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Norm Inequalities for Potential-Type Operators

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Introduction

The purpose of this paper is to derive norm inequalities for potentials of the form

$$Tf(x) = \int_{\mathbb{R}^n} f(y)K(x,y)\,dy, \qquad x \in \mathbb{R}^n,$$

when K is a kernel which satisfies estimates like those that hold for the Green function associated with the degenerate elliptic equations studied in [3] and [4]. Thus, for $0 \leqslant r < \infty$, $x \in \mathbb{R}^n$, and a nonnegative function $a(r,x)$ to be specified, we assume that

$$\text{(i)} \qquad |K(x,y)| \leqslant C \frac{a(|x-y|,x)}{|x-y|^n},$$

and, in some cases, that there exists ϵ_0, $0 < \epsilon_0 \leqslant 1$, such that

$$\text{(ii)} \qquad |K(x,y) - K(x,z)| \leqslant C\left(\frac{|y-z|}{|x-y|}\right)^{\epsilon_0} \frac{a(|x-y|,x)}{|x-y|^n}$$

if

$$|y-z| < \frac{1}{2}|x-y|.$$

It will be convenient to think of $a(r, x)$ as a function of the ball $B = B(r, x)$ with radius r and center x, and to write

$$a(B) = a(B(r, x)) = a(r, x).$$

We then require that there is a constant $C > 0$ such that

$$\text{(iii)} \qquad a(B_1) \leqslant Ca(B_2) \quad \text{if} \quad B_1 \subset B_2, \quad \text{and}$$

(iv) there exist $\mu, d > 0$ so that if tB denotes the ball concentric with B whose radius is t times that of B, then

$$C^{-1}t^{\mu}a(B) \leqslant a(tB) \leqslant Ct^{nd}a(B), \qquad t > 1.$$

Conditions (iii) and (iv) can be weakened in some of our results: see the comments later in this section. We also remark that the results which require condition (ii), which is a first-order smoothness condition, have analogues which reflect any higher order smoothness that K may have.

The simplest examples of such kernels are the classical fractional integral kernels $K(x, y) = 1/|x - y|^{n-\alpha}$, $\alpha > 0$. In this case, $a(r, x) = r^{\alpha}$ and (ii) holds with $\epsilon_0 = 1$. A more typical example is

$$(*) \qquad a(r, x) = \frac{r^{n+\alpha}}{w(B(r, x))},$$

where w is a weight function, *i.e.*, a nonnegative locally integrable function on $\mathbb{R}^n$, and $w(B) = \int_B w(x)\,dx$. In this case, estimate (i) becomes

$$|K(x, y)| \leqslant C\frac{|x - y|^{\alpha}}{w(B(|x - y|, x))}.$$

Such kernels with $\alpha = 2$ arise naturally if we consider a bounded domain Ω and a divergence form differential operator

$$L = -\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)$$

whose coefficient matrix $A(x) = (a_{ij}(x))$ satisfies

$$c_1 w(x)|\xi|^2 \leqslant \langle A(x)\xi, \xi\rangle \leqslant c_2 w(x)|\xi|^2, \qquad \xi \in \mathbb{R}^n.$$

Here, $\langle\cdot,\cdot\rangle$ denotes the usual dot product in $\mathbb{R}^n$ and $0 < c_1 < c_2 < \infty$. In [4], estimates are derived for the Green function $G(x, y)$ associated with such an operator. If w is suitably restricted (see the comments later in this section), these estimates imply that in the interior of Ω, $G(x, y)$ satisfies (i) and (ii) for some $\epsilon_0 > 0$, with $a(B)$ given by $(*)$ and $\alpha = 2$.

If $0 < p < \infty$ and v is a weight function, let

$$L_v^p = \left\{f\colon \|f\|_{L_v^p} = \left(\int_{\mathbb{R}^n} |f(x)|^p v(x)\,dx\right)^{1/p} < \infty\right\}.$$

We also consider the Hardy space H_v^p defined as follows. Let $\mathcal{S}$ denote the Schwartz class of rapidly decreasing functions on $\mathbb{R}^n$, and let $\mathcal{S}'$ denote the class of tempered distributions. For $0 < p < \infty$,

$$H_v^p = \{f \in \mathcal{S}' \colon \|f\|_{H_v^p} = \|N(f)\|_{L_v^p} < \infty\},$$

where

$$N(f)(x) = N_\phi(f)(x) = \sup_{t>0} |(f * \phi_t)(x)|$$

for $\phi \in \mathcal{S}$, $\int_{\mathbb{R}^n} \phi\,dx \neq 0$, $\phi_t(x) = t^{-n}\phi(x/t)$. By [10], the finiteness of $\|N(f)\|_{L_v^p}$ is independent of ϕ if v satisfies the doubling condition $v(2B) \leqslant cv(B)$. We write $v \in D_\infty$ for such v.

