weaker $L_z \in A^2(\mu) \hat{\otimes} A^2(\mu)$ with norm bounded by $\varkappa K(z,z)$ for each z. However, we do not know of any example that satisfies this condition but not V2. (Cf. the Appendices, written much later).

We collect the main results of Section 2.

Proposition 3.1. *Suppose that* V0 *holds and* $A^2(\mu) \neq \{0\}$. *Suppose further that* $G(\mu)$ *is transitive and that* (2.6) *holds for some* $z \in \Omega$. *Then* V1, V2 *and* V3 *hold. If furthermore* (2.8) *holds for some* $z \in \Omega$, *then* V4 *holds too.*

PROOF. V1 and V2 follow by Lemma 2.1 and Theorem 2.1. Since $G(\mu)$ is transitive, (2.6) holds for every z. Since $G(\mu) \subset G(\nu)$ by Theorem 2.1 and, as was shown above, $L(z,z) > 0$, V3 follows by Theorem 2.2 applied to ν and L. V4 follows from (2.8) by Theorem 2.3. □

This proposition gives us the only non-trivial examples satisfying V0 – V3 that we know. After these preliminaries, we show that the «Bergman» projection Q can be extended to L^p_ω for any $p \in [1,\infty]$. Note that this contrasts to the classical case of $H^2(\mathbb{T})$, where the analytic projection is a bounded operator in L^p for $1 < p < \infty$, but not for $p = 1$ or $p = \infty$. Recall (cf. (1.11)) that if $f \in L^2_\omega = L^2(\nu)$,

$$Qf(z) = \int_\Omega L(z,w) f(w)\,d\nu(w). \qquad (3.6)$$

We use this formula to extend the domain of Q.

Theorem 3.1. *Suppose that* V0 – V3 *hold. Then*

(*a*) (3.6) *defines* Q *as a bounded linear operator* $L^1_\omega + L^\infty_\omega \to A^\infty_\omega$,
(*b*) Q *is a bounded linear projection of* L^p_ω *onto* A^p_ω *for every* p, $1 \leqslant p \leqslant \infty$.

PROOF. If $f \in L^1_\omega$, then, by V2 and (1.6),

$$\begin{aligned}\int |L(z,w)f(w)|\,d\nu(w) &= \varkappa \int |K(z,w)^2 f(w)|\omega(w)^2\,d\lambda(w)\\ &\leqslant \varkappa \int K(z,z)|f(w)|\omega(w)\,d\lambda(w) \qquad (3.7)\\ &= \varkappa K(z,z)\|f\|_{L^1_\omega}.\end{aligned}$$

Hence $Qf(z)$ is well defined and $Qf \in L^\infty_\omega$. Next we observe that, by V2 and (1.5),

$$\begin{aligned}\int |L(z,w)|\omega(w)\,d\lambda(w) &= \varkappa \int |K(w,z)|^2\omega(w)\,d\lambda(w)\\ &= \varkappa \int |K_z|^2\,d\mu = \varkappa K(z,z)\end{aligned} \qquad (3.8)$$

Hence, if $f \in L^\infty_\omega$,

$$\begin{aligned}\int |L(z,w)f(w)|\,d\nu(w) &= \int |f(w)\omega(w)|\,|L(z,w)|\omega(w)\,d\lambda(w)\\ &\leqslant \|f\|_{L^\infty_\omega}\varkappa K(z,z),\end{aligned} \qquad (3.9)$$

whence $Qf \in L^\infty_\omega$.

It follows that Q maps $L^1_\omega + L^\infty_\omega$ into L^∞_ω. Furthermore, by Morera's theorem (using Fubini's theorem and the estimates (3.7) and (3.9)), Qf is analytic if $f \in L^1_\omega + L^\infty_\omega$, i.e. $Qf \in A^\infty_\omega$. This proves (*a*). We have proved in (3.9) that $Q: L^\infty_\omega \to L^\infty_\omega$. Dually, if $f \in L^1_\omega$, then by (3.8),

$$\begin{aligned}\|Qf\|_{L^1_\omega} &\leqslant \iint |L(z,w)f(w)|\,d\nu(w)\omega(z)\,d\lambda(z)\\ &= \iint |L(w,z)|\omega(z)\,d\lambda(z)|f(w)|\,d\nu(w) \qquad (3.10)\\ &= \varkappa \int |f(w)|K(w,w)\,d\nu(w)\\ &= \varkappa\|f\|_{L^1_\omega}.\end{aligned}$$

Hence $Q: L^1_\omega \to L^1_\omega$.

By interpolation, $Q: L^p_\omega \to L^p_\omega$ for every $p \in [1,\infty]$. Since Qf is analytic for any $f \in L^p_\omega \subset L^1_\omega + L^\infty_\omega$ by (*a*), $Q: L^p_\omega \to A^p_\omega$. Finally, V3, (3.7) and (3.9) show that if $f \in L^1_\omega + L^\infty_\omega$ is analytic (in particular, if $f \in A^p_\omega$ for some $p \in [1,\infty]$), then $Qf = f$. □

Remark 3.2. The proof shows that the norm of Q as an operator in L^p_ω, $1 \leqslant p \leqslant \infty$, is at most $\varkappa$. It is easily seen that this norm equals $\varkappa$ for $p = 1$ and $p = \infty$. On the other hand, when $p = 2$ the norm is 1. Interpolation yields better estimates for $p \neq 1, 2, \infty$, but these estimates are presumably not sharp. The norm is strictly greater that 1 for any $p \neq 2$ (unless $A^2(\mu)$ is one-dimensional), since otherwise a result of Strichartz (1986) would imply that the norm would equal 1 for all p, $1 \leqslant p \leqslant \infty$, which contradicts the fact that the norm for $p = 1$ is $\varkappa > 1$.

Corollary 3.1. *If* $1 \leqslant p \leqslant q \leqslant \infty$, *then* $A^p_\omega \subset A^q_\omega$.

PROOF. If $f \in A^p_\omega$ then $f = Qf \in L^p_\omega \cap L^\infty_\omega \subset L^q_\omega$. □

Corollary 3.2. *The spaces* A^p_ω, $1 \leqslant p \leqslant \infty$, *interpolate as expected for the real and complex methods*:

$$[A^{p_0}_\omega, A^{p_1}_\omega]_\theta = (A^{p_0}_\omega, A^{p_1}_\omega)_{\theta p_\theta} = A^{p_\theta}_\omega, \quad \textit{where} \quad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}\cdot \quad \Box$$

It is obvious that, if $1 \leqslant p < \infty$, $(L^p_\omega)^* = L^{p'}_\omega$ $(1/p + 1/p' = 1$ as usual) with the pairing $\langle \omega f, \omega g\rangle_\lambda = \langle f, g\rangle_\nu$.

Corollary 3.3. *Q is self-adjoint in the sense that if* $f \in L^p_\omega$ *and* $g \in L^{p'}_\omega$, $1 \leqslant p \leqslant \infty$, *then*

$$\langle Qf, g\rangle_\nu = \langle f, Qg\rangle_\nu. \tag{3.11}$$

PROOF. By (1.4) and Fubini's theorem, justified by (3.7) and (3.9). □

Corollary 3.4. *If* $1 \leqslant p < \infty$, *then* $(A^p_\omega)^* \cong A^{p'}_\omega$ *with the pairing* $\langle \quad \rangle_\nu$. □

Corollary 3.5. *The linear span of* $\{L_z\}$ *is a dense subspace of* A^p_ω *for every* p, $1 \leqslant p < \infty$.

PROOF. $L_z \in A^1_\omega \subset A^p_\omega$ by (3.8) and Corollary 3.1. If $g \in (A^p_\omega)^* = A^{p'}_\omega$ is orthogonal to every L_z, then $g(z) = Qg(z) = \langle g, L_z\rangle = 0$ for every z. □

Corollaries 3.4 and 3.5 fail for $p = \infty$, but we have the following substitute. Define a^∞_ω as the closed linear span of $\{L_z\}$ in A^∞_ω.

Corollary 3.6. *If* $1 \leqslant p < \infty$, *then* $A^p_\omega \subset a^\infty_\omega$ *densely.* $(a^\infty_\omega)^* \cong A^1_\omega$ *with the pairing* $\langle \quad \rangle_\nu$.

PROOF. The first assertion follows by Corollaries 3.1 and 3.5. Thus, if $\chi \in (a^\infty_\omega)^*$ there exists $g \in (A^2_\omega)^* = A^2_\omega$ such that $\chi(f) = \langle f, g\rangle_\nu$ for every $f \in A^2_\omega \subset a^\infty_\omega$. Hence, if $f \in L^2_\omega \cap L^\infty_\omega$, by Theorem 3.1 and Corollary 3.3,

$$\langle f, g\rangle_\nu = \langle f, Qg\rangle_\nu = \langle Qf, g\rangle_\nu = \chi(Qf)$$

and

$$\left|\int_\Omega \bar{f} g\, d\nu\right| = |\langle f, g\rangle_\nu| \leqslant \|\chi\| \, \|Qf\|_{A^\infty_\omega} \leqslant C\|f\|_{L^\infty_\omega}.$$

This implies $g \in L^1_\omega$, i.e. $g \in A^1_\omega$. The rest is easy. □

Corollary 3.7. *A^p_ω is reflexive when $1 < p < \infty$. $(a^\infty_\omega)^{**} = A^\infty_\omega$. Thus A^1_ω and A^∞_ω are reflexive iff $a^\infty_\omega = A^\infty_\omega$.* □

Corollary 3.8. *Suppose that also* V4 *holds. Then*

$$a^\infty_\omega = \{f \in \mathcal{H}(\Omega): \omega f \in C_o(\Omega)\}.$$

PROOF. $A^\infty_\omega \cap \omega^{-1} C_o(\Omega)$ is a closed subspace of A^∞_ω, which by assumption contains every $L_z = \varkappa K^2_z$ and thus a^∞_ω. On the other hand, if $f \in A^\infty_\omega \cap \omega^{-1} C_o(\Omega)$, let $f_j = \chi_{K_j} f$, where $\{K_j\}$ is an increasing sequence of compact subsets of Ω with $\bigcup_1^\infty \operatorname{int}(K_j) = \Omega$. Then $f_j \in L^2_\omega \cap L^\infty_\omega$ and $f_j \to f$ in L^∞_ω, whence $Qf_j \in A^2_\omega \subset a^\infty_\omega$ and $Qf_j \to Qf = f$ in A^∞_ω. □

4. Hankel Forms

We assume throughout this section that the conditions V0 – V3 are satisfied. We continue to use the notations introduced in §§0 and 3.

As explained in the introduction, we will in this section study Hankel forms on $A^2(\mu)$ with symbols taken with respect to ν, i.e.

$$\Gamma_b(f, g) = \langle fg, b\rangle_\nu = \int_\Omega \bar{b} f g\, d\nu, \tag{4.1}$$

where $f, g \in A^2(\mu)$.

Theorem 4.1. *Let $b \in L^1_\omega + L^\infty_\omega$. Then Γ_b is a bounded bilinear form on $A^2(\mu)$, $\Gamma_b = \Gamma_{Qb}$ and*

$$\varkappa^{-1}\|Qb\|_{A^\infty_\omega} \leqslant \|\Gamma_b\| \leqslant \|Q_b\|_{A^\infty_\omega} \tag{4.2}$$

PROOF. By Hölder's inequality,

$$\|fg\|_{L^1_\omega} = \|fg\|_{L^1(\mu)} \leqslant \|f\|_{A^2(\mu)} \|g\|_{A^2(\mu)} \tag{4.3}$$

Thus $fg \in A^1_\omega \subset L^1_\omega \cap L^\infty_\omega$ (Corollary 3.1) which proves the first assertion, and

$$\|\Gamma_b\| \leqslant \|b\|_{L^\infty_\omega}. \tag{4.4}$$

By Corollary 3.3 and Theorem 3.1,

$$\Gamma_{Qb}(f, g) = \langle fg, Qb\rangle_\nu = \langle Q(fg), b\rangle_\nu = \langle fg, b\rangle_\nu = \Gamma_b(f, g),$$

which proves the second assertion, and

$$\|\Gamma_b\| = \|\Gamma_{Qb}\| \leqslant \|Qb\|_{A^\infty_\omega}.$$

Finally, if $z \in \Omega$,

$$\begin{aligned} |\varkappa^{-1}Qb(z)| &= |\varkappa^{-1}\langle b, L_z\rangle_\nu| = |\langle b, K_z^2\rangle_\nu| \\ &= |\Gamma_b(K_z, K_z)| \leqslant \|\Gamma_b\| \cdot \|K_z\|^2 \\ &= \|\Gamma_b\| K(z, z). \quad \square \end{aligned}$$

We proceed to the Schatten-von Neumann theory. We define an anti-linear operator $\Gamma\colon L^1_\omega + L^\infty_\omega \to S_\infty$ by $\Gamma(b) = \Gamma_b$.

Theorem 4.2. *If $1 \leqslant p \leqslant \infty$ and $b \in L^p_\omega$, then $\Gamma_b \in S_p$ and $\|\Gamma_b\|_{S_p} \leqslant \|b\|_{L^p_\omega}$.*

PROOF. If suffices to prove the result for $p = 1$, since the general case then follows by (4.4) and interpolation. Thus, assume that $b \in L^1_\omega$. The Banach space valued integral $\int b(z)L_z\, d\nu(z)$ then converges in $A^2(\nu) = A^2_\omega$, because the integrand is measurable (recall that $z \to L_z$ is continuous by Corollary 1.2 applied to ν) and

$$\begin{aligned} \int |b(z)|\, \|L_z\|_{A^2(\nu)}\, d\nu &= \int |b(z)| L(z, z)^{1/2}\, d\nu \\ &= \int \varkappa^{1/2} |b(z)| \omega(z)^{-1}\, d\nu \\ &= \varkappa^{1/2} \|b\|_{L^1_\omega} < \infty. \end{aligned} \tag{4.5}$$

Evaluating the integral pointwise by (1.2) and (3.6), we obtain

$$Qb = \int_\Omega b(z) L_z\, d\nu(z). \tag{4.6}$$

Since, by Theorem 4.1, Γ is a bounded anti-linear operator:

$$A^2_\omega \subset L^1_\omega + L^\infty_\omega \to S_\infty,$$

this yields

$$\Gamma(b) = \Gamma(Qb) = \int_\Omega \overline{b(z)}\, \Gamma(L_z)\, d\nu(z) \tag{4.7}$$

with the integral convergent in S_∞.

However,

$$\Gamma(L_z)(f,g) = \langle fg, L_z\rangle_\nu = fg(z) = \langle f, K_z\rangle_\mu \langle g, K_z\rangle_\mu. \tag{4.8}$$

Thus $\Gamma(L_z)$ is a bilinear form of rank 1, and

$$\|\Gamma(L_z)\|_{S_1} = \|\Gamma(L_z)\|_{S_\infty} = \|K_z\|^2_{A^2(\mu)} = K(z,z). \tag{4.9}$$

Thus

$$\int \|\overline{b(z)}\Gamma(L_z)\|_{S_1}\, d\nu = \int |b(z)|K(z,z)\, d\nu = \|b\|_{L^1_\omega} < \infty. \tag{4.10}$$

Furthermore, since $\Gamma(L_z) - \Gamma(L_w)$ has rank at most 2,

$$\|\Gamma(L_z) - \Gamma(L_w)\|_{S_1} \leqslant 2\|\Gamma(L_z) - \Gamma(L_w)\|_{S_\infty} \leqslant C\|L_z - L_w\|_{A^2(\nu)} \to 0$$

as $z \to w$, whence $z \to \Gamma(L_z)$ is a continuous map of Ω into S_1 and $z \to \overline{b(z)}\Gamma(L_z)$ is a measurable map into S_1. Consequently the integral (4.7) converges in S_1 as well and $\Gamma(b) \in S_1$ with norm bounded by (4.10). □

Next we define, for every bounded bilinear form T on $A^2(\mu)$,

$$\Gamma^*(T)(z) = \overline{T(K_z, K_z)}. \tag{4.11}$$

Cf. (for operators) Aronszajn (1950) and Berezin (1975).

Theorem 4.3. Γ^* *is a bounded anti-linear mapping of* S_∞ *into* A^∞_ω *that maps* S_p *into* A^p_ω *with*

$$\|\Gamma^*(T)\|_{A^p_\omega} \leqslant \|T\|_{S_p}, \qquad 1 \leqslant p \leqslant \infty. \tag{4.12}$$

PROOF. Fix $w \in \Omega$. Since $f \to T(f, K_w)$ is a bounded linear form on $A^2(\mu)$, there exists $g \in A^2(\mu)$ such that $T(f, K_w) = \langle f, g\rangle$. Thus

$$\overline{T(K_z, K_w)} = \overline{\langle K_z, g\rangle} = \langle g, K_z\rangle = g(z)$$

is analytic in z. By symmetry, $\overline{T(K_z, K_w)}$ is analytic in w too, whence it is analytic in (z, w) by Hartogs's theorem. In particular, $\Gamma^*(T)(z)$ is analytic.

It remains to prove that $\|\Gamma^*(T)\|_{L^p_\omega} \leqslant \|T\|_{S_p}$. By interpolation, it suffices to consider $p = \infty$ and $p = 1$. The case $p = \infty$ follows by

$$|\Gamma^* T(z)| \leqslant \|T\|_{S_\infty}\|K_z\|^2_{A^2(\mu)} = \|T\|_{S_\infty}K(z,z) = \|T\|_{S_\infty}\omega(z)^{-1}.$$

Next, if T is of rank 1, say $T(f,g) = \langle f, \phi\rangle_\mu \langle g, \psi\rangle_\mu$, then

$$\Gamma^* T(z) = \overline{\langle K_z, \phi\rangle_\mu}\,\overline{\langle K_z, \psi\rangle_\mu} = \phi(z)\psi(z) \tag{4.13}$$

and thus by Hölder's inequality

$$\|\Gamma^* T\|_{L^1_\omega} = \|\phi\psi\|_{L^1(\mu)} \leqslant \|\phi\|_{A^2(\mu)} \|\psi\|_{A^2(\mu)} = \|T\|_{S_1}.$$

Since S_1 is spanned by forms of rank 1, the case $p = 1$ follows. □

Next we prove that, as our notation suggests, Γ^* is the adjoint of Γ. (Recall that the operators are anti-linear which explains the form of (4.14)).

Theorem 4.4. *If* $1 \leqslant p \leqslant \infty$ *and* $1/p + 1/p' = 1$, *then*

$$\langle T, \Gamma_b \rangle = \langle b, \Gamma^* T \rangle_\nu, \qquad T \in S_{p'}, \qquad b \in L^p_\omega \tag{4.14}$$

PROOF. We study two cases separately. If $1 < p \leqslant \infty$, then forms of finite rank are dense in $S_{p'}$. Since both scalar products in (4.14) are bounded bilinear forms on $S_{p'} \times L^p_\omega$ (by Theorems 4.2 and 4.3), it suffices to prove (4.14) when T has rank one, say $T(f, g) = \langle f, \phi \rangle_\mu \langle g, \psi \rangle_\mu$. In this case $\Gamma^* T = \phi\psi$ by (4.13), and

$$\langle T, \Gamma_b \rangle = \overline{\Gamma_b(\phi, \psi)} = \overline{\langle \phi\psi, b \rangle_\nu} = \langle b, \phi\psi \rangle_\nu = \langle b, \Gamma^* T \rangle_\nu.$$

If $p = 1$ we use the representation (4.7)

$$\Gamma_b = \int \overline{b(z)} \Gamma(L_z) \, d\nu,$$

which converges in S_1 by the proof of Theorem 4.2. Since

$$\Gamma(L_z)(f, g) = \langle f, K_z \rangle_\mu \langle g, K_z \rangle_\mu$$

by (4.8),

$$\langle T, \Gamma(L_z) \rangle = T(K_z, K_z) = \overline{\Gamma^* T(z)},$$

and

$$\langle T, \Gamma_b \rangle = \int b(z) \langle T, \Gamma(L_z) \rangle \, d\nu = \int b(z) \overline{\Gamma^* T(z)} \, d\nu = \langle b, \Gamma^* T \rangle_\nu. \quad \square$$

We proceed to study $\Gamma^*\Gamma$ and $\Gamma\Gamma^*$.

