

The Relation Between the Porous Medium and the Eikonal Equations in Several Space Dimensions

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Abstract

We study the relation between the porous medium equation $u_t = \Delta(u^m)$, $m > 1$, and the eikonal equation $v_t = |Dv|^2$. Under quite general assumptions, we prove that the pressure and the interface of the solution of the Cauchy problem for the porous medium equation converge as $m \downarrow 1$ to the viscosity solution and the interface of the Cauchy problem for the eikonal equation. We also address the same questions for the case of the Dirichlet boundary value problem.

Introduction

In this paper we investigate the relation between the porous medium equation

$$(0.1) \quad u_t = \Delta(u^m), \quad m > 1,$$

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and the eikonal equation

$$(0.2) \quad v_t = |Dv|^2,$$

for appropriate initial and boundary data. Here $Dv = (v_{x_1}, \dots, v_{x_N})$ denote the spatial gradient of v and Δ is the Laplace operator. The connection between these equations is made apparent when we perform the change of variables

$$(0.3) \quad v = \frac{m}{m-1} u^{m-1}$$

which transforms (0.1) into the «pressure» equation

$$(0.4) \quad v_t = (m-1)v\Delta v + |Dv|^2.$$

Letting now $m \downarrow 1$ we formally obtain (0.2).

Equation (0.1) arises naturally as a mathematical model in several areas of applications (e.g. percolation of gas through porous media [33], radiative heat transfer in ionized plasmas [34], thin liquid films spreading under gravity [12] crowd-avoiding population spreading [26], etc.). Equation (0.2), which is a special case of a Hamilton-Jacobi equation, is of main interest in optimal control theory [29], the theory of geometrical optics [29], where it describes the propagation of wave fronts [23], etc.

As far as mathematical properties are concerned, (0.1) exhibits both parabolic and hyperbolic behavior. In particular, at all points where $u > 0$, u is smooth. Moreover, it is known ([9], [10]), that the solution u of (0.1) depends continuously in appropriate norms on both the initial data and on m . Thus, as $m \downarrow 1$, (0.1) can be regarded as a perturbation of the heat equation. The hyperbolic behavior of (0.1) is manifested by the existence of a finite speed of propagation and the development of interfaces. (For a detailed discussion of the above as well as a complete list of references, see [37]). On the other hand, (0.2) is a hyperbolic equation with only locally defined smooth solutions but with globally defined weak solutions, namely the viscosity solutions [17]. A common method for approximating viscosity solutions is the method of artificial viscosity [19], [35]. This method, however, does not give any information whatsoever about the interface of the hyperbolic problem. In order to control the interface one needs to use approximations which exhibit the interface. In this context a natural question is whether (0.4) can be regarded as a degenerate viscosity (or diffusion) approximation to (0.2).

In [5] D. G. Aronson and J. L. Vázquez explored the convergence, as $m \downarrow 1$ of the solutions of (0.4) to (0.2) in the case of the Cauchy problem in one space dimension. They proved that not only the solutions but also the interfaces of the solutions of (0.4) converge to the solution and the interface

respectively of (0.2), if the initial data are continuous, nonnegative and converge locally uniformly. The proofs rely on estimates that are very particular to the one-dimensional setting.

In this paper we consider the convergence in N space dimensions with general initial data both for the Cauchy problem in \mathbb{R}^N and for the Dirichlet problem in a bounded domain $O \subset \mathbb{R}^N$. For the Cauchy problem we prove that solutions of (0.4) converge to the unique viscosity solutions of (0.2) (Section 1, Theorem 1). Moreover, we show that the positivity sets of solutions of (0.4) converge (in the sense of sets) to the positivity sets of solutions of (0.2) (Section 2, Theorem 2). The main point for the convergence of the solutions is a new type estimates for the gradient. Gradient estimates are easy to obtain in the case of the one space dimension but not obvious at all in higher dimensions (cf. [1], [14], etc.). For the interfaces we also need some new information. This follows from an important result of L. Caffarelli and A. Friedman [13]. In the case of the Dirichlet problem we investigate the convergence of the solutions. We show that the limit takes on natural boundary conditions, thus giving rise to a boundary layer (Section 3, Theorem 3). Finally, the Appendix is a short survey on (0.2). We examine the existence and uniqueness of viscosity solutions under optimal initial conditions as well as some of their properties (e.g. regularity, growth at infinity, interfaces etc.).

1. The Cauchy Problem

Let us consider the following two problems

$$(1.1) \quad \begin{cases} v_{mt} = (m - 1)v_m \Delta v_m + |Dv_m|^2 & \text{in } \mathbb{R}^N \times (0, T_m) \\ v_m = v_{m0} & \text{on } \mathbb{R}^N \times \{t = 0\} \end{cases}$$

and

$$(1.2) \quad \begin{cases} v_t = |Dv|^2 & \text{in } \mathbb{R}^N \times (0, T) \\ v = v_0 & \text{on } \mathbb{R}^N \times \{t = 0\} \end{cases}$$

with nonnegative initial data $v_{m0}, v_0 \in C(\mathbb{R}^N)$. Here T_m and T denote the maximal time of existence for equations (1.1) and (1.2) respectively.

Problem (1.2) has a unique viscosity solution defined in a time interval $(0, T)$ if the initial data satisfy a quadratic growth condition of the form

$$(1.3) \quad v_0(x) \leq a|x|^2 + b$$

with $a, b \geq 0$. Moreover if

$$(1.4) \quad \alpha = \limsup_{|x| \rightarrow \infty} \frac{v_0(x)}{|x|^2},$$

we have

$$(1.5) \quad T = 1/4\alpha,$$

so that a global solution exists if and only if $\alpha = 0$. On the other hand, a growth condition like (1.3) on v_{m0} ensures the existence of a unique continuous weak solution of problem (1.1) ([11], [21]) for a time

$$(1.6) \quad T_m \geq \frac{1}{2[N(m-1) + 2]\alpha}.$$

Again v_m is global in time if and only if $\alpha = 0$. Observe that $\liminf_{m \rightarrow 1} T_m \geq$

Our first Theorem states the convergence of solutions of (1.1) to the solution of (1.2) as $m \downarrow 1$.

Theorem 1. *Assume that for m close to 1 we are given nonnegative initial data $v_{m0} \in C(\mathbb{R}^N)$ satisfying (1.3) uniformly in m and such that as $m \rightarrow 1$, $v_{m0} \rightarrow v_0$ locally uniformly in \mathbb{R}^N . Let v_m and v be the solutions to problems (1.1) and (1.2). Then $v_m \rightarrow v$ as $m \rightarrow 1$ locally uniformly in $\mathbb{R}^N \times [0, T)$.*

The proof of this result relies on obtaining gradient estimates in the case where the solutions are uniformly bounded from below away from zero and a series of delicate approximations which use the uniqueness and continuous dependence on the initial data of the solutions to problems (1.1) and (1.2). We begin with the gradient estimate. We state the result in a generality that will be also useful in Section 3, when dealing with problems in bounded domains. The proof is based on a variation of Bernstein's trick ([22], [24], [29]).

Lemma 1.1. *For $m > 1$ let v_m be a smooth solution of the equation*

$$(v_m)_t = (m-1)v_m \Delta v_m + |Dv_m|^2$$

in O , where O is an open subset of $\mathbb{R}^N \times (0, T]$. Assume that

$$(1.7) \quad \beta \geq \sup_O v_m \geq \inf_O v_m \geq \gamma > 0$$

with β and γ independent of m . Then for every compact subset K of O and for $m-1$ sufficiently small depending on K, β and γ , there exists a constant $C = C(K, \beta, \gamma)$ such that

$$(1.8) \quad |Dv_m| \leq C \text{ in } K.$$

If Dv_{m0} is locally bounded, the above estimate holds down to $t = 0$, i.e.

PROOF. Let ζ be a cut-off function supported in O such that: $0 \leq \zeta \leq 1$ and $\zeta \equiv 1$ on K . We consider the function

$$(1.9) \quad Z = \zeta^2 |Dv|^2 + \lambda v$$

where λ is a constant to be chosen later. Here for notational simplicity we have dropped the subscript m from v_m . If Z has a maximum at some point (x_0, t_0) such that $\zeta(x_0, t_0) > 0$, then at (x_0, t_0) we have

$$\begin{aligned} Z_t &= 2\zeta\zeta_t|Dv|^2 + 2\zeta^2 v_{x_i} v_{x_i t} + \lambda v_t \geq 0, \\ Z_{x_k} &= 2\zeta\zeta_{x_k}|Dv|^2 + 2\zeta^2 v_{x_i} v_{x_i x_k} + \lambda v_{x_k} = 0, \quad k = 1, \dots, N, \\ Z_{x_k x_k} &= (2\zeta_{x_k}^2 + 2\zeta\zeta_{x_k x_k})|Dv|^2 + 4\zeta\zeta_{x_k} v_{x_i} v_{x_i x_k} \\ &\quad + 2\zeta^2 (v_{x_i x_k})^2 + 2\zeta^2 v_{x_i} v_{x_i x_k x_k} + \lambda v_{x_k x_k} \leq 0, \quad k = 1, \dots, N, \end{aligned}$$

and

$$0 \leq Z_t - (m-1)v\Delta Z - 2Dv \cdot DZ.$$

Substituting in the last inequality and using the equation we obtain

$$\begin{aligned} 0 \leq & 2(m-1)\zeta^2 |Dv|^2 \Delta v - \lambda |Dv|^2 - 2(m-1)v(|D\zeta|^2 + \zeta\Delta\zeta)|Dv|^2 \\ & + 2\zeta\zeta_t |Dv|^2 - 2(m-1)v\zeta^2 v_{x_i x_k}^2 - 4\zeta\zeta_{x_k} v_{x_k} |Dv|^2 \\ & - 8(m-1)\zeta\zeta_{x_k} v v_{x_i} v_{x_i x_k}. \end{aligned}$$

Applying the Cauchy-Schwartz inequality together with the elementary inequality

$$(\Delta v)^2 \leq N \sum_{i,j=1}^N \left[\frac{\partial^2 v}{\partial x_i \partial x_j} \right]^2$$

we get

$$\lambda |Dv|^2 \leq C\zeta |Dv|^3 + C|Dv|^2 + (m-1)\zeta^2 \left(2|Dv|^2 \Delta v - \frac{\gamma}{N} (\Delta v)^2 \right)$$

where γ is from (1.7) and C stands for a constant which depends only on $\|D\zeta\|_\infty, \|\Delta\zeta\|_\infty, \|\zeta_t\|_\infty$ and β from (1.7), and may change from line to line. The last inequality can be transformed into

$$\lambda |Dv|^2 \leq C\zeta |Dv|^3 + C|Dv|^2 + (m-1) \frac{N}{\gamma} \zeta^2 |Dv|^4$$

with all the functions evaluated at (x_0, t_0) . Let

$$(1.10) \quad \lambda = \mu \left[\max_O (\zeta^2 |Dv|^2) + 1 \right]$$

where $\mu > 0$ is to be chosen. Substituting in the above inequality we obtain

$$\mu \left[\max_O \zeta^2 |Dv|^2 + 1 \right] |Dv|^2 \leq C \left[\frac{1}{\gamma} (m-1) \left(\max_O \zeta^2 |Dv|^2 + 1 \right)^{1/2} + 1 \right] \cdot \left(\max_O \zeta^2 |Dv|^2 + 1 \right)^{1/2} |Dv|^2.$$

Now, if $|Dv|^2(x_0, t_0) \neq 0$, then

$$\left(\mu - \frac{(m-1)C}{\gamma} \right) (\max \zeta^2 |Dv|^2 + 1)^{1/2} \leq C,$$

so that, if $\mu > (m-1)C/\gamma$, we have

$$(1.11) \quad |Dv|^2 \leq \left[\frac{C}{\mu - \frac{(m-1)C}{\gamma}} \right]^2 \quad \text{for every } (x, t) \in K.$$

On the other hand, if $|Dv|^2(x_0, t_0) = 0$, then by the definitions of Z and (x_0, t_0) we have

$$(1.12) \quad \zeta^2(\bar{x}, \bar{t}) |Dv|^2(\bar{x}, \bar{t}) \leq \lambda \beta$$

where

$$\zeta^2(\bar{x}, \bar{t}) |Dv|^2(\bar{x}, \bar{t}) = \max_O \zeta^2 |Dv|^2.$$

Using (1.10) and (1.12) we get

$$(1.13) \quad |Dv|^2(x, t) \leq \frac{\beta \mu}{1 - \mu \beta} \quad \text{for every } (x, t) \in K,$$

provided that $1 - \mu \beta > 0$. Choosing μ such that $\mu \beta = 1/2$, then for $m - 1$ sufficiently small we have $\mu > (m-1)C/\gamma$, therefore the result follows in the case where O is a subset of $\mathbb{R}^N \times (0, T)$. If O intersects the set $\mathbb{R}^N \times \{0\}$ the maximum of Z may take place at $t_0 = 0$. In that case we obtain a local bound $|Dv|$ depending only on β, λ and the sup of $|Dv_{m_0}|$ on $K \cap (\mathbb{R}^N \times \{0\})$.

