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# A Harnack Inequality Approach to the Regularity of Free Boundaries. Part I: Lipschitz Free Boundaries are $C^{1,\alpha}$

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## Introduction

1. This is the first in a series of papers where we intend to show, in several steps, the existence of «classical» (or as classical as possible) solutions to a general two-phase free-boundary problem.

2. We plan to do so by

(*a*) constructing rather weak generalized solutions of the free-boundary problems,
(*b*) showing that the free boundary of such solutions have nice measure theoretical properties (i.e., finite $(n-1)$-dimensional Hausdorff measure and the associated differentiability properties),
(*c*) showing that near a «flat» point free boundaries are Lipschitz graphs and
(*d*) showing that Lipschitz free boundaries are really $C^{1,\alpha}$.

From then on, the theory of regularity developed by Kinderlehrer-Nirenberg and Spruck applies.

We start here with the last part of the project, that is, to show that Lipschitz free boundaries are $C^{1,\alpha}$, mainly for two reasons: the first because many of the ideas in this part reappear in a much more entangled way than in the others, and the second, because this part is of immediate interest, since the existence of solutions to which these theorems will apply has been obtained already in many cases by different means.

An heuristic discussion of this paper can be found in [C]. The ideas presented here originated in a joint work with J. Athanasopoulus (see [At-C]).

## Notion of Weak Solution

We denote a point in $\mathbb{R}^{n+1}$ as $X$ or $(x, y)$, with $x = (x_1, \ldots, x_n)$. To state the simplest version of our results, let us define what we mean by a weak solution of a free-boundary problem.

**Definition 1.** *In the unit cylinder $C_1 = B_1 \times [-1, 1]$ of $\mathbb{R}^{n+1}$, we are given a continuous function $u$ satisfying*

(i) $\Delta u = 0$ *on* $\Omega^+ = \{u > 0\}$,

(ii) $\Delta u = 0$ *on* $\Omega^- = \{u \leqslant 0\}^0$,

(iii) *(The weak free-boundary condition). Along $F = \partial\{u > 0\}$ $u$ satisfies the free-boundary condition*

$$u_{\nu+} = G(u_{\nu-})$$

*in the following sense.*

*If $X_0 \in F$ and $F$ has a one-sided tangent ball at $X_0$ (i.e. $\exists B_p(Y)$ such that $X_0 \in \partial B_p(Y)$ and $B_p(Y)$ is contained either in $\Omega^+$ or $\Omega^-$) then*

$$u(X) = \alpha\langle X - X_0, \nu\rangle^+ - \beta\langle X - X_0, \nu\rangle^- + o(|X - X_0|)$$

*and $\alpha = G(\beta)$.*

The basic requirements on $G$ will be strict monotonicity and continuity in $u_\nu$.

**Theorem 1.** *Let $u$ be a continuous function in the unit ball. Assume that $u$ satisfies*

(i) $\Delta u = 0$ *in* $\Omega^+ = \{u > 0\}$ *and* $\Omega^- = \{u \leqslant 0\}^0$.

(ii) $\Omega^+ = \{(x, y) : y > f(x)\}$, *with $f(x)$ a Lipschitz continuous function.*

(iii) $0 \in F = \partial\Omega^+$ *and along $F$, the free-boundary condition $u_\nu^+ = G(u_\nu^-)$ is satisfied in the sense described above.*

*Assume further that $G(s)$ is strictly increasing and for some $C$ large, $s^{-C}G(s)$ is decreasing. Then, on $B_{1/2}$, $f$ is a $C^{1,\alpha}$ function.*

## 1. Some Properties of Harmonic Functions in a Lipschitz Domain

In this section we recall some properties of nonnegative harmonic functions in a Lipschitz domain.

**Lemma 1.** (Dahlberg, see [D], see also [C-F-M-S]). *Let $u_1, u_2$ be two nonnegative harmonic functions in a (Lipschitz) domain $D$ of $\mathbb{R}^{n+1}$ of the form*

$$D = \{|x| < 1, |y| < M, y > f(x)\}$$

*with $f$ a Lipschitz function with constant less than $M$ and $f(0) = 0$. Assume further that $u_1$ and $u_2$ take continuously the value $u_1 = u_2 = 0$ along the graph of $f$. Then, on the domain*

$$D_{1/2} = \left\{|x| < \frac{1}{2}, |y| < \frac{M}{2}, y < f(x)\right\},$$

*we have*

$$0 < C_1 \leqslant \frac{u_1(x,y)}{u_2(x,y)} \cdot \frac{u_2\left(0, \frac{M}{2}\right)}{u_1\left(0, \frac{M}{2}\right)} \leqslant C_2$$

*with $C_1, C_2$ depending only on $M$. In particular, if*

$$\frac{u_2(0, M/2)}{u_1(0, M/2)} = 1$$

*we get*

$$0 < C_1 \leqslant \frac{u_1(x,y)}{u_2(x,y)} \leqslant C_2.$$

**Lemma 2** (Jerison and Kenig [J-K], see also [At-C]). *Let $D$, $u_1$ and $u_2$ be as in Lemma 1. Assume further that*

$$\frac{u_1(0, M/2)}{u_2(0, M/2)} = 1.$$

*Then, $u_1(x,y)/u_2(x,y)$ is Hölder continuous in $\bar{D}_{1/2}$ (i.e. up to the graph of $f(x)$) for some coefficient $\alpha$, both $\alpha$ and the $C^\alpha$ norm of $u_1/u_2$ depending only on $M$.*

**Lemma 3** (Dahlberg [D], see also [C-F-M-S]). *Let $u$ be as $u_1$ (or $u_2$) above. Then, there exists a constant $\delta = \delta(M)$ such that for*

$$D_\delta = \{|x| < \delta, |y| < \delta M, y > f(x)\}$$

*we have*

$$u|_{D_\delta} \leqslant \frac{1}{2} u\left(0, \frac{M}{2}\right).$$

**Lemma 4.** *Let $u$ be as in Lemma 3. Assume further that $D_y u \geqslant 0$ on $D$. Then,*

$$0 < C_1 \leqslant \frac{D_y u\left(0, \frac{M}{2}\right)}{u\left(0, \frac{M}{2}\right)} \leqslant C_2.$$

*As usual $C_i = C_i(M)$.*

PROOF. From Lemma 3,

$$\frac{1}{2} u\left(0, \frac{M}{2}\right) \leqslant \int_\delta^{M/2} D_y u(0, t)\, dt \leqslant u\left(0, \frac{M}{2}\right).$$

But $D_y$ is positive and harmonic in $\Omega$. Therefore, by Harnack's inequality, all the values along the segment of integration are comparable, and the formula with $d = M/2$ follows. For $0 < d < M/2$ we may use rescaling. □

**Lemma 5.** *Let $u$ be as in Lemma 3. Then, in $D_\delta$, for some $\delta(M)$, $D_y u \geqslant 0$.*

PROOF. Let $u_1 = u$ and $u_2$ be the (bounded) auxiliary function

$$\begin{cases} u_2 = C > 0, & \text{on } \partial D \backslash \operatorname{graph} f \\ u_2 = 0, & \text{on } \operatorname{graph} f \\ \Delta u_2 = 0, & \text{on } D. \end{cases}$$