Theorem 4.5. $\Gamma^*\Gamma(b) = \varkappa^{-1} Q b$ *for every* $b \in L^1_\omega + L^\infty_\omega$.

PROOF.

$$\Gamma^*\Gamma(b)(z) = \overline{\Gamma_b(K_z, K_z)} = \overline{\langle K_z^2, b \rangle_\nu} = \langle b, \varkappa^{-1} L_z \rangle = \varkappa^{-1} Q b(z). \quad \square \tag{4.15}$$

Theorems 4.1, 4.2, 4.3 and 4.5 yield one of our main results.

Theorem 4.6. *Let $b \in L^1_\omega + L^\infty_\omega$ and $1 \leqslant p \leqslant \infty$. Then $\Gamma_b \in S_p$ if and only if $Qb \in A^p_\omega$.* □

Theorem 4.7. *Let $1 \leqslant p \leqslant \infty$. Then Γ is an anti-linear isomorphism mapping A^p_ω onto the set of Hankel forms in S_p. The inverse is given by $\varkappa\Gamma^*$. If $p = 2$ then $\sqrt{\varkappa}\,\Gamma$ is an anti-linear isometry.*

PROOF. The first assertions follow immediately. That $\varkappa^{1/2}\Gamma$ is an isometry was proved in the introduction (0.7), and follows also by Theorems 4.4 and 4.5. □

Remark 4.1. The proof of Theorem 4.6 yields the estimates

$$\|\Gamma_b\|_{S_p} \leqslant \|Q_b\|_{A^p_\omega} \leqslant \varkappa\|\Gamma_b\|_{S_p}, \qquad 1 \leqslant p \leqslant \infty,$$

but Theorem 4.7 shows that improved estimates can be obtained for $1 < p < \infty$ by interpolating with the case $p = 2$.

Theorem 4.5 yields $\varkappa\Gamma\Gamma^*\Gamma = \Gamma Q = \Gamma$ and $\varkappa\Gamma^*\Gamma\Gamma = \Gamma^*$. The results above now give the following results on the Hankel projection.

Theorem 4.8. *$\varkappa\Gamma\Gamma^*$ is a linear projection of S_∞ onto the subspace of Hankel forms. $\varkappa\Gamma\Gamma^*$ is bounded on S_p, $1 \leqslant p \leqslant \infty$, and $\langle \varkappa\Gamma\Gamma^* S, T\rangle = \langle S, \varkappa\Gamma\Gamma^* T\rangle$ for $S \in S_p$, $T \in S_{p'}$, $1/p + 1/p' = 1$. In particular, the restriction of $\varkappa\Gamma\Gamma^*$ to S_2 is the orthogonal projection onto the space of Hilbert-Schmidt Hankel forms.* □

Remark 4.2. This contrasts to the classical case $H^2(\mathbb{T})$, where the Hankel projection is bounded when $1 < p < \infty$, but not at the endpoints, see e.g. Peller (1980).

Corresponding results for compactness are easily obtained using the fact that the space of compact forms equals the closed hull of S_2 in S_∞ together with Corollary 3.6.

Theorem 4.9. *Γ_b is compact if and only if $Qb \in a^\infty_\omega$. Γ maps a^∞_ω onto the set of compact Hankel forms. The Hankel projection $\varkappa\Gamma\Gamma^*$ maps compact forms to compact Hankel forms.* □

5. Multilinear Hankel Forms

The theory above for bilinear Hankel forms is easily generalized to multilinear forms. We will here sketch this generalization omitting most of the details.

Let $m > 2$ be an integer ($m = 2$ gives the results of the preceding sections) and define the measure $d\nu_m = \omega^m \, d\lambda$ and, for $f_1, \ldots, f_m \in A^2(\mu)$ and b a suitable function on Ω,

$$\Gamma_b(f_1, \ldots, f_m) = \int_\Omega \bar{b} f_1 \cdot \ldots \cdot f_m \, d\nu_m. \tag{5.1}$$

The weight ω^m in the definition of ν_m makes the expression (5.1) gauge invariant, with b transforming as $b \to \phi^m b$ («weight m») under an analytic change of gauge (1.12).

Let L_m denote the reproducing kernel in $A^2(\nu_m)$. We assume throughout this section that V0 and V1 hold, that $L_m(z, w) = \varkappa_m K(z, w)^m$ for some constant $\varkappa_m$, and that

$$\int L_m(z, w) f(w) \, d\nu_m(w) = f(z), \qquad z \in \Omega \tag{5.2}$$

for every analytic function f such that the left-hand side is defined for all z. (The natural generalizations of V2 and V3 to the present situation). Note that these conditions are satisfied, for every m, whenever V0 holds, $A^2(\mu) \neq \{0\}$, $G(\mu)$ is transitive and (2.6) holds for some (and thus all) $z \in \Omega$. (Because the proof of Proposition 3.1 extends immediately, using Theorem 2.1 with $r = m - 1$).

Define Q_m by

$$Q_m f(z) = \int L_m(z, w) f(w) \, d\nu_m(w)$$

and let

$$L^p_{\omega^{m/2}} = \{ f \colon \omega^{m/2} f \in L^p(\lambda) \}, \qquad A^p_{\omega^{m/2}} = L^p_{\omega^{m/2}} \cap \mathcal{H}(\Omega).$$

Note that if $f \in A^2(\mu)$, then

$$\| \omega^{1/2} f \|_{L^2(\lambda)} = \| f \|_{A^2(\mu)}$$

and, by (1.7),

$$\| \omega^{1/2} f \|_{L^\infty(\lambda)} \leqslant \| f \|_{A^2(\mu)}.$$

Hence also

$$\| \omega^{1/2} f \|_{L^m(\lambda)} \leqslant \| f \|_{A^2(\mu)}$$

and, by Hölder's inequality,

$$\| \omega^{m/2} f_1 \cdot \ldots \cdot f_m \|_{L^1(\lambda)} \leqslant \| f_1 \|_{A^2(\mu)} \cdot \ldots \cdot \| f_m \|_{A^2(\mu)} \tag{5.3}$$

$$\| \omega^{m/2} f_1 \cdot \ldots \cdot f_m \|_{L^\infty(\lambda)} \leqslant \| f_1 \|_{A^2(\mu)} \cdot \ldots \cdot \| f_m \|_{A^2(\mu)}. \tag{5.4}$$

Consequently,

$$f_1 \cdot \ldots \cdot f_m \in A^1_{\omega m/2} \cap A^\infty_{\omega m/2} \quad \text{if} \quad f_1, \ldots, f_m \in A^2(\mu).$$

In particular,

$$L_{m,z} = \varkappa_m K_z^m \in A^1_{\omega m/2} \cap A^\infty_{\omega m/2}.$$

It is now easily seen, as in Section 3, that Q_m is a projection of $L^p_{\omega m/2}$ onto $A^p_{\omega m/2}$ for $1 \leqslant p \leqslant \infty$, and that the analogues of Corollaries 3.1-3.8 hold. Equipped with these results, we proceed to study the multilinear Hankel form defined by (5.1). Theorem 4.1 extends easily.

Theorem 5.1 *Let $b \in L^1_{\omega m/2} + L^\infty_{\omega m/2}$. Then Γ_b is a bounded multilinear form on $A^2(\mu)$, $\Gamma_b = \Gamma_{Q_m b}$ and*

$$\varkappa_m^{-1} \| Q_m b \|_{A^\infty_{\omega m/2}} \leqslant \| \Gamma_b \| \leqslant \| Q_m b \|_{A^\infty_{\omega m/2}} \tag{5.5}$$

PROOF. We prove (5.5) and leave the rest to the reader. We may assume that $b = Q_m b$. Then, if $\|f_1\|_{A^2(\mu)}, \ldots, \|f_m\|_{A^2(\mu)} \leqslant 1$, (5.3) yields

$$|\Gamma_b(f_1, \ldots, f_m)| = \left| \int \omega^{m/2} \bar{b} \omega^{m/2} f_1 \ldots f_m \, d\lambda \right| \leqslant \| \omega^{m/2} \bar{b} \|_{L^\infty(\lambda)} = \| b \|_{A^\infty_{\omega m/2}}$$

which proves the right inequality. The left inequality follows by

$$\begin{aligned} |Q_m b(z)| &= \left| \int \bar{b} L_{m,z} \, d\nu_m \right| = \left| \varkappa_m \int \bar{b} K_z^m d\nu_m \right| = \varkappa_m |\Gamma_b(K_z, \ldots, K_z)| \\ &\leqslant \varkappa_m \| \Gamma_b \| \, \| K_z \|^m_{A^2(\mu)} \\ &= \varkappa_m \| \Gamma_b \| \omega(z)^{-m/2}. \quad \square \end{aligned}$$

Also the S_p-results of Section 4 extend. However, as far as we know, no theory is so far developed for S_p-classes of multilinear forms on a Hilbert space $\mathcal{H}$. Hence we confine ourselves to the case $p = 1, 2, \infty$.

Let S_∞ be the space of all bounded multilinear forms on $\mathcal{H} \times \cdots \times \mathcal{H}$. Let S_1 be the space of nuclear forms, i.e.

$$\left\{ (x_1, \ldots, x_m) \to \sum_{j=1}^{\infty} a_j \prod_{i=1}^{m} \langle x_i, y_{ij} \rangle : \sum_j |a_j| \prod_i \| y_{ij} \|_{\mathcal{H}} < \infty \right\};$$

S_1 is the m-fold projective tensor product $\mathcal{H} \hat{\otimes} \cdots \hat{\otimes} \mathcal{H}$ (identyfying $\mathcal{H}$ and its dual).

Let S_2 be the space of Hilbert-Schmidt forms; S_2 is the Hilbert tensor product $\mathcal{H} \hat{\otimes}_2 \cdots \hat{\otimes}_2 \mathcal{H}$.

It follows that $S_1 \subset S_2 \subset S_\infty$, and $S_1^* \cong S_\infty$.

Furthermore,

$$[S_1, S_\infty]_{1/2} = (S_1, S_\infty)_{1/2,2} = S_2 .$$

Remark 5.1. This is an instance of the general principle that interpolation between a space and its dual (by one of these two interpolation methods) gives a Hilbert space. (We call this a «principle», not a «theorem», because it is not yet proved in a complete generality, see Janson (1986)).

Let $\Gamma(b) = \Gamma_b$. $\Gamma(L_{m,z})$ is a multilinear form of rank 1 and it follows as in Theorem 4.2 that $b \in L^1_{\omega m/2} \to \Gamma_b \in S_1$. For the converse we define for any multilinear form T on $A^2(\mu)$,

$$\Gamma^* T(z) = \overline{T(K_z, \ldots, K_z)}$$

(cf. (4.11)) and obtain as in Theorem 4.3, using (5.3), that Γ^* maps S_∞ into $A^\infty_{\omega m/2}$ and S_1 into $A^1_{\omega m/2}$. Furthermore, $\varkappa_m \Gamma^* \Gamma = Q_m$. Hence we obtain the extension of Theorem 4.6.

Theorem 5.2. *If* $p = 1, 2, \infty$, *then*

$$\Gamma_b \in S_p \quad \textit{if and only if} \quad Q_m b \in A^p_{\omega m/2} . \quad \square$$

It is also easy to treat the case S_2 directly as in §0.4.

Remark 5.2. If we *define* S_p for $1 < p < \infty$ by (real or complex) interpolation between S_1 and S_∞, Theorem 5.2 holds for every $p \geqslant 1$. Indeed, this seems to be the only reasonable definition one can think of if one wants to carry over the usual theorems on Hankel forms (operators) to the multilinear situation (see Peetre (1985)).

6. Miscellaneous Complements

We assume that V0 – V3 hold.

6.1. More general symbols. In §4 we assumed for technical reasons that $b \in L^1_\omega + L^\infty_\omega$, which made all occurring integrals finite. In the next subsection we have to consider more general symbols, which may be done as follows. Recalling that the linear span of $\{K_z\}$ is dense in $A^2(\mu)$, we say that Γ_b exists when

$$\int |b K_z K_w| \, d\nu < \infty \quad \text{for all} \quad z, w,$$

and

$$\left| \int \bar{b} f g \, d\nu \right| \leqslant C \|f\| \, \|g\| \quad \text{for all} \quad f, g \in \operatorname{span}\{K_z\}. \tag{6.1}$$

Then Γ_b is defined on $A^2(\mu) \times A^2(\mu)$ by continuity. The assumption (6.1) implies that Qb is well-defined and

$$Qb(z) = \int b\bar{L}_z \, d\nu = \varkappa \int b\bar{K}_z^2 \, d\nu = \varkappa \Gamma^* \Gamma_b \in A_\omega^\infty$$

by Theorem 4.3, whence the preceding theory applies to Γ_{Qb}.

If we assume that $\Gamma_b = \Gamma_{Qb}$ for all b such that (6.1) holds, then the results of §4 can be carried over to this enlarged classs of symbols; in particular, it follows that $\Gamma_b \in S_p$ if and only if $Qb \in A_\omega^p$.

An alternative formulation of this assumption is:

If (6.1) holds and

$$\int \bar{b} K_z^2 \, d\nu = 0 \quad \text{for every } z$$

then

$$\int \bar{b} K_z K_w \, d\nu = 0 \quad \text{for all} \quad z, w \tag{6.2}$$

To see this equivalence, notice that

$$\int (\bar{b} - \overline{Qb}) K_z^2 \, d\nu = 0.$$

Thus, by (6.2),

$$\int \bar{b} K_z K_w \, d\nu = \int \overline{Qb} K_z K_w \, d\nu,$$

that is, $\Gamma_b = \Gamma_{Qb}$.

Unfortunately, we have not been able to prove (6.2) in general, but it is easily verified in the examples in §§7-13. Note that (6.2) is gauge invariant.

6.2. Symbols with respect to other measures. The time has come to treat Hankel forms with symbols with respect to general absolutely continuous measures. We recall the notations, cf. §0.2,

$$H_b^\xi(f, g) = \int \bar{b} f g \, d\xi \tag{6.3}$$

and, as a special case,

$$H_b = H_b^\mu. \tag{6.4}$$

More precisely, we say that H_b^ξ exists if $\int \bar{b} f g \, d\xi$ is absolutely convergent and defines a bounded form for $f, g \in \operatorname{span}\{K_z\}$.

If follows from the definition that

$$H_b^\xi = \Gamma\left(b \frac{d\xi}{d\nu}\right) \tag{6.5}$$

where the two sides are defined (in the sense of this section) for the same set of b. Hence the previous results for Γ can be transfered. The condition (6.2) is equivalent to

If

$$\left|\int \bar{b} f g \, d\xi\right| \leqslant C\|f\| \, \|g\|, \qquad f, g \in \operatorname{span}\{K_z\}, \quad \text{and} \quad \int \bar{b} K_z^2 \, d\xi = 0 \tag{6.6}$$

for every z, then

$$\int \bar{b} K_z K_w \, d\xi = 0$$

for all z, w.

The discussion above and Theorem 4.6 and 4.9 yield:

Corollary 6.1. *Suppose that* (6.6) *holds. Then*

$$H_b^\xi \in S_p \quad \textit{if and only if} \quad Q\left(b \frac{d\xi}{d\nu}\right) \in A_\omega^p, \qquad 1 \leqslant p \leqslant \infty.$$

$$H_b^\xi \quad \textit{is compact if and only if} \quad Q\left(b \frac{d\xi}{d\nu}\right) \in a_\omega^p.$$

Note that, for $\xi = \mu (H_b^\xi = H_b)$, $Q(b \cdot d\mu/d\nu) = Q(\omega^{-1} b)$.

It remains to identify $Q(b \cdot d\xi/d\nu)$, in particular for analytic symbols b. Here the general theory fails us (even for $\xi = \mu$), and this has to be done by a separate analysis in each case (because $b \mapsto Q(b \cdot d\mu/d\nu) = Q(b(z)K(z, z))$ is *not* gauge-invariant, cf. §§7 and 11). We observe nevertheless the formula

$$Q\left(b \frac{d\xi}{d\nu}\right)(z) = \int b \frac{d\xi}{d\nu} \bar{L}_z \, d\nu = \int b \bar{L}_z \, d\xi \tag{6.7}$$

and that $b \mapsto Q(b \cdot d\xi/d\nu)$ formally is the adjoint of the (possibly unbounded) identity map $A^2(\nu) \to A^2(\xi)$ because, for $f \in A^2(\nu)$,

$$\left\langle Q\left(b \frac{d\xi}{d\nu}\right), f\right\rangle_\nu = \left\langle b \frac{d\xi}{d\nu}, f\right\rangle_\nu = \langle b, f\rangle_\xi. \tag{6.8}$$

6.3. Minimal and maximal invariant spaces. The representation $f = \int f(z) L_z \, d\nu$ expresses any function in A_ω^1 as a continuous linear combination of $\{L_z\}$. There is also a discrete counterpart.

Theorem 6.1. $f \in A_\omega^1$ *if and only if* $f = \sum_1^\infty a_i \omega(z_i) L_{z_i}$ *for some sequences* $\{z_i\} \subset \Omega$ *and* $\{a_i\} \in l^1$. $\|f\|_{A_\omega^1}$ *is equivalent to the infimum of* $\sum |a_i|$ *extended over all such representations.*

PROOF. By (3.8),

$$\|\omega(z)L_z\|_{A^1_\omega} = \omega(z)\varkappa K(z,z) = \varkappa \quad \text{for every} \quad z \in \Omega.$$

Hence we may define a linear operator $T: l^1(\Omega) \to A^1_\omega$ (with norm $\varkappa$) by

$$T\{a_z\} = \sum_{z\in\Omega} a_z \omega(z) L_z.$$

The adjoint T^* maps $A^\infty_\omega \cong (A^1_\omega)^*$ (cf. Corollary 3.4) into $l^\infty(\Omega) \cong (l^1(\Omega))^*$. Let $g \in A^\infty_\omega$. Since

$$\langle g, L_z\rangle_\nu = Qg(z) = g(z),$$

it is easily seen that $T^*g = \{\omega(z)g(z)\}$, and thus $\|T^*g\|_{l^\infty(\Omega)} = \|g\|_{A^\infty_\omega}$.

Consequently T^* is an isomorphism into, and T is onto. □

In other terms, A^1_ω is the smallest Banach space that contains all L_z with norms bounded by some constant times $K(z,z)$.