PROOF OF THEOREM.

Step 1. We assume that $0 < \gamma \leq v_{m_0}(x)$ with γ independent of m . It follows from known properties of the porous medium equation [1], [11] that for every m , $v_m(x, t) \geq \gamma$, $v_m \in C^\infty(\mathbb{R}^N \times (0, T_m))$ and the v_m 's are locally bounded in $\mathbb{R}^N \times [0, T_m)$ uniformly in m . Therefore we can apply Lemma 1 on any compact subset K of $\mathbb{R}^N \times (0, T)$ and obtain a bound for $|Dv_m|$ on K that is uniform in m for m sufficiently close to 1. By [25] it follows that

v_m 's are also locally Hölder-continuous in t with exponent $1/2$ and coefficient independent of m if m is again sufficiently close to 1. The family $\{v_m\}_{m>1}$ is therefore relatively compact in $C(K)$. By a standard diagonal process we may extract a subsequence from every sequence $m_n \rightarrow 1$, which we again denote by m_n for simplicity, such that the v_{m_n} 's converge locally uniformly in $\mathbb{R}^N \times (0, T)$ to a function $v \in C(\mathbb{R}^N \times (0, T))$, which is locally Lipschitz continuous in x , Hölder continuous with exponent $1/2$ in t and a viscosity solution of (0.2) (cf. [15], [17]).

If, moreover, $Dv_{m_0} \in L^\infty_{loc}(\mathbb{R}^N)$ uniformly in m , then the gradient estimates hold in compact subsets of $\mathbb{R}^N \times [0, T_m)$ and the same argument implies that the convergence $v_{m_n} \rightarrow v$ holds locally uniformly in $\mathbb{R}^N \times [0, T)$. Since $v_{m_0} \rightarrow v_0$ locally in \mathbb{R}^N we conclude that $v \in C(\mathbb{R}^N \times [0, T))$ takes on the initial value v_0 . Therefore, in view of Theorem A.1 of the Appendix, v is the unique viscosity solution of problem (1.2) and the whole family $\{v_m\}_{m>1}$ converges to v .

To prove that v is continuous down to $t = 0$ and $v(x, 0) = v_0(x)$ for $x \in \mathbb{R}^N$ in the case where we do not have a control on $|Dv_{m_0}|$ we proceed by approximation. Indeed, we approximate v_{m_0} by sequences $\{v_{m_0}^n\}, \{v_{m_0,n}\}$ such that:

- (i) the functions $v_{m_0}^n$ and $v_{m_0,n}$ are smooth in \mathbb{R}^N , and for fixed n the gradients are locally bounded in \mathbb{R}^N uniformly in m .
- (ii) for each fixed m we have the monotone convergence $v_{m_0}^n \downarrow v_{m_0}$ and $v_{m_0,n} \uparrow v_{m_0}$ uniformly in m and $x \in \mathbb{R}^N$.

(Such approximations can be easily obtained by partition of unity and convolution with a smooth kernel).

We conclude as follows: For each fixed n , $v_{m_0}^n$ and $v_{m_0,n}$ converge along subsequences to some functions v_0^n and $v_{0,n}$ respectively, which have gradients locally bounded in \mathbb{R}^N and converge, as $n \rightarrow \infty$, locally uniformly to v_0 . The ordering properties of the porous medium equation imply that $v_m^n \geq v_m \geq v_{m,n}$ in $\mathbb{R}^N \times [0, T_m)$ where v_m^n and $v_{m,n}$ are the solutions of problem (1.1) in $\mathbb{R}^N \times [0, T_m)$ with initial data $v_{m_0}^n$ and $v_{m_0,n}$ respectively. The argument above then implies that for each fixed n , as $m \downarrow 1$, $v_m^n \rightarrow v^n$ and $v_{m,n} \rightarrow v_n$ locally uniformly in $\mathbb{R}^N \times [0, T)$ where v^n, v_n are the unique viscosity solutions of problem (1.2) in $\mathbb{R}^N \times [0, T]$ with initial data v_0^n and $v_{0,n}$ respectively. Moreover,

$$v^n \geq \overline{\lim}_{m \downarrow 1} v_m \geq \liminf_{m \downarrow 1} v_m \geq v_n \quad \text{in } \mathbb{R}^N \times [0, T).$$

Letting $n \rightarrow \infty$ and using the uniqueness result of Theorem A.1 we obtain

$$\lim_{n \rightarrow \infty} v^n = \lim_{n \rightarrow \infty} v_n = v$$

where $v \in C(\mathbb{R}^N \times [0, T])$ is the unique viscosity solution of (1.2) in $\mathbb{R}^N \times [0, T)$. The result follows.

Step 2. The general case. Let

$$v_{m0}^n = v_{m0} + 1/n.$$

If v_m^n is the solution of (0.1) in $\mathbb{R}^N \times [0, T_m)$, then the maximum principle yields $v_m \leq \overline{\lim}_{m \rightarrow 1} v_m^n$ in $\mathbb{R}^N \times [0, T)$. Using Step 1 and Theorem A.1 we get

$$\overline{\lim}_{m \rightarrow 1} v_m \leq v$$

uniformly on compact subsets of $\mathbb{R}^N \times [0, T)$. To conclude, we need to establish the inequality

$$(1.14) \quad \underline{\lim}_{m \rightarrow 1} v_m \geq v \quad \text{locally uniformly in } \mathbb{R}^N \times [0, T).$$

We first prove (1.14) in the case where the v_{m0} 's satisfy the inequalities

$$0 \leq v_{m0} \leq C \quad \text{and} \quad \Delta v_{m0} \geq -C \quad \text{in } \mathbb{R}^N$$

where C is a constant independent of m .

Let v_m^n be the solution of (1.1) with initial data $v_{m0} + 1/n$. Then $v_m^n \in C^\infty(\mathbb{R}^N \times [0, T_m))$, $0 < v_m^n \leq C + 1/n$ and $\Delta v_m^n \geq -C$ in $\mathbb{R}^N \times [0, T)$. Using (1.1) we see that the function

$$w_m^n = v_m^n + C \left(C + \frac{1}{n} \right) (m-1)t$$

is a smooth solution of

$$\begin{cases} w_{mt}^n \geq |Dw_m^n|^2 & \text{in } \mathbb{R}^N \times [0, T_m) \\ w_m^n \geq v_{m0} & \text{on } \mathbb{R}^N \times \{t = 0\}. \end{cases}$$

It then follows that $w_m^n \geq V_m$, the solution of (1.2) with initial data v_{m0} in $\mathbb{R}^N \times [0, T_m)$. Now we let $n \rightarrow \infty$. The continuous dependence of the solution of the porous medium equation on the initial data ([11]) yields

$$v_m(x, t) + C^2(m-1)t \geq v(x, t).$$

Letting $m \downarrow 1$ and using Proposition A.10 we obtain (1.14).

Next we prove (1.14) under only the assumption that v_0 is bounded. $\delta > 0$. We can find functions $\tilde{v}_0, \tilde{v}_{m0}$ bounded in $W^{2,\infty}(\mathbb{R}^N)$ uniformly in m such that $\tilde{v}_0, \tilde{v}_{m0} \geq 0$, $\tilde{v}_{m0} \leq v_{m0}$, $\tilde{v}_{m0} \geq \tilde{v}_0$ in \mathbb{R}^N and $\tilde{v}_0(x) \geq v_0(x) - \delta$ in \mathbb{R}^N . Then

$$\underline{\lim}_{m \downarrow 1} v_m \geq \underline{\lim}_{m \downarrow 1} \tilde{v}_m \geq \tilde{v} \quad \text{locally uniformly in } \mathbb{R}^N \times [0, T).$$

Since $v - \delta$ is a solution of (1.2) with data $v_0 - \delta \leq \bar{v}_0$ we have $v - \delta \leq \bar{v}$. Letting $\delta \rightarrow 0$ we conclude (1.14).

For the general unbounded case we truncate the initial data at height n . If v_m^n and v^n are the solutions of (1.1) and (1.2) with the truncated initial data, the above and the maximum principle yield

$$\lim_{m \rightarrow 1} v_m \geq \lim_{m \rightarrow 1} v_m^n \geq v^n \quad \text{locally uniformly in } \mathbb{R}^N \times [0, T).$$

Letting $n \rightarrow \infty$ we obtain $v^n \rightarrow v$. The result follows. \square

We continue with a remark concerning Lemma 1.1. In fact gradient estimates can be obtained in a similar way for general classes of equations like for instance

$$(1.15) \quad u_t^\epsilon - \epsilon F(x, t, u^\epsilon, Du^\epsilon, D^2u^\epsilon) + H(x, t, u^\epsilon, Du^\epsilon) = 0$$

under suitable assumptions on F and H and provided that the family of smooth solutions $\{u^\epsilon\}_{\epsilon > 0}$ is locally bounded from above and below away from zero uniformly in ϵ . Such bounds allow to pass to the limit $\epsilon \rightarrow 0$ and thus obtain viscosity solutions of the limit problem

$$(1.16) \quad u_t + H(x, t, u, Du) = 0.$$

General equations of the form (1.15) have a certain usefulness. For instance, in some numerical codes the approximation of shocks is improved with the addition of some nonlinear artificial viscosity (so called numerical viscosity, cf. [32]). The assumptions that one has to make on F and H are rather cumbersome although quite general. We leave it to the reader to fill in the details in particular applications.

Our next remark deals with an alternative and simpler proof of the gradient estimate of Lemma 1.1. Though it needs stronger assumptions on the initial data, it can be of interest for some applications.