If we compare $u_2$ with vertical translations in their common domain of definition, we obtain

$$D_y u_2 > 0 \quad \text{on} \quad D.$$

Let us adjust $C$ so that

$$\frac{u_1\left(0, \frac{M}{2}\right)}{u_2\left(0, \frac{M}{2}\right)} = 1.$$

Then, from Lemma 2, on $D_{1/2}$

$$0 < C_1 \leqslant \frac{u_1(x,y)}{u_2(x,y)} \leqslant C_2$$

and further, from Lemma 3

$$\left| \frac{u_1(x,y)}{u_2(x,y)} - \frac{u_1(\bar{x},\bar{y})}{u_2(\bar{x},\bar{y})} \right| \leqslant C(|x-\bar{x}| + |y-\bar{y}|)^\alpha.$$

In particular, if we freeze $(\bar{x},\bar{y})$, at distance $d$ from graph of $f$, and let $(x,y)$ vary in a $d/2$-neighborhood of $(\bar{x},\bar{y})$, we get

$$\begin{aligned} \left| u_1(x,y) - u_2(x,y)\left[\frac{u_1(\bar{x},\bar{y})}{u_2(\bar{x},\bar{y})}\right] \right| &< Cu_2(x,y)(|x-\bar{x}| + |y-\bar{y}|)^\alpha \\ &\leqslant Cu_2(\bar{x},\bar{y})d^\alpha \\ &\leqslant CD_y u_2(\bar{x},\bar{y})d^{\alpha+1} \end{aligned}$$

(we may substitute $u_2(x,y)$ by $u_2(\bar{x},\bar{y})$ by Harnack's inequality, and $u_2(\bar{x},\bar{y})$ by $d(D_y u_2(\bar{x},\bar{y}))$, because of Lemma 4). Therefore, taking $D_y$ derivative on the unfrozen variable $y$, and evaluating at $\bar{y}$, we get, from standard interior *a priori* estimates for $w = u_1 - u_2 k$, $k = u_1(\bar{x},\bar{y})/u_2(\bar{x},\bar{y})$

$$\left| D_y u_1(\bar{x},\bar{y}) - \left[\frac{u_1(\bar{x},\bar{y})}{u_2(\bar{x},\bar{y})}\right] D_y u_2(\bar{x},\bar{y}) \right| \leqslant CD_y u_2(\bar{x},\bar{y}) \cdot d^\alpha$$

that is

$$D_y u_1(\bar{x},\bar{y}) \geqslant \left\{ \left[\frac{u_1(\bar{x},\bar{y})}{u_2(\bar{x},\bar{y})}\right] - Cd^\alpha \right\} \cdot D_y u_2(\bar{x},\bar{y}).$$

And this last term is positive if $d^\alpha$ is small enough. □

## 2. Subsolutions to Our Free-Boundary Problems and Comparison Principles

In this section we define weak subsolutions to our free-boundary problem, and discuss a comparison principle.

We start by defining the notion of a weak subsolution.

**Definition 2.** *The continuous function $v(X)$ is a subsolution to our free-boundary problem in $\Omega$ if*

(i) $\Delta v \geqslant 0$ *both in* $\Omega^+ = \{v > 0\}$ *and* $\Omega^- = \{v \leqslant 0\}^0$

(ii) *let* $X_0 \in F = (\partial\Omega^+) \cap \Omega$, *assume that at* $X_0$, *F has a tangent ball* $B_\epsilon$ *from the* $\Omega^+$ *side* (*i.e.* $B_\epsilon \subset \Omega^+$, $X_0 \in \partial B_\epsilon \cap F$). *Then, for some* $\alpha \geqslant 0$, $\beta = G(\alpha)$, $\nu$ *the unit inner radial direction of* $\partial B_\epsilon$ *at* $X_0$,

$$v(X) \geqslant \beta\langle X - X_0, \nu\rangle^+ - \alpha\langle X - X_0, \nu\rangle^- + o(|X - X_0|).$$

**Definition 3.** *Given a subsolution v to our F.B. Problem, a point* $X_0 \in F$, *at which F has a tangent ball from* $\Omega^+$ (*as in Definition* 2(ii)) *will be called a regular point.*

We now state a strong comparison principle.

**Lemma 6.** *Let* $v \leqslant u$ *be two continuous functions in* $\Omega$, $v < u$ *in* $\Omega^+(v)$, *v a subsolution and u a solution. Let* $X_0 \in F(v) \cap F(u)$ (*the free boundaries of v and u*). *Then* $X_0$ *cannot be a regular point for* $F(v)$.

PROOF. Since $\Omega^+(v) \subset \Omega^+(u)$, $X_0$ automatically will be a point for which $u$ has the desired asymptotic development (Definition 1)

$$u(X) = \beta\langle X - X_0, \nu\rangle^+ - \alpha\langle X - X_0, \nu\rangle^- + o(|X - X_0|)$$

with $\beta = G(\alpha)$

$$v(X) \geqslant \bar{\beta}\langle X - X_0, \nu\rangle^+ - \bar{\alpha}\langle X - X_0, \nu\rangle^- + o(|X - X_0|)$$

with $\bar{\beta} = G(\bar{\alpha})$. This implies that $\beta \geqslant \bar{\beta}$ and $\alpha \leqslant \bar{\alpha}$.

Since $G$ is assumed to be monotone $\alpha = \bar{\alpha}$ and $\beta = \bar{\beta}$. But $u - v$ is a positive superharmonic function in $\Omega^+(v)$. By Hopf principle, since $X_0$ is regular

$$(u - v)(X) \geqslant \epsilon|X - X_0|$$

radially into $\Omega^+(v)$, along $\nu$ from $X_0$. □

We refine the previous lemma to a continuous family of subsolutions.

**Lemma 7.** *Let* $v_t$, *for* $0 \leqslant t \leqslant 1$, *be a continuous family of subsolutions in* $\Omega$ (*continuous in* $\bar{\Omega} \times [0, 1]$). *Let u be a solution in* $\Omega$, *continuous in* $\bar{\Omega}$. *Assume that*

(i) $v_0 \leqslant u$ *in* $\Omega$.
(ii) $v_t \leqslant u$ *on* $\partial\Omega$ *and* $v_t < u$ *in* $[\overline{\Omega^+(v_t)} \cap \partial\Omega]$ *for* $0 \leqslant t \leqslant 1$.
(iii) *every point* $X_0 \in F(v_t)$ *is regular and*
(iv) *the family* $\Omega^+(v_t)$ *is continuous, that is* $\Omega^+(v_{t_1}) \subset N_\epsilon(\Omega^+(v_{t_2}))$ *whenever* $|t_1 - t_2| < \delta(\epsilon)$ ($N_\epsilon$ *denotes the* $\epsilon$*-neighborhood of the set*).