It is easily seen that $G^*(\mu)$ acts isometrically in each A^p_ω by the action

$$R_{(\gamma,\phi)}f(z) = \phi(z)^2 f(\gamma^{-1}(z)) \tag{6.9}$$

(Cf. (2.2) and recall that A^p_ω transforms with weight 2). Using (2.3), it follows easily that

$$R_{(\gamma,\phi)}L_w(z) = \phi(z)^2 L(\gamma^{-1}(z), w) = \overline{\phi(\gamma(w))}^{-2} L(z, \gamma(w))$$

and

$$R_{(\gamma,\phi)}(\omega(w)L_w) = \omega(w)\overline{\phi(\gamma(w))}^{-2} L_{\gamma(w)} = \operatorname{sign}\phi(\gamma(w))^2 \omega(\gamma(w)) L_{\gamma(w)}.$$

Hence, if G is a transitive subgroup of $G(\mu)$ and G^* is the corresponding subgroup of $G^*(\mu)$, the theorem above shows that A^1_ω is the smallest G^*-invariant (under the action (6.9)) Banach space that contains some L_z.

Dually, A^∞_ω is the largest G^*-invariant Banach space of analytic functions in Ω admitting continuous evaluation at some point. This follows because, if $|f(z_0)| \leqslant C\|f\|$ for every function in the space, $(\gamma, \phi) \in G^*$ implies by (2.3) and (6.5)

$$\begin{aligned}
\omega(\gamma^{-1}(z_0))|f(\gamma^{-1}(z_0))| &= |\phi(z_0)|^2 \omega(z_0)|f(\gamma^{-1}(z_0))| \\
&= \omega(z_0)|R_{(\gamma,\phi)}f(z_0)| \\
&\leqslant C\omega(z_0)\|R_{(\gamma,\phi)}f\| \\
&= C\omega(z_0)\|f\|.
\end{aligned}$$

We will not pursue the investigation of invariant spaces here, but refer to the surveys Arazy and Fisher (1984) and Peetre (1984), (1985).

6.4. Factorization.

Theorem 6.2. *$f \in A^1_\omega$ if and only if $f = \sum_1^\infty a_i g_i h_i$ for some sequences $\{g_i\}$ and $\{h_i\}$ in the unit ball of $A^2(\mu)$ and $\{a_i\} \in l^1$. $\|f\|_{A^1_\omega}$ is equivalent to the infimum of $\sum |a_i|$ over all such representations.*

PROOF. Hölder's inequality yields

$$\left\| \sum_1^\infty a_i g_i h_i \right\|_{A^1_\omega} \leqslant \sum |a_i| \, \|g_i\|_{A^2(\mu)} \|h_i\|_{A^2(\mu)}.$$

The existence of representations follows by duality as in the proof of Theorem 6.1, or alternatively, from Theorem 1 by taking $g_i = h_i = K_{z_i}/\|K_{z_i}\|$ and replacing a_i by $\varkappa a_i$ (because then

$$\varkappa g_i h_i = \varkappa K^2_{z_i} / \|K_{z_i}\|^2 = \omega(z_i) L_{z_i}). \quad \square$$

This is a so-called weak factorization. We do not know whether a similar strong factorization is valid in general, i.e. whether each $f \in A^1_\omega$ can be factorized as gh with g and h in $A^2(\mu)$ and $\|g\| \, \|h\| \leqslant C \|f\|_{A^1_\omega}$. For the special case of Bergman spaces in the disc, Horowitz (1977) proved strong factorization.

6.5. Another S_p criterion. Let $k_z = K_z / \|K_z\|$ be the normalized reproducing kernels. If T is any bounded linear operator of $A^2(\mu)$ into a Hilbert space $\mathcal{H}_1$, then if $\{e_\alpha\}$ is an ON-basis in $\mathcal{H}_1$,

$$\begin{aligned} \int \|Tk_z\|^2 \, d\lambda(z) &= \int \|TK_z\|^2 \, d\mu(z) = \int \sum_\alpha |\langle TK_z, e_\alpha \rangle|^2 \, d\mu(z) \\ &= \sum_\alpha \int |\langle K_z, T^* e_\alpha \rangle|^2 \, d\mu(z) = \sum_\alpha \int |T^* e_\alpha(z)|^2 \, d\mu(z) \\ &= \sum \|T^* e_\alpha\|^2 = \|T^*\|^2_{S_2} \\ &= \|T\|^2_{S_2} \end{aligned} \tag{6.10}$$

It follows, by interpolation with $p = \infty$, that

$$\| \, \|Tk_z\|_{\mathcal{H}_1} \|_{L^p(\lambda)} \leqslant \|T\|_{S_p}, \qquad 2 \leqslant p \leqslant \infty \tag{6.11}$$

and, by duality,

$$\| \, \|Tk_z\|_{\mathcal{H}_1} \|_{L^p(\lambda)} \geqslant \|T\|_{S_p}, \qquad 1 \leqslant p \leqslant 2. \tag{6.12}$$

For Hankel operators there exist converses to (6.11) and (6.12) (within constants). Let $\tilde{\Gamma}_b$ be the operator corresponding to the form Γ_b as in (0.1). We may assume that $b \in A_\omega^\infty$. We begin with the case $p \geqslant 2$, where there are no problems.

Theorem 6.3. *If* $2 \leqslant p \leqslant \infty$, *then*

$$\tilde{\Gamma}_b \in S_p \quad \textit{if and only if} \quad \|\tilde{\Gamma}_b k_z\| \in L^p(\lambda). \tag{6.13}$$

PROOF.

$$\|\tilde{\Gamma}_b k_z\| \geqslant |\langle \tilde{\Gamma}_b k_z, k_z\rangle| = |\Gamma_b(k_z, k_z)| = \omega(z)|\Gamma_b(K_z, K_z)| = \varkappa^{-1}\omega(z)|b(z)|,$$

cf. (4.15). Thus, by Theorem 4.2,

$$\|\tilde{\Gamma}_b\|_{S_p} = \|\Gamma_b\|_{S_p} \leqslant \|\omega b\|_{L^p(\lambda)} \leqslant \varkappa \|\, \|\tilde{\Gamma}_b k_z\| \,\|_{L^p(\lambda)}. \quad \square$$

The converse for $p < 2$ only holds in some cases, however.

Theorem 6.4. *Suppose that*

$$\sup_{w \in \Omega} \int \frac{|K(z, w)|}{(K(z, z)K(w, w))^{1/2}}\, d\lambda(z) < \infty. \tag{6.14}$$

Then, for every $1 \leqslant p \leqslant \infty$,

$$\tilde{\Gamma}_b \in S_p \quad \textit{if and only if} \quad \|\tilde{\Gamma}_b k_z\| \in L^p(\lambda). \tag{6.15}$$

Conversely, if (6.15) *holds for* $p = 1$, *then* (6.14) *holds.*

PROOF. By Theorem 6.3 and Theorem 4.6, it suffices to prove that if (6.14) holds, $b \in A_\omega^p$ implies $\|\tilde{\Gamma}_b k_z\| \in L^p(\lambda)$. By interpolation we may assume $p = 1$, and by Theorem 6.1 this implication is equivalent to

$$\sup \{ \|\Gamma_b k_z\|_{L^1(\lambda)} : b = \omega(w) L_w \} < \infty. \tag{6.16}$$

A simple calculation shows that (6.16) is the same as (6.14). $\square$

If $G(\mu)$ is transitive, then, by Corollaries 2.1 and 2.2 the integral in (6.14) is independent of w, whence it is sufficient that it is finite for some w. Hence (take $w = 0$) (6.14) and (6.15) hold for the Fock space (§§7-11), and for the Bergman spaces (§§12, 13) with parameter $\gamma > n - 1$, but (6.14) does not hold for Bergman spaces with $\gamma \leqslant n - 1$. (Presumably, (6.13) holds for some $p < 2$ even in the latter case; more research is needed).

The corresponding result for $H^2(\mathbb{T})$ and $p = \infty$ is given by Bonsall (1984); it is equivalent to an oscillation condition.

7. Fock Space

The general theory will now be applied to the Fock space. Let, as in the introduction, $\Omega = \mathbb{C}^n$ ($n = 1, 2, \ldots$ will be fixed in the sequel) and, for $\alpha > 0$,

$$d\mu_\alpha = (\alpha/\pi)^n e^{-\alpha|z|^2} dm. \tag{7.1}$$

We define F_α^2 (Fock space) as the Hilbert space $A^2(\mu_\alpha)$.

More generally, let L_α^p be the space of measurable functions f on $\mathbb{C}^n$ such that $f(z)e^{-\alpha|z|^2/2} \in L^p(m)$, and let F_α^p be the subspace of entire functions. (We normalize the norms so that $\|1\| = 1$. In any case, the results below in general hold only up to equivalence of norms).

Remark 7.1. Note that L_α^p is not the same as $L^p(\mu_\alpha)$ unless $p = 2$; in fact, $L^p(\mu_\alpha) = L_{2\alpha/p}^p$. The parametrization L_α^p is, as we will see, very natural. We return to $L^p(\mu_\alpha)$ is Section 9.

Remark 7.2. In our analysis it is natural to consider the whole scale of spaces F_α^p at this time. The parameter α which plays something like the rôle of Planck's constant, is of course devoid of intrinsic interest. Notice that the dilation $f \mapsto f((\beta/\alpha)^{1/2}z)$ maps F_α^p into F_β^p isometrically. This is exploited several times below.

Whenever necessary, we add a subscript α to the notation. Thus

$$\langle f, g \rangle_\alpha = \int f\bar{g}\, d\mu_\alpha,$$

K_α is the reproducing kernel in F_α^2, etc.

It is easy to see that $\{z^\gamma\}$, where γ ranges over all multi-indices, is an orthogonal basis in F_α^2 and that $\|z^\gamma\|_\alpha^2 = \alpha^{-|\gamma|}\gamma!$. Hence, by (1.9),

$$K_\alpha(z, w) = \sum z^\gamma \bar{w}^\gamma \alpha^\gamma/\gamma! = e^{\alpha\langle z, w\rangle}. \tag{7.2}$$

($\langle z, w \rangle = \sum_1^n z_i \bar{w}_i$ is the scalar product in $\mathbb{C}^n$).

It is easy to see that, for each $w \in \mathbb{C}^n$, the mapping $C_\alpha(w)$ defined by

$$C_\alpha(w)f(z) = f(z - w)e^{\alpha\langle z, w\rangle - \alpha|w|^2/2} \tag{7.3}$$

is an isometry of F_α^p (and L_α^p) onto itself, $1 \leqslant p \leqslant \infty$. Further,

$$C_\alpha(w_1 + w_2) = C_\alpha(w_1)C_\alpha(w_2)e^{i\alpha \operatorname{Im}\langle w_1, w_2\rangle}. \tag{7.4}$$

Hence $(w,t) \to e^{i\alpha t}C_\alpha(w)$ is a unitary representation of the Heisenberg group in F^2_α. (Recall that the Heisenberg group is $\mathbb{C}^n \times \mathbb{R}$ with the group law $(z,t) \circ (w,s) = (z+w, t+s-\operatorname{Im}\langle z,w\rangle)$).

In the notation of Section 2, $G(\mu_\alpha)$ contains the group of translations of $\mathbb{C}^n$, and the corresponding subgroup of $G^*(\mu_\alpha)$ is essentially the Heisenberg group. (It is the quotient group $\mathbb{C}^n \times \mathbb{T} \cong (\mathbb{C}^n \times \mathbb{R})/2\pi\mathbb{Z}$). In particular, $G(\mu_\alpha)$ is transitive. Furthermore, $G(\mu_\alpha)$ obviously contains the group $U(n)$ of linear isometries, which satisfies (2.6) for $z = 0$.

Proposition 3.1 shows that V0 – V4 holds, so our theory is applicable.

Let us identify the notations in §3. $\mu = \mu_\alpha$ and $K = K_\alpha$ are given above. Hence

$$d\lambda = e^{\alpha|z|^2}\,d\mu_\alpha = (\alpha/\pi)^n\,dm,$$

a constant multiple of the Lebesgue measure, and

$$d\nu = e^{-\alpha|z|^2}d\mu_\alpha = (\alpha/\pi)^n e^{-2\alpha|z|^2}\,dm = 2^{-n}\,d\mu_{2\alpha}. \tag{7.5}$$

Thus, e.g. by Proposition 1.2, the reproducing kernel for ν is

$$L = 2^n K_{2\alpha} = 2^n K^2_\alpha \tag{7.6}$$

which gives a direct proof of V2 and shows that $\varkappa = 2^n$. Q is the orthogonal projection onto

$$A^2(\mu) = F^2_{2\mu}; \tag{7.7}$$

hence $Q = P_{2\alpha}$. By (7.2), $w(z) = e^{-\alpha|z|^2}$, and thus $L^p_\omega = L^p_{2\alpha}$ and $A^p_\omega = F^p_{2\alpha}$. If we write $f^\infty_\alpha = \{f \in \mathcal{H}(\mathbb{C}^n): f(z) = o(e^{\alpha|z|^2/2}) \text{ as } |z| \to \infty\}$, then $a^\infty_\omega = f^\infty_{2\alpha}$ by Corollary 3.8.

Thus translating, and replacing α by $\alpha/2$, the results of Section 3 yield the following for every $\alpha > 0$.

Theorem 7.1. *P_α, defined by*

$$P_\alpha f(z) = \int e^{\alpha\langle z,w\rangle} f(w)\,d\mu_\alpha(w), \tag{7.8}$$

is a bounded self-adjoint projection of

$$L^p_\alpha \text{ onto } F^p_\alpha, \qquad 1 \leqslant p \leqslant \infty. \quad \square$$

Theorem 7.2. *If $1 \leqslant p \leqslant q \leqslant \infty$, then*

$$F^p_\alpha \subset F^q_\alpha \subset f^\infty_\alpha \subset F^\infty_\alpha.$$

This first and second inclusions have dense ranges. $\square$

Theorem 7.3. *If* $1 \leqslant p_0 \leqslant p_1 \leqslant \infty$ *and* $0 \leqslant \theta \leqslant 1$,

$$[F_\alpha^{p_0}, F^{p_1}]_\theta = (F_\alpha^{p_0}, F_\alpha^{p_1})_{\theta p_\theta} = F_\alpha^{p_\theta},$$

where $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$. □

Theorem 7.4. $(F_\alpha^p)^* \cong F_\alpha^{p'}$, $1 \leqslant p < \infty$, *and* $(f_\alpha^\infty)^* \cong F_\alpha^1$ *with the pairing* $\langle\ \rangle_\alpha$. □

Since $e^{\alpha z^2/2} \in F_\alpha^\infty \backslash f_\alpha^\infty$, the spaces F_α^1, F_α^∞ and f_α^∞ are not reflexive. It is also easily seen that $F_\alpha^p \neq F_\alpha^q$ when $p \neq q$.

We turn to Hankel forms in F_α^2. Let H_b^β, $\beta > 0$, denote the Hankel form with symbol b with respect to μ_β (i.e. H_b^ξ with $\xi = \mu_\beta$ in out general notation);

$$H_b^\beta(f, g) = \int \bar{b} f g \, d\mu_\beta \tag{7.9}$$

(suitably interpreted).

Thus $H_b = H_b^\alpha$ and, by (7.5), $\Gamma_b = 2^{-n} H_b^{2\alpha}$. Note that (6.2) and (6.6) hold because $K_z K_w = K^2_{(z+w)/2}$. Furthermore, by (6.7), (7.6), (7.2) and (7.8),

$$\begin{aligned} Q\left(b \frac{d\mu_\beta}{d\nu}\right)(z) &= \int b \bar{L}_z \, d\mu_\beta = \int b(w) 2^n e^{2\alpha\langle z, w\rangle} \, d\mu_\beta(w) \\ &= 2^n \int b(w) K_\beta\left(\frac{2\alpha}{\beta} z, w\right) d\mu_\beta(w) \\ &= 2^n P_\beta b\left(\frac{2\alpha}{\beta} z\right). \end{aligned} \tag{7.10}$$

Thus, if we restrict attention to analytic b,

$$Q\left(b \frac{d\mu_\beta}{d\nu}\right) \in A_\omega^p = F_{2\alpha}^p \Leftrightarrow b\left(\frac{2\alpha}{\beta} z\right) \in F_{2\alpha}^p \Leftrightarrow b \in F^p_{\beta^2/2\alpha}. \tag{7.11}$$

Consequently, the results of §§4 and 6 yield

Theorem 7.5. *Suppose that b is an entire function on $\mathbb{C}^n$ and $\alpha > 0$, $\beta > 0$, $1 \leqslant p \leqslant \infty$. Then*

(*a*) *$H_b^\beta \in S_p(F_\alpha^2)$ if and only if $b \in F^p_{\beta^2/2\alpha}$, i.e. if and only if $b(z) e^{-(\beta^2/4\alpha)|z|^2} \in L^p(dm)$. The respective norms are equivalent within constants.*

(*b*) *H_b^β is compact if and only if $b \in f^\infty_{\beta^2/2\alpha}$, i.e. if and only if $b(z) = o(e^{(\beta^2/4\alpha)|z|^2})$.*

(*c*) *The Hankel projection is bounded in every S_p.*

In particular, $\Gamma_b \in S_p$ if and only if $b \in F_{2\alpha}^p$, and $H_b \in S_p$ if and only if $b \in F^p_{\alpha/2}$. □

(As the family $\{H_b^\beta\}$ is independent of β, there is only one Hankel projection in $S_\infty(F_\alpha^2)$). Note the formula

$$H_b^\beta = H^\gamma_{b((\gamma/\beta)z)}, \qquad \beta, \gamma > 0, \tag{7.12}$$

which is proved as (7.10), or by checking $f = K_{\beta,z}$, $g = K_{\beta,w}$.

Remark 7.3. Strictly speaking, the argument above presumes $\beta < 4\alpha$, because otherwise e.g. the integral

$$\int \bar{b} K_z K_w \, d\mu_\beta$$

may diverge for $b \in F^\infty_{\beta 2/2\alpha}$. Theorem 7.5 is true for all β with a suitable interpretation of H_b^β (e.g. by (7.12)).

We may also study the Hankel form (7.9) when f and g are in two different Fock spaces $F^2_{\alpha_1}$ and $F^2_{\alpha_2}$. (Cf. Feldman and Rochberg (1986)).

Theorem 7.6. *Let $1 \leqslant p \leqslant \infty$, $\alpha_1 > 0$, $\alpha_2 > 0$, $\beta > 0$ and assume that b is an entire function. Then $H_b^\beta \in S_p(F^2_{\alpha_1} \times F^2_{\alpha_2})$ if and only if $b \in F^p_{\beta 2/(\alpha_1+\alpha_2)}$.*

PROOF. The case $\beta = \alpha_1 + \alpha_2$ is proved exactly as in §4, using the fact that $K_{\alpha_1+\alpha_2} = K_{\alpha_1} K_{\alpha_2}$. The general case follows by (7.12). □

Theorem 7.5 also generalizes to multi-linear forms as is shown in Section 5.