Lemma 1.2. *Assume that for every $m > 1$ the continuous functions v_{m0} satisfy $0 < \gamma \leq v_{m0} \leq \beta$ and $|Dv_{m0}| \leq M_0$ where β, γ and M_0 are positive constants. Then there exists a bound for $|Dv_m|$ of the form*

$$(1.17) \quad |Dv_m(x, t)|^2 \leq \frac{M_0}{1 - (t/T_m)} \quad \text{for } 0 \leq t < T_m$$

where

$$\tau = \frac{2\gamma}{M_0}.$$

PROOF. Let $w_m = |Dv_m|^2$. Using equation (1.1) we obtain

$$\begin{aligned} w_t - (m - 1)v \Delta w - 2Dv \cdot Dw &= 2(m - 1)w \Delta v - 2(m - 1)v \sum_{i,j=1}^N \left(\frac{\partial^2 v}{\partial x_i \partial x_j} \right) \\ &\leq 2(m - 1)w \Delta v - \frac{2(m - 1)\gamma}{N} (\Delta v)^2 \\ &\leq \frac{(m - 1)N}{2\gamma} w^2, \end{aligned}$$

where we have dropped the m 's for simplicity and have used the inequality

$$(\Delta v)^2 \leq N \sum_{i,j=1}^N \left(\frac{\partial^2 v}{\partial x_i \partial x_j} \right)^2.$$

We compare w_m to the explicit solution

$$W_m(t) = \frac{M_0 C}{C - t}, \quad 0 < t < C = \frac{2\gamma}{M_0 N(m - 1)}$$

of the problem

$$\begin{cases} W' = \frac{N(m - 1)}{2\gamma} W^2 \\ W(0) = M_0 \end{cases}$$

The maximum principle implies that $w_m \leq W_m$ in $\mathbb{R}^N \times [0, C)$. If γ, β and α do not depend on m , then $C \uparrow \infty$ as $m \downarrow 1$. Thus we obtain a bound for $|Dv_m|$ on bounded time intervals which is uniform in m for m close to 1. \square

We conclude with a further remark about a sharper gradient estimate for solutions of (1.1) under the assumption that $0 < \gamma \leq v_{m0} \leq \beta$. Indeed a result of Ph. Bénilan [8] implies that if u_m is a solution of (0.1) and $m \leq 1 + (N - 1)\alpha$ then there exists a gradient bound of the form

$$(1.18) \quad |D(u_m)^{m-1/2}| \leq Ct^{-1/2}$$

where C depends on β but not on m . If $v_{m0} \geq \gamma > 0$, we conclude that

$$|Dv_m|^2 \leq \tilde{c}_1 t^{-1}$$

where \tilde{c}_1 depends on β and γ .

2. Convergence of Interfaces

In this section we prove that under the assumptions of Theorem 1 the interface

of the solution of problem (1.1) converges to the interface of the solution of (1.2) as $m \downarrow 1$. The interfaces are described in terms of the functions

$$(2.1) \quad S_m(x) = \inf \{ t \geq 0 : v_m(x, t) > 0 \},$$

and

$$(2.2) \quad S(x) = \inf \{ t \geq 0 : v(x, t) > 0 \},$$

The so-called retention property implies that $v_m(x, t) > 0$ for every $t > S_m(x)$. On the other hand it is proved in [12] that, if v_{m0} has compact support, the function S_m is Hölder continuous in the open set

$$(2.3) \quad A_m = \mathbb{R}^N \setminus \bar{\Omega}_m(0),$$

where for $m > 1$ and $t \geq 0$, $\Omega_m(t) = \{x \in \mathbb{R}^N : v_m(x, t) > 0\}$. The above restriction is essential in view of the fact that S_m is discontinuous at points of the boundary of $\Omega_m(0)$ whenever a positive waiting time occurs. This phenomenon may appear even in one space dimension, (cf. [3], [4]). The restriction of compact support in [13] is inessential. The interface to problem (1.2) has similar properties as we show in the Appendix.

In view of these observations we prove the convergence of S_m to S away from the initial sets $\Omega_m(0)$. More precisely, let

$$(2.4) \quad A = \mathbb{R}^N \setminus \bigcap_{\epsilon > 0} \text{closure} \left(\bigcup_{1 < m < 1 + \epsilon} \Omega_m(0) \right)$$

The set A consists of points $x \in \mathbb{R}^N$ such that for some $\epsilon > 0$ and $r > 0$ and all $1 < m < 1 + \epsilon$, v_{m0} vanishes identically on $B(x, r)$, the open ball centered at x with radius r .

Theorem 2. S_m converges to S as $m \downarrow 1$ uniformly on compact subsets of A .

PROOF. Let K be a compact subset of A and suppose that S_m does not converge uniformly to S on K . Then there exist $\epsilon > 0$, $x_n \in K$ and $m_n \downarrow 1$ such that

$$(2.5) \quad |S_{m_n}(x_n) - S(x_n)| \geq \epsilon.$$

Suppose first (upon passing to a subsequence if necessary) that we have

$$(2.6.a) \quad S_{m_n}(x_n) \geq S(x_n) + \epsilon.$$

A contradiction follows then easily from the uniform convergence of v_m to v on K and the continuity of S (see Appendix). In fact the definition of S_m (2.6.a) implies

$$v_{m_n}(x_n, S(x_n) + \epsilon) = 0,$$

so that for any limit point \bar{x} of $\{x_n\}$ we have

$$v(\bar{x}, S(\bar{x}) + \epsilon) = 0,$$

against the definition of S .

Let us now discuss the case where, as $m_n \downarrow 1$,

$$(2.6.b) \quad S_{m_n}(x_n) \leq S(x_n) - \epsilon.$$

This case is significantly more difficult. To exclude it, we need to use a precise information about the growth of solutions and interfaces of (1.1) based on the results of [13]. This information is summarized in the following lemmata.

Lemma 2.1. *Let K be a compact set where $v_{m_0} = 0$, $1 < m < 1 + \epsilon \leq 2$. The for every compact set $K' \subset K$ there exists a $\tau > 0$ depending only on K, K' but not on m or ϵ , such that for every $m \in (1, 1 + \epsilon)$*

$$v_m \equiv 0 \quad \text{on} \quad K' \times [0, \tau].$$

Lemma 2.2. *Let $0 < \tau < t_0, x_0 \in \mathbb{R}^N$ and $R_0 > 0$. There exist $\delta = \delta(\tau, R_0) > 0$ and $R = R(\tau, t_0, R_0) > 0$ such that whenever $v_m(\cdot, \tau) \equiv 0$ on $\overline{B(x_0, R_0)}$ and $v_m \leq \delta$ on $\overline{B(x_0, R_0)} \times [\tau, t_0]$, then $v_m(\cdot, t_0) \equiv 0$ on $B(x_0, R)$.*

We postpone the proof of the lemmata to the end of the section and continue with the proof of Theorem 2. Let \bar{x} be a limit point of $\{x_n\}$. The properties of S (cf. Appendix) imply that there exists $t_0 \geq S(\bar{x}) - \epsilon/2$ and $r > 0$ such that $v(\cdot, t_0) \equiv 0$ on $B(\bar{x}, r)$ and, since $v_t \geq 0$ by (1.2),

$$v \equiv 0 \quad \text{on} \quad \overline{B(\bar{x}, r)} \times [0, t_0].$$

On the other hand, Lemma 2.1 yields the existence of a $\tau = \tau(\bar{x}, r) < t_0$ such that, for m_n sufficiently close to 1,

$$v_{m_n} \equiv 0 \quad \text{on} \quad B\left(\bar{x}, \frac{r}{2}\right) \times \{\tau\}.$$

Finally, in view of Theorem 1 and the above there exists n_0 such that for $n \geq n_0$

$$v_{m_n} \leq \delta \quad \text{on} \quad \overline{B(\bar{x}, r)} \times [0, t_0],$$

where $\delta = \delta(\tau, r)$ is given by Lemma 2.2. Since all the assumptions of Lemma 2.2 are satisfied, we obtain

$$v_{m_n}(\cdot, t_0) \equiv 0 \quad \text{on} \quad \overline{B(x, r')},$$

where $r' = r'(\tau, t_0, r)$. This contradicts (2.6.b). \square

We continue with a discussion of the convergence in the complement of the set A . Firstly, for every $\bar{x} \in \Omega(0)$ we have $v_0(x) > 0$ and $v_{m_0}(x) > 0$ for m near 1 and x near \bar{x} . Therefore $S(x) = S_m(x) = 0$; i.e., $S_m \rightarrow S$ locally uniformly in $\Omega(0)$. In the case where $\Omega_m(0) \subset \bar{\Omega}(0)$ for all m near 1, then $A \supset \mathbb{R}^N \setminus \bar{\Omega}(0)$. So the only place where the convergence may fail is the boundary of $\Omega(0)$. It is easy to construct examples with waiting times where this happens. Finally, we cannot expect convergence on the set

$$B = \limsup_{m \downarrow 1} \Omega_m(0) \setminus \bar{\Omega}(0).$$

In fact for each $x \in B$ there exists a subsequence $m_n \downarrow 1$ such that $v_{m_n 0}(x) > 0$ and $S_{m_n}(x) = 0$. However, $S(x) > 0$. In particular, it may happen that $\Omega_m(0) = \mathbb{R}^N$ for every $m > 1$ so that $B = \mathbb{R}^N \setminus \bar{\Omega}(0)$ and the only convergence that we get is the trivial convergence on $\Omega(0)$.

We next formulate the convergence of the interfaces in terms of the positivity sets $\Omega_m(t)$ and $\Omega(t)$. Since the proof of this result is only a variation of the proof of Theorem 2 we leave it up to the reader to fill in the details.

Theorem 2'. *Under the assumptions of Theorem 1 we have*

- (i) $\liminf_{m \downarrow 1} \Omega_m(t) \supset \Omega(t)$
- (ii) $\limsup_{m \downarrow 1} \Omega_m(t) \subset \bar{\Omega}(t) \cup (\mathbb{R}^N \setminus A)$.

As explained above an inclusion of the type

$$\limsup_{m \downarrow 1} \Omega_m(t) \subset \bar{\Omega}(t)$$

cannot be true in general. It may happen e.g. that $\Omega_m(t) = \mathbb{R}^N$ for every $m > 1$, $t > 0$, while $\Omega(t)$ is bounded.

We conclude with the proof of the lemmata.

PROOF OF LEMMA 2.1. Without any loss of generality we may assume that $K = \bar{B}(0, R_1)$ and $K' = \bar{B}(0, R)$ with $R < R_1$.

We proceed by constructing a barrier function $V: B(0, R_1) \times [0, \tau] \rightarrow \mathbb{R}$ for an appropriate choice of τ . It is given by the formula

$$(2.7) \quad V(r, t) = \lambda[a^2 t + a(r - R - \theta)]^+$$

where $r = |x|$, $a^+ = \max\{a, 0\}$, $R < R_1$, $A < R$ and $a, \lambda > 0$. We choose a, λ

(i) Whenever $V > 0$, V satisfies

$$(2.8) \quad V_t \geq (m - 1)V\Delta V + |DV|^2$$

(ii) $V(r, \tau) = 0$ for $r \leq R$.

(iii) $V(R_1, t) \geq A$ for $t \in [0, \tau]$, where A is the L^∞ -bound of v_m on $K \times [0, T - \epsilon]$, which in view of the proof of Theorem 1, is independent of m .

If all the above are satisfied, setting

$$U = \left(\frac{m - 1}{m} V\right)^{1/(m-1)}, \quad u = \left(\frac{m - 1}{m} v_m\right)^{1/(m-1)}$$

we have $U_t \geq \Delta U^m$, $u_t = \Delta u^m$ in $B(0, R_1) \times (0, \tau]$ and $U \geq u$ on the parabolic boundary of $B(0, R_1) \times (0, \tau]$. By the standard comparison principle for the porous medium equation, it follows that $U \geq u$ throughout $B(0, R_1) \times [0, \tau]$ hence, in particular, $v(x, \tau) \leq V(|x|, \tau) = 0$ if $|x| < R$ and thus the result.