*Then* $v_t \leqslant u$ *in* $\Omega$ *for any t.*

PROOF. The set of $t$'s for which $v_t \leqslant u$ is obviously closed. Let us show that it is open: first, if $v_{t_0} \leqslant u$, it follows from (ii) and the strong maximum principle, that $v_{t_0} < u$ in $\Omega^+(v_{t_0})$. And since every point of $F(v_{t_0})$ is regular (assumption (iii)), it follows that $\overline{[\Omega^+(v_{t_0})]}$ is compactly contained in $\Omega^+(u)$ (up to $\partial\Omega$, from assumption (ii)). From assumption (iv), the openness follows. □

*Remark.* Since $u$ may be the solution of a one-phase problem, that is $u|_{\Omega^-(u)} \equiv 0$, assumption (iv) is necessary (an easy counterexample where $\Omega^+(v_t) = \Omega$ for $t > 0$, can be constructed).

## 3. Continuous Families of Subsolutions

In this section we construct particular families of subsolutions, starting from a given solution. The simplest family is the following:

**Lemma 8.** *Let u, a continuous function in Ω, be a weak solution of our F. B. Problem. Let*

$$v_t(X) = \sup_{B_t(X)} u(Y), \qquad t > 0.$$

*Then $v_t$ is a subsolution of our F. B. Problem in its domain of definition. Furthermore, any point of $F(v_t)$ is regular.*

PROOF. $v_t$ is the supremum of a family of translations of $u$, and as such, $v$ is subharmonic both in $\Omega^+(v_t)$ and $\Omega^-(v_t)$. Let now $X_0 \in F(v_t)$. That means that $B_t(X_0)$ is tangent from $\Omega^-(u)$ to $F(u)$ at a point $Y_0$. Therefore

(*a*) $X_0$ is regular since $B_t(Y_0) \subset \Omega^+(v)$ and is tangent to $F(v)$ at $X_0$.
(*b*) At $Y_0$, $u$ has the asymptotic behavior

$$u = \beta\langle X - Y_0, \nu\rangle^+ - \alpha\langle X - Y_0, \nu\rangle^- + o(|X - Y_0|),$$

with $\beta = G(\alpha)$, and $\nu$ the outer normal to $\partial B_t(X_0)$ at $Y_0$, and hence

$$v \geqslant \beta\langle X - X_0, \nu\rangle^+ - \alpha\langle X - X_0, \nu\rangle^- + o(|X - X_0|). \quad \square$$

The family $v_t$ on the previous lemma is an admissible family for the comparison lemma (Lemma 7) and as such it can be used for a comparison principle that says: «If $u_1$ and $u_2$ are two weak solutions, with $u_1 \leqslant u_2$ and near $\partial\Omega$, $\sup_{B_\epsilon(X)} u_1 \leqslant u_2(X)$, then also in the interior of $\Omega$ $\sup_{B_\epsilon(X)} u_1 \leqslant u_2(X)$», keeping, in particular $F(u_2)$, $\epsilon$-away from $F(u_1)$.

This family has the problem of being too rigid. If $u_2$ is, for instance, much larger than $u_1$ in some section of $\partial\Omega$, one cannot exploit that fact. Therefore, we will now introduce a more delicate family of perturbations, where we make the radius of the ball $B_t(X_0)$ dependent on $X_0$ itself ($t = t(X_0)$).

The key lemma is the following.

**Lemma 9.** *Let $\varphi(x)$ be a $C^2$-positive function satisfying*

$$\Delta\varphi \geqslant \frac{C|\nabla\varphi|^2}{|\varphi|}$$

*(for C large enough) in $B_1(0)$ (the unit ball of $\mathbb{R}^n$). Let u be continuous, defined in a domain $\Omega$ large enough so that the following function be defined in $B_1(0)$*

$$w(X) = \sup_{|\nu|=1} u(X + \varphi(x)\nu).$$

*Then, if u is harmonic in $\{u > 0\}$, w is subharmonic in $w > 0$.*

PROOF. Assume $w(0)$ to be positive. We will show that

$$\lim_{r\to 0} \frac{1}{r^2}\left[\fint_{B_1(0)} (w(X) - w(0))\,dx\right] \geqslant 0.$$

For that purpose, we will estimate $w(x)$ by below near 0, choosing an appropriate value for $\nu = \nu(X)$: Choose the system of coordinates so

(1) $w(0) = u(\varphi(0)e_n)$

(2) $\nabla\varphi(0) = \alpha e_1 + \beta e_n$.

We evaluate $w$ by below by choosing $\nu(X) = \nu^*/|\nu^*|$ with

$$(3)\quad \nu^*(X) = e_n + \frac{(\beta x_1 - \alpha x_n)}{\varphi(0)} e_1 + \frac{\gamma}{\varphi(0)}\left(\sum_2^{n-1} x_i e_i\right).$$

Here $\gamma$ is chosen so that

(4) $(1 + \gamma)^2 = (1 + \beta)^2 + \alpha^2$.

Let us examine the point $Y$ obtained by such a choice.

$$Y = X + \left\{\varphi(0) + \nabla\varphi(0)X + \frac{1}{2}(D_{ij}\varphi)x_i x_j + o(|X|^2)\right\}$$
$$\left\{\left[e_n + \frac{(\beta x_1 - \alpha x_n)}{\varphi(0)} e_1 + \frac{\gamma}{\varphi(0)} \sum_2^{n-1} x_i e_i\right]\right.$$
$$\left.\left[1 - \frac{(\beta x_1 - \alpha x_n)^2}{\varphi^2(0)} - \left(\frac{\gamma}{\varphi}\right)^2 \sum_2^{n-1} x_i^2 + o(|X|^4)\right]\right\}.$$

The above expression has a constant (translation) term $\varphi(0)e_n$. A first-order term

$$Y^* - \varphi(0)e_n = X + (\alpha x_1 + \beta x_n)e_n + (\beta x_1 - \alpha x_n)e_1 + \gamma \sum x_i e_i$$

than can be thought as a rotation followed by and expansion by $1 + \gamma$ since

$$\frac{1}{1+\gamma}[Y^* - \varphi(0)e_n] = \begin{bmatrix} \frac{1+\beta}{1+\gamma} & \cdots & \frac{-\alpha}{1+\gamma} \\ \vdots & \ddots & \vdots \\ \frac{\alpha}{1+\gamma} & \cdots & \frac{1+\beta}{1+\gamma} \end{bmatrix} X = MX.$$

where $M$ is a rotation in the $e_1$, $e_n$ plane (by the definition of $\gamma$) and a quadratic term

$$Y - Y^* = \left[\frac{1}{2}(D_{ij}\varphi)x_i x_j - \frac{(\beta x_1 - \alpha x_n)^2}{\varphi(0)} + \frac{\gamma^2}{\varphi(0)}\sum_2^{n-1} x_i^2\right]e_n + O\left(\frac{|\nabla\varphi|^2}{\varphi}|X|^2\right)\mu$$

with $\mu \perp e_n$ and $|\mu| = 1$. Hence

$$\begin{aligned} ⨍ w(X) - w(0) &\geqslant ⨍ u(Y(X)) - u(Y(0)) \\ &= ⨍ u(Y(X)) - u(Y^*(X)) + \int u(Y^*(X)) - u(Y(0)) \\ &= ⨍ u(Y(X)) - u(Y^*(X)). \end{aligned}$$