Theorem 7.7. *Let $m > 2$, $\alpha > 0$, $\beta > 0$, $p = 1, 2$ or ∞, and $b \in \mathcal{H}(\mathbb{C}^n)$. Then $\int \bar{b} f_1 \cdot \ldots \cdot f_m \, d\mu_\beta$ is an S_p multilinear form on F^2_α if and only if $b \in F^p_{\beta 2/m\alpha}$.*

PROOF. By Theorem 5.2 if $\beta = m\alpha$; the general case follows by a multilinear version of (7.12). □

We end this section with some remarks on the norms in Theorem 7.5 and their dependence on n. For simplicity we take $\beta = \alpha$; the general case is covered by (7.12). We obtain from the estimates in §4 (cf. Remark 4.1) by straight-forward computations.

$$A_p \|b\|_{F^p_{\alpha/2}} \leqslant \|H_b\|_{S_p} \leqslant B_p \|b\|_{F^p_{\alpha/2}} \tag{7.13}$$

with

$$A_p = 2^{n \min(1/p,\, 1-1/p)} p^{-n/p}$$

and

$$B_p = 2^{n \max(1/p,\, 1-1/p)} p^{-n/p}.$$

In particular, $\|H_b\|_{S_2} = \|b\|_{F^2_{\alpha/2}}$. However, if $p \neq 2$,

$$b_p / A_p = 2^{n|1-2/p|} \to \infty \quad \text{as} \quad n \to \infty$$

(a consequence of the fact that $\varkappa = 2^n \to \infty$ as $n \to \infty$). The constants given above are not best possible, as we soon will see for $p = 4$, but simple examples (take $b = 1, z, \ldots$) show that $\|b\|$ and $\|H_b\|$ are not strictly proportional for any $p \neq 2, 4$, even when $n = 1$. By considering symbols of the type $b(z_1) \cdot b(z_2) \cdot \ldots \cdot b(z_n)$ (i.e. tensor products), we then easily see that it is impossible to have A_p and B_p in (7.13) independent of n, except when $p = 2$ or 4. Surprisingly, however, there is an exact result for $p = 4$.

Theorem 7.8. *It b is entire, then*

$$\|H_b\|_{S_2} = \|b\|_{F^2_{\alpha/2}} = \|b\|_{L^2(\mu_{\alpha/2})}$$

and

$$\|H_b\|_{S_4} = \|b\|_{F^4_{\alpha/2}} = \|b\|_{L^4(\mu_\alpha)}.$$

PROOF. The S_2 result was given above. Let us for notational convenience assume $n = 1$ and $\alpha = 1$, and let

$$b(z) = \sum_0^N b_j z^j$$

be a polynomial. $\{z^j(j!)^{-1/2}\}_0^\infty$ is an ON-basis in F^2_1. In this basis, H_b has the matrix representation $\{h_{jk}\}_{j,k \geq 0}$ with

$$h_{jk} = H_b(z^j(j!)^{-1/2}, z^k(k!)^{-1/2}) = \bar{b}_{j+k}(j+k)!(j!)^{-1/2}(k!)^{-1/2}.$$

Let $\tilde{H}_b$ be the corresponding linear operator $F^2_1 \to (F^2_1)^*$. $\tilde{H}^*_b\tilde{H}_b$ corresponds to the matrix $\{\sum h_{ji}\bar{h}_{jk}\}_{ik}$. Thus

$$\begin{aligned} \|H_b\|^4_{S_4} &= \|\tilde{H}^*_b\tilde{H}_b\|^2_{S_2} = Tr\tilde{H}^*_b\tilde{H}_b\tilde{H}^*_b\tilde{H}_b \\ &= \sum_{ijkl} h_{ji}\bar{h}_{jk}h_{lk}\bar{h}_{li} \qquad\qquad (7.14) \\ &= \sum_{ijkl} \bar{b}_{i+j}b_{j+k}\bar{b}_{k+l}b_{l+i}\frac{(i+j)!(j+k)!(k+l)!(l+i)!}{i!j!k!l!}. \end{aligned}$$

Let $a_i = b_i \cdot i!$. Then (7.14) may be written

$$\|H_b\|^4_{S_4} = \sum_{m=0}^\infty \frac{1}{m!}\Sigma_m,$$

with

$$\Sigma_m = \sum_{i+j+k+l=m} \frac{m!}{i!j!k!l!}\bar{a}_{i+j}a_{j+k}\bar{a}_{k+1}a_{l+i}.$$

A combinatorial argument, which we omit, shows that

$$\Sigma_m = \left| \sum_{p=0}^{m} \binom{m}{p} a_p a_{m-p} \right|^2 = \left| \sum_{p=0}^{m} m! b_p b_{m-p} \right|^2.$$

Hence

$$\|H_b\|_{S_4}^4 = \sum_{m=0}^{\infty} m! \left| \sum_{p=0}^{\infty} b_p b_{m-p} \right|^2 = \left\| \sum_{m=0}^{\infty} \left(\sum_{p=0}^{\infty} b_p b_{m-p} \right) z^m \right\|_{L^2(\mu_1)}^2$$

$$= \|b^2\|_{L^2(\mu_1)}^2 = \|b\|_{L^4(\mu_1)}^4 \quad \square$$

If we study the Fock space in infinitely many dimensions (a well-known object in physics), we obtain (at least formally, ignoring all questions of definition etc.)

$$H_b \in S_2 \Leftrightarrow b \in L^2(\mu_{\alpha/2}) \quad \text{and} \quad H_b \in S_4 \Leftrightarrow b \in L^2(\mu_\alpha).$$

We repeat that Theorem 7.8 does not extend to any other p. (In particular, interpolation between $p = 2$ and $p = 4$ is not possible!)

Problem. What happens on the infinite-dimensional Fock space for $p \neq 2, 4$ (in particular for $p = \infty$)?

8. Decomposition, Approximation and (Pointwise) Interpolation

Theorem 6.1 yields, replacing α by $\alpha/2$, the following.

Theorem 8.1. *$f \in F_\alpha^1$ if and only if*

$$f(z) = \sum_1^{\infty} a_j e^{\alpha\langle z, z_j\rangle - \alpha|z_j|^2/2},$$

for some sequences $\{z_j\} \subset \mathbb{C}^n$ and $\{a_j\} \in l^1$. $\square$

Let the Heisenberg group act on functions on $\mathbb{C}^n$ by $(w, t) \to e^{iat}C_\alpha(w)$, cf. (7.3)-(7.4). Then, by Section 6, we have the following.

Corollary 8.1. *F_α^1 is the smallest Heisenberg invariant Banach space that contains the constant functions. F_α^∞ is the largest Heisenberg invariant space such that $f \to f(0)$ is continuous.* $\square$

Theorem 8.1 says that the functions $k_z = K_z / \|K_z\|$ are atoms in F_α^1. We will show that suitable subsets of them can be employed as atoms in F_α^p also for $p > 1$.

We will call a set of points $\{z_j\} \subset \mathbb{C}^n$ ϵ-dense if every point of $\mathbb{C}^n$ is within distance ϵ of some z_j, i.e. if every ball with radius ϵ contains at least one z_j. We call the set separated if there exists a constant M such that any ball with radius 1 contains at most M points. (Any other fixed radius would do as well). In particular $\{z_j\}$ is separated if $\inf_{i \neq j} |z_i - z_j| > 0$. The lattice $\epsilon d^{-1/2}\mathbb{Z}^{2n}$ is ϵ-dense and separated.

Theorem 8.2. *There exists $\epsilon_0 > 0$ such that if $\{z_j\}$ is ϵ-dense with $\epsilon < \epsilon_0 \alpha^{-1/2}$ and separated, and $1 \leqslant p \leqslant \infty$, then $f \in F^p_\alpha$ if and only iff*

$$f(z) = \sum_1^\infty a_j e^{\alpha\langle z, z_j\rangle - \alpha|z_j|^2/2} \tag{8.1}$$

with $\{a_j\} \in l^p$ (and similarly for f^∞_α and c_0). The norm $\|f\|_{F^p_\alpha}$ is equivalent to $\inf \|\{a_j\}\|_{l^p}$ *within constants depending on α, ϵ and the constant in the separation definition.*

Remark 8.1. The coefficients a_j are not unique, but the proof shows that they may be chosen as continuous linear functions of f.

Remark 8.2. A characterization of the lattices $\{z_i\}$ for which $\{e^{\alpha\langle z, z_j\rangle}\}$ span F^2_α is given by Bargmann et al. (1971).

PROOF. We assume, without loss of generality, that $\alpha = 1$. Let $G = \mathbb{C}^n \times \mathbb{T}$ be the quotient group of the Heisenberg group defined by

$$(z, u) \circ (w, v) = (z + w, uv \exp(-i\, Im\langle z, w\rangle))$$

cf. the discussion after (7.4). As Haar measure on G we choose $\mathrm{dm}(z)|du|/2\pi^{n+1}$.

Given a function f on $\mathbb{C}^n$ we define Tf on G by

$$Tf(z, u) = uf(z)e^{-|z|^2/2}, \qquad (z, u) \in G = \mathbb{C}^n \times \mathbb{T}.$$

T is a linear isometry of F^p_1 onto a subspace of $L^p(G)$ (with the norm in F^p_1 suitably renormalized).

Let $\phi = T1$. We write in this proof

$$g = (z, u) \quad \text{and} \quad h = (w, v).$$

Thus $\phi(g) = ue^{-|z|^2/2}$ and

$$\begin{aligned}\phi(gh^{-1}) &= \phi(z - w, uv^{-1}e^{i\, Im\langle z, w\rangle}) = uv^{-1}e^{i\, Im\langle z, w\rangle - |z - w|^2/2} \\ &= uv^{-1}e^{\langle z, w\rangle - |z|^2/2 - |w|^2/2}.\end{aligned} \tag{8.2}$$

Consequently, if $F = Tf$, the reproducing formula (Theorem 7.1) yields,

$$\begin{aligned}\phi * F(g) &= \int_G \phi(gh^{-1})F(h)\,dh \\ &= \int_G uv^{-1}e^{\langle z,w\rangle - |z|^2/2 - |w|^2/2} v f(w) e^{-|w|^2/2}\,dm(w)|dv|/2\pi^{n+1} \\ &= ue^{-|z|^2/2}\int f(w)e^{\langle z,w\rangle}\,d\mu_1(w) \\ &= ue^{-|z|^2/2}f(z) \\ &= F(g).\end{aligned} \tag{8.3}$$

Let $N = [2\pi/\epsilon] + 1$ and $h_{jk} = (z_j, e^{2\pi ik/N})$, $1 \leqslant j < \infty$, $1 \leqslant k \leqslant N$. Partition G into disjoint sets G_{jk} such that $|hh_{jk}^{-1} - (0,1)| \leqslant 2\epsilon$ when $h \in G_{jk}$. ($(0,1)$ is the unity in G).

Define

$$RF = \left\{\int_{G_{jk}} F(g)\,dg\right\}_{1 \le j \le \infty,\, 1 \le k \le N}$$

and

$$\begin{aligned}S(\{a_{jk}\})(z) &= \sum_{j,k} a_{jk} e^{-2\pi ik/N + \langle z, z_j\rangle - |z_j|^2/2} \\ &= \sum_{j=1}^{\infty}\left(\sum_{k=1}^{N} a_{jk}e^{-2\pi ik/N}\right)e^{\langle z,z_j\rangle - |z_j|^2/2}.\end{aligned} \tag{8.4}$$

It is easily seen that $R\colon L^p(G) \to l^p$. Since

$$\begin{aligned}|S(\{a_{jk}\})(z)e^{-|z|^2/2}| &\leqslant \sum_{j,k}|a_{jk}|e^{-|z-z_j|^2/2} \\ &\leqslant N\sum_j e^{-|z-z_j|^2/2}\sup_{j,k}|a_{jk}| \\ &\leqslant C\sup|a_{jk}|,\end{aligned}$$

because $\{z_j\}$ is separated, $S(\{a_{jk}\}) \in F_1^\infty$ when $\{a_{jk}\} \in l^\infty$. Also,

$$\begin{aligned}\|S(\{a_{jk}\})\|_{F_1^1} &\leqslant \sum_{j,k}|a_{jk}|\,\|e^{\langle z,z_j\rangle}\|_{F_1^1}e^{-|z_j|^2/2} \\ &= C\|\{a_{jk}\}\|_{l^1}.\end{aligned}$$

By interpolation, $S\colon l^p \to F_1^p$, $1 \leqslant p \leqslant \infty$. Next we observe that, by (8.2),

$$\begin{aligned}TS(\{a_{jk}\})(g) &= \sum_{j,k} a_{jk}ue^{-2\pi ik/N}e^{\langle z,z_j\rangle - |z_j|^2/2 - |z|^2/2} \\ &= \sum_{j,k} a_{jk}\phi(gh_{jk}^{-1}).\end{aligned}$$

Hence, if $f \in F_1^\infty$ and $F = Tf$,

$$TSRF(g) = \sum_{j,k} \int_{G_{jk}} F(h)\, dh \phi(gh_{jk}^{-1})$$

and, by (8.3),

$$F(g) - TSRF(g) = \sum_{j,k} \int_{G_{jk}} (\phi(gh^{-1}) - \phi(gh_{jk}^{-1}))F(h)\, dh. \tag{8.5}$$

Define

$$\delta(g, \epsilon) = \sup \{ |\phi(g) - \phi(h)| : |g^{-1}h - (0, 1)| \leqslant \epsilon \}$$

and

$$\delta(\epsilon) = \int_G \delta(g, \epsilon)\, dg.$$

Note that $\delta(g, \epsilon) \to 0$ as $\epsilon \to 0$ for every g, and thus, by dominated convergence, $\delta(\epsilon) \to 0$ as $\epsilon \to 0$. (8.5) now yields

$$|F(g) - TSRF(g)| \leqslant \sum_{j,k} \int_{G_{jk}} \delta(gh^{-1}, 2\epsilon)|F(h)|\, dh \leqslant \delta(2\epsilon) \| F \|_{L^\infty}.$$

Thus

$$\| Tf - TSRTf \|_{L^\infty} \leqslant \delta(2\epsilon) \| Tf \|_{L^\infty}.$$

Since T is an isometry of F_1^∞ into $L^\infty(G)$, this gives

$$\| I - SRT \|_{F_1^\infty} \leqslant \delta(2\epsilon).$$

Similarly, if $f \in F_1^1$,

$$\begin{aligned} \| F - TSRF \|_{L^1(G)} &\leqslant \iint \delta(gh^{-1}, 2\epsilon)|F(h)|\, dh\, dg \\ &= \int \delta(2\epsilon)|F(h)|\, dh \\ &= \delta(2\epsilon) \| F \|_{L^1(G)} \end{aligned}$$

and

$$\| I - SRT \|_{F_1^1} \leqslant \delta(2\epsilon).$$

It follows by interpolation (Theorem 7.3), that if ϵ is sufficiently small, $\| I - SRT \|_{F_1^p} \leqslant 1$, whence SRT is invertible and S maps l^p onto F_1^p, for every p, $1 \leqslant p < \infty$. The conclusion of the theorem follows easily. □

As a corollary we obtain an approximation definition of F_α^p. Let P_N be the set of entire functions of the type $\sum_1^N a_j e^{\langle z, z_j \rangle}$, with $a_j \in \mathbb{C}$ and $z_j \in \mathbb{C}^n$,

$j = 1, \ldots, N$. (P_N is not a linear space!). If $b \in P_N$, then H_b has rank $\leqslant N$; the converse is almost true, see Section 14.

Theorem 8.3. *Let $\alpha > 0$ and $1 \leqslant p < \infty$. Then $f \in F^p_\alpha$ if and only if $f \in F^\infty_\alpha$ and the sequence $\{d_N\}_0^\infty \in l^p$, where $d_N = \inf\{\|f - g\|_{F^\infty_\alpha} : g \in P_N\}$ is the distance from f to P_N in F^∞_α.*

PROOF. If $f \in F^p_\alpha$, then (8.1) holds for suitable sequences $\{z_j\}$ and $\{a_j\} \in l^p$. Reordering the sequences simultaneously, we may assume that $\{a_j\}$ is decreasing. Then

$$d_N \leqslant \left\| \sum_{N+1}^{\infty} a_j e^{\alpha\langle z, z_j\rangle - \alpha|z_j|^2/2} \right\|_{F^\infty_\alpha} \leqslant C\|\{a_j\}_{N+1}^\infty\|_{l^\infty} = C|a_{N+1}|,$$

by another application of Theorem 8.2. Thus $\{d_N\} \in l^p$. Conversely, if $g \in P_N$ then H_g has rank $\leqslant N$ as a bilinear form on $F^2_{2\alpha}$. Thus, on that Hilbert space, $s_N(H_f) \leqslant \|H_f - H_g\|_{S_\infty(F^2_{2\alpha})} \leqslant C\|f - g\|_{F^\infty_\alpha}$, because of Theorem 7.5. Thus $s_N(H_f) \leqslant Cd_N$. Hence $\{d_N\} \in l^p$ implies that $H_f \in S_p(F^2_{2\alpha})$ and, by Theorem 7.5 again, $f \in F^p_\alpha$. □

In the classical case $H^2(\mathbb{T})$, Adamjan, Arov and Kreĭn (1971) have proved that $s_N(H_f) = \inf\{\|H_f - H_g\| : H_g$ is a *Hankel* operator of rank $\leqslant N\}$. Theorems 8.3 and 7.5 suggest that something similar may be true on the Fock space too, possibly in a weaker version such as

$$S_N(H_f) \leqslant C_1 \inf\{\|H_f - H_g\| : g \in P_{C_2 N}\}.$$

Another consequence of Theorem 8.2 is the following weak factorization, cf. Theorem 6.2. We do not know whether strong factorization is possible (as for Bergman spaces by Horowitz (1977)).

Corollary 8.2. *Let $\alpha = \alpha_0 + \alpha_1$ and $1/p = 1/p_0 + 1/p_1$ with $\alpha_0, \alpha_1 > 0$, $p_0, p_1 \leqslant \infty$ and $1 \leqslant p < \infty$. Then $f \in F^p_\alpha$ if and only if $f = \sum_1^\infty a_i g_i h_i$, for some sequences $\{g_i\}$ and $\{h_i\}$ in the unit balls of $F^{p_0}_{\alpha_0}$ and $F^{p_1}_{\alpha_1}$, respectively. The norm of f is equivalent to $\inf \sum |a_i|$, extended over all such representations.*

PROOF. Let $\{z_j\}$ be an ϵ-dense lattice, with ϵ sufficiently small. Any $f \in F^p_\alpha$ has a representation of the type (8.1), and it is easily seen that it suffices to consider functions that has a finite representation

$$f = \sum_1^N a_j e^{\alpha\langle z, z_j\rangle - \alpha|z_j|^2/2}.$$

Let $b_j = |a_j|^{p/p_0} \operatorname{sign}(a_j)$ and $c_j = |a_j|^{p/p_1}$, and define, for each sequence $I = (\iota_1, \ldots, \iota_N)$ with each $\iota_j = \pm 1$,

$$g_I = \sum_j \iota_j b_j e^{\alpha_0 \langle z, z_j \rangle - \alpha_0 |z_j|^2/2}, \qquad h_I = \sum_j \iota_j c_j e^{\alpha_1 \langle z, z_j \rangle - \alpha_1 |z_j|^2/2}.$$

It is easily seen that

$$f = 2^{-N} \sum_I g_I h_I$$

and, by Theorem 8.2 again, for each I,

$$\| g_I \|_{F^{p_0}_{\alpha_0}} \leqslant C \| \{ b_j \} \|_{l^{p_0}} = C \| \{ a_j \} \|_{l^p}^{p/p_0} \quad \text{and} \quad \| h_I \|_{F^{p_1}_{\alpha_1}} \leqslant C \| \{ a_j \} \|_{l^p}^{p/p_1}.$$

The remaining, simple, details are left to the reader. □

Theorem 8.2 has also an interesting dual.

Theorem 8.4. *Let $\{z_j\}$ be ϵ-dense and separated with $\epsilon < \epsilon_0 \alpha^{-1/2}$. If $1 \leqslant p \leqslant \infty$ and $f \in F^p_\alpha$, then*

$$C_1 \| \{ f(z_j) e^{-\alpha |z_j|^2/2} \} \|_{l^p} \leqslant \| f \|_{F^p_\alpha} \leqslant C_2 \| \{ f(z_j) e^{-\alpha |z_j|^2/2} \} \|_{l^p}$$

with C_1 and C_2 depending only on α, ϵ and the constant in the separation definition.