We conclude by establishing (i), (ii) and (iii) above. We begin by observing that $V > 0$ if and only if

$$(2.9) \quad r \geq R + \theta - at.$$

To satisfy (ii) it suffices to have

$$(2.10) \quad a\tau \leq \theta$$

For (iii) we need

$$\lambda[a^2t + a(R_1 - (R + \theta))] \geq A$$

which requires

$$(2.11) \quad \lambda a \geq \frac{A}{R_1 - (R + \theta)}.$$

Finally, V satisfies (2.8), whenever $V > 0$, if and only if

$$\lambda \left[(m - 1)(N - 1) \frac{at + r - R - \theta}{r} + 1 \right] \leq 1.$$

For the latter to be satisfied, in view of (2.9) and (2.10), it suffices to have

$$(2.12) \quad \lambda \leq \frac{1}{1 + N(1 - (R/R_1))}.$$

To conclude we choose λ so that the inequality holds in (2.12). Then for a sufficiently large and τ sufficiently small (2.10) and (2.11) can be achieved. \square

For the proof of Lemma 2.2 we need the following result.

Lemma 2.3 [13]. *For any $\tau > 0$ and $m > 1$ there exist positive constants η, c depending only on m, N and τ such that the following is true: Let $t_0 > \tau, R > 0, 0 < \sigma < \eta$. If*

$$(2.13) \quad v_m(\cdot, t_0) \equiv 0 \quad \text{on} \quad B(x_0; R) \quad x_0 \in \mathbb{R}^N$$

and

$$(2.14) \quad \int_{B(x_0, R)} v_m(x, t_0 + \sigma) dx \leq \frac{cR^2}{\sigma},$$

then

$$(2.15) \quad v_m(\cdot, t_0 + \sigma) \equiv 0 \quad \text{on} \quad B(x_0, R/6),$$

where $\int_{B(x_0, R)} v_m(x, s) dx$ denotes the average of $v_m(\cdot, s)$ over the ball $B(x_0, R)$.

A careful scrutiny of the proof shows that the constants *do not depend* on m in the range $1 < m < 2$.

PROOF OF LEMMA 2.2. Let η, c be the constants which correspond to τ via Lemma 2.3, let M be so large that $\sigma = (t_0 - \tau)/M < \eta$ and let $\delta > 0$ be such that $\delta < c 6^{-2(M-1)}R_0^2/\sigma$. For every $i = 1, \dots, M$, we then have

$$\int_{B(x_0, 6^{-(i-1)}R_0)} v_m(x_0, \tau + i\sigma) dx \leq \frac{c}{\sigma} 6^{-2(i-1)}R_0^2.$$

Using Lemma 2.3 and arguing inductively we obtain

$$v_m(\cdot, t_0) \equiv 0 \quad \text{on} \quad \overline{B(x_0, 6^{-M}R_0)}. \quad \square$$

3. The Initial-Boundary Value Problem

Here we focus our attention to the initial-boundary value problems

$$(3.1) \quad \begin{cases} v_{mt} = (m-1)v_m \Delta v_m + |Dv_m|^2 & \text{in } O \times (0, T] \\ v_m = 0 & \text{on } \partial O \times [0, T] \\ v_m = v_{m0} & \text{on } O \times \{t = 0\} \end{cases}$$

and

$$(3.2) \quad \begin{cases} v_t = |Dv|^2 & \text{in } O \times (0, T] \\ v = 0 & \text{on } \partial O \times [0, T] \\ v = v_0 & \text{on } O \times \{t = 0\}. \end{cases}$$

Problem (3.2) does not have a globally defined viscosity solution which is continuous up to the boundary. There exists, however, a *minimal* viscosity solution v (the value function of the underlying control problem) which assumes some natural boundary conditions, not necessarily zero ([27]). We begin discussing this minimal viscosity solution. To make some of the formulas clearer, we will occasionally refer to their form when O is convex. To this end for $x, y \in O$ and $t > 0$ we define

$$(3.3) \quad L(x, y, t) = \inf \left\{ \int_0^t \frac{1}{4} |\dot{\xi}_s|^2 ds : \xi(0) = x, \xi(t) = y, \xi(s) \in \bar{O} \text{ for } s \in [0, t] \right\}$$

If O is convex, then it is easy to see that

$$(3.4) \quad L(x, y, t) = \frac{|x - y|^2}{4t}.$$

In order to have a viscosity solution of (3.2) which is continuous up to the boundary, one needs certain compatibility conditions which restrict the class of allowed initial data and the time of existence [29]. In particular, to have a viscosity solution $v \in C(\bar{O} \times [0, T])$ of the problem

$$(3.5) \quad \begin{cases} v_t = |Dv|^2 & \text{in } O \times (0, T] \\ v = \phi & \text{on } \partial O \times (0, T] \\ v = v_0 & \text{on } O \times \{t = 0\} \end{cases}$$

we need to assume

$$(3.6) \quad \begin{cases} \phi(x, t) \geq \phi(y, s) - L(x, y, t - s) & \text{for all } x, y \in \partial O, t \geq s > 0 \\ \phi(x, t) \geq v_0(y) - L(x, y, t) & \text{for all } x \in \partial O, t > 0 \text{ and } y \in O \end{cases}$$

Next we define

$$(3.7) \quad v(x, t) = \sup_{y \in \bar{O}} \{v_0(y) - L(x, y, t)\}.$$

Arguments similar to the ones of [29, Chapter 11] yield that v is a viscosity solution of $v_t = |Dv|^2$ in $O \times (0, \infty)$. More precisely, (cf. [29]) v is the minimum element of the set of Lipschitz-continuous solutions of

$$(3.8) \quad \begin{cases} v_t - |Dv|^2 = 0 & \text{in } O \times (0, \infty), \\ v \geq 0 & \text{on } \partial O \times (0, \infty), \\ v = v_0 & \text{on } O \times \{t = 0\}. \end{cases}$$

Moreover, on $\partial O \times (0, \infty)$ $v = \tilde{\phi}$, where $\tilde{\phi}$ is the minimum element of the set of functions $\psi \in C([\partial O \times (0, \infty)] \cup [O \times \{t = 0\}])$ satisfying (3.6) and $\psi = v_0$ on $O \times [0, \infty)$.

In the case when O is convex and $\phi = 0$, (3.4) and (3.6) yield

$$(3.9) \quad v_0(x) \leq \frac{1}{4T} \text{dist}(x, \partial O)^2.$$

Then v defined by (3.7) satisfies $v = 0$ on $\partial O \times [0, T]$ and it is the unique viscosity solution of (3.2) in $O \times (0, T)$. The maximal time T for which such a solution exists is given by

$$(3.10) \quad T^* = \inf_{x \in \bar{O}} \frac{\text{dist}(x, \partial O)^2}{4v_0(x)}.$$

We remark that this is precisely the waiting time T for the interface of the Cauchy problem in \mathbb{R}^N (cf. Proposition A.15. See also (1.5)). In general we say that v is the minimal viscosity solution to (3.2).

The relation between (3.1) and (3.2) in the interior of $\bar{O} \times [0, T]$ is the same as the one of (1.1) and (1.2). At the boundary, however, *boundary layers* appear. This is due to the fact that although we are forcing Dirichlet data on (3.1), the solution of (3.2) takes on natural boundary values as explained above.

Theorem. 3. *Assume that $v_{m_0} \rightarrow v_0$ uniformly on \bar{O} as $m \downarrow 1$ and let v be the viscosity solution of (3.2) in $O \times [0, \infty)$ given by (3.7) above. Then, as $m \downarrow 1$, $v_m \rightarrow v$ locally uniformly in $O \times [0, \infty)$.*

PROOF.

Step 1. We begin by assuming that $v_0 > 0$ in O . We may also assume that the v_{m_0} 's are Lipschitz continuous with gradients bounded uniformly in m ; the general case follows by approximating v_{m_0} from above and below by Lipschitz-continuous functions much as in Theorem 1. Let $B(0, R)$ be a ball strictly included in O , and let $\bar{B}(0, R_1) \subset O$ for some $R_1 > R$. Since $v_{m_0} \rightarrow v_0$ uniformly on $\bar{B}(0, R_1)$ as $m \downarrow 1$ and $v_0 > 0$ in O , there exist $m_0 = m_0(R_1)$ and $\gamma > 0$ such that

$$\min_{\bar{B}(0, R_1)} v_{m_0} > \gamma > 0 \quad \text{for } m < m_0.$$

We claim that for every $T > 0$ there exist $m_1 = m_1(T) > 1$ and $\beta = \beta(R, R_1) > 0$ such that for $m < m_1$

$$(3.11) \quad v_m \geq \beta \quad \text{on } \bar{B}(0, R) \times [0, T].$$

Indeed we consider the similarity solutions

$$V_m(x, t; a, \tau) = \frac{1}{2[N(m-1)+2]} \frac{1}{(t+\tau)} (a^2(t+\tau)^{2/(N(m-1)+2)} - |x|^2)^+$$

of $v_t = (m - 1)v \Delta v + |Dv|^2$ in $\mathbb{R}^N \times (0, \infty)$, where $a, \tau > 0$. We choose a and τ so that for m near 1 we have

$$\begin{cases} \text{supp } V_m(\cdot, t) \subset B(0, R_1) & \text{for } t \in [0, T] \\ \overline{B(0, R)} \subset \text{supp } V_m(\cdot, 0) \\ V_m \leq \gamma & \text{on } \overline{B(0, R_1)} \times \{t = 0\}. \end{cases}$$

But then

$$v_m \geq V_m \quad \text{on } (\{|x| = R_1\} \times [0, T]) \cup (\{|x| \leq R_1\} \times \{t = 0\}),$$

therefore

$$v_m \geq V_m \quad \text{on } B(0, R_1) \times [0, T].$$

We conclude by observing that there exists a constant $\beta > 0$, independent m , such that

$$\min V_m \geq \frac{1}{2[N(m - 1) + 2]} \frac{1}{T + \tau} (a^2 \tau^{2/(N(m - 1) + 2)} - R^2) \geq \beta.$$

Using Lemma 1.1 and the results fo [25], we obtain that, along subsequen-
 $m \downarrow 1$, $v_m \rightarrow \bar{v} \geq 0$ locally uniformly in $O \times [0, \infty)$, where \bar{v} is a viscosity solu-
of (3.8). Since the function v given by (3.7) is the minimal viscosity solu-
of (3.8), we immediately have $\bar{v} \geq v$.