(Since the last term is zero, due to the fact that $u$ is harmonic and $Y^*$ is a rigid rotation plus a dilation of $X$). We now point out that, by the definition of $w$, $\nabla u$ must point in the direction of $e_n$ at $Y(0)$. Hence

$$\begin{aligned} u(Y) - u(Y^*) &= \nabla u \circ (Y - Y^*) + O(|Y - Y^*|)^2 \\ &= |\nabla u|\left[\frac{1}{2}D_{ij}\varphi x_i x_j - \frac{(\beta x_1 - \alpha x_n)^2}{\varphi(0)} + \frac{\gamma^2}{\varphi(0)}\sum x_i^2\right] + O(|X|^4) \end{aligned}$$

and hence

$$\frac{1}{r^2}\int u(Y) - u(Y^*) + O(|X|^2) =$$

$$= |\nabla u(Y(0))|\left\{\frac{1}{n}\left(\Delta\varphi - [\beta^2 + \alpha^2 + (n-2)\gamma^2]\frac{1}{\varphi(0)}\right)\right\} \geqslant 0$$

if

$$\Delta\varphi > C\frac{|\Delta\varphi|^2}{\varphi}.$$

*Remark.*

$$\Delta\varphi \geqslant C\frac{|\nabla\varphi|^2}{\varphi}$$

if $\varphi^{1-c}$ is superharmonic.

We now study a more flexible family of perturbations, namely, given a solution $u$ of our F. B. Problem and a function $\varphi$ satisfying the properties of Lemma 9, we want to consider $v = v_\varphi$ defined by

$$v(x) = \sup_{B_{\varphi(X)}(X)} u(y).$$

We start with the asymptotic behavior of $v$ at the free boundary.

**Lemma 10.** *Let $u$ be a continuous function and*

$$v(X) = \sup_{B_{\varphi(X)}(X)} u(Y).$$

*with $\varphi$ a positive $C^2$ function, and $|\nabla\varphi| < 1$. Assume that*

$$X_0 \in \partial\Omega^+(v), \qquad Y_0 \in \partial\Omega^+(u)$$

*and that they are related by the fact that*

$$Y_0 \in \partial B_{\varphi(X_0)}(X_0).$$

*Then*

(*a*) *$X_0$ is a regular point for $F(v)$.*
(*b*) *If near $Y_0$, $u^+$ (resp. $u^-$) has the asymptotic behavior*

$$u^+ (\text{resp } u^-) = \alpha\langle Y - Y_0, \nu\rangle^+ + o(|Y - Y_0|)$$

*then*

$$v^+ \geqslant \alpha\langle X - X_0, \nu + \nabla\varphi\rangle^+ + o(|X - X_0|)$$

(*resp. $v^- \leqslant \alpha\langle X - X_0, \nu + \nabla\varphi\rangle + o(|X - X_0|)$*).
(*c*) *If $F(u)$ is a Lipschitz graph, and $|\nabla\varphi|$ is small enough (depending on the Lipschitz norm, $\lambda$, of $F(u)$*), then *$F(v)$ is a Lipschitz graph with Lipschitz norm*

$$\lambda' \leqslant \lambda + C \sup|\nabla\varphi|.$$

PROOF. To prove (*a*), we notice that $\Omega^+(v)$ contains the set

$$\Theta = \{|X - Y_0|^2 < \varphi^2(X)\}.$$

The boundary of this set is a smooth ($C^2$) surface, since

$$\nabla(|X - Y_0|^2 - \varphi^2(X)) = 2(X - Y_0 - \varphi(X)\nabla\varphi(X)) \neq 0$$

along the surface. Since this surface goes through $X_0$, (*a*) is proven.

To prove (*b*) we use the fact that near $X_0$,

$$\varphi(X) \geqslant \varphi(X_0) + \langle X - X_0 \cdot \nabla\varphi(X_0)\rangle + o(|X - X_0|^2).$$

Hence

$$v^+(X) \geqslant \alpha\langle X - X_0, \nu + \nabla\varphi(X_0)\rangle^+ + o(|X - X_0|)$$

and

$$v^-(X) \leqslant \alpha\langle X - X_0, \nu + \nabla\varphi(X_0)\rangle^- + o(|X - X_0|)$$

respectively.

To prove (*c*) it is enough to assume that $\Omega^+(u)$ is above the graph of a smooth convex cone $f(x)$, since the general case is a union of such sets. Then if $X_0$ and $Y_0$ are as before, $Y_0 - X_0$ is by definition parallel to the inner unit normal $\nu$ to a supporting plane to $F(u)$ at $Y_0$. About $\nu$ we can say that it must lie in a cone of apperture $\arctan\lambda$ around $e_{n+1}$. On the other hand at $X_0$, $F(v)$ has the upper and lower envelopes the implicit surfaces

$$S_1 = \{|X - Y_0|^2 - \varphi^2(X) = 0\}$$

and

$$S_2 = \{d(X, \pi)^2 - \varphi^2(X) = 0\}$$

where $\pi$ is the support plane to $F(u)$ at $Y_0$. Both surfaces are smooth with unit normal vector, $\bar{\nu}$, parallel to

$$Y_0 - X_0 + \varphi(X_0)\nabla\varphi(X_0)$$

or to

$$\nu + \nabla\varphi(X_0).$$

Therefore, the angle between $\bar{\nu}$ and $e_{n+1}$ is less than

$$\arctan\lambda + |\nabla\varphi|.$$

If $|\nabla\varphi|$ is small enough depending on $\lambda$, more precisely $|\nabla\varphi|$, a small multiple of $1/(1+\lambda)$, the angle between $\bar{\nu}$ and $e_{n+1}$ is less than

$$\arctan(\lambda(1+(c+\lambda)|\nabla\varphi|))$$

i.e., $F(v)$ is Lipschitz, with Lipschitz constant

$$\lambda' = \lambda(1+(c+\lambda)|\nabla\varphi|).$$

An important corollary is our next lemma.