In case $1 < p < \infty$ and $v \in A_p$, *i.e.*, if

$$\left(\frac{1}{|B|}\int_B v\,dx\right)\left(\frac{1}{|B|}\int_B v^{-1/(p-1)}\,dx\right)^{p-1} \leqslant c$$

for all balls B, it is well-known that $H_v^p \equiv L_v^p$ with equivalence of norms. This identification is also valid for certain other v's (see [1] and [11]) and is important for some of our results. We mention here that the class $\mathcal{S}_{0,0}$ of Schwartz functions whose Fourier transforms have compact support not containing the origin is dense in all the spaces L_v^p and H_v^p which we will consider. Note that all the moments of an $f \in \mathcal{S}_{0,0}$; in particular, $\int_{\mathbb{R}^n} f\,dx = 0$ for such f.

In order to state our main results, we now introduce several conditions on a pair of weight functions u and v. Let χ_E denote the characteristic function of a set $E \subset \mathbb{R}^n$. For $0 < p, q < \infty$, we consider the following kinds of conditions:

(1) $$a(B)u(B)^{1/q} \leqslant cv(B)^{1/p}$$

for all balls B;

(1)′ $$a(B)^p\left(\frac{1}{|B|}\int_B u^s\,dx\right)^{1/s} \leqslant c\frac{v(B)}{|B|}$$

for some $s > 1$ and all balls B;

(2) $$\|\Sigma a(B_k)\lambda_k\chi_{B_k}\|_{L_u^q} \leqslant c\|\Sigma\lambda_k\chi_{B_k}\|_{L_v^p}$$

for all sequences $\{\lambda_k\}$, $\lambda_k > 0$, and all balls B_k.

Note that (2) is stronger than (1) since it reduces to (1) in the case of a single ball. Also note that (1)′ with $s = 1$ is the same as (1) with $q = p$; thus, (1)′ represents a strengthening of (1) in case $q = p$. Some other relations between these conditions are given in Theorems 5 and 6, and an alternate form of (1)′ is given in Lemma 6.5.

We shall also use the condition

$$(3) \qquad \|\Sigma\lambda_k \chi_{tB_k}\|_{L^p_v} \leqslant ct^\delta \|\Sigma\lambda_k \chi_{B_k}\|_{L^p_v}, \qquad t > 1.$$

This is related to the doubling condition: in fact, if $v \in D_\sigma$, by which we mean

$$v(tB) \leqslant ct^{n\sigma} v(B), \qquad t > 1,$$

then (3) holds with $\delta = n\sigma$ and $1 \leqslant p < \infty$ (see [12]). Moreover, as shown in [10], if $v \in A_r \cap D_\sigma$ then (3) holds with

$$\delta = \frac{n}{p}\left(\frac{r-p}{r-1}\right)(\sigma - 1) + n, \qquad r \geqslant p.$$

It is clear by considering a single term that (3) implies that $v \in D_{\delta p/n}$.

The principal results to be proved are as follows.

Theorem 1. *Let* (i)-(iv) *hold for a kernel* $K(x, y)$. *Let* $0 < p$, $q < \infty$, *u and v satisfy* (2), *and v satisfy* (3) *for some* $\delta < n + \epsilon_0$. *Then if* $f \in \mathcal{S}_{0,0}$, *the operator defined by*

$$Tf(x) = \int_{\mathbb{R}^n} f(y) K(x, y)\, dy$$

satisfies $\|Tf\|_{L^q_u} \leqslant c\|f\|_{H^p_v}$ *with c independent of f.*

Without assuming the smoothness condition (ii), we have

Theorem 2. *Let* (i), (iii) *and* (iv) *hold for a kernel* $K(x, y)$. *Let* $1 < p < \infty$, *u and v satisfy* (2), *and* $v \in A_p$. *Then if Tf is defined as above,*

$$\|Tf\|_{L^q_u} \leqslant c\|f\|_{L^p_v} \quad \textit{for all} \quad f \in L^p_v.$$