For the other inequality, $v \geq \bar{v}$, we use several approximations. To this e-
let O^δ be the δ -neighborhood of O defined by

$$(3.12) \quad O^\delta = \{x \in \mathbb{R}^N: \text{dist}(x, \bar{O}) \leq \delta\}.$$

We extend v_{m0}, v_0 to be zero in $O^\delta \setminus O$ and we denote by $\tilde{v}_{m0}, \tilde{v}_0$ their ex-
sions respectively. Let \underline{v}_m^δ be the minimal viscosity solution of

$$(3.13) \quad \begin{cases} \underline{v}_{mt}^\delta = |D\underline{v}_m^\delta|^2 & \text{in } O^\delta \times (0, \infty) \\ \underline{v}_m^\delta = \delta & \text{on } \partial O^\delta \times (0, \infty) \\ \underline{v}_m^\delta = \tilde{v}_{m0} + \delta & \text{on } O^\delta \times \{t = 0\}, \end{cases}$$

and define the function $w: O^{\delta/2} \times [0, \infty) \rightarrow \mathbb{R}$

$$(3.14) \quad w = \underline{v}_m^\delta * \rho_\alpha = (\underline{v}_m^\delta)_\alpha$$

where $*$ denotes the standard convolution with a smooth kernel ρ_α .
immediate that, for $\alpha < \alpha_0 = \alpha_0(\delta)$, w satisfies

$$\begin{cases} w_t \geq (m - 1)w \Delta w - (m - 1)C_\alpha w & \text{in } O \times (0, T) \\ w \geq v_{m0} & \text{on } O \times \{t = 0\} \\ w > 0 & \text{on } \partial O \times (0, \infty), \end{cases}$$

where C_α is such that $\Delta w \leq C_\alpha$ on $O \times [0, T]$. Next for $\mu > 0$ we define

$$z = e^{\mu t} w \left(x, \frac{e^{\mu t} - 1}{\mu} \right).$$

A simple calculation shows that, for $\alpha < \alpha_0(\delta)$, z satisfies

$$z_t - (m - 1)z \Delta z - |Dz|^2 \geq (\mu - (m - 1)C_\alpha e^{\mu t})z.$$

For any $T > 0$ choose m_1 so small that $(m - 1)C_\alpha e^T < 1$ for $m < m_1$. Then there exists $\mu = \mu(\alpha, m_1)$ such that $\mu \geq (m - 1)C_\alpha e^{\mu T}$. Using the standard comparison argument for the porous medium equation we obtain

$$v_m \leq e^{\mu t} (\underline{v}_m^\delta)_\alpha \left(x, \frac{e^{\mu t} - 1}{\mu} \right) \quad \text{in } O \times [0, T].$$

Now we let $m \downarrow 1$ keeping μ, α, δ fixed and we get

$$(3.15) \quad \bar{v} \leq e^{\mu t} (\underline{v}^\delta)_\alpha \left(x, \frac{e^{\mu t} - 1}{\mu} \right) \quad \text{in } O \times [0, T],$$

where \underline{v}^δ is the minimal viscosity solution of

$$\begin{cases} \underline{v}_t^\delta = |D\underline{v}^\delta|^2 & \text{in } O^\delta \times [0, \infty) \\ \underline{v}^\delta = \delta & \text{on } \partial O^\delta \times [0, \infty), \\ \underline{v}^\delta = \tilde{v}_0 + \delta & \text{on } O^\delta \times \{t = 0\}. \end{cases}$$

Sending $\mu \rightarrow 0, \alpha \rightarrow 0$ and $\delta \rightarrow 0$ we obtain the following sequence of inequalities in $O \times [0, T]$

$$\bar{v} \leq (\underline{v}^\delta)_\alpha, \quad \bar{v} \leq \underline{v}^\delta, \quad \bar{v} \leq v.$$

The result follows.

Step 2. We next consider the general case where $v_0 \geq 0$ in \bar{O} . The main problem here is that, since we cannot bound the v_m 's from below away from zero, we are unable to obtain local Lipschitz estimates. To circumvent this difficulty (i.e. the apparent lack of estimates), we will employ some of the recent ideas of H. Ishii [27] and G. Barles and B. Perthame [7]. To this end, we define the functions

$$v_*(x, t) = \liminf_{\substack{m \downarrow 1 \\ (y, s) \rightarrow (x, t)}} v_m(y, s)$$

and

$$v^*(x, t) = \limsup_{\substack{m \downarrow 1 \\ (y, s) \rightarrow (x, t)}} v_m(y, s).$$

It is known ([7]) that v_* is a lower-semicontinuous viscosity solution of

$$(3.16) \quad \begin{cases} v_{*t} - |Dv_*|^2 \geq 0 & \text{in } O \times (0, \infty) \\ \max(v_{*t} - |Dv_*|^2, v_* - \psi) \geq 0 & \text{on } [\partial O \times (0, \infty)] \cup [\bar{O} \times \{t = 0\}] \end{cases}$$

and v^* is an upper semicontinuous viscosity solution of

$$(3.17) \quad \begin{cases} v_t^* - |Dv^*|^2 \leq 0 & \text{in } O \times (0, \infty) \\ \min(v_t^* - |Dv^*|^2, v^* - \psi) \leq 0 & \text{on } [\partial O \times (0, \infty)] \cup [\bar{O} \times \{t = 0\}] \end{cases}$$

where $\psi: [\partial O \times (0, \infty)] \cup [\bar{O} \times \{t = 0\}] \rightarrow \mathbb{R}$ is given by

$$(3.18) \quad \psi = \begin{cases} 0 & \text{on } \partial O \times (0, \infty) \\ v_0 & \text{on } \bar{O} \times \{t = 0\}. \end{cases}$$

Our goal is to show that

$$v^* = v_* = v \quad \text{in } O \times (0, \infty).$$

Since, by definition, $v_* \leq v^*$, we only have to show that

$$(3.19) \quad v \leq v_* \quad \text{and} \quad v^* \leq v \quad \text{in } O \times (0, \infty).$$

We begin with the right-hand side of (3.19), which is more or less immediate. Indeed, let $v_0^n > 0$ be such that $v_{m0}, v_0 \leq v_0^n$ and $v_0^n \rightarrow v_0$ as $n \rightarrow \infty$ uniformly on \bar{O} . If v_m^n and v^n are the solutions of (3.1) and (3.2) with initial datum v_0^n , then the first part of this proof yields

$$\lim_{m \downarrow 1} v_m^n = v^n \quad \text{uniformly on compact subsets of } O \times (0, \infty).$$

By the maximum principle we have that $v_m \leq v_m^n$ on $\bar{O} \times [0, \infty)$. Moreover it follows from the formulae that $v^n \rightarrow v$ uniformly on $O \times (0, \infty)$ as $n \rightarrow \infty$. Combining all the above we obtain $v^* \leq v$ in $O \times (0, \infty)$.

To obtain the left-hand side of (3.19) we have to work a bit harder. We begin by regularizing the v_* 's using the *inf-convolutions* introduced by J. Lasry and P.-L. Lions [28]. For $\alpha > 0$, let $O_\alpha = \{x \in O: \text{dist}(x, \partial O) \geq \alpha\}$; consider the functions

$$(3.20) \quad v_{*\alpha}(x, t) = \inf_{(y, s) \in \bar{O} \times (0, \infty)} \left\{ v_*(y, s) + \frac{|x - y|^2 + |t - s|^2}{2\alpha} \right\}.$$

It turns out (cf. P.-L. Lions and P. E. Souganidis [31]) that for each $\alpha > 0$, $v_{*\alpha}$ is a Lipschitz continuous viscosity solution of $w_t - |Dw|^2 \geq 0$ in $O_\alpha \times (0, \infty)$, and $v_{*\alpha} \downarrow v_*$ as $\alpha \downarrow 0$. Next we consider the minimal viscosity solution of problem

$$\begin{cases} w_t - |Dw|^2 = 0 & \text{in } O_\alpha \times (\alpha, \infty) \\ w = 0 & \text{on } \partial O_\alpha \times (\alpha, \infty) \\ w = v_{*\alpha}^n & \text{on } \bar{O}_\alpha \times \{t = \alpha\}, \end{cases}$$

where $v_{*\alpha}^n \in C(\bar{O}_\alpha)$, $v_{*\alpha}^n|_{\partial O_\alpha} = 0$ and $v_{*\alpha}^n \uparrow v_{* \alpha}(\cdot, t)$ as $n \rightarrow \infty$. The definition of w (given at the beginning of this section) yields

$$(3.21) \quad v_{* \alpha} \geq w \quad \text{in } O_\alpha \times (\alpha, \infty).$$

Let $(x, t) \in O_\alpha \times (\alpha, \infty)$. Since

$$w(x, t) = \sup_{y \in \bar{O}_\alpha} \{v_{*\alpha}^n(y, \alpha) - L(x, y, t - \alpha)\},$$

the properties of $v_{* \alpha}$ and (3.21) yield

$$v_*(x, t) \geq v_{* \alpha}(x, t) \geq v_{*\alpha}^n(x, \alpha) - L(x, y, t - \alpha) \quad \text{for all } y \in \bar{O}_\alpha.$$

Upon letting $n \rightarrow \infty$ we obtain

$$v_*(x, t) \geq v_{* \alpha}(x, t) \geq v_{* \alpha}(x, \alpha) - L(x, y, t - \alpha) \quad \text{for all } y \in \bar{O}_\alpha.$$

To conclude, we need to examine the behaviour of $v_{* \alpha}(x, \alpha)$ as $\alpha \downarrow 0$. Since $v_{*t} \geq 0$, (3.21) yields

$$v_{* \alpha}(x, \alpha) \geq \inf_{(y, s) \in \bar{O} \times [0, \infty)} \left\{ v_*(y, 0) + \frac{|x - y|^2 + |\alpha - s|^2}{\alpha} \right\}.$$

Using the lower semicontinuity of $v_*(\cdot, 0)$ we then see that

$$\lim_{\alpha \downarrow 0} v_{* \alpha}(s, \alpha) \geq v_*(x, 0).$$

Combining all the above we get

$$(3.22) \quad v_*(x, t) \geq v_*(x, 0) - L(x, y, t) \quad \text{for all } y \in \bar{O}.$$

Finally, it follows from (3.16) and the definition of v_* (cf. [7]) that

$$v_*(\cdot, 0) = v_0 \quad \text{on } \bar{O}.$$

This together with (3.22) yields

$$\begin{aligned} v_*(x, t) &\geq \sup_{y \in \bar{O}} \{v_0(y) - L(x, y, t)\} \\ &= v(x, t); \end{aligned}$$

APPENDIX

We consider the questions of existence and uniqueness as well as some qualitative properties of viscosity solutions of $v_t = |Dv|^2$ defined in a set $Q_T = \mathbb{R}^N \times (0, T)$ for some $T > 0$. The uniqueness results we obtain generalize the results of [16] and [18], in the sense that they allow more general initial data. Some of the properties of the viscosity solutions we are interested in here are growth at infinity, regularization effects, domain of dependence, interface, etc. Several of the results presented have also appeared in similar form in ([6], [16], [18], [29], [30], etc.); for this reason a lot of proofs are rather sketchy.

We recall here for the reader's convenience the definition of *viscosity solution*. A continuous function u defined in a domain $\Omega \subset \mathbb{R}^{N+1}$ is called a viscosity solution of equation $v_t - |Dv|^2 = 0$ if for any function $\varphi \in C^1(\Omega)$ we have $\varphi_t - |D\varphi|^2 \leq 0$ at all points $P_0 = (x_0, t_0) \in \Omega$ at which $v - \varphi$ attains a local maximum and $\varphi_t - |D\varphi|^2 \geq 0$ where $v - \varphi$ attains a local minimum. We refer the interested reader to the references at the end of this paper, especially [17], for the theory of viscosity solutions.