PROOF. If $p > 1$, let p' be the conjugate exponent. The linear mapping

$$\{ a_j \} \to \sum a_j e^{-\alpha |z_j|^2/2} K_{z_j}$$

is by Theorem 8.2 a quotient mapping of $l^{p'}$ onto $F^{p'}_\alpha$, whence the adjoint map, which maps f to $\{ f(z_j) e^{-\alpha |z_j|^2/2} \}$ is an isomorphism of $F^p_\alpha \cong (F^{p'}_\alpha)^*$ (cf. Theorem 7.4) into l^p. If $p = 1$, we use c_0 and f^∞_α. □

In fact, the left inequality in (8.6) holds as soon as $\{z_j\}$ is separated and the right inequality holds as soon as $\{z_j\}$ is ϵ-dense, because then $\{z_j\}$ can be enlarged or reduced, respectively, to become both separated and ϵ_1-dense (for any $\epsilon_1 > \epsilon$). In particular, if $f \in F^p_\alpha$ vanishes on an ϵ-dense set (ϵ small enough), then it vanishes identically.

The right inequality in (8.6) is not valid without some a priori assumption on f. (E.g., if $n = 1$, Weierstrass' theorem shows that f may vanish at every z_j but not elsewhere). It is easy to show that the condition $f \in F^p_\alpha$ may be relaxed to $f \in F^\infty_\alpha$. We have proved that the mapping $f \to \{ f(z_j) e^{-\alpha |z_j|^2/2} \}$ maps F^p_α into l^p, if (and, as it is easily seen, only if) $\{z_j\}$ is separated. If the set is sufficiently well separated, this map is also onto.

Theorem 8.5. *There exists $D < \infty$ such that, if $\{z_j\}$ is a sequence in $\mathbb{C}^n$ with $\inf_{i \neq j} |z_i - z_j| > D\alpha^{-1/2}$, and $1 \leqslant p \leqslant \infty$, a sequence $\{a_j\}$ of complex numbers equals $\{f(z_j)\}$ for some $f \in F^p_\alpha$ if and only if $\{a_j e^{-\alpha|z_j|^2/2}\} \in l^p$.*

PROOF. We may assume that $\alpha = 2$. Define

$$Tf = \{f(z_j)e^{-|z_j|^2}\} \quad \text{and} \quad S\{a_j\} = \sum a_j e^{2\langle z, z_j\rangle - |z_j|^2}.$$

Let

$$\delta = \inf_{i \neq j} |z_i - z_j| > 0.$$

T maps F^p_α into l^p by Theorem 8.4, and it is easy to see, interpolating between $p = 1$ and $p = \infty$, that S maps l^p into F^p_α. Furthermore,

$$TS\{a_j\} = \left\{\sum_i a_i e^{2\langle z_j, z_i\rangle - |z_i|^2 - |z_j|^2}\right\}$$

and thus, again interpolating between $p = 1$ and $p = \infty$,

$$\|I - TS\|_{l^p} \leqslant \sup_i \sum_{j \neq i} |e^{2\langle z_j, z_i\rangle - |z_i|^2 - |z_j|^2}| = \sup_i \sum_{j \neq i} e^{-|z_i - z_j|^2} < 1,$$

provided δ in large enough. Hence TS is invertible and T is onto. □

9. More on Projection, Duality and (Abstract) Interpolation

We have shown that the projection P_α is a bounded operator in $L^p_\alpha (1 \leqslant p \leqslant \infty)$, but it is also of interest to study the action of P_α on L^p_β when $\beta \neq \alpha$. In particular, this applies to the spaces $L^p(\mu_\alpha) = L^p_{2\alpha/p}$.

Theorem 9.1. *Let $\alpha > 0$, $\beta < 2\alpha$, $1 \leqslant p \leqslant \infty$. Then P_α maps L^p_β onto F^p_γ with $1/\gamma = 2/\alpha - \beta/\alpha^2$. P_α is not bounded on L^p_β unless $\beta = \alpha$.*

Remark 9.1. We may here allow $\beta \leqslant 0$. In particular, $P_\alpha(L^p) = F^p_{\alpha/2}$.

PROOF. We introduce explicitly the dilation and multiplication operators defined by

$$D_\delta f(z) = f(\delta z) \tag{9.1}$$
$$E_\epsilon f(z) = e^{\epsilon|z|^2} f(z), \tag{9.2}$$

the idea being that D_δ maps F^p_α isometrically onto $F^p_{\delta^2\alpha}$ $(\alpha, \delta > 0)$ and E_ϵ maps L^p_β isometrically onto $L^p_{\beta + 2\epsilon}$ $(\beta, \epsilon \in \mathbb{R})$. Furthermore, it follows by the same argument as in (7.10) that, for any $\alpha, \beta > 0$,

$$P_\alpha E_{\alpha-\beta} f(z) = \left(\frac{\alpha}{\beta}\right)^n P_\beta f\left(\frac{\alpha}{\beta} z\right),$$

(at least when $f \in L_\gamma^\infty$ for some $\gamma < 2\beta$), i.e.

$$P_\alpha E_{\alpha-\beta} = \left(\frac{\alpha}{\beta}\right)^n D_{\alpha/\beta} P_\beta. \tag{9.3}$$

Hence, substituting $2\alpha - \beta$ for β in (9.3),

$$\begin{aligned} P_\alpha(L_\beta^p) &= P_\alpha E_{\beta-\alpha}(L_{2\alpha-\beta}^p) = D_{\alpha/(2\alpha-\beta)} P_{2\alpha-\beta}(L_{2\alpha-\beta}^p) \\ &= D_{\alpha/(2\alpha-\beta)}(F_{2\alpha-\beta}^p) = F_{\alpha^2/(2\alpha-\beta)}^p. \end{aligned}$$

The last statement follows because $\gamma > \beta$ unless $\beta = \alpha$. □

Applying the theorem to $L^p(\mu_\alpha)$, we obtain (and refine) a result by Sjögren (1976).

Corollary 9.1. *Let $\alpha > 0$, $1 < p \leqslant \infty$ and $1/p + 1/p' = 1$. Then P_α maps $L^p(\mu_\alpha)$ onto $F_{p'\alpha/2}^p$. Hence P_α maps $L^p(\mu_\alpha)$ into $L^q(\mu_\alpha)$ $(0 < q \leqslant \infty)$ if and only if either $q < 4/p'$ or $p = q = 2$. P_α does not map $L^p(\mu_\alpha)$ into itself unless $p = 2$.*

PROOF. We may assume that $\alpha = 1$. Theorem 9.1 shows that P_1 maps $L^p(\mu_1) = L_{2/p}^p$ onto F_γ^p with $\gamma = (2 - 2/p)^{-1} = p'/2$, which is contained in $L^q(\mu_1) = L_{2/q}^q$ if and only if $2/q > \gamma = p'/2$ or $2/q = p'/2$ and $q \geqslant p$. Since $pp' > 4$ unless $p = 2$ (e.g. by the inequality between geometric and harmonic means), the latter case entails $p = q = 2$. □

Let $A^p(\mu_\alpha)$ be the space of analytic functions in $L^p(\mu_\alpha)$. Thus $A^p(\mu_\alpha) = F_{2\alpha/p}^p$. Sjögren (1976) used the above result to show that the dual of $A^p(\mu_\alpha)$ (for the pairing $\langle\ \rangle_\alpha$) is strictly larger than $A^{p'}(\mu_\alpha)$, unless $p = 2$. More generally, now we can prove the following.

Theorem 9.2. *Let $\alpha, \beta > 0$, $1 \leqslant p < \infty$ and $1/p + 1/p' = 1$. Then $(F_\beta^p)^* \cong F_{\alpha^2/\beta}^{p'}$ with the pairing $\langle\ \rangle_\alpha$. Similarly, $(f_\beta^\infty)^* \cong F_{\alpha^2/\beta}^1$.*

Remark 9.2. $\int f\bar{g}\, d\mu_\alpha$ does not necessarily converge when $f \in F_\beta^p$, $g \in F_{\alpha^2/\beta}^{p'}$, but $\langle\ \rangle_\alpha$ is easily extended. We omit the details.

PROOF. Since

$$\langle f, g\rangle_\beta = \left(\frac{\beta}{\alpha}\right)^n \langle f, E_{\alpha-\beta} g\rangle_\alpha, \qquad (L^p_\beta)^* \cong E_{\alpha-\beta} L^{p'}_\beta$$

with this pairing. The Hahn-Banach theorem and the fact that

$$\langle f, g\rangle_\alpha = \langle f, P_\alpha g\rangle_\alpha$$

shows, using (7.3), that

$$(F^p_\beta)^* \cong P_\alpha E_{\alpha-\beta}(L^{p'}_\beta) = D_{\alpha/\beta} P_\beta (L^{p'}_\beta) = D_{\alpha/\beta} F^{p'}_\beta = F^{p'}_{\alpha^2/\beta}.$$

$(f^\infty_\beta)^* \cong F^1_{\alpha^2/\beta}$ is proved similarly (using $C_0^* = M$), or by letting $p \to \infty$, noting that all constants stay bounded. □

Corollary 9.2. *If* $1 \leqslant p < \infty$, *then* $A^p(\mu_\alpha)^* \cong F^{p'}_{p\alpha/2}$, *with the pairing* $\langle\ \rangle_\alpha$. □

If $p \neq 2$, $p\alpha/2 > 2\alpha/p'$, whence $F^{p'}_{p\alpha/2} \supsetneqq F^{p'}_{2\alpha/p'} = A^{p'}(\mu_\alpha)$, and we recover Sjögren's result.

We may now extend the interpolation theorem (Theorem 7.3) for the complex method. Since Fock spaces with different values of α are related by dilations, this is related to interpolation between spaces of functions defined in different discs, cf. Lions and Peetre (1964).

Theorem 9.3. *Let* $\alpha_0, \alpha_1 > 0$, $1 \leqslant p_0, p_1 \leqslant \infty$, *and* $0 < \theta < 1$. *Let* $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$ *and* $\alpha_\theta = \alpha_0^{1-\theta}\alpha_1^\theta$. *Then*

$$[F^{p_0}_{\alpha_0}, F^{p_1}_{\alpha_1}]_\theta = F^{p_\theta}_{\alpha_\theta}, \qquad p_\theta < \infty, \tag{9.4}$$

$$[F^\infty_{\alpha_0}, F^\infty_{\alpha_1}]_\theta = f^\infty_{\alpha_\theta}, \qquad \alpha_0 \neq \alpha_1, \tag{9.5}$$

(*Note that* $p_\theta = \infty$ *if and only if* $p_0 = p_1 = \infty$). (9.4) *and* (9.5) *hold also with* $F^\infty_{\alpha_j}$ *replaced by* $f^\infty_{\alpha_j}$ ($j = 0$ *or* 1) *on the left hand side.*

PROOF. Define, for $\zeta \in \mathbb{C}$, $T_\zeta = D_{\alpha_0/\alpha_1}(\zeta - \theta)/2$, i.e.

$$T\zeta f(z) = f\left(\left(\frac{\alpha_0}{\alpha_1}\right)^{(\zeta-\theta)/2} z\right). \tag{9.6}$$

T_ζ is an isometry of $F^{p_0}_{\alpha_0}$ onto $F^{p_0}_{\alpha_\theta}$ when $\operatorname{Re}\zeta = 0$, and of $F^{p_1}_{\alpha_1}$ onto $F^{p_1}_{\alpha_\theta}$ when $\operatorname{Re}\zeta = 1$.

Furthermore, $T_\zeta f$ and $T_\zeta^{-1} f = T_{-\zeta} f$ are analytic in ζ (when f is analytic). Hence, by the abstract Stein interpolation therorem, see e.g. Cwikel and Janson (1984), Theorem 1, $T_\theta = I$ is an isometry of $[F^{p_0}_{\alpha_0}, F^{p_1}_{\alpha_1}]_\theta$ onto

$$[F^{p_0}_{\alpha_\theta}, F^{p_1}_{\alpha_\theta}]_\theta = F^{p_\theta}_{\alpha_\theta}$$

(using Theorem 7.3), provided p_0 and $p_1 < \infty$.

The same argument holds if $p_0 = \infty$ or $p_1 = \infty$, provided we use f^∞ instead of F^∞. However, an extra argument is needed for F^∞, because e.g. $t \to T_{it}f$ is, in general, not a continuous map of $\mathbb{R}$ into $F^\infty_{\alpha_0}$ when $f \in F^\infty_{\alpha_0}$, cf. Cwikel and Janson (1984). One possibility is to use duality (Theorem 9.2) and the result just proved (with p_0', α_0^{-1} etc.) to conclude that, see e.g. Bergh & Löfström (1976), Theorem 4.5.1,

$$[F^{p_0}_{\alpha_0}, F^{p_1}_{\alpha_1}]^\theta = F^{p_\theta}_{\alpha_\theta}, \qquad 1 \leqslant p_0, p_1 \leqslant \infty. \tag{9.7}$$

Berg (1979) proved that, for any Banach couple $[X_0, X_1]_\theta$ equals the closed hull of $X_0 \cap X_1$ in $[X_0, X_1]^\theta$, which gives (9.4) and (9.5) from (9.7). □

Remark 9.3. Also (9.7) remains valid if $F^\infty_{\alpha_j}$ is replaced by $f^\infty_{\alpha_j}$ on the left hand side (except in the trivial case $p_0 = p_1 = \infty$, $\alpha_0 = \alpha_1$). This follows from (9.4) and (9.7) when p_0 or p_1 is finite. That

$$[f^\infty_{\alpha_0}, f^\infty_{\alpha_1}] = F^\infty_{\alpha_\theta} \qquad (\alpha_0 \neq \alpha_1)$$

follows directly from the definition (Bergh & Löfström (1976), Chapter 4), taking

$$g(w) = \int_0^w T_{-\zeta} f \, d\zeta$$

with T as in (9.6) and $f \in F^\infty_{\alpha_\theta}$.

Note that α_θ is the (weighted) geometric mean of α_0 and α_1, while

$$[L^{p_0}_{\alpha_0}, L^{p_1}_{\alpha_1}]_\theta = L^{p_\theta}_{(1-\theta)\alpha_0 + \theta\alpha_1}$$

with the arithmetic mean of α_0 and α_1.

Exercise 9.1. Let us review Theorem 9.1 in the light of Theorem 9.3. First, Theorem 9.1 plainly may be restated as follows: Given α, the set of pairs (β, γ) such that $P_\alpha : L^p_\beta \to F^p_\gamma$ is precisely given by the inequality $\alpha^2/\gamma \leqslant 2\alpha - \beta$.

We leave it to the reader to show that this region cannot be enlarged by the interpolation theorem.

10. Addenda

10.1. Convolutions. The Hankel operator, which is a modified multiplication, is surprisingly also a convolution operator on the Fock space.

Theorem 10.1. *If $b \in F^{\infty}_{\alpha/2}$ and $f, g \in F^2_\alpha$, then*

$$\tilde{H}_b f(z) = \langle b(z + \cdot), f \rangle_\alpha = \int b(z + w)\overline{f(w)}\, d\mu_\alpha(w) \tag{10.1}$$

and

$$H_b(f, g) = \iint \overline{b(z + w)} f(z) g(w)\, d\mu_\alpha(z)\, d\mu_\alpha(w). \tag{10.2}$$

PROOF. Since both sides of (10.1) are continuous anti-linear functionals of f, it suffices to verify the formula when $f = K_\zeta$ for some ζ. Then

$$\begin{aligned}\tilde{H}_b K_\zeta(z) &= \langle \tilde{H}_b K_\zeta, K_z \rangle = \overline{H_b(K_z, K_\zeta)} = \langle b, K_z K_\zeta \rangle_\alpha \\ &= \langle b, K_{z+\zeta} \rangle = b(z + \zeta) \\ &= \int b(z + w) K_\zeta(w)\, d\mu_\alpha(w).\end{aligned}$$

(10.2) follows. □

10.2. Finite rank. It follows from (10.1) that (for $b \in F^{\infty}_{\alpha/2}$) $\tilde{H}_b K_w = b(\cdot + w)$. Hence the linear span of the set of translates of b is a dense subspace of the range of $\tilde{H}_b$. In particular, invoking our general Kronecker's theorem (Corollary 14.1):

Theorem 10.2. *The following are equivalent* (*for b entire*).

(i) *H_b has finite rank.*
(ii) *Span $\{b(\cdot + w)\}$ has finite dimension.*
(iii) $b(z) = \sum_{j=1}^{m} \sum_{|\nu| \le k_j} c_{j\nu} z^\nu e^{z\bar{w}_j}$ *for some m, k_j, w_j.*
(iv) *$b \in \bar{P}_N$ for some N, with P_N as in §8.* (*Closure e.g. in F^∞_α*).

If $n = 1$, we can add:

(v) *$Db = 0$ for some constant coefficient linear differential operator D.*

10.3. An abstract characterization. Let H be any Hankel form on $F^2_\alpha(\mathbb{C})$. Then it is abstractly characterized by (cf. the introduction) the property

$$H(zf, g) = H(f, zg).$$

Notice that in terms of the associated *anti-linear* Hankel operator $\tilde{H}$ this can be written as

$$\tilde{H}A^* = A\tilde{H}$$

where, taking $\alpha = 1$, A^* and A are the creation and annihilation operator respectively, $Af = f'$, $A*f = zf$. This should be contrasted with the well-known abstract characterization of Hankel operators $\tilde{H}_b\colon H^2(\mathbb{T}) \to H^2(\mathbb{T})$ using the shift operator S, $Sf = zf$, and its compression to $H^2_-(\mathbb{T})$ (cf. e.g. Nikol'skii (1986), p. 180).

10.4. Other Gaussian measures. Let A be a positive definite matrix. Then we may define

$$\mu_A = \frac{\det A}{\pi^N} e^{-\langle Az, z\rangle}\, dm$$

and $F^2_A = A^2(\mu_A)$. The results of §7 extend immediately (because F^2_A is mapped onto F^2_α by a (linear) change of variables). The results of §9 extend too

$$(F^p_A)^* \cong F^{p'}_{A^{-1}} \quad \text{for} \quad 1 \leqslant p < \infty \qquad (\text{while } L^p_A \cong L^{p'}_{2-A})$$

in the duality $\langle\ \rangle_{\mu_1}$, and

$$[F^{p_0}_{A_0}, F^{p_1}_{A_1}]_\theta = F^{p_\theta}_{A_\theta} \qquad (p_0 < \infty)$$

for some A_θ (a «geometric mean» of A_0 and A_1).

11. Fock Space with a Different Gauge

In this section we for simplicity consider only the case $n = 1$. We write $z = x + iy$ and $w = u + iv$.