Theorem 2 turns out to be a corollary of Theorem 1 and the fact that $H^p_v \equiv L^p_v$ if $v \in A_p$. We can use Theorem 1 to derive results of the L^q_u, L^p_v type when $v \notin A_p$ provided $H^p_v \equiv L^p_v$, although we need the smoothness condition (ii) in this case. The next theorem is an example of such a result. It includes the power weights $v(x) = |x|^\beta$ for $n(p-1) < \beta < n(p-1) + \epsilon_0 p$; the range of β for which $|x|^\beta \in A_p$ is $-n < \beta < n(p-1)$. More general theorems of this type can also be derived.

Theorem 3. *Let* (i)-(iv) *hold for a kernel* $K(x, y)$. *Let* $1 < p < \infty$, $0 < q < \infty$, *u and v satisfy* (2), $|x|^{-\epsilon p} v \in A_p$ *for some* ϵ *with* $0 < \epsilon \leqslant \epsilon_0$, *and* $|x|^{-np} v \in A_p$. *If S is defined by*

$$Sf(x) = \int_{\mathbb{R}^n} f(y)[K(x, y) - K(x, 0)]\, dy$$

then $\| Sf \|_{L^q_u} \leqslant c \| f \|_{L^p_v}$ *for all* $f \in L^p_v$.

Moreover, the integral defining Sf converges absolutely a.e. Of course, $Sf = Tf$ *if* $\int f\, dy = 0$.

The next three theorems are intended to help understand condition (2) and show its relation to (1) and (1)′. For $1 < p < \infty$, p' is defined by $1/p + 1/p' = 1$.

Theorem 4. *Let* (iii) *hold,* $v \in D_\infty$ *and*

$$Mf(x) = \sup_{B \ni x} \frac{a(B)}{v(B)} \int_B |f(y)| u(y)\, dy.$$

If $1 < p$, $q < \infty$, *then* (2) *holds if and only if* $\|Mf\|_{L^{p'}_v} \leqslant c \| f \|_{L^{q'}_u}$.

Theorem 5. *Let* (iii) *and* (iv) *hold,* $1 < p < q < \infty$, $u \in D_\infty$ *and* $v \in A_p$. *Then* (1) *implies* (2).

Theorem 6. *Let* (iii) *and* (iv) *hold,* $1 < p < \infty$ *and* $v \in A_p$. *Then* (1)′ *implies* (2) *with* $q = p$.

By combining Theorems 2, 5 and 6, we see that if (i), (iii) and (iv) hold, $u \in D_\infty$, $1 < p < \infty$ and $v \in A_p$, then $\| Tf \|_{L^q_u} \leqslant \| f \|_{L^p_v}$ provided either

(*a*) $p < q < \infty$ and (1) holds, or
(*b*) $p = q$ and (1)′ holds.

Theorems 1-6 are proved in §1-6 respectively. Some of the methods of the proofs are related to those in [2], [6] and [12]. We also use results about H^p_v from [1] and [10], such as theorems stating when H^p_v and L^p_v can be identified, the atomic decomposition, and the fact that if $v \in D_\infty$ ($= \bigcup_{\sigma \geq 1} D_\sigma$), then $\| f \|_{H^p_v}$ is equivalent to the L^p_v norm of the «grand» maximal function f^* of f defined in [5].

In addition to the notation already introduced, we write $v \in A_\infty$ if $v \in A_p$ for some p, and $v \in RD_\nu$ (reserve doubling), $\nu > 0$, if

$$v(tB) \geqslant ct^\nu v(B), \qquad t > 1,$$

for some $c > 0$ independent of t and B. We will use the same letter c to denote different constants, and we often write $\int f$ for $\int_{\mathbb{R}^n} f(x)\, dx$.