1. Growth at Infinity and Initial Trace

Proposition A.1. *Let v be a viscosity solution of (0.2) defined in Q_T . For every $(x_1, t_1), (x_2, t_2) \in Q_T$ with $0 < t_1 < t_2 < T$ we have*

$$(A.1) \quad v(x_1, t_1) \leq v(x_2, t_2) + \frac{|x_1 - x_2|^2}{4(t_2 - t_1)}.$$

Therefore, if $t \in (0, T)$, then

$$(A.2) \quad \limsup_{|x| \rightarrow \infty} \frac{v(x, t)}{|x|^2} \leq \frac{1}{4(T - t)}$$

and

$$(A.3) \quad \liminf_{|x| \rightarrow \infty} \frac{v(x, t)}{|x|^2} \geq -\frac{1}{4t}.$$

PROOF. We begin assuming that v is bounded below. Let $C, \delta > 0$ and define the function $\phi \in C^\infty(\mathbb{R}^N \times [t_1, T))$ by

$$\phi(x, t) = v(x_1, t_1) - \frac{|x - x_1|^2}{4(t - t_1 + \delta)} - C(t + 1).$$

If we fix $C > 0$ and choose δ small enough, it is immediate that $\phi(\cdot, t_1) < v(x, t_1)$ on \mathbb{R}^N . We want to prove that $v \geq \phi$ in $\Omega = \mathbb{R}^N \times [t_1, t_2]$. In fact if $v - \phi$ attains a minimum in Ω at a point (\bar{x}, \bar{t}) , then, by the definition of the viscosity solution, we must have $\phi_t - |D\phi|^2 \geq 0$ at (\bar{x}, \bar{t}) . However, $\phi_t - |D\phi|^2 = -C < 0$ in Ω . Therefore the minimum of $v - \phi$ either is attained at $t = t_1$ and then $v > \phi$ in Ω or it is approached as $|x| \rightarrow \infty$. But v is bounded from below and $\phi \rightarrow -\infty$ as $|x| \rightarrow \infty$, therefore the latter cannot happen. Letting first $\delta \downarrow 0$ and then $C \downarrow 0$ we obtain (A.1), from which (A.2) and (A.3) follow easily.

If v is not bounded from below we have to suitably modify our test function ϕ . To simplify notation we assume that $t_1 = v(x_1, t_1)$, $x_1 = 0$ and $|x_2| \leq 1$. We consider the rectangle $R = \{(x, t) : |x| \leq 2, 0 \leq t \leq t_2\}$ and we define the function

$$\phi(x, t) = -\frac{1}{4} \psi\left(\frac{x^2}{t + \delta}\right) - C(t + 1)$$

where C and δ are positive constants and $\psi \in C^\infty(\mathbb{R}^+)$ satisfies $\psi(0) = 0$, $\psi'(s) \geq 1$ for $s > 0$, $\psi(s) = s$ for $0 < s \leq s_2 = 1/(t_2 + \delta)$ and $-(1/4)\psi(s) \leq v(x, t)$ for every $|x| = 2$, $0 \leq t \leq t_2$ and $s = 4/t + \delta$. With these assumptions ϕ satisfies $\phi_t - |D\phi|^2 \leq -C < 0$ in R . Repeating the argument of the first part of the proof, we see that the minimum of $v - \phi$ is attained either at $t = 0$ or at $|x| = 2$. It follows from the properties of ψ that $v - \phi \geq 0$ for $|x| = 2$, $0 \leq t \leq t_2$. Moreover, choosing δ very small for fixed $C > 0$ we have $v(x, 0) \geq \phi(x, 0)$. Therefore $v \geq \phi$ in R and, in particular,

$$v(x_2, t_2) \geq \phi(x_2, t_2) = -C(t_2 + 1) - \frac{1}{4} \psi\left(\frac{|x_2|^2}{t_2 + \delta}\right).$$

The properties of ψ imply, however, that

$$\psi\left(\frac{|x_2|^2}{t_2 + \delta}\right) = \frac{|x_2|^2}{t_2 + \delta},$$

hence letting first $\delta \downarrow 0$ and then $C \downarrow 0$ we conclude. \square

Next we turn our attention to the question of the initial trace of viscosity solutions of (0.2). Since $v_t \geq 0$, the family $\{v(\cdot, t)\}_t$ is nondecreasing as $t \downarrow 0$ ([17]). Therefore the initial trace

$$v_0(\cdot) = \lim_{t \downarrow 0} v(\cdot, t)$$

exists. The following proposition is immediate.

Proposition A.2. *Every viscosity solution of (0.2) in Q_T has an initial trace v_0 , which is an upper semicontinuous function $v_0: \mathbb{R}^N \rightarrow \{-\infty\} \cup \mathbb{R}$ satisfy*

$$(A.4) \quad \limsup_{|x| \rightarrow \infty} \frac{v_0(x)}{|x|^2} \leq \frac{1}{4T}.$$

2. Existence of Solutions

It follows from Proposition A.1 that for every viscosity solution v we have

$$(A.5) \quad v \geq \underline{v} \quad \text{on} \quad Q_T$$

where \underline{v} is given by the Lax-Oleinik formula

$$(A.6) \quad \underline{v}(x, t) = \sup_{y \in \mathbb{R}^N} \left\{ v_0(y) - \frac{|x - y|^2}{4t} \right\}.$$

In fact, as we will see below, this last formula provides with the unique solution of the Cauchy problem (1.2) in $\mathbb{R}^N \times [0, T)$, where T depends on

$$\limsup_{|x| \rightarrow \infty} v_0(x)|x|^{-2}.$$

The Lax-Oleinik formula has been studied rather extensively at least in case where v_0 is bounded ([6], [29], [30]). Next, in a series of propositions summarize the properties of (A.6) under assumption (A.4). The proofs of a lot of these propositions are slight modifications of the ones for bounded v_0 therefore we omit them.

Proposition A.3. *For every function $v_0: \mathbb{R}^N \rightarrow \mathbb{R} \cup \{-\infty\}$ such that*

$$(A.7) \quad -\infty \neq v_0(x) \leq A|x|^2 + B \quad \text{in} \quad \mathbb{R}^N$$

for some $A, B > 0$, the Lax-Oleinik formula (A.6) provides with a continuous viscosity solution of (1.2) in Q_T , where the maximal T (blow-up time) is given by

$$(A.8) \quad T = 1/4\alpha$$

with

$$(A.9) \quad \alpha = \limsup_{|x| \rightarrow \infty} \{ v_0(x)|x|^{-2} \}.$$

In particular, v exists for all time if and only if $v_0(x) \leq o(|x|^2)$. Moreover, every $t \in (0, T)$, $v(\cdot, t) \geq v_0(\cdot)$ and

$$(A.10) \quad \liminf_{|x| \rightarrow \infty} \frac{v(x, t)}{|x|^2} \geq -\frac{1}{4t} \quad \text{and} \quad \limsup_{|x| \rightarrow \infty} \frac{v(x, t)}{|x|^2} \leq \frac{\alpha}{1 - 4\alpha t}.$$

Let \mathcal{F}_α be the set of functions $v_0: \mathbb{R}^N \rightarrow \{-\infty\} \cup \mathbb{R}$ such that

$$-\infty \neq v_0(x) \leq A|x|^2 + B \quad \text{for some } A, B > 0$$

and

$$\limsup_{|x| \rightarrow \infty} v_0(x)|x|^{-2} \leq \alpha.$$

For $t \in (0, 1/4\alpha)$ and $\beta = \alpha(1 - 4\alpha t)^{-1}$, let $L_t: \mathcal{F}_\alpha \rightarrow \mathcal{F}_\beta$ be the nonlinear operator defined by the Lax-Oleinik formula.

Proposition A.4. *Let $\alpha > 0$, $t \in (0, 1/4\alpha)$, $\beta = \alpha/(1 - 4\alpha t)$ and $s \in (0, 1/4\beta)$. For any $v_0 \in \mathcal{F}_\alpha$ we have*

$$L_s(L_t v_0) = L_{g+s}(v_0)$$

i.e. L_t has the semigroup property.

We have remarked in Proposition A.2 that a viscosity solution can only take on upper-semicontinuous initial data. On the other hand, we have not discussed yet about whether $L_t v_0$ assumes the initial datum v_0 . The next proposition addresses this question and gives more precise information.

Proposition A.5. *Let v_0 be an upper-semicontinuous function in \mathcal{F}_α for some α and let v given by (A.6). Then v takes on the initial value v_0 . More precisely.*

$$(A.11) \quad \limsup_{(x, t) \rightarrow (x_1, 0)} v(x, t) \leq v_0(x_1)$$

If $v_0(x_1) > -\infty$, then

$$(A.12) \quad \liminf_{\substack{(x, t) \rightarrow (x_1, 0) \\ |x - x_1| = 0(t^{1/2})}} v(x, t) \geq v_0(x_1).$$

PROOF. We only prove (A.11). For every $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x - x_1| \leq \delta$ then

$$v_0(x) \leq v_0(x_1) + \epsilon.$$

If $|x - x_1| < \delta/2$ and $|y - x| \leq \delta/2$, then for $t > 0$ we have

$$v_0(y) - \frac{|x - y|^2}{4t} \leq v_0(x_1) + \epsilon.$$

On the other hand, if $|x - x_1| < \delta/2$ and $|y - x| \geq \delta/2$, then

$$v_0(y) - \frac{|x - y|^2}{4t} \downarrow -\infty$$

uniformly in y as $t \downarrow 0$. \square

Corollary. *If $v_0 \in \mathcal{F}_\alpha$ the $L_t v_0$ converges as $t \downarrow 0$ to the upper semicontinuous envelope of v_0 i.e. the minimal of the upper semicontinuous functions $w: \mathbb{R}^N \rightarrow \{-\infty\} \cup \mathbb{R}$ which are larger than v_0 .*

3. Regularity properties of the Lax-Oleinik Formula

Proposition A.6. *Let $v_0 \in \mathcal{F}_\alpha$ for some $\alpha > 0$. Then for every $\tau > 0$, v is Lipschitz continuous uniformly on compact subsets of $\mathbb{R}^N \times (\tau, T)$. If $v_0(x) \leq (a|x| + b)^2$ in \mathbb{R}^N , then for almost every $x \in \mathbb{R}^N$ and $t \in (0, 1/4\alpha)$*

$$(A.13) \quad v_t = |Dv|^2 \leq \frac{(a|x| + b)^2}{t(1 - 2at^{1/2})^2}.$$

On the other hand, if v_0 is bounded from above by M , then

$$(A.14) \quad v_t = |Dv|^2 \leq \frac{M - v}{t} \leq \frac{M - v_0(x)}{t}.$$

The proofs are easy consequences of (A.6). Another regularity type question is related to the optimality of the bounds (A.10).

Proposition A.7. *For every $t \in (0, T)$, we have ($\alpha = 1/4T$)*

$$(A.15) \quad \limsup_{|x| \rightarrow \infty} \frac{v(x, t)}{|x|^2} = \frac{\alpha}{1 - 4\alpha t} = \frac{1}{4(T - t)}.$$

PROOF. The inequality \leq was proved in Proposition A.2. For the converse assume that for some $t_1 \in (0, T)$ we have

$$\limsup_{|x| \rightarrow \infty} \frac{v(x, t_1)}{|x|^2} = \alpha_1 < \frac{\alpha}{1 - 4\alpha t_1}.$$

Then the solution with initial value $v(\cdot, t_1)$ exists for a time

$$t_2 = \frac{1}{4\alpha_1} > \frac{1 - 4\alpha t_1}{4\alpha}.$$

By the semigroup property the original solution would then exist for a time

$$t_1 + t_2 > \frac{1}{4\alpha} = T,$$

which is a contradiction. \square

As far as lower bounds are concerned we have the following result.

Proposition A.8. *Let $\beta \in [-\infty, \infty)$ be defined by*

$$(A.16) \quad \beta = \liminf_{|x| \rightarrow \infty} \frac{v_0(x)}{|x|^2}.$$

For every $t \in (0, T)$ we have

$$(A.17) \quad \liminf_{|x| \rightarrow \infty} \frac{v(x, t)}{|x|^2} \geq \frac{\beta}{1 - 4t\beta}.$$

The equality is false in general.