The gauge transformation $f(z) \to e^{-\alpha z^2/2} f(z)$ maps F^p_α onto the space of entire functions g such that

$$|g(z)| e^{\operatorname{Re} \alpha z^2/2 - \alpha|z|^2/2} = |g(z)| e^{-\alpha y^2} \in L^p(dx\, dy).$$

We denote this space by G^p_α. In particular, G^2_α is the Hilbert space

$$A^2\left(\frac{\alpha}{\pi} e^{-2\alpha y^2}\, dx\, dy\right).$$

The reproducing kernel for G^2_α is, by Proposition 1.2 and (7.2),

$$K(z, w) = e^{-\alpha z^2/2 - \alpha \bar{w}^2/2 + \alpha z \bar{w}} = e^{-\alpha(z - \bar{w})^2/2} \tag{11.1}$$

Thus $K(z, z) = e^{-\alpha(2iy)^2/2} = e^{2\alpha y^2}$ and $\omega(z) = e^{-2\alpha y^2}$. Consequently, $A^p_\omega = G^p_{2\alpha}$, $1 \leqslant p \leqslant \infty$, cf. §7. It follows from (7.3) that the Heisenberg group acts isometrically in every G^p_α by $(w, t) \to e^{i\alpha t} C'_\alpha(w)$, with

$$\begin{aligned} C'_\alpha(w) g(z) &= e^{-\alpha z^2/2} e^{\alpha(z - w)^2/2} g(z - w) e^{\alpha z \bar{w} - \alpha|w|^2/2} \\ &= g(z - w) e^{i\alpha v(w - 2z)} \end{aligned} \tag{11.2}$$

Remark 11.1 From the group theory point of view this is something which lives «half way» between the Bargmann-Segal representation of the Heisenberg group, which lives in Fock space $F^2_\alpha(\mathbb{C})$, and the Heisenberg representation of the same group, which lives in $L^2(\mathbb{R})$.

Let $g \in G^2_\alpha$, let $g_y(x) = g(x + iy)$ and let $\hat{g} \in S'(\mathbb{R})$ be the Fourier transform of $g(x)$, $x \in \mathbb{R}$. Then $\hat{g}_y(\xi) = e^{-y\xi}\hat{g}(\xi)$ and, by Plancherel's theorem,

$$\begin{aligned}\|g\|^2_{G^2_\alpha} &= \frac{\alpha}{\pi}\int |f(z)|^2 e^{-2\alpha y^2}\,dx\,dy \\ &= \frac{\alpha}{\pi}\int \|g_y\|^2 e^{-2\alpha y^2}\,dy \\ &= \frac{\alpha}{2\pi^2}\int e^{-2y\xi}|\hat{g}(\xi)|^2 e^{-2\alpha y^2}\,d\xi\,dy \\ &= \frac{\alpha^{1/2}}{(2\pi)^{3/2}}\int |\hat{g}(\xi)|^2 e^{\xi^2/2\alpha}\,d\xi.\end{aligned} \tag{11.3}$$

Conversely, it is easily seen that if

$$\int |h(\xi)|^2 e^{\xi^2/2\alpha}\,d\xi < \infty,$$

then $h = \hat{g}$ for some $g \in G^2_\alpha$. Thus, $g \to \hat{g}(\xi)e^{\xi^2/4\alpha}$ is an isomorphism of G^2_α onto $L^2(\mathbb{R})$; with a suitable constant factor we obtain an isometry.

Remark 11.2. The composition of the isometry $F^2_\alpha \to G^2_\alpha$ given above and this isometry is the Bargmann transform $F^2_\alpha \to L^2(\mathbb{R})$ which maps the function z^k to the k:th Hermite function h_k ($e^{-\xi^2/4\alpha}$ times the k:th Hermite polynomial; the normalization depends on α). Now let $g \in G^p_\alpha$ for some p, $1 \leqslant p \leqslant \infty$, let $\gamma(\xi) = \hat{g}(\xi)e^{\xi^2/4\alpha}$ and $\phi(\xi) = e^{-\xi^2/4\alpha}$. Then

$$\hat{g}_y(\xi) = \hat{g}(\xi)e^{-y\xi} = \gamma(\xi)\phi(\xi + 2\alpha y)e^{\alpha y^2}$$

and thus

$$\begin{aligned}\|g\|_{G^p_\alpha} &= \|\, \|g_y\|_{L^p(dx)} e^{-\alpha y^2}\|_{L^2(dy)} \\ &= \|\, \|\gamma(\xi)\phi(\xi + 2\alpha y)\|_{L^p}\|_{L^p} \\ &= (2\alpha)^{-1/p}\|\, \|\gamma(\xi)\phi(\xi + y)\|_{L^p}\|_{L^p}.\end{aligned}$$

Again the converse holds, i.e. the mapping $g \to \gamma$ maps G^p_α onto the space of all distributions γ such that $\|\, \|\gamma \cdot \phi(\bullet + y)\|_{L^p}\|_{L^p} < \infty$. Furthermore, the latter space is a kind of generalized Besov space defined by translations instead of dilations of the kernel ϕ (cf. Peetre (1976), Chapter 10 with, formally, $Af = f(\bullet + i)$ on $L^2(\mathbb{R})$), and it is seen, just as for ordinary Besov spaces, that the space remains the same if ϕ is replaced by an arbitrary test function (except 0). Also, it suffices to restrict y to the integers, provided some non-degeneracy condition holds. We omit the details.

Theorem 11.1. *Let $\phi \in C_0^\infty(\mathbb{R})$ with $\phi \not\equiv 0$. Then the space*

$$E_p = \{\gamma \in \mathfrak{D}' : \| \, \|\phi(\cdot + y)\gamma\|_{\hat{L}^p} \|_{L^p(dy)} < \infty\} \tag{11.4}$$

does not depend on the choice of ϕ, and if $\phi \neq 0$ on [0, 1],

$$E_p = \{\gamma \in \mathfrak{D}' : \| \, \|\phi(\cdot + n)\gamma\|_{\hat{L}^p} \|_{l^p} < \infty\}.$$

The mapping $g \to \hat{g}(\xi)e^{\xi^2/4\alpha}$ is an isomorphism of G_α^p onto E_p, $1 \leqslant p \leqslant \infty$. □

Note that E_p does not depend on α. Define mappings M_a by

$$\widehat{M_a g(\xi)} = \hat{g}(\xi)e^{a\xi^2}.$$

Corollary 11.1. *Let $\alpha, \beta > 0$ and $a = 1/4\alpha - 1/4\beta$. Then M_a is a isomorphism of G_α^p onto G_β^p, $1 \leqslant p \leqslant \infty$.* □

We are now prepared to deal with the Hankel forms Γ_b and H_b on G_α^2. Cf. Theorem 7.5 and recall that the results for Γ_b are gauge invariant, but as we see here, not those for H_b.

Theorem 11.2. *Let b be entire and $1 \leqslant p \leqslant \infty$. Then $\Gamma_b \in S_p$ if and only if $b \in G_{2\alpha}^p$, i.e. if and only if $b(z)e^{-2\alpha y^2} \in L^p(dx\,dy)$, and $H_b \in S_p$ if and only if $b \in G_{2\alpha/3}^p$, i.e. if and only if $b(z)e^{-2\alpha y^2/3} \in L^p(dx\,dy)$.*

PROOF. The result for Γ_b follows by Theorem 4.6. The result for H_b follows by Corollary 6.1 once we note that $Q(\omega^{-1}b) = cM_{1/4\alpha}b$ for some constant c and thus $Q(\omega^{-1}b) \in G_{2\alpha}^p$ if and only if $b \in G_{2\alpha/3}^p$ by Corollary 11.1. The latter formula follows by (6.8) and (11.3):

$$\begin{aligned}\langle Q(\omega^{-1}b), f\rangle_\nu = \langle b, f\rangle_\mu &= C_1 \int \hat{b}(\xi)\overline{\hat{f}(\xi)}e^{\xi^2/2\alpha}\,d\xi \\ &= C_1 \int (M_{1/4\alpha}b)^\wedge(\xi)\overline{\hat{f}(\xi)}e^{\xi^2/4\alpha}\,d\xi = C_2\langle M_{1/4\alpha}b, f\rangle_\nu. \quad \square\end{aligned}$$

One can similarly show, more generally, that if $0 < \beta < 4\alpha$, then the Hankel form H_b^ξ with $d\xi = e^{-2\beta y^2}\,dx\,dy$ belongs to $S_p(G_\alpha^2)$ if and only if $b \in G_\gamma^p$ with $1/\gamma = 2/\beta - 1/2\alpha$.

Remark 11.3 The spaces E_p are interesting in their own right . They are essentially special cases of more general function spaces studied intensively (also in the context of general locally compact Abelian groups) by Feichtinger. Here we briefly recapitulate some of their salient properties. They form an increasing scale of spaces of distributions, with $E_2 = L^2$. E_1 is the minimal strongly character invariant Segal algebra, see, e.g., Feichtinger (1981*a*), (1981*b*). The spaces are translation invariant (isometrically) and dilation invariant. They are also preserved by (Feichtinger) the Fourier and (new!) Mehler transforms. This follows because F^p_α is mapped onto E_p by the Bargmann transform (see Remark 11.2), which maps z^j to h_j with, for $\alpha = 1/2$, $\hat{h}_j = (2\pi)^{1/2}(-i)^j h_j$. Hence the Bargmann transform intertwines the rotation $f \mapsto f(iz)$ (which obviously preserves $F^p_{1/2}$) and the Fourier transform on E_p. (In particular, we may take a Fourier transform in (11.4) and obtain the definition of E_p given in the introduction). More generally, the Mehler transform $h_j \to \zeta^j h_j$ (ζ is fixed with $|\zeta| \leqslant 1$) corresponds to $f \mapsto f(\zeta z)$ in $F^p_{1/2}$. (Cf. Peetre (1980)).

Note also that Theorems 7.3 and 7.4 imply (already in Feichtinger (1981*b*))

$$[E_{p_0}, E_{p_1}]_\theta = (E_{p_0}, E_{p_1})_{\theta p_\theta} = E_{p_\theta}, \tag{11.5}$$

and $(E_p)^* \cong E_{p'}$ $(1 \leqslant p < \infty)$, with the usual pairing on $\mathbb{R}$.

Remark 11.4. An argument similar to the proof of Theorem 9.3, using the operators M_a with a complex, can be employed to show that (11.5) implies (for $p_\theta < \infty$)

$$[G^{p_0}_{\alpha_0}, G^{p_1}_{\alpha_1}]_\theta = G^{p_\theta}_{\alpha_\theta} \tag{11.6}$$

with p_θ as before and $1/\alpha_\theta = (1-\theta)/\alpha_0 + \theta/\alpha_1$.

Thus α_θ is the harmonic mean of α_0 and α_1, while we obtain the geometric mean for F^p_α (Theorem 9.3) and the arithmetic mean for L^p_α. In fact, both these results can be understood from the point of view of the Shale-Weil representation of the so-called metaplectic groups. (Cf. again Peetre (1980)).

12. Bergman Spaces in a Ball

In this section we study the case $\Omega =$ the open unit ball in $\mathbb{C}^n$ and

$$d\mu = c(1 - |z|^2)^\gamma \, dm, \tag{12.1}$$

where $\gamma > -1$ is fixed, m is the Lebesgue measure, and

$$c = \pi^{-n}\Gamma(n + \gamma + 1)/\Gamma(\gamma + 1)$$

is a normalization constant making $\mu(\Omega) = 1$. Thus $A^2(\mu)$ is the (weighted) Bergman space in the unit ball.

It is easily seen that $\{z^\alpha\}$, where α ranges over the set of multiindices, is an orthogonal basis in $A^2(\mu)$. An integration shows that

$$\|z^\alpha\|^2 = \frac{\alpha!\Gamma(\gamma+n+1)}{\Gamma(|\alpha|+\gamma+n+1)} \tag{12.2}$$

and thus the reproducing kernel is

$$\begin{aligned} K(z, w) &= \sum_\alpha z^\alpha \bar{w}^\alpha / \|z^\alpha\|^2 \\ &= \sum_{m=0}^{\infty} \langle z, w\rangle^m \Gamma(m+\gamma+n+1)/\Gamma(\gamma+n+1) \\ &= (1-\langle z, w\rangle)^{-\gamma-n-1} \end{aligned} \tag{12.3}$$

(Cf. Rudin (1980), Chapter 7.1.)

The group $\operatorname{Aut}(\Omega)$ is the Möbius group $PSU(n, 1)$ (Cf. Rudin (1980), Chapter 2). Every automorphism acts on μ as an analytic gauge transformation, i.e. $G(\mu) = \operatorname{Aut}(\Omega) = PSU(n, 1)$. Since the Möbius group is transitive and isotropic in the sense of (2.5) (choose $z = 0$ for convenience), the results Sections 2-6 apply. The invariant measure is by Corollary 2.1 and (12.3)

$$d\lambda(z) = K(z, z)\, d\mu(z) = c(1-|z|^2)^{-n-1}\, dm(z). \tag{12.4}$$

(Conversely, since this measure can be shown directly to be invariant, Proposition 1.5 yields an alternative proof of (12.3)). We observe that

$$\omega(z) = (1-\|z|^2)^{\gamma+n+1}. \tag{12.5}$$

An elementary computation yields

$$\varkappa = \prod_{j=1}^{n} \frac{n+2\gamma+1+j}{\gamma+j}$$

We will, after some preliminaries, apply the general theory to the Hankel forms H_b and Γ_b on $A^2(\mu)$. The same argument yields similar results for Hankel forms H_b^ξ with $\xi = c(1-|z|^2)^\beta$ for any $\beta > 0$, but that is left as an exercise for the reader. See also Burbea (1986), where (independently) this type of Hankel operator is treated by a different method.

It is convenient to relate the Bergman spaces to the (analytic) Besov spaces on the unit sphere $S^{2n-1} \subset \mathbb{C}^n$. These Besov spaces can be defined as follows, in complete analogy with the standard Besov spaces on $\mathbb{R}$, cf. e.g. Peetre (1976), Ahlmann (1984), Mitchell and Hahn (1976). Let $\phi \in L^1(\mathbb{R})$ with $\hat{\phi} \in C_0^\infty(0, \infty)$ and define, for any analytic function (or formal power series) $f(z) = \sum \hat{f}(\alpha) z^\alpha$,

$$\phi * f = \int \phi(e^{is}) f(e^{-is}z)\, ds = \sum \hat{\phi}(|\alpha|)\hat{f}(\alpha)z^{\alpha}. \tag{12.6}$$

Define $\phi_t(x) = t^{-1}\phi(x/t)$; thus $\hat{\phi}_t(\xi) = \hat{\phi}(t\xi)$, and

$$B_s^{pq} = \left\{ f\colon \int_0^\infty (t^{-s}\|\phi_t * f\|_{L^p(S^{2n-1})})^q \frac{dt}{t} < \infty \right\}. \tag{12.7}$$

Hence $-\infty < s < \infty$, $1 \leqslant p \leqslant \infty$ and $1 \leqslant q \leqslant \infty$ (with the standard modification if $q = \infty$). These spaces are independent of the choice of ϕ; furthermore, it is equivalent to use only the discrete values $\{2^{-k}\}_0^\infty$ for t. In the sequel we let $B_s^p = B_s^{pp}$.

When $s < 0$, we may in the definition of B_p^{sq} allow $\hat{\phi}(\xi) = e^{-\xi}$, $\xi > 0$ (and $\hat{\phi}(0) = 0$; thus $\phi \notin L^1$, but that does not matter). This choice gives

$$\phi_t * f(z) = \sum_{\alpha \neq 0} e^{-t|\alpha|}\hat{f}(\alpha)z^{\alpha} = f(e^{-t}z) - f(0).$$

Restricting attention to $t \leqslant 1$, as we may do, and changing variables, we find that, when $s < 0$,

$$\begin{aligned} f \in B_s^{pq} &\Leftrightarrow t^{-s}\|f(e^{-t}z)\|_{L^p(S^{2n-1})} \in L^q((0,1), dt/t) \\ &\Leftrightarrow (1-r)^{-s}\|f(rz)\|_{L^p(S^{2n-1})} \in L^q((0,1),(1-r)^{-1}\,dr). \end{aligned} \tag{12.8}$$

Hence these Besov spaces coincide with the weighted Bergman spaces in the unit ball. In particular, in the important case $q = p$,

$$B_s^p = \{ f\colon (1-|z|^2)^{-s} f(z) \in L^p(\Omega, (1-|z|^2)^{-1}\,dm)\}, \tag{12.9}$$

provided $s < 0$.

We now see that $A^2(\mu) = B^2_{-(\gamma+1)/2}$ and, cf. the definitions in §3,

$$A_\omega^p = \{ f\colon (1-|z|^2)^{\gamma+n+1} f(z) \in L^p((1-|z|^2)^{-n-1}\,dm)\} = B^p_{n/p-n-1-\gamma}.$$

Define

$$Df = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}.$$

Thus $Dz^\alpha = |\alpha| z^\alpha$. The Taylor coefficient multiplier D^u gives an isomorphism of B_s^{pq} onto B_{s-u}^{pq} (modulo constants) for any real s, u and $1 \leqslant p,\ q \leqslant \infty$. Similarly, $f \in B_s^{pq}$ if and only if $\partial f/\partial z_j \in B_{s-1}^{pq}$, $j = 1, \ldots, n$. This yields a way to extend (12.8) and (12.9) to $s \geqslant 0$; e.g.

$$f \in B_0^\infty \Leftrightarrow (1-|z|^2)|Df(z)| \leqslant C \Leftrightarrow (1-|z|^2)|\nabla f(z)| \leqslant C$$

(B_0^∞ is the n-dimensional Bloch space).

We will also consider a related multiplier. Taking $f = z^\alpha$ (and $\xi = \mu$) in (6.8), we see that

$$\langle Q(\omega^{-1}b), z^\alpha \rangle_\nu = \langle b, z^\alpha \rangle_\mu = \hat{b}(\alpha) \|z^\alpha\|_\mu^2,$$

and thus

$$Q(\omega^{-1}b) = \sum_\alpha \psi(\alpha)\hat{b}(\alpha)z^\alpha,$$

with

$$\psi(\alpha) = \|z^\alpha\|_\mu / \|z^\alpha\|_\nu^2 = \Gamma(|\alpha| + 2\gamma + 2n + 2) / \Gamma(|\alpha| + \gamma + n + 1)$$

(cf. (12.2)). It is easy to show that the multiplier $\psi(\alpha)|\alpha|^{-\gamma-n-1}$ maps any space B_s^{pq} onto itself. Hence,

$$Q(\omega^{-1}b) \in A_w^p = B_{n/p-n-1-\gamma}^p$$

if and only if

$$D^{n+1+\gamma} b \in B_{n/p-n-1-\gamma}^p$$

if and only if

$$b \in B_{n/p}^p.$$

The results of §§4 and 6 thus yields the following. (To see that (6.6) holds, let $g(z, w) = \langle f, K_z K_w \rangle$ and note that g is analytic and

$$\begin{aligned} D_z^\alpha D_w^\beta g(z, w)|_{w=z} &= \text{const} \langle f(\zeta), \zeta^\alpha (1 - \langle \zeta, z \rangle)^{-n-1-\gamma-|\alpha|} \\ &\qquad \zeta^\beta (1 - \langle \zeta, z \rangle)^{-n-1-\gamma-|\beta|} \rangle \\ &= \text{const}\, D_z^{\alpha+\beta} g(z, z).) \end{aligned}$$

Theorem 12.1 *Let b be analytic and $1 \leqslant p \leqslant \infty$. Then $\Gamma_b \in S_p$ if and only if $b \in B_{n/p-n-1-\gamma}^p$ if and only if $(1 - |z|^2)^{n+1+\gamma} b(z) \in L^p((1 - |z|^2)^{-n-1} dm)$ and $H_b \in S_p$ if and only if $b \in B_{n/p}^p$. In particular, H_b is bounded if and only if $|\nabla b(z)| = O((1 - |z|^2)^{-1})$, Γ_b is compact if and only if $|\nabla b(z)| = o((1 - |z|^2)^{-n-1-\gamma})$ and H_b is compact if and only if $|\nabla b(z)| = o((1 - |z|^2)^{-1})$ as $|z| \to 1$.* □

When $n = 1$, this result is due to Peller (1982). Note that $H^2(\mathbb{T})$ formally is the limit of $A^2(\mu)$ as $\gamma \to -1$, and recall that the result above for H_b is valid on $H^2(\mathbb{T})$ too, provided $p < \infty$, see Peller (1980), (1982).