**Lemma 11.** *Let $u$ be a solution of our F. B. Problem and both $\varphi$ and $v = v_\varphi$ be the functions of Lemmas* 9 *and* 10 *(i.e. $\varphi$ satisfies the hypothesis of both lemmas). Then*

*(a) $v$ is subharmonic in $\Omega^+(v)$ and $\Omega^-(v)$.*
*(b) Every point of $F(v)$ is regular.*
*(c) At every point of $F(v)$, $v$ satisfies the asymptotic inequality*

$$v(X) \geqslant \beta\langle X - X_0, \nu\rangle^+ - \alpha\langle X - X_0, \nu\rangle^- + o(|X - X_0|)$$

*with*

$$\frac{\beta}{1-|\nabla\varphi|} \geqslant G\left(\frac{\alpha}{1+|\nabla\varphi|}\right).$$

## 4. Main Harnack

In this section we develop the basis of our iteration technique. First, two preliminary lemmas:

**Lemma 12.** *Let $0 \leqslant u_1 \leqslant u_2$ be harmonic functions in $B_\lambda(0)$. Let $\epsilon < \lambda/8$ and assume that on $B_{\lambda-\epsilon}(0)$*

$$v_\epsilon(X) = \sup_{B_\epsilon(X)} u_1(Y) \leqslant u_2(X)$$

*and further*

$$u_2(0) - v_\epsilon(0) \geqslant \sigma\epsilon u_2(0).$$

*Then, for some $\bar{C} = \bar{C}(\lambda)$, $\mu = \mu(\lambda) > 0$, we have in $B_{(3/4)\lambda}$*

$$u_2(X) - v_{(1+\mu\sigma)\epsilon}(X) \geqslant \bar{C}\sigma\epsilon u_2(0).$$

PROOF. For any $|\nu| < 1$

$$w(X) = u_2(X) - u_1(X + \epsilon\nu)$$

is harmonic and positive in $B_{\lambda - \epsilon}$. By Harnack's inequality in $B_{3\lambda/4}$

$$w(X) \geqslant Cw(0) \geqslant C\sigma\epsilon u_2(0).$$

Also, both

$$|\nabla u_i(X)| \leqslant \frac{C}{\lambda} u_i(0) \leqslant \frac{C}{\lambda} u_2(0)$$

on $B_{3\lambda/4}$. It follows that

$$\begin{aligned} u_2(X) - u_1(X + (1 + \sigma\mu)\epsilon\nu) &= w(X) + u_1(X + \epsilon\nu) - u_1(X + (1 + \sigma\mu)\epsilon\nu) \\ &\geqslant \sigma\epsilon u_2(0) - \frac{C\mu\sigma}{\lambda} \epsilon u_2(0) \\ &\geqslant \bar{C}\sigma\epsilon u_2(0) \end{aligned}$$

if $\mu$ is chosen small. □

**Lemma 13.** *Let $0 < \lambda < 1/8$, then there exists a $\theta$ and a $\mu > 0$, $(\mu(\lambda), \theta(\lambda))$ and a $C^2$ family of functions $\varphi_t$ $(0 \leqslant t \leqslant 1)$ defined in $\bar{B}_1 \backslash B_{\lambda/2}(0, 3/4)$, such that*

(i) $1 \leqslant \varphi_t \leqslant 1 + t\mu$
(ii) $\varphi\Delta\varphi \geqslant C|\nabla\varphi|^2$
(iii) $\varphi \equiv 1$ *outside of* $B_{7/8}$
(iv) $\varphi|_{B_{1/2}} \geqslant 1 + \theta t\mu$
(v) $|\nabla\varphi| < Ct\mu$.

PROOF. It is not hard to construct a smooth function $\psi_0$ in $B_1 \backslash B_{\lambda/2}(0, 3/4)$ such that

$$\begin{cases} 0 \leqslant \psi_0 \leqslant 1 \\ \psi_0 \equiv 0 \quad \text{outside} \quad B_{7/8}(0) \\ |\nabla\psi_0| < C\Delta\psi_0, \quad \text{for some } C \text{ large} \\ \psi_0|_{B_{1/2}} \geqslant \gamma > 0. \end{cases}$$

Then $\varphi_t = 1 + t\mu\psi_0$ is our desired function, provided that $\mu$ is small enough. □

Now, a comparison theorem:

**Lemma 14.** *Let $u_1 \leqslant u_2$ be two solutions of our free-boundary problem in $B_1 \subset \mathbb{R}^{n+1}$ with $F_2 = F(u_2)$ a Lipschitz free boundary through the origin. Assume further that*

$$v_\epsilon(x) = \sup_{B_\epsilon(x)} u_1(y) \leqslant u_2(x)$$

*in $B_{1-\epsilon}$, that*

$$v_\epsilon\left(0, \frac{3}{4}\right) \leqslant (1 - \sigma\epsilon) u_2\left(0, \frac{3}{4}\right)$$

*and that*

$$B_\lambda\left(0, \frac{3}{4}\right) \subset \Omega^+(u_1).$$

*Then, for $\epsilon$ small enough, there exists a $\delta$, depending only on $\lambda$ and the various constants C, such that on $B_{1/2}$*

$$v_{(1+\delta\sigma)\epsilon}(x) = \sup_{B_{(1+\delta\sigma)\epsilon}(x)} u_1(y) \leqslant u_2(x).$$

PROOF. We construct a continuous family of subsolutions $\bar{v}_t$, such that $\bar{v}_0 \leqslant u_2$, $\bar{v}_1|_{B_{1/2}} \geqslant v_{(1+\delta)\epsilon}$, and for which the comparison lemma (Lemma 7), applies. More precisely

$$\bar{v}_t(x) = \sup_{B_{\epsilon\varphi_{\sigma t}}(x)} u_1(y) + C\sigma\epsilon w_t \equiv v_t(x) + C\sigma\epsilon w_t$$

for a small constant $C > 0$, with $w_t$ a continuous function in

$$\Omega = B_{9/10} - B_{\lambda/2}(0, 3/4)$$

defined by

$$\begin{cases} \Delta w_t = 0 \quad \text{in} \quad \Omega^+(v_t) \cap \Omega = \Omega_1 \\ w_t|_{\partial(\Omega^+(v_t) \cap B_{9/10})} = 0 \\ w_t|_{\partial B_{\lambda/2}(0,3/4)} = u_2(0, 3/4). \end{cases}$$

Let us check that $\bar{v}_t$ satisfies the hypothesis of Lemma 7 in $\Omega$ with respect to $u = u_2$:

(i) comparison in $B_{9/10} - \Omega^+(v_0)$ is clear. In $\Omega_1$ we compare the boundary values of $\bar{v}_0$ and $u_2$ thanks to Lemma 12

(ii) follows from our hypothesis and Lemma 12, provided that $\mu = \mu(\lambda)$ is kept small (we should really replace $\epsilon$ by any smaller $\epsilon^1$, to ensure the validity of (ii) along $\partial B_1$, but that is a minor detail)

(iii) follows from part (*a*) of Lemma 10
(iv) is by constrution.

It only remains to check the fact that $\bar{v}_t$ are indeed subsolutions.