PROOF. If $\beta = -\infty$, (A.17) reduces to (A.10). If $\beta \in (-\infty, 0)$, then for every $\epsilon > 0$ there exists a $B_\epsilon \in \mathbb{R}$ such that $v_0(x) \geq (\beta - \epsilon)|x|^2 - B_\epsilon$. We compare $v(x, t)$ to the explicit solution

$$\phi(x, t) = -B_\epsilon - \frac{|x|^2}{4t + \tau}$$

with $\tau = -1/(\beta - \epsilon)$ and $\beta - \epsilon \neq 0$. Using the Lax-Oleinik formula we conclude that $v \geq \phi$ in Q_T , hence as $\epsilon \rightarrow 0$ we obtain the inequality \geq in (A.18). To show that equality does not hold in general we consider a v_0 defined as follows: Let $B(y_n, r_n)$ be a sequence of balls such that $r_n \rightarrow 0$ and $y_n \rightarrow \infty$ where v_0 is negative, continuous and $v(y_n)|y_n|^{-2} \rightarrow -\infty$. Outside these balls $v_0 \equiv 0$. Therefore $\beta = -\infty$. If $(x, t) \in \mathbb{R}^N \times (0, \infty)$ we have $v(x, t) = 0$ if $x \notin \bigcup_n B(y_n, r_n)$. If $x \in B(y_n, r_n)$, then there exists $y = y(x)$ such that $v_0(y) = 0$ and $|y - x| = r_n$. Hence

$$v(x, t) \geq v_0(y) - \frac{|x - y|^2}{4t} \geq -\frac{r_n^2}{4t}.$$

Therefore,

$$\lim_{|x| \rightarrow \infty} \frac{v(x, t)}{|x|^2} = 0. \quad \square$$

We conclude the presentation of regularity-related properties of the L Oleinik formula with a result concerning their semiconvexity. Since th properties are an immediate consequence of the formula, we again omit proof.

Proposition A.9. (1) *Let $\chi \in \mathbb{R}^N$, $|\chi| = 1$. Then*

$$(i) \quad \frac{\partial^2 v}{\partial \chi^2} \geq -\frac{1}{2t}.$$

(ii) *If for every $\chi \in \mathbb{R}^N$, $|\chi| = 1$, $\partial^2 v_0 / \partial \chi^2 \geq -\alpha$ then for every $t > 0$,*

$$\frac{\partial^2 v}{\partial \chi^2} \geq -\frac{\alpha}{1 + 2\alpha t}.$$

(2) *If $\Delta v_0 \geq -\alpha$ then for $t > 0$,*

$$\Delta v(x, t) \geq -\frac{N\alpha}{N + 2\alpha t}.$$

All the above inequalities should be interpreted in the sense of distributio

4. Uniqueness and continuous dependence

We begin with a proposition concerning the domain of dependence of Lax-Oleinik formula. The proof of this result is based on the gradi estimates from Proposition A.6 and the proofs of M. G. Crandall and Newcomb [20] and P. E. Souganidis [36] concerning viscosity solutions on boundary. See [5] for $N = 1$. Since it is a long exercise, we omit it.

Proposition A.10. *Let v_{01}, v_{02} be two initial data in \mathbb{R}^N such that*

$$v_{01}(x), v_{02}(x) \leq (a|x| + b)^2$$

for every $x \in \mathbb{R}^N$ and some constants $a, b \geq 0$. Let $t \in (0, 1/4a^2)$. Then

$$v_1(0, t) - v_2(0, t) \leq \sup_{y \in I_0} \{v_{01}(y) - v_{02}(y)\}$$

where

$$I_0 = \left\{ y \in \mathbb{R}^N: |y| \leq \frac{b}{a} \left[\exp\left(\frac{\lambda}{1-\lambda}\right) - 1 \right], \lambda = 2at^{1/2} \right\}$$

if $a, b > 0$ and

$$I_0 = \{y \in \mathbb{R}^N: |y| \leq \sqrt{4bt}\}$$

if $a = 0$ and $b > 0$.

Corollary. *If $v_0^n \rightarrow v_0$ locally uniformly in \mathbb{R}^N , then $v^n \rightarrow v$ locally uniformly in $\mathbb{R}^N \times [0, T)$.*

Next we prove the uniqueness of viscosity solutions of (A.7) with upper-semicontinuous initial datum $v_0: \mathbb{R}^N \rightarrow \mathbb{R} \cup \{-\infty\}$. This implies that the viscosity solution of (A.7) is given by the Lax-Oleinik formula, therefore it enjoys all the regularity presented above.

Theorem A.1. *The viscosity solution of (1.2) in Q_T with upper-semicontinuous initial datum $v_0: \mathbb{R}^N \rightarrow \mathbb{R} \cup \{-\infty\}$ is unique.*

PROOF. If $\underline{v}(x, t) = L_t(v_0)(x)$, in view of (A.5), we only have to show that $v \leq \underline{v}$. We argue as follows: Since v is defined in Q_T , for every $t \in [0, T)$ we have

$$\limsup_{|x| \rightarrow \infty} \frac{v(x, t)}{|x|^2} \leq \frac{1}{4(T-t)}.$$

Let $v_{0n} \in C(\mathbb{R}^N, \mathbb{R})$ be such that $v_{0n} > v_0$ and

$$\limsup_{|x| \rightarrow \infty} \frac{v_{0n}(x)}{|x|^2} = \frac{1}{4\left(T - \frac{1}{n}\right)}.$$

Then $v_n = L_t(v_{0n})$ exists for a time $T_n = T - 1/n$. Moreover, for $t \in (0, T_n)$,

$$\limsup_{|x| \rightarrow \infty} \frac{v_n(x, t)}{|x|^2} = \frac{1}{4\left(T - \frac{1}{n} - t\right)}.$$

If $v_{n\epsilon} = v_n * \rho_\epsilon$, where $*$ denotes the standard convolution, then for $t > \epsilon$ we have

$$(v_{n\epsilon})_t = |Dv_n|^2 * \rho_\epsilon \geq |Dv_{n\epsilon}|^2.$$

Let $w_{\epsilon n}: \mathbb{R}^N \times [0, T_n - \epsilon]$ be defined by

$$w_{n\epsilon}(x, t) = v_{n\epsilon}(x, t + \epsilon) + \epsilon t.$$

The sup of $v - w_{n\epsilon}$ in $\mathbb{R}^N \times [0, t_1)$ with $t_1 < t_n - \epsilon$ cannot be taken in the interior of $\mathbb{R}^N \times [0, t_1)$. Therefore, either it is approached as $|x| \rightarrow \infty$ or taken at $t = 0$. In either case, it is negative. It then follows that

$$v < w_{n\epsilon} \quad \text{in } \mathbb{R}^N \times [0, T_n - \epsilon].$$

Letting $\epsilon \rightarrow 0$ yields

$$v \leq v_n \quad \text{in } \mathbb{R}^N \times [0, T_n).$$

Sending $n \rightarrow \infty$ and using the continuous dependence of the Lax-Oleinik conditions on the initial data in local norms we conclude. \square

5. Free Boundaries

In the case of solutions which are bounded from either above or below it makes sense to consider the boundary of the sets where the largest or smallest values are attained. Let us consider first the case of an upper-semicontinuous initial datum $v_0: \mathbb{R}^N \rightarrow \mathbb{R} \cup \{-\infty\}$ which is bounded from above by a constant M . Let

$$(A.18) \quad \begin{aligned} D_+ &= D_+(v_0) \\ &= \{x \in \mathbb{R}^N: v_0(x) = M\}. \end{aligned}$$

This is a closed, possibly empty, set. It is immediate from $v_t \geq 0$ that if $x \in D_+$ then for every $t > 0$, $v(x, t) = M$. Therefore the set D_+ is invariant in time so is its boundary. On the contrary, if v_0 is bounded from below by a constant which without any loss of generality we may assume to be zero, then the

$$(A.19) \quad \Omega_0 = \{x \in \mathbb{R}^N: v_0(x) > 0\}$$

is not necessarily open or closed. We define:

$$(A.20) \quad \begin{cases} \Omega = \{(x, t) \in \mathbb{R}^N \times [0, T): v(x, t) > 0\} \\ \Omega(t) = \{x \in \mathbb{R}^N: (x, t) \in \Omega\} \\ \Gamma = \text{boundary of } \Omega \text{ in } \mathbb{R}^N \times [0, T) \\ \Gamma(t) = \{x \in \mathbb{R}^N: (x, t) \in \Gamma\}. \end{cases}$$

Γ is called the *free boundary* of v . Since $v_t \geq 0$, the following result is immediate.

Proposition A.11. *For every $t_2 > t_1$ in $(0, T)$, $\Omega_0 \subset \Omega(t_1) \subset \Omega(t_2)$.*

Next we examine the behavior of v on Ω .

Proposition A.12. *Let $t \in (0, T)$. For almost every $x \in \Omega(t) \setminus \Omega_0$ there exists a point $y = y(x) \in \Omega_0$ such that*

$$Dv = -\frac{x - y}{2t}.$$

For all points $x \in \Omega(t) \setminus \Omega_0$, $-\frac{x - y}{2t}$ is a subdifferential of v at x .

PROOF. Since $v(x, t) > 0$ there exist $y_n \in \mathbb{R}^N$ such that

$$v_0(y_n) - \frac{|x - y_n|^2}{4t} \uparrow v(x, t).$$

It follows that $v_0(y_n) > 0$, i.e. $y_n \in \Omega_0$ and

$$(A.21) \quad |x - y_n|^2 \leq 4tv_0(y_n) < 4t(\alpha + \epsilon)(|y_n| + b_\epsilon)^2.$$

Since $4t(\alpha + \epsilon) < 1$ if ϵ is small enough, $|y_n| \leq C$ and, upon passing to a subsequence, we may assume that $y_n \rightarrow y$. The upper semicontinuity of v_0 yields $y \in \Omega_0$ and

$$(A.22) \quad v(x, t) = v_0(y) - \frac{|x - y|^2}{4t},$$

Next let $h \in \mathbb{R}^N$ with $|h|$ small. Then

$$\begin{aligned} v(x + h, t) - v(x, t) &\geq v_0(y) - \frac{|x + h - y|^2}{4t} - v_0(y) - \frac{|x - y|^2}{4t} \\ &\geq -\frac{1}{2t} h \cdot (x - y) - \frac{|h|^2}{4t}. \quad \square \end{aligned}$$

Since $-Dv$ is the local velocity of propagation of the solutions of (0.2), this result controls the speed with which the interface moves. In fact the interface consists of a stationary part Γ_0 , a union of vertical segments $\{(x, t): 0 \leq t \leq t_1\}$ with $x \in \partial\Omega_0$ fixed, and the moving interface

$$\Gamma_1 = \{(x, t) \in \Gamma: x \notin \Omega_0\}.$$

Proposition A.13. *The moving interface Γ_1 can be described by a Lipschitz continuous function $t = S(x)$ for $x \in \mathbb{R}^N \setminus \bar{\Omega}_0$. More precisely, for every $(\bar{x}, \bar{t}) \in \Gamma_1$ there is a conical region $K = \{(x, t): |x - \bar{x}| < h, |x - \bar{x}| < c|t - \bar{t}|\}$ with $0 < c < \text{dist}(\bar{x}, \Omega_0)(2\bar{t})^{-1}$ and h small depending on c, \bar{x} , such that*

$$K_+ = \{(x, t) \in K: t > \bar{t}\} \subset \Omega \quad \text{and} \quad K_- = \{(x, t) \in K: t < \bar{t}\}$$

is disjoint with Ω

PROOF. We only prove the result concerning K_+ . The result about K follows in a similar way. To this end, let $(x, t) \in K_+$ and set $x - \bar{x} = z$ at $t - \bar{t} = \tau$. If there exists a $y \in \Omega_0$ with the properties described in Proposition A.12 (since $v(\bar{x}, \bar{t}) = 0$ this is not necessarily the case) we have

$$\begin{aligned} v(x, t) &\geq v_0(y) - \frac{|x - y|^2}{4t} \\ &= \frac{|\bar{x} - y|^2}{4\bar{t}} - \frac{|x - y|^2}{4t} \\ &= \frac{|\bar{x} - y|^2}{4\bar{t}} - \frac{|\bar{x} - y|^2}{4t} - \frac{(\bar{x} - y) \cdot z}{2t} - \frac{|z|^2}{4t} \\ &= \frac{|\bar{x} - y|}{4t} \left(\frac{|\bar{x} - y|}{\bar{t}} \tau - 2z \cdot \theta \right) - 0(|z|^2) \end{aligned}$$

where $\theta = \bar{x} - y/|\bar{x} - y|$. Therefore if $z/\tau \leq d(\bar{x}, D_0)/2\bar{t} \leq |\bar{x} - y|/2\bar{t}$ and $0 < |z| < h$ with h small we have $v(x, t) > 0$.