For multilinear Hankel forms we obtain by Theorem 5.2 and arguments as above the following, with $H_b(f_1, \ldots, f_m) = \int \bar{b} f_1 \cdot \ldots \cdot f_m \, d\mu$.

Theorem 12.2. *Let $m > 2$ and $p = 1, 2$ or ∞. Then, if b is analytic, the m-linear form $\Gamma_b \in S_p$ if and only if $(1 - |z|^2)^{m(n+1+\gamma)/2} \in L^p((1 - |z|^2)^{-n-1} \, dm)$ if and only if $b \in B^p_{-m(n+1+\gamma)/2+n/p}$ and $H_b \in S_p$ if and only if $b \in B^p_{(m/2-1)(\gamma+n+1)+n/p}$.* □

The same result is true for H_b on $H^2(\mathbb{T})$ too, cf. Peetre (1985), Lecture 5.

Theorems on decomposition, approximation and interpolation for Bergman spaces are given by Coifman and Rochberg (1980) and Rochberg (1985).

Theorem 3.1 shows that the Bergman projection Q is bounded on L^p_ω. The problem of telling exactly when such a projection is bounded on $L^p(\Omega, m)$ is solved by the Forelli-Rudin Theorem, see Rudin (1980), Chapter 7.1.

13. Bergman Spaces in a Half Plane

In this section we consider Ω = the upper half plane ($n = 1$) and $d\mu = y^\gamma \, dx \, dy$, $\gamma > -1$. Thus, by Plancherel's formula,

$$\begin{aligned} \|f\|^2_{A^2(\mu)} &= \iint |f(z)|^2 y^\gamma \, dx \, dy = \int_0^\infty \int_0^\infty |e^{-\xi y} \hat{f}(\xi)|^2 y^\gamma \, dy \, d\xi \\ &= c_\gamma \int_0^\infty |\hat{f}(\xi)|^2 \xi^{-\gamma-1} \, d\xi \qquad (z = x + iy). \end{aligned} \tag{13.1}$$

Hence the (weighted) Bergman space $A^2(\mu)$ equals the analytic Besov space $B^2_{-(\gamma+1)/2}$. (See Peetre (1976) for definition and properties of Besov spaces. We do not distinguish between functions in Ω and their boundary values on $\mathbb{R}$).

The change of variables $z \to (z - i)/(z + i)$ maps Ω onto the unit disc, and the measure $c(1 - |z|^2) \, dx \, dy$ ($c = (\gamma + 1)/\pi$) on the disc corresponds to $c4^{\gamma+1}|z + i|^{-2\gamma-4} y^\gamma \, dx \, dy$ on Ω, which is mapped to $y^\gamma \, dx \, dy$ by the change of gauge $f \to 2^{\gamma+1}\sqrt{c}(1 - iz)^{\gamma+2} f(z)$.

It follows from (12.3) and Proposition 1.3 and 1.2 that the reproducing kernel in $A^2(\mu)$ is

$$K(z, w) = \frac{\gamma + 1}{4\pi} \left(\frac{z - \bar{w}}{2i} \right)^{-\gamma-2}. \tag{13.2}$$

Thus, the invariant measure

$$K(z, z) \, d\mu = \frac{\gamma + 1}{4\pi} y^{-2} \, dx \, dy.$$

We, again, for simplicity consider only the Hankel forms Γ_b and H_b. We obtain by §§4 and 6, if b is analytic in Ω,

$$\Gamma_b \in S_p \Leftrightarrow y^{\gamma+2}b(x+iy) \in L^p(y^{-2}\,dx\,dy) \Leftrightarrow b \in B^p_{-\gamma-2+1/p}.$$

If $g = Q(\omega^{-1}b)$, then by (13.1) and (6.8), for suitable f,

$$\langle g,f\rangle_\nu = C_1\int_0^\infty \hat{g}(\xi)\overline{\hat{f}(\xi)}\xi^{-2\gamma-3}\,d\xi = \langle b,f\rangle_\mu = C_2\int_0^\infty \hat{b}(\xi)\hat{f}(\xi)\xi^{-\gamma-1}\,d\xi.$$

Hence $\hat{g}(\xi) = \xi^{\gamma+2}\hat{b}(\xi)$, and $Q(\omega^{-1}b) \in B^p_s$ if and only if $b \in B^p_{s+\gamma+2}$, $-\infty < s < \infty$. We thus obtain

Theorem 13.1. *Let b be analytic in the upper half plane and $1 \leqslant p \leqslant \infty$. Then $\Gamma_b \in S_p$ if and only if $b \in B^p_{-\gamma-2+1/p}$ and $H_b \in S_p$ if and only if $b \in B^p_{1/p}$.* □

For this and similar results, see Peller (1982), Rochberg (1982), Semmes (1984), Janson and Peetre (1985).

Note that $H^2(\mathbb{R})$ formally is a limit of $B^2_{-(\gamma+1)/2}$ as $\gamma \to -1$.

Remark 13.1. By conformal mapping followed by a suitable change of gauge one can also get interesting formulations in other domains (not necessarily (generalized) discs). For instance in the case of the standard strip $0 < \operatorname{Im} z < 1$, the condition on the symbol takes the form

$$\int_{-\infty}^{\infty}\int_0^1 |b(z)|^p(\sin \pi y)^a\,dm(z) < \infty. \tag{13.3}$$

Such a condition can be made more explicit using Besov spaces on the boundary lines $\operatorname{Im} z = 0, 1$. Some limiting cases are likewise of interest. If one writes (13.3) for the strip $0 < \operatorname{Im} z < s$ then $s \to \infty$ gives back the upper halfplane. Similary taking the strip $-s/2 < \operatorname{Im} z < s/2$, so that we have the weight $(\cos \pi y/s)^a$, then if we let $a = \alpha\lambda$, $\pi^2/2s^2 = \lambda^{-1}$, $\lambda \to \infty$ then Fock space evolves once more (use $\cos \pi y/s \approx 1 - \pi^2 y^2/2s^2 = 1 - y^2/\lambda$).

14. A General Kroneckers's Theorem

The classical Kronecker's theorem asserts that an ordinary Hankel form (or operator), in the Hardy space $H^2(\mathbb{T})$, is of finite rank if and only if its symbol is a rational function. Here we wish to establish an analogous result in maximal generality. A closely related multivariable Kronecker theorem, containing the algebraic part of the proof below, is otherwise in Power (1982*b*).

We consider the following set-up, which differs considerably from the one used in the main part of the paper; in particular, we do not any longer require the Hankel forms to be defined on a Hilbert space.

Ω is a domain in $\mathbb{C}^n$. Let R be the ring of all polynomial functions in Ω. We say that a bilinear form H defined on a space X that contains R (or, more generally, or a pair X, Y of such spaces) is a Hankel form if

$$H(f, g) = H(fg, 1), \qquad f, g \in R. \tag{14.1}$$

We will study spaces X that satisfy the following assumptions.

W1: X is a topological vector space of analytic functions in Ω.
W2: X contains R as a dense subspace.
W3: The inclusion $X \subset H(\Omega)$ is continuous.
W4: If $z \in \mathbb{C}^n \setminus \Omega$, then the mapping $f \to f(z)$, $f \in R$, has no continuous extension to X.

Note that W3 implies that the mapping $f \to D^\nu f(z)$ is continuous for every multi-index ν and every $z \in \Omega$. Conversely, if e.g. X is a Banach space, it follows from the Banach-Steinhaus theorem that W3 is equivalent to

W3′: If $z \in \Omega$, then $f \to f(z)$ is continuous on X.

Hence, in that case, W3 and W4 may informally be summarized by «$f \to f(z)$ is continuous if and only if $z \in \Omega$».

The Fock spaces in Section 7-10 and the Bergman spaces in Section 12 are examples where these assumptions are satisfied (also for $p \neq 2$ as long as $p < \infty$), but W2 fails to hold for the related spaces in Sections 11 and 13. Another example where the assumptions hold is the classical Hardy space $H^2(\mathbb{T})$ (a limiting case of Bergman space hitherto not permitted).

Theorem 14.1. *Assume that X and Y are two vector spaces such that X satisfies* W1-W4 *and Y satisfies* W1-W3. *Then every (separately) continuous Hankel form H on $X \times Y$ of finite rank is given by*

$$H(f, g) = \sum_{j=1}^{N} \sum_{|\nu| \leq k_j} c_{j\nu} D^\nu(fg)(z_j), \tag{14.2}$$

for some finite sequence $\{z_j\}_1^N$ in Ω, integers k_j and constants $c_{j\nu}$. Conversely, (14.2) *defines a continuous Hankel form for any $\{z_j\}_1^N \subset \Omega$, k_j and $c_{j\nu}$.*

PROOF. The last statement is obvious by Leibniz' rule. In order to prove that a Hankel form H has the sought representation, it is by continuity sufficient to show that (14.2) holds for all $f, g \in R$. Hence we will study the restriction of H to R, and the remainder of the proof will be almost purely algebraic. Let

$$J = \{f \in R : H(f, g) = 0 \quad \text{for all} \quad g \in R\}.$$

If $f \in J$ and $h, g \in R$, then

$$H(fh, g) = H(fhg, 1) = H(f, hg) = 0.$$

Thus $fh \in J$. This proves that J is an ideal in R. Furthermore, J has finite codimension because H has finite rank. (In fact, $\dim(R/J)$ equals the rank of H).

To fix the ideas, let us first study the case $n = 1$. The structure of the ideals in $R = \mathbb{C}[z]$ is well-known and, since $J \neq 0$, we conclude that here exist $z_1, \ldots, z_N \in \mathbb{C}$ and integers k_j such that

$$J = \{f \in R: D^\nu f(z_j) = 0, 0 \leqslant \nu \leqslant k_j, j = 1, \ldots, N\}. \tag{14.3}$$

Since J thus is described by finitely many linear functionals, and the linear functional $f \to H(f, 1), f \in R$, vanishes on J, there exist contants $c_{j\nu}$ such that

$$H(f, 1) = \sum_{j=1}^{N} \sum_{\nu=0}^{k_j} c_{j\nu} D^\nu f(z_j), \qquad f \in R. \tag{14.4}$$

The formula (14.2) follows by (14.1), but it remains to show that $z_j \in \Omega$. We may assume that $c_{jk_j} \neq 0$ for $j = 1, \ldots, N$. Define, for $i \leqslant N$,

$$g_i(z) = \prod_{j=1}^{N} (z - z_j)^{k_j + 1 - \delta_{ij}}.$$

Then, by (14.1) and (14.4),

$$H(f, g_i) = H(fg_i, 1) = \sum_{j=1}^{N} \sum_{\nu=0}^{k_j} c_{j\nu} D^\nu (fg_i)(z_j) = c_{ik_i} k_i! f(z_i), \qquad f \in R.$$

Consequently the mapping $f \to f(z_i)$ is continuous, and $z_i \in \Omega$ follows by W4. This completes the proof when $n = 1$.

When $n > 1$, we will require the following result which is an exercise in commutative algebra, see Power (1982*b*). For completeness we will supply the details of the proof. Our reference will be van der Waerden (1959).

Remark 14.1 When one of the authors was a young student he bought a copy of that venerable treatise. Now after many years he has finally got use for it.

Lemma 14.1. *If J is an ideal of finite codimension in $R = \mathbb{C}[\zeta_1, \ldots, \zeta_n]$, then there exist finitely many points $z_1, \ldots, z_N \in \mathbb{C}^n$ and integers $k_1, \ldots, k_N$ such that*

$$J \supset \{f \in R: D^\nu f(z_j) = 0, |\nu| \leqslant k_j, j = 1, \ldots, N\}. \tag{14.5}$$

PROOF. Let $V = \{z \in \mathbb{C}^n: f(z) = 0 \text{ for all } f \in J\}$ be the algebraic variety corresponding to J. If V is an infinite set, let $\{z_j\}_1^\infty$ be distinct points in V and pick $f_1, f_2, \ldots$ in R such that

$$f_i(z_1) = \cdots = f_i(z_{i-1}) = 0, \qquad f_i(z_i) = 1.$$

Then $f_1, f_2, \ldots$ are linearly independent mod J, which contradicts the assumption that J has finite codimension. Hence V is finite, $V = \{z_1, \ldots, z_N\}$.

Let us now invoke the primary decomposition: since R is Noetherian, J is a finite intersection of primary ideals (van der Waerden (1959), p. 73). Thereby it suffices to prove the lemma for primary ideals (of finite codimension).

Claim. If J is primary, then V is a point.

PROOF. Suppose that $N > 1$. Choose f in R such that $f(z_1) = 1$, $f(z_2) = f(z_3) = \cdots = f(z_N) = 0$. Then $f(1 - f)$ vanishes on V. Thus, by Hilbert's Nullstellensatz (van der Waerden (1959), p. 102), $f^m(1 - f)^m \in J$ for some $m \geqslant 1$. Since $(1 - f)^m \notin J$ and J is primary, $f^{mk} \in J$ for some $k \geqslant 1$; a contradiction. The claim is proved.

Let now J be primary and $V = \{z\}$. Let M be the maximal ideal $\{f \in R : f(z) = 0\}$. If $f \in M$, then, by the Nullstellensatz again, some power of f lies in J. It follows that, for some k,

$$J \supset M^k = \{f \in R : D^\nu f(z) = 0, |\nu| < k\}$$

(van der Waerden (1959), p. 70). The lemma is proved for primary ideals, and thus in general. □

We may now complete the proof of Theorem 14.1 as in the case $n = 1$. It follows by (14.5) that $H(f, 1) = \sum\sum c_{j\nu} D^\nu f(z_j)$. Hence, using (14.1), (14.2) holds for some $\{z_j\}_1^N \subset \mathbb{C}^n$. Fix j. We may assume that $c_{j\nu} \neq 0$ for some ν with $|\nu| = k_j$. Let $g = (z - z_j)^\nu h^m$ where $h(z_j) = 1$, $h(z_i) = 0$ for $i \neq j$, and $m > \max k_i$. Then, by (14.2), $H(f, g) = c_{j\nu} \nu! f(z_j)$, and W4 implies that $z_j \in \Omega$. □

Let us now specialize to the case when X is a Hilbert space. Define the symbol of the Hankel form H as the function $b \in X$ which satisfies $H(f, 1) = \langle f, b \rangle$, $f \in X$. Equivalently, by (14.1) $H(f, g) = \langle fg, b \rangle$, $f, g \in R$.

Let K_z be the reproducing kernel defined in Section 1. Then K_z is the symbol of the Hankel form $(f, g) \to f(z)g(z)$. Recalling that K_z is an antianalytic X-valued function in Ω, we obtain the following.

Corollary 14.1. *Assume that X is a Hilbert space which satisfies* W1-W4. *Then a continuous Hankel form with finite rank on X has a symbol of the form*

$$b(w) = \sum_{j=1}^{N} \sum_{|\nu| \leq k_j} c_{j\nu} (\partial / \partial \bar{z})^\nu K_z(w), \tag{14.6}$$

with $z_1, \ldots, z_N \in \Omega$, and every such symbol defines a continuous Hankel form with finite rank. □

There is no problem to extend the results above to multilinear Hankel forms of finite rank. We leave the details to the reader.

Remark 14.2. Theorem 14.1 implies that any finite rank continuous Hankel form H is a limit (pointwise, and uniformly on bounded subsets of $X \times Y$) of Hankel forms

$$H_m(f, g) = \sum_{j=1}^{N_m} c_{mj} fg(z_{mj}),$$

$z_m \in \Omega$, with $\{N_m\}$ bounded $\left(\text{e.g. } N_m \leqslant \sum_1^N (k_j + 1)^n\right)$. The converse is obvious. We conjecture that it is possible to take N_m as the rank of H, i.e. that the set of continuous Hankel forms of rank $\leqslant r$ coincides with the closure (in any reasonable topology) of the set $\left\{ (f, g) \to \sum_1^r c_j fg(z_j)\colon c_1, \ldots, c_r \in \mathbb{C},\ z_1, \ldots, z_r \in \Omega \right\}$. (For $n = 1$, this follows easily from (14.3), but we have been unable to find a proof in higher dimension). For the Fock space, this would imply, in the notation of Section 8, that H_b has rank $\leqslant r$ if and only if $b \in \overline{P_r}$ (e.g. in F_α^∞).

Remark 14.3. What can be said about the kernel $\{f\colon H(f, g) = 0 \text{ for all } g\}$ of a general Hankel form (not of finite rank)? Again restricting attention to R, we see that the kernel is an ideal J. Let V be the corresponding subvariety of Ω. Then the Hankel form is «concentrated» on V. Examples of such forms, given V, are those of the type $\int_V \bar{\beta} fg\, d\sigma$, where σ is (e.g) the are measure. What can be said about the boundedness or smoothness of such forms?

We end with another open question: How can the results of this section be extended to the case of a general complex manifold! Is there an analogue of the polynomial ring R?

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APPENDICES by Jaak Peetre

Appendix I. Hankel Forms in Weaker Assumptions

In the main body of the paper, referred to below as [JPR], a general theory of Hankel forms is developed but in rather severe restrictions (essentially a homogeneous situation). As the title indicates, the aim of this note is to establish all the essential general results of that paper in much weaker assumptions. We assume that the reader is somewhat familiar with the contents of [JPR] so we repeat only the most rudimentary notions.

Let Ω be a domain in $\mathbb{C}^d$ and μ a positive measure on Ω. Denote by $A^2(\Omega, \mu)$ the space of analytic functions over Ω which are square integrable with respect to μ and let $K(z, \bar{w})$ be the reproducing function in $A^2(\Omega, \mu)$, $L(z, \bar{w})$ the one in $A^2(\Omega, \nu)$, where ν is the «measure» associated to ν

$$d\nu(z) = \omega(z)\, d\mu(z) \quad \text{where} \quad \omega(z) = 1/K(z, \bar{z}).$$

We make the following hypothesis:

$$\text{(weak-V)} \qquad \begin{cases} \forall w \in \Omega \quad \text{we can write} \quad L_w = \Sigma_s u_s v_s \\ \text{with} \quad \Sigma_s \|u_s\|_{A^2(\Omega,\mu)} \|v_s\|_{A^2(\Omega,\mu)} \leqslant C/\omega(w), \end{cases}$$

(that is, L_w is in the image of $A^2(\Omega, \mu) \hat{\otimes} A^2(\Omega, \mu)$ with norm $\leqslant C/\omega(w)$).

Then we have in particular

$$(1) \qquad \int_\Omega |L(z, \bar{w})|\, d\mu(z) \leqslant C \cdot K(w, \bar{w}).$$

(PROOF. Just use Schwarz's inequality).