The subharmonicity in $\Omega^+$ and $\Omega^-$ follows from Lemma 9. About the asymptotic behavior, we write

$$\bar{v}_t = v_t + C\sigma\epsilon w_t.$$

From Lemma 11, $v_t$ satisfies the asymptotic inequality (*c*) with

$$\frac{\beta}{1 - \epsilon|\nabla\varphi_{\sigma t}|} \geqslant G\left(\frac{\alpha}{1 + \epsilon|\nabla\varphi_{\sigma t}|}\right).$$

Since outside $B_{7/8}$, $|\nabla\varphi| \equiv 0$ the right inequality is satisfied by $v_t$ and hence by $\bar{v}_t$ since $w_t$ is positive. Inside $B_{7/8} \cap \Omega^+(v_t)$, we notice that by Dahlberg's theorem (Lemma 1) $(w_t/v_t) \geqslant C$, provided that $\epsilon\mu$ and hence $\epsilon|\nabla\varphi|$, is kept small to make sure that the $F(v_t)$ are uniformly Lipschitz domains (see Lemma 10(*c*)). Therefore, from the asymptotic development of Lemma 11(*c*), we may say that

$$(v_t + C\sigma\epsilon w_t)^+ \geqslant \bar{\beta}\langle X - X_0, \nu\rangle^+ + o(|X - X_0|)$$

with $\bar{\beta} \geqslant (1 + C\sigma\epsilon)\beta$. Therefore, to complete the proof of the theorem, we must prove that, for $\mu$ in the definition of $\varphi_{\sigma t}$ small enough,

$$\bar{\beta} \geqslant G(\alpha).$$

From the properties of $G(s)$, $s^{-C}G(s)$ is decreasing. Hence

$$\alpha^{-C}G(\alpha) \leqslant \left[\frac{\alpha}{1 + \epsilon|\nabla\varphi_{\sigma t}|}\right]^{-C} G\left(\frac{\alpha}{1 + \epsilon|\nabla\varphi_{\sigma t}|}\right)$$

or

$$G(\alpha) \leqslant (1 + C\epsilon|\nabla\varphi_{\sigma t}|)G\left(\frac{\alpha}{1 + \epsilon|\nabla\varphi_{\sigma t}|}\right) \leqslant \frac{1 + C\epsilon|\nabla\varphi_{\sigma t}|}{1 - \epsilon|\nabla\varphi_{\sigma t}|}\beta$$

$$\leqslant \frac{1 + C\epsilon|\nabla\varphi_{\sigma t}|}{1 - \epsilon|\nabla\varphi_{\sigma t}|}\,\frac{\bar{\beta}}{1 + C\epsilon}.$$

Since $|\nabla\varphi_{\sigma t}| \leqslant C\mu t$, the proof is complete for $\mu$ small. □

## 5. Intermediate Cones

In this section we state an auxiliary lemma about cones in $\mathbb{R}^n$.

We denote by $\alpha(e, f)$ the angle between the vectors $e$ and $f$, and by $\Gamma(\theta, e)$ the cone of axis $e$ and apperture $\theta$, i.e.

$$\Gamma(\theta, e) = \{\tau: \alpha(\tau, e) < \theta\}.$$

**Lemma 16.** *Let* $0 < \theta_0 < \theta < \pi/2$ *and let*

$$\Gamma(\theta, e) \subset \Gamma\left(\frac{\pi}{2}, \nu\right) = H(\nu).$$

*For* $\tau \in \Gamma(\theta/2, e)$, *let*

$$E(\tau) = \frac{\pi}{2} - \left(\alpha(\tau, \nu) + \frac{\theta}{2}\right)$$

*and for* $\mu$ *small, define*

$$\rho(\tau) = |\tau| \sin\left(\frac{\theta}{2} + \mu E(\sigma)\right).$$

*Finally, let*

$$S_\mu = \bigcup_{\tau \in \Gamma(\theta/2, e)} B_{\rho(\tau)}(\tau).$$

*Then,* $\exists \bar{\theta}$, $\bar{e}$ *such that*

$$\Gamma(\theta, e) \subset \Gamma(\bar{\theta}, \bar{e}) \subset S_\mu$$

*and*

$$\frac{\bar{\theta} - \theta}{\pi/2 - \theta} \geqslant Q(\theta_0, \mu) > 0.$$

PROOF. We reduce it to a problem in the plane through stereographic projection. We first restrict ourselves to the sphere, and then project using $\nu$ as the north pole. By symmetry, the lemma reduces to the following question in the plane (changing slightly $\theta, \theta_0, \mu$)

Let $D_\theta(e)$ be a disc in $\mathbb{R}$ of radius $\theta > \theta_0 > 0$. Assume that $D_\theta \subset D_1$, the unit disc. For $0 < \lambda_0 < \lambda < \lambda_1 < 1$, for any $\tau \in D_{\lambda\theta}(e)$, define

$$E(\tau) = (1 - [|\tau| + (1 - \lambda)\theta])$$

(note that $E(\tau) > 0$, since $D_\theta \subset D_1$) and $\rho(\tau) = (1 - \lambda)\theta + \mu E(\tau)$ $(0 < \mu < 1)$. Then

$$S_\mu = \bigcup_{\tau \in D_{\lambda\theta}(e)} B_{\rho(\tau)}(\tau) \supset D_{\bar{\theta}}(\bar{e}) \supset D_\theta(e)$$

with

$$\frac{\bar{\theta} - \theta}{1 - \theta} \geqslant Q(\mu, \theta_0, \lambda_0, \lambda_1) > 0.$$

The proof is an elementary computation. □

## 6. The Basic Iteration

We are now ready to prove our basic iterative lemmas.

**Lemma 17.** *Let u be a weak solution of our F. B. Problem on $B_1$. Assume that, for some $0 < \theta_0 < \theta \leqslant \pi/2$, u is monotonically increasing for any direction $\tau \in \Gamma(\theta, e_m)$. Then, $\exists \mu < 1$, $(\mu(\theta_0))$ and e a unit vector such that, for*

$$\bar{\theta} - \pi/2 = \mu(\theta - \pi/2),$$

*the cone*

$$\Gamma(\bar{\theta}, e) \supset \Gamma(\theta, e_n)$$

*and, on $B_{1/2}$, u is monotonically increasing for any direction $\tau \in \Gamma(\bar{\theta}, e)$.*

PROOF. We first point out that $B_{1/4 \sin \theta_0}\left(\frac{3}{4} e_n\right)$ is all contained in $\Omega^+$ by the monotonicity of $u$. Let $\nu$ be the direction of $\nabla u$ at $\frac{3}{4} e_n$. Then for any $\tau \in \Gamma(\theta, e_n)$, we have that on $B_{1/4 \sin \theta_0}\left(\frac{3}{4} e_n\right)$, $D_\tau u$ is harmonic and nonnegative, and

$$D_\tau u\left(\frac{3}{4} e_n\right) = \langle \nabla u, \tau \rangle = |\nabla u| \langle \nu, \tau \rangle.$$

From Lemma 4 and Harnack's inequality applied to both $D_\tau u$ and $u$ in $B_{1/4 \sin \theta_0}\left(\frac{3}{4} e_n\right)$, we get

$$D_\tau u|_{B_{1/4 \sin \theta_0}(3e_n/4)} \geqslant C(\sup |\nabla u|) \langle \tau, \nu \rangle \geqslant \sup D_{\epsilon_n} u \geqslant C\Big(\sup_{B_{1/8 \sin \theta_0}(3e_n/4)} u\Big) \langle \nu, \tau \rangle.$$