If such a y does not exist we select a sequence of points $x_n \in \Omega_t$, $x_n \rightarrow x$, and find $y_n \in \Omega_0$, construct a cone K_n with vertex (x_n, \bar{t}) and let $n \rightarrow \infty$ to obtain K_+ . We define

$$(A.23) \quad S(x) = \sup \{t \geq 0: v(x, t) = 0\}.$$

The Lipschitz continuity of S at (\bar{x}, \bar{t}) follows from the fact for every x such that $|x - \bar{x}| < h$ then $(x, t) \in \Gamma_1$ implies $t - \bar{t} \leq C|x - \bar{x}|$. \square

Corollary. Let $x \in \Omega_t \setminus \Omega_0$. If $d(x) = \text{dist}(x, \Omega_0)$, then

$$(A.24) \quad \frac{d(x)}{2t} < |Dv| < \frac{c(|x|)}{2t}$$

where $c(r)$ is a continuous function of r .

PROOF. Take $y \in \Omega_0$ as in (A.22). We have $|x - y| \geq d(x)$ and, from (A.2)

$$|y| \leq \frac{|x| + b_\epsilon k}{1 - k}, \quad k = (4t(\alpha + \epsilon))^{1/2}.$$

We conclude. \square

The above proof also shows that at every point x where S is differentiable we have $|DS| < 1/c$ for any c as in Proposition A.13. Therefore

$$(A.25) \quad |DS| \cdot \frac{d(x)}{2t} < 1.$$

In other words, $(2t)^{-1}d(x)$ is a lower bound for the velocity with which Ω_t grows. Thus if $\text{dist}(\Omega_t, \Omega_0) = d > 0$ then for every $\tau > 0$ small enough $\Omega_{t+\tau}$ contains an ϵ -neighborhood of Ω_t , if $\epsilon < d\tau/2t$. In fact Ω_t moves with speed bounded from above. More precisely, we have:

Proposition A.14. *If $\tau > 0$ is small enough, then for every $\bar{x} \in \partial Q_{\bar{t}+\tau}$ there exists $C = C(|\bar{x}|)$ such that*

$$(A.26) \quad \text{dist}(\bar{x}, \Omega_{\bar{t}}) \leq \frac{C\tau}{2t}.$$

Moreover, if v_0 is bounded from above, the bound on (A.26) is independent of \bar{x} and

$$(A.27) \quad \text{dist}(\Gamma_{t+\tau}, \Omega_t) \leq \frac{C\tau}{2t}.$$

PROOF. Let $r = \text{dist}(\bar{x}, \Omega_{\bar{t}})$ and c be an upper bound on v in $B(x, 2r)$ (which should be separated from Ω_0). The function

$$V(x, t) = c(c(t - \bar{t}) + |x - \bar{x}| - r)^+$$

is a supersolution of (1.2). The result follows. \square

We conclude by characterizing the existence of a stationary interface Γ_0 . A careful look at the Lax-Oleinik formula yields the following proposition.

Proposition A.15. *Let $x \in \Omega_0$. Then $v(x, t) = 0$ for $t \in [0, t^*]$ if and only if the quantity*

$$(A.28) \quad \gamma(x) = \sup_{y \in \mathbb{R}^N} \frac{v_0(y)}{|x - y|^2}$$

is finite. The starting time t^* is given by $1/4\gamma$.

6. Generalizations

All the above can be easily generalized to the Cauchy problems

$$(A.29) \quad \begin{cases} v_t = |Dv|^p & \text{in } \mathbb{R}^N \times (0, T) \\ v = v_0(x) & \text{on } \mathbb{R}^N \times \{t = 0\}, \end{cases}$$

with $p > 1$. The formula for viscosity solutions of (A.29) is

$$(A.30) \quad \underline{v}(x) = \sup \left\{ v_0(y) - C_p \frac{|x - y|^{p/(p-1)}}{t^{1/(p-1)}} \right\}$$

for $C_p = (p - 1)p^{-p/(p-1)}$.

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References

- [1] Aronson, D. G. The porous medium equation, in Nonlinear Diffusion Problems, A. Fasano and M. Primicerio eds., *Lecture Notes in Math.* 1224, Springer, 1986.
- [2] Aronson, D. G. and Bénéilan, Ph. Régularité des solutions de l'équation milieux poreux dans \mathbb{R}^N , *C.R. Acad. Sci. Paris* **288**(1979), 103-105.
- [3] Aronson, D. G., Caffarelli, L. A. and Kamin, S. How an initially stationary interface begins to move in porous medium flow, *SIAM J. Math. Anal.* **14**(1982), 639-658.
- [4] Aronson, D. G., Caffarelli, L. A. and Vázquez, J. L. Interfaces with a contact point in one-dimensional porous medium flow, *Comm. Pure Appl. Math.* **38**(1985), 375-404.
- [5] Aronson, D. G. and Vázquez, J. L. The porous medium equation as a first order approximation to a Hamilton-Jacobi equation, *J. Anal. Nonlinéaire, Inst. H. Poincaré*, **4**(1987), 203-330.
- [6] Bardi, M. and Evans, L. C. On Hopf's formulas for solutions of Hamilton-Jacobi equations, *Nonlinear Anal. TMA.* **8**(1984), 1373-1381.
- [7] Barles, G. and Perthame, B. Discontinuous Solutions of deterministic optimal stopping time problems, *Math. Math. Anal. Num.*, to appear.
- [8] Bénéilan, Ph. Evolution equations and accretive operators, *Lecture Notes in Math.* University of Kentucky, Spring 1981.
- [9] Bénéilan, Ph. and Crandall, M. G. The continuous dependence on ϕ of the solutions of $u_t - \Delta\phi(u) = 0$, *Indiana U. Math. J.* **30**(1981), 161-177.
- [10] Bénéilan, Ph. and Crandall, M. G. Regularizing effects of homogeneous reaction equations, in *Contributions to Analysis and Geometry*, Edited by I. Clark, G. Peceili, R. Sacksteder, The Johns Hopkins University Press, (1982), 23-39.
- [11] Bénéilan, Ph., Crandall, M. G. and Pierre, M. Solutions of the porous medium equation in \mathbb{R}^N under optimal conditions on initial values, *Indiana U. Math. J.* **33**(1984), 51-87.

- [12] Buckmaster, J. Viscous sheets advancing over dry bed, *J. Fluid Mech.* **81** (1977), 735-756.
- [13] Caffarelli, L. A. and Friedman, A. Regularity of the free boundary of a gas flow in an n -dimensional porous medium, *Indiana U. Math. J.* **29** (1980), 361-391.
- [14] Caffarelli, L. A., Vázquez, J. L. and Wolansky, N. Lipschitz continuity of solutions and interfaces of the N -dimensional porous medium equation, *Indiana U. Math. J.* **36** (1987), 373-401.
- [15] Crandall, M. G., Evans, L. C. and Lions, P.-L. Some properties of viscosity solutions of Hamilton-Jacobi equations, *Trans. Amer. Math. Soc.* **282**(1984), 487-502.
- [16] Crandall, M. G., Ishii, H. and Lions, P.-L. Uniqueness of viscosity solutions revisited, *Math. Soc. of Japan*, to appear.
- [17] Crandall, M. G. and Lions, P.-L. Viscosity solutions of Hamilton-Jacobi equations, *Trans. Amer. Math. Soc.* **277**(1983), 1-42.
- [18] Crandall, M. G. and Lions, P.-L. On existence and uniqueness of solutions of Hamilton-Jacobi equations, *Nonlinear Anal.* **10**(1986), 353-370.
- [19] Crandall, M. G. and Lions, P.-L. Two approximations of solutions of Hamilton-Jacobi equations, *Math. Comp.* **43**(1984), 1-19.
- [20] Crandall, M. G. and Newcomb, R. Viscosity solutions of Hamilton-Jacobi equations at the boundary, *Proc. Amer. Math. Soc.* **94**(1985), 283-290.
- [21] Dahlberg, B. E. and Kenig, C. E. Nonnegative solutions of the porous medium equation, *Comm. PDE.* **9**(1984), 409-437.
- [22] Evans, L. C. and Ishii, H. Personal communication.
- [23] Evans, L. C. and Souganidis, P. E. A PDE approach to geometric optics for certain reaction-diffusion equations, *Indiana U. Math. J.*, to appear.
- [24] Fleming, W. H. and Souganidis, P. E. PDE-viscosity solution approach to some problems of large deviations, *Ann. Scuola Norm. Sup.* **13**(1986), 171-192.
- [25] Gilding, B. H. Hölder continuity of solutions of parabolic equations, *J. London Math. Soc.* **13** (1976), 103-106.
- [26] Gurtin, M. and MacCamy, R. C. On the diffusion of biological populations, *Math. Biosc.* **33**(1977), 35-49.
- [27] Ishii, H. Hamilton-Jacobi equations with discontinuous Hamiltonians on arbitrary open subsets.
- [28] Lasry, J. M. and Lions, P.-L. A remark on regularization on Hilbert spaces, *Israel J. Math.*, to appear.
- [29] Lions, P.-L. *Generalized Solutions of Hamilton-Jacobi Equations*, Pitman, Boston, 1982, to appear
- [30] Lions, P.-L. and Rochet, J.-C. Hopf formula and multi-time Hamilton-Jacobi equations, *Proc. Amer. Math. Soc.* **96**(1986), 79-84.
- [31] Lions, P.-L. and Souganidis, P. E. Viscosity solutions of second-order equations, stochastic control and stochastic differential games, Proceedings of Workshop on Stochastic Control and PDE's, IMA, June 1986.
- [32] MacHyman, personal communication with P.-L. Lions.
- [33] Muskat, M. *The Flow of Homogeneous Fluids Through Porous Media*, McGrawHill, New York, 1937.
- [34] Raizer, Yu. P., and Zeldovich, Ya. B. *Physics of Shock-Waves and High-Temperature hydrodynamic phenomena*, Vol. II, Academic Press, New York, 1966.
- [35] Souganidis, P. E. Existence of viscosity solutions of Hamilton-Jacobi equations, *J. Diff. Equations.* **56**(1985), 345-390.

- [36] Souganidis, P. E. A remark about viscosity solutions of Hamilton-Jacobi equations at the boundary, *Proc. Amer. Math. Soc.* **96**(1986), 323-330.
- [37] Vázquez, J. L. Hyperbolic aspects in the theory of the porous medium equation. *Metastability and Incompletely Posed Problems*. IMA Volumes in Mathematics and Applications No. 3, S. Antman et al. eds., Springer 1987, 325-342.

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