We consider Hankel forms Γ_b with (usually) analytic symbol b with respect to ν:

$$\Gamma_b(f, g) = \int_\Omega \overline{b(z)} f(z) g(z)\, d\nu(z).$$

We require the following spaces of symbols

$$\mathcal{L}^p(\Omega, \nu) = \{b\colon b \text{ locally integrable, } \omega b \in L^p(\Omega, \sigma)\},$$
$$\mathcal{A}^p(\Omega, \nu) = \mathcal{L}^p(\Omega, \nu) \cap \{b\colon b \text{ analytic}\}.$$

Here σ is the «invariant» measure corresponding to μ, ν:

$$d\sigma(z) = d\mu(z)/\omega(z) = d\nu(z)/\omega(z)^2.$$

It is clear that

$$\mathcal{L}^2(\Omega, \nu) = L^2(\Omega, \nu),$$
$$\mathcal{L}^1(\Omega, \nu) = L^1(\Omega, \mu)$$

and, generally,

$$\mathcal{L}^p(\Omega, \nu) = L^p(\Omega, \omega^{p-2}\nu) = L^p(\Omega, \omega^{p-1}\mu).$$

Similarly with $\mathfrak{A}$ and A instead of $\mathcal{L}$ and L.

Proposition. *The projections Q: $\mathcal{L}^1(\Omega, \nu) \to \mathfrak{A}^1(\Omega, \nu)$ and Q: $\mathcal{L}^\infty(\Omega, \nu) \to \mathfrak{A}^\infty(\Omega, \nu)$ are continuous.*

PROOF. As the kernel of Q is $L(z, \bar{w})$,

$$Qf(z) = \int_\Omega L(z, \bar{w}) f(w)\, d\nu(w),$$

this follows from the estimate (1); the latter can also be rewritten as

$$\int_\Omega |L(z, \bar{w})| / \omega(z)\, d\nu(z) \leqslant C/\omega(w). \quad \square$$

By interpolation (real or complex) we obtain

Corollary. *The projections Q: $\mathcal{L}^p(\Omega, \nu) \to \mathfrak{A}^p(\Omega, \nu)$, $1 \leqslant p \leqslant \infty$, are continuous.* □

Corollary. $(\mathfrak{A}^{p_0}, \mathfrak{A}^{p_1})_{\theta p} = [\mathfrak{A}^{p_0}, \mathfrak{A}^{p_1}]_\theta = \mathfrak{A}^p$ *if* $1/p = (1-\theta)/p_0 + \theta/p_1$ $(0 < \theta < 1)$. □

We can now prove

Proposition. *Γ_b is bounded (on $A^2(\Omega, \mu) \times A^2(\Omega, \mu)$) if and only if $b \in \mathfrak{A}^\infty(\Omega, \nu)$.*

PROOF. $\Leftarrow$ If $b \in \mathfrak{A}^\infty(\Omega, \nu)$ then $|b(z)| \leqslant C\omega(z)^{-1}$. Therefore

$$\begin{aligned} |\Gamma_b(f, g)| &\leqslant \int_\Omega |b(z)|\, |f(z)|\, |g(z)|\, d\nu(z) \\ &\leqslant \int_\Omega |f(z)|\, |g(z)|\, d\mu(z) \\ &\leqslant C\|f\|_{A^2(\Omega,\mu)} \|g\|_{A^2(\Omega,\mu)}, \end{aligned}$$

where we in the last step used Schwarz's inequality.

$\Rightarrow$ Assume that Γ_b is bounded. We may write

$$|b(w)| \leqslant \|\Gamma_b\| \sum_s \|u_s\|_{A^2(\Omega,\mu)} \|v_s\|_{A^2(\Omega,\mu)} \leqslant C\|\Gamma_b\|/\omega(w),$$

completing the proof. □

Without «any» assumptions we can prove

Proposition. *$b \in \mathfrak{A}^1(\Omega, \nu)$ implies $\Gamma_b \in S_1$.*

PROOF. For each $w \in \Omega$ introduce the Hankel form

$$\Gamma_w(f, g) = f(w)g(w) = \langle f, K_w \rangle \langle g, K_w \rangle.$$

It is clear that

$$\|\Gamma_w\|_1 \leqslant \|K_w\|^2_{\mathfrak{A}^2(\Omega, \mu)} = K(w, \bar{w}) = 1/\omega(w).$$

Now formally we may write

$$\Gamma_b = \int_\Omega \bar{b}(w)\Gamma_b \, d\nu(w).$$

Therefore, if is legitimate to use Minkowski's inequality in this situation, we get

$$\begin{aligned} \|\Gamma_b\|_1 &\leqslant \int_\Omega |b(w)| \, \|\Gamma_b\|_1 \, d\nu(w) \\ &\leqslant \int_\Omega |b(w)| \cdot (1/\omega(w)) \cdot \omega(w) \, d\mu(w) \\ &= \int_\Omega |b(w)| \, d\mu(w) \\ &= \|b\|_{\mathfrak{A}^1(\Omega, \nu)}. \quad \square \end{aligned}$$

Remark. Notice that the constant in this imbedding is 1.

By interpolation we obtain at once

Corollary. $b \in \mathfrak{A}^p(\Omega, \nu)$ implies $\Gamma_b \in S_p (1 < p < \infty)$. $\square$

Remark. In particular thus $b \in \mathfrak{A}^2(\Omega, \nu) \Rightarrow \Gamma_b \in S_2$ (= Hilbert-Schmidt (H. S.) forms). Is it possible to prove this directly (without using interpolation)?

So far we have only proved «direct» results (except for $p = \infty$). We now come to the «converse».

We have the following formula

$$(*) \qquad \langle \Gamma_b, \Gamma_c \rangle_{H.S.} = \int_\Omega \overline{b(z)} \tilde{c}(z) \, d\nu(z),$$

where $\tilde{c}$ is determined from c via the formula

$$\tilde{c}(z) = \int_\Omega K(z, \bar{\zeta})^2 c(\zeta) \, d\nu(\zeta).$$

PROOF OF (*). The Hilbert space $A^2(\Omega, \mu)$ admits the «continuous» basis

$$\{K_w / \|K_w\|_{A^2(\Omega, \mu)}\}_{w \in \Omega}.$$

Therefore, for any bilinear forms B, C on $A^2(\Omega, \mu) \times A^2(\Omega, \mu)$ one has

$$\langle B, C\rangle_{H.S.} = \int_{\Omega \times \Omega} B(K_w, K_{w'})\overline{C(K_w, K_{w'})}\, d\mu(w)\, d\mu(w').$$

For a Hankel form $B = \Gamma_b$ the «matrix elements» in this basis are given by

$$\Gamma_b(K_w, K_{w'}) = \int_\Omega \overline{b(z)} K(z, \bar{w}) K(z, \bar{w}')\, d\nu(z).$$

Similarly for $C = \Gamma_c$. This gives

$$\begin{aligned}\langle \Gamma_b, \Gamma_c\rangle_{H.S.} &= \iint_{\Omega \times \Omega} \overline{b(z)} c(\zeta) \int_\Omega K(z, \bar{w})\overline{K(\zeta, \bar{w})}\, d\mu(w) \cdot \\ &\quad \cdot \int_\Omega K(z, \bar{w}')\overline{K(\zeta, \bar{w}')}\, d\mu(w') \cdot d\nu(z)\, d\nu(\zeta) \\ &= \iint_{\Omega \times \Omega} \overline{b(z)} c(\zeta) K(z, \bar{\zeta})^2\, d\nu(z)\, d\nu(\zeta),\end{aligned}$$

which is the «bilinear» form of formula (*). □

By a standard duality reasoning we now obtain

Proposition. *$\Gamma_c \in S_p$ implies $\tilde{c} \in \mathfrak{A}^p(\Omega, \nu)$.*

PROOF. If $\Gamma_c \in S_p$ then by a previous proposition $\langle \Gamma_c, \Gamma_b\rangle_{H.S.}$ makes sense for any $b \in \mathfrak{A}^{p'}(\nu)$ (where $1/p + 1/p' = 1$). In other words we have a continuous linear functional $b \mapsto \langle \Gamma_b, \Gamma_c\rangle_{H.S.}$ on $\mathfrak{A}^{p'}(\Omega, \nu)$. By one of the corollaries then $\tilde{c} \in \mathfrak{A}^p(\Omega, \nu)$. □

Let us introduce an operator $\mathfrak{J}$ by the relation $\mathfrak{J}c = \tilde{c}$ and let us make the new assumption, supplementing the previous assumption (weak-V),

(I) $\mathfrak{J}$ is invertible in each of the spaces $\mathfrak{A}^p(\Omega, \nu)$ $(1 \leqslant p < \infty)$.

Then we can summarize our findings in an elegant

Theorem. *$\Gamma_b \in S_p (1 \leqslant p \leqslant \infty)$ if and only if $b \in \mathfrak{A}^p(\Omega, \nu)$ (or if and only if $\tilde{b} \in \mathfrak{A}^p(\Omega, \nu)$).* □

Remark. In [JPR], apparently, the case $\tilde{b} = \varkappa^{-1} b$ was considered, that is $\mathfrak{J} = \varkappa^{-1}$. (Identity operator) so hypothesis (I) is trivially fulfilled. (Also in this case the strong(er) hypothesis (V), implying our present (weak-V), is fulfilled). We know as yet no other cases when (I) is fulfilled.

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Appendix II. Recent Progress in Hankel Forms

On these pages I would like to report very briefly on work done by me —in one instance, jointly with Svante Janson— since last summer ('86). One of my main objectives has been to push beyond the limitations on the entire theory put in [JPR]. (It is assumed that the reader us somewhat familiar with the main ideas of that paper).

Here is an appropriate quotation: «Then, English, French, and mere Spanish will disappear from this planet. The world will be Tlön.» ([B], p. 35).

1. Weak factorization and boundedness. Let Ω be a domain in $\mathbb{C}^d$. If μ is a positive measure on Ω we denote by $A^p(\Omega, \mu)$ the subspace of $L^p(\Omega, \mu)$ consisting of analytic functions. Let $K = K(z, \bar{w})$ denote the reproducing kernel in $A^2(\Omega, \mu)$ and $L = L(z, \bar{w})$ the one in $A^2(\Omega, \mu)$ where ν is the measure «associated» with μ (definition:

$$d\nu(z) = \omega(z)\, d\mu(z) \quad \text{where} \quad \omega(z) = 1/K(z, \bar{z}))$$

or, possibly, an equivalent measure. The basic hypothesis in [JPR] is («factorization» of the reproducing kernel):

$$\text{(V)} \qquad L = \varkappa K^2 \qquad (\varkappa \text{ a constant} > 1).$$

But already there the following weaker hypotehsis is mentioned («weak factorization»)

$$\text{(weak-V)} \qquad \begin{cases} \forall \in \Omega \quad \text{one can write} \quad L_w = \sum_s u_s v_s \\ \text{where} \quad \sum_s \|u_s\|_{A^2(\Omega,\mu)} \|v_s\|_{A^2(\Omega,\mu)} \leqslant c/\omega(w). \end{cases}$$

(Then sum may be finite or infinite). It is shown in Appendix 1 that, under the hypothesis of (weak-V), holds:

$$\Gamma_b \quad \text{is bounded on} \quad A^2(\Omega, \mu) \times A^2(\Omega, \mu) \quad \textit{if and only if} \quad b \in \mathfrak{A}^\infty(\Omega, \nu).$$

Here Γ_b is the Hankel form with (usually) analytic symbol b with respect to ν,

$$\Gamma_b(f, g) = \int_\Omega \overline{b(z)} f(z) g(z)\, d\nu(z),$$

and, generally speaking, the symbol class $\mathfrak{A}^p(\Omega, \nu)$, $0 < p \leqslant \infty$, is defined as

$$\{b \colon b \text{ analytic}, \ \omega b \in L^p(\Omega, \sigma)\},$$

where again σ is the «invariant» measure (definition:

$$d\sigma(z) = d\mu(z)/\omega(z) = d\nu(z)/\omega(z)^2).$$

The condition (weak-V) has been verified in several concrete cases.

In [P1] the case $d = 1$, $\Omega =$ an annulus $\{z: 1 < |z| < R\}$ is treated. In this case there is a natural family of measures $\mu = \mu_\alpha$ $(\alpha > -1)$ to be considered: $d\mu_\alpha(z) = \lambda(z)^\alpha\, dE(z)$ where $ds = |dz|/\lambda(z)$ is the Poincaré metric on Ω and E is the Euclidean area measure $dE(z) = dx\,dy = i/2 \cdot dz\,d\bar{z})$. (This construction applies to any plane domain Ω bounded by finitely many smooth arcs (a «regular» domain); if Ω is the unit disk then $\lambda(a) = 1 - |z|^2$ so one gets back the usual weighted Bergman (or Dzhrbashyan) spaces). For α integer ($\alpha = 0, 1, 2, \ldots$) the weak factorization can be verified on the basis of the fact that the kernel L (and K) can be expressed in terms of *elliptic functions*. Here it is natural to take $\nu = \mu_\beta$, $\beta = 2\alpha + 2$, so it is not exactly the associated measure, only equivalent to it.

In [P2] I plan to extend the analysis in [P1] to the case of arbitrary regular planar domains. My idea is to invoke the Shottky double $\hat{\Omega}$ of Ω (= set theoretically the union $\Omega \cup \hat{\Omega} \cup \partial\Omega$, where $\hat{\Omega}$ is Ω with the «opposite» complex structure) and thus the theory of «symmetric» compact Riemann surfaces (= real algebraic curves). However, if the genus is >1, this cannot be done as explitly as in the above case of genus 1, because no such nice tool as the theory of elliptic functions is available.

Let me also remark that the case α not an integer is entirely open, also in genus 1.

In [JP] the case of «periodic» Fock space is considered, that is, entire periodic functions (with period, say, 2π) which are square integrable with respect to the measure $e^{-y^2}\,dx\,dy$ (if «Planck's constant» is taken to be 1/2). The basic fact about this case is now that the reproducing kernel can be written in terms of *theta functions* so the desired weak factorization can be obtained by just looking up in the literature the appropriate formulae for theta functions.

Again [P3] is addressed to the case of subspaces of Fock space singled out by symmetries. Example: $f(-z) = f(z)$ (even functions), $f(-z) = -f(z)$ (odd functions). In this case the reproducing kernel is expressed in terms of *hyperbolic functions* (cosh, sinh) and the weak factorization follows from the duplication formulae for the latter (viz. $\cosh 2x = \cosh^2 x + \sinh^2 x$, $\sinh 2x = 2\sinh x \cosh x$). In a more general situation one similarly requires generalized hyperbolic functions.

I have assigned to a student the task of extending the analysis in [P3] to the case of weighted Bergman spaces. This seems to involve a sort of *generalization* of the generalized hyperbolic functions.

In [P4] I determine explicitly the reproducing kernels for certain Hilbert spaces of holomorphic tensor fields over the unit ball in $\mathbb{C}^d$ (the «Rudin ball»). Of course, these are then tensorial too. In this case «strong factorization» holds true (an appropriate tensor version of the previous condition (V)), so a corresponding boundedness result for Hankel like forms can be established.

Again the case of the ball is just the simplest case of a symmetric domain (essentially the rank one case). It is conceivable that one has similar results for other symmetric domains in É. Cartan's list.

Finally, in [P5] I have reformulated the relevant portions of [JPR], that is, as far as the issue of boundedness goes, in the language of holomorphic line bundles and, more generally, holomorphic vector bundles.

2. S_p Theory. In [P6] I address myself to the question of generalizing the S_p-theory in [JPR]-carried out in the hypothesis of condition (V)-to a more general setting. It turns out that besides condition (weak-V), which seems to be virtually indispensable, one requires basically only one more assumption. To formulate it let us introduce the «square operator» $\mathcal{J}$ on $\mathfrak{A}^2(\Omega, \nu)$ (Ω and μ are general now, as in the beginning of Sec. 1), defined by

$$\mathcal{J}f(z) = \int_\Omega K(z, \bar{w})^2 f(w)\, d\mu(w).$$

The relevant hypothesis is then

$$\text{(I)} \qquad \mathcal{J} \quad \text{is invertible in} \quad \mathfrak{A}^p(\Omega, \nu).$$

In this hypotheses ((I) + (weak-V)) it is easy to establish that

$$\Gamma_b \in S_p \quad \text{if and only if} \quad b \in \mathfrak{A}^p(\Omega, \nu), \qquad 1 \leqslant p < \infty.$$

It is trivial that (V) $\Rightarrow$ (I). Indeed, if (V) is fulfilled then clearly $\mathcal{J} = \varkappa^{-1}$ (identity) where $\varkappa$ is a constant $\geqslant 1$. So far I have no non-trivial case when (I) is fulfilled but it should not be difficult to establish it in some of the simpler cases mentioned in Sec. 1. Work is in progress! (*Note* (added Jan. 88). See [JP1].)

3. Some related investigations. In [P7] I study the action of the metaplectic group on the spaces $F^p_\alpha(\mathbb{C})$, which are the natural L^p symbol classes corresponding to the scale of Fock spaces $F^p_\alpha(\mathbb{C})$; see [JPR] for details. (Similar results as those now described are expected in $\mathbb{C}^d$, $d > 1$). In particular I verify that this action is bounded continuous but not isometric, which is a result at least implicit in the work of Feichtinger (see e.g. [F1], [F2], [FG]). To get an isometric action one has to consider a new «caloric» representation of the Heisenberg group (and the *metaplectic* group as well), a «caloric Fock space». This leads also to the idea of a «caloric Bloch space» and a «caloric minimal space», which ought to be studied more. In this context, «caloric» means that the elements of the spaces are (analytic) solutions of the heat equation.

Turning to the situation of regular planar domains treated in [P1], [P2] (see Sec. 1), there is for any fixed halfinteger $l \in 1/2\mathbb{N}$ a natural duality between the following type of objects

$$\text{holomorphic } (1-l)\text{-forms } F(z)(dz)^{1-l} - \text{«integrals»}$$

and

$$\text{holomorphic } l\text{-forms } g(z)(dz)^{l} - \text{«differentials»},$$

embodied in the presence of the pairing

$$[F,f] = \int_{\partial\Omega} F(z)(dz)^{1-l} \cdot \overline{g(z)(dz)^{l}}.$$

In particular, it gives a possibility to lift invariant Hilbert metrics for differentials (for instance, the Dzrbashyan metric mentioned in Sec. 1) to a metric for integrals. One can then define corresponding «minimal» and «maximal» spaces, thus obtaining a new opportunity to extend Arazy's great program for Möbius invariant spaces (cf. [P8]). This connects also with lots of interesting notions such as (real) projective structures of Riemann surfaces, uniformization, Eichler cohomology, Schottky double, real algebraic curves etc. I have started a cooperation on these matters with Björn Gustafsson (Stockholm), who is a specialist on quadrature domains (see e.g. [G]). In particular, we have begun to study invariant differential operators on compact Riemann surfaces equipped with a projective structure (generalizing the classical Schwartz derivative).

It is not clear how much, if anything, of the above can be extended to several variables but on the whole I am about to believe that there must exist interesting illustrations to the theory of Hankel forms with higher dimensional algebraic varieties, especially algebraic surfaces (complex dimension 2).

4. A small selection of open problems.

4.1. To extend the AAK theorem beyond its classical $H^2(T)$-setting (see e.g. [N], App. 4).

4.2. The S_p-theory in [JPR] and its extension indicated in Sec. 3 below is confined to the case $1 \leqslant p < \infty$. It would be interesting to have any general results for $0 < p < 1$ too. In the case of Fock space rather complete results have been obtained by Svante Janson's student Robert Wallstén [W].

4.3. To extend the theory of higher weight Hankel forms in [JP2] to more general cases, for instance the unit ball in $\mathbb{C}^d$. This is basically a question of *Invariant Theory*.

4.4. Is there an analogue of the metaplectic group in the case of Bergman space (cf. [P6])?

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