Let $\tau$ be a small vector in $\Gamma(\theta/2, e_n)$, and let $\bar{u}(x) = u(x - \tau)$. We now apply the main Harnack-type Lemma 14 with

$$u_1(x) = \bar{u}(x)$$

$$u_2(x) = u(x)$$

$$\epsilon = |\tau| \sin \frac{\theta}{2}$$

and $\sigma$ defined as

$$\sigma = C\left(\frac{\pi}{2} - \left(\alpha(\tau, \nu) + \frac{\theta}{2}\right)\right) \sim C\cos\left(\alpha(\tau, \nu) + \frac{\theta}{2}\right)$$

($C$ to be chosen). Then, the only nontrivial hypothesis is that

$$v_\epsilon\left(0, \frac{3}{4}\right) \leqslant (1 - \sigma\epsilon)u_2\left(0, \frac{3}{4}\right).$$

Let $Y \in B_\epsilon(X)$, $u_1(Y) = u(Y - \tau) = u(X - \tau - (X - Y)) = u(X - \bar{\tau})$ with

$$\alpha(\bar{\tau}, \tau) \leqslant \theta/2$$

(since $|\bar{\tau} - \tau| = |X - Y| \leqslant |\tau| \sin\theta/2$). Also

$$|\bar{\tau}| \geqslant |\tau| - |\tau| \sin\frac{\theta}{2} \geqslant \frac{1}{2}|\tau|.$$

since, $\bar{\tau} \in \theta/2 < \pi/4$. It follows that

$$\begin{aligned} \inf_{B_{1/8}(3/4\, e_n)} D_{\bar{\tau}}u &\geqslant C\Big[\sup_{B_{1/8}(3/4\, e_e)} u\Big]\langle \nu, \bar{\tau}\rangle \\ &= C(\sup u)|\bar{\tau}| \cos\alpha(\nu, \bar{\tau}) \\ &\overset{\text{def}}{\geqslant} \sigma\epsilon(\sup u). \end{aligned}$$

(Here we chose $C$ in the definition of $\sigma$). Hence

$$u(X - \bar{\tau}) \leqslant u(X) - D_{\bar{\tau}}u(\tilde{X}) \geqslant (1 - \sigma\epsilon)u(X)$$

and the hypotheses of the Harnack lemma (Lemma 14) are satisfied.

It follows that on $B_{1/2}$

$$\sup_{B_{(1+\delta\tau)\epsilon}(x)} u(y - \tau) \leqslant u(x).$$

Recalling that $\epsilon = |\tau| \sin\theta/2$, $\sigma = C(\pi/2 - (\alpha(\tau, \nu) + \theta/2))$, we get, for any $\tau$ in $\Gamma(\theta/2, e_n)$, that

$$(1 + \delta\sigma)\epsilon = |\tau|\left(\sin\frac{\theta}{2}\right)(1 + \delta C E(\tau))$$

(in the notation of Lemma 16) and for $\theta_0 < \theta < \pi/2$ we get

$$(1 + \delta\sigma)\epsilon \geqslant |\tau| \sin\left(\frac{\theta}{2} + \mu E(\tau)\right), \qquad \mu = \delta C\frac{\theta_0}{2}.$$

The statement above then translates into saying, for any $Z$ of the form $Z = Y - \tau$ (for $Y \in B_{|\tau| \sin(\theta/2 + \mu E(\tau))}(X)) = X - (Y - X) - \tau = X - \tau$, with $\tau$ in $S_\mu$ (of Lemma 16), we have

$$u(Z) \leqslant u(X).$$

That is, $u$ is monotone for any direction $\tau$ in $S_\mu$, and in particular in the intermediate cone $\Gamma(\bar{\theta}, \bar{e})$.

The proof of the lemma is now complete.

PROOF OF THEOREM 1. To prove Theorem 1, we repeat inductively Lemma 17, (notice that if $u$ is a solution of our F. B. Problem; $u(\lambda X)/\lambda$ is also a solution in the corresponding domain). We get that if $u$ is a weak solution as in Theorem 1, then on $B_{2^{-k}}$, $u$ is monotone in a cone of directions

$$\Gamma(\theta_k, e_k)$$

with

$$\Gamma(\theta_{k+1}, e_{k+1}) \supset \Gamma(\theta_k, e_k)$$

and

$$\frac{\pi/2 - \theta_{k+1}}{\pi/2 - \theta_k} = \mu < 1.$$

It follows that $\pi/2 - \theta_k \leqslant b^k$ and hence the fact that the free boundary is $C^{1,\alpha}$ at the origin for some $\alpha(b) > 0$. (Note: the first step in the inductive process, i.e. the free boundary being Lipschitz implies $u$ to be monotone in a cone of directions, follows from Lemma 5). □

## 7. A Generalization

In this last section, we show how to treat the case in which $X$ and $\nu$ dependence is introduced in the free-boundary relation and how the restriction on $G$ at infinity are unnecessary. That is, we now consider weak solutions to the free-boundary problem

$$u_\nu^+ = G(u_\nu^+, X, \nu)$$

in the same sense as before, i.e. whenever $X_0$ has a tangent ball from $\Omega^+$ or $\Omega^-$

$$u = \beta\langle X - X_0, \nu\rangle^+ - \alpha\langle X - X_0, \nu\rangle^-$$

with

$$\beta = G(\alpha, X_0, \nu)$$

($\nu$ given by the radial direction of the tangent ball at $X_0$) and assume that

(*a*) $\log G$ is Lipschitz continuous on $X$ and $\nu$ for bounded values of $u_\nu^-$,
(*b*) for $u_\nu^-$ in a compact interval $[0, M]$, $G$ is strictly monotone in $u_\nu^-$ and $s^{-C}G(s, X, \nu)$ is decreasing in $s$, $(C = C(M))$.

Then we have

**Theorem 2.** *Same geometric situation as in Theorem* 1, *u and G satisfying now the conditions above, the same conclusion as in Theorem* 1 *holds.*

In order to prove Theorem 2, we must do two things. First, to show that $u$ is Lipschitz continuous, eliminating the need to impose conditions at infinity on $G$. Second, to verify that the dependence in $X$ and $\nu$ introduce controllable perturbations in our argument. The first step is achieved by the following monotonicity formula, due to Alt, Friedman and myself.

**Lemma 18.** (See [A-C-F]). *Let u be a continuous function in* $B_1$, $u(0) = 0$. *Assume that on* $\{u > 0\}$, $\Delta u \geqslant 0$ *and on* $\{u < 0\}$, $\Delta u \leqslant 0$. *Then*, ($\rho, \sigma$ *are radial and spherical coordinates in* $\mathbb{R}^n$)

$$g(r) = \frac{\int_{B_r} (\nabla u^+)^2 \rho \, d\rho \, d\sigma \int_{B_r} (\nabla u^-)^2 \rho \, d\rho \, d\sigma}{r^4}$$

*is an increasing function of r.*

*Remark.* $g$ is shown to be finite from the continuity of $u$ by an approximation of say, $u^+$, by a smooth function and the fact that

$$(\nabla u^+)^2 \leqslant (\nabla u^+)^2 + u^+ \Delta u^+ = \frac{1}{2} \Delta (u^+)^2$$

and

$$\rho \, d\rho \, d\sigma = \frac{1}{|X|^{n-2}} dx.$$

By integrating by parts, this allows us to control $g(r)$ for say, $r < 1/2$, by $(\sup_{B_1} |u|)^4$.

**Lemma 19.** (Corollary to Lemma 18). *Let u be a weak solution as in Theorem* 2. *Then u is Lipschitz continuous in* (*say*) $B_{1/4}$.

PROOF. It is enough to prove that $|u(X)| < Cd(X, F)$.

From Lemma 18 and from the remark following it,

$$g(r) \leqslant C\Big(\sup_{B_1} |u|\Big)^4$$

for any $r < 1/2$ and taking as origin any point $X_0$ on $F \cap B_{1/4}$. We consider two cases:

(*a*) $u|_{\Omega^-} \equiv 0$ or,
(*b*) $u|_{\Omega^-}$ is never zero.

In Case (*a*) let $X \in \Omega^+$, $u(X) = \sigma$, $d(X, F) = \rho$, and $X_0 \in \partial B_\rho(X) \cap F \cap B_{1/2}$. Then by Harnack's inequality, $u|_{B_{\rho/2}(X)} \geqslant C\sigma$ and hence

$$u|_{B_\rho(X)} \geqslant h$$

where $h$ is the auxiliary radially symmetric harmonic function on $B_\rho(X) - B_{\rho/2}(X)$ with values $h|_{\partial B_\rho(X)} = 0$ and $h|_{\partial B_{\rho/2}(X)} = C\sigma$. Since $h$ has linear behaviour

$$h = C\frac{\sigma}{\rho}\langle X - X_0, \nu\rangle$$

near $X_0$ and

$$\begin{aligned} u &= \alpha\langle X - X_0, \nu\rangle^+ - \beta\langle X - X_0, \nu\rangle^- + o(|X - X_0|) \\ &= G(0, X_0, \nu)\langle X - X_0, \nu\rangle^+ + o(|X - X_0|) \end{aligned}$$

we get

$$C\frac{\sigma}{\rho} \leqslant G(0, X_0, \nu) \leqslant C,$$

or

$$\sigma \leqslant C\rho$$

and Case (*a*) is complete.

Case (*b*) (we only prove it for $u^+$). We proceed as in Case (*a*) and we obtain at $X_0$ the estimate

$$u(X) = \alpha\langle X - X_0, \nu\rangle^+ + \beta\langle X - X_0, \nu\rangle^- + o(|X - X_0|)$$

with

$$C\frac{\sigma}{\rho} \leqslant \alpha$$

and

$$\alpha = G(\beta, X_0, \nu).$$

We now bring into play the monotonicity formula by pointing out that

$$g(O^+) \geqslant C\alpha^2\beta^2.$$

(Indeed, in any non-tangential domain, $|\langle X - X_0, \nu\rangle| > \delta$, $\nabla u$ converges to $\alpha\nu$ (resp. $\beta\nu$)). Therefore,

$$\alpha^2\beta^2 \leqslant C\|u\|^4_{L^\infty(B_1)}$$

and

$$\alpha = G(\beta, X_0, \nu).$$

Since $G$ is monotone in $\beta$, and

$$G(1, X_0, \nu) \geqslant \mu_0 > 0$$
$$\beta G(\beta, X_0, \nu) \geqslant \mu_0\beta.$$

Therefore,

$$\beta \leqslant C\|u\|^2_{L^\infty(B_1)} \leqslant C$$

and hence

$$\alpha \leqslant C.$$

It follows that $\sigma/\rho \leqslant C\alpha \leqslant C$ and Case (*b*) is also proven.

To complete the proof of the theorem, we only need to prove

**Lemma 20.** *Let $\bar{v}_t$ be the one parameter family of functions constructed in the proof of Lemma* 14. *There exists a $\theta > 0$, depending only on $\lambda$ and the various constants C such that if*

$$|\log(\alpha, X, \nu) - \log(\alpha, Y, \nu)| \leqslant \theta\sigma|X - Y|$$

*for any $\alpha \leqslant \|\nabla u\|_{L^\infty}$ for any $\nu \in S_1$, then $\bar{v}_t$ is still a subsolution of our generalized free boundary problem.*

PROOF. We estimate once more the coefficients in the asymptotic inequality (*c*) of Lemma 11, satisfied by $v_t$ at $X_0$ in $F(v)$. For that, we go back to Lemma 10, and with the notation there employed, we now have that $v$ satisfies there the assymptotic inequality

$$v(X) \geqslant \alpha\langle X - X_0, \nu\rangle^+ - \beta\langle X - X_0, \nu\rangle^- + \sigma(|X - X_0|)$$

with

$$\frac{\beta}{1 - C\epsilon|\nabla\varphi_{\sigma t}|} \geqslant G\left(\frac{\alpha}{1 + C\epsilon|\nabla\varphi_{\sigma t}|}, Y_0, \nu_0\right)$$

where

(*a*) $Y_0 \in \partial B_{\epsilon\varphi_t}(X_0)$

(*b*) $\nu_0 = \dfrac{Y_0 - X_0}{|Y_0 - X_0|}$ and

(*c*) $\nu$ is parallel to $\nu + \epsilon\nabla\varphi_{\sigma t}$.

It follows that

$$|Y_0 - X_0| \leqslant C\epsilon$$

and

$$|\nu - \nu_0| \leqslant \epsilon|\nabla\varphi_t|.$$

Therefore

$$\begin{aligned} \log\beta - \log 1 + C\epsilon|\nabla\varphi_{\sigma t}| &\geqslant \log G\left(\frac{\alpha}{1 + C\epsilon|\nabla\varphi_t|}, Y_0, \nu\right) \\ &\geqslant \log G\left(\frac{\alpha}{1 + C\epsilon|\nabla\varphi_t|}, X_0, \nu\right) - \theta\sigma\epsilon - C\epsilon|\nabla\varphi_{\sigma t}| \\ &\geqslant \log G(\alpha, X_0, \nu) - C\epsilon|\nabla\varphi_{\sigma t}| - C\theta\sigma\epsilon. \end{aligned}$$

But $\log\bar\beta \geqslant \log\beta + C\sigma\epsilon$ ($\beta$ and $\bar\beta$ being bounded). The proof of the lemma is complete.

PROOF OF THEOREM 2. To prove Theorem 2, we now want to apply the equivalent of Lemma 17 inductively. We want, therefore, to make sure that the hypothesis of the Harnack type Lemma 14 (now Lemma 20) holds. This follows from the fact that after a first Lipschitz expansion,

$$\tilde{u}(X) = \frac{1}{\lambda}u(\lambda X),$$

the Lipschitz norm of $\log G$ in $X$ becomes as small as we wish, ($= \theta$) and that after a $k^{\text{th}}$ expansion, $\|\log\|_{\Lambda'(X)} \leqslant \theta 2^{-nk}$ and $\sigma_k$ can be chosen $\geqslant 2^{-nk}$.

*Remark.* Only a Hölder condition in $X$ and $\nu$ is necessary, but this requires a more careful argument.

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