Finally, we make a few comments on two points raised earlier. First, it is not hard to see from the estimates derived in [4] that the Green function mentioned above satisfies (i) and (ii), for some $\epsilon_0 > 0$ and $a(r, x) = r^{n+2}/w(B(r, x))$, provided $w \in A_2 \cap RD_\nu$ for some $\nu > 2$. Furthermore, in this case, (iii) and (iv) hold if $w \in A_{1+2/n}$, or more generally, if $w \in D_\sigma$ for some $\sigma < 1 + 2/n$. Second, some of our results can be proved under weaker conditions than those listed in (i)-(iv). Generally speaking, we only use the full force of condition (iii) as well as of the first inequality in condition (iv) when we prove Theorems 5 and 6. Elsewhere we can weaken these by requiring only that $a(r, x) \leqslant ca(s, y)$ when $r \approx s$ and $|x - y| \leqslant cr$, provided we also require some local integrability of K. For example, Theorem 1 remains true if (i)-(iv) are replaced by assuming that $K(x, y)(1 + |y|)^{-L}$ is an integrable function of y for some L, and

$$\int_{B(r,z)} |K(x, y)|\, dy \leqslant ca(r, z) \quad \text{if} \quad x \in B(4r, z),$$

and

$$\int_{B(r,z)} |K(x, y) - K(x, z)|\, dy \leqslant c\left(\frac{r}{|x - z|}\right)^{n+\epsilon_0} a(|x - z|, z)$$

if $x \notin B(4r, z)$, where $a(\cdot, \cdot)$ is a function which satisfies $a(r, x) \leqslant ca(s, x)$ when $r \approx s$.

1. Proof of Theorem 1

Let f be an atom associated with $B(r, y_0)$, *i.e.*, let $|f| \leqslant 1$, supp $(f) \subset B(r, y_0)$, $\int f = 0$. Write

$$\begin{aligned} Tf(x) &= \int f(y)K(x, y)dy \\ &= \int f(y)[K(x, y) - K(x, y_0)]\, dy. \end{aligned}$$

Note that Tf converges absolutely since $K(x, y)$ is locally integrable as a function of y: by (i) and (iv),

$$\begin{aligned} |K(x, y)| &\leqslant c\frac{a(|x - y|, x)}{|x - y|^n} \\ &\leqslant c\frac{a(1, x)}{|x - y|^{n-\mu}} \end{aligned}$$

if $|x - y| < 1$. Also, if $|x - y| > 1$, $|K(x, y)| \leqslant ca(1, x)|x - y|^{n(d-1)}$ by (iv).
We want to estimate the size of $|Tf(x)|$.

Case 1. $x \in B(4r, y_0)$. By (i) and the first representation of Tf above,

$$\begin{aligned}
|Tf(x)| &\leqslant c \int_{B(r,y_0)} |f(y)| \frac{a(|x-y|, x)}{|x-y|^n}\, dy \\
&\leqslant c \int_{|x-y|<5r} \frac{(|x-y|/r)^{\mu} a(r,x)}{|x-y|^n}\, dy \quad \text{by (iv)} \\
&= ca(r,x) \\
&\leqslant ca(r, y_0).
\end{aligned}$$

Case 2. $x \notin B(4r, y_0)$. By (ii) and the second representation of Tf above,

$$\begin{aligned}
|Tf(x)| &\leqslant c \int_{B(r,y_0)} |f(y)| \left(\frac{|y-y_0|}{|x-y_0|} \right)^{\epsilon_0} \frac{a(|x-y_0|, y_0)}{|x-y_0|^n}\, dy \\
&\leqslant \frac{cr^{n+\epsilon_0}}{|x-y_0|^{n+\epsilon_0}} a(|x-y_0|, y_0)
\end{aligned}$$

since $|f| \leqslant 1$.

Thus, in any case,

$$|Tf(x)| \leqslant c \sum_{j=1}^{\infty} 2^{-j(n+\epsilon_0)} a(2^j r, y_0) \chi_{B(2^j r, y_0)}(x).$$

Now let

$$f = \Sigma \lambda_k g_k$$

be a finite sum, $\lambda_k > 0$, and g_k be an atom associated with $B(r_k, y_k)$. Then

$$|Tf(x)| \leqslant c \sum_j 2^{-j(n+\epsilon_0)} \sum_k \lambda_k a(2^j r_k, y_k) \chi_{B(2^j r_k, y_k)}(x).$$

For $q \geqslant 1$, by Minkowski's inequality,

$$\begin{aligned}
\|Tf\|_{L^q_u} &\leqslant c \sum_j 2^{-j(n+\epsilon_0)} \left\| \sum_k \lambda_k a(2^j r_k, y_k) \chi_{B(2^j r_k, y_k)} \right\|_{L^q_u} \\
&\leqslant c \sum_j 2^{-j(n+\epsilon_0)} \left\| \sum_k \lambda_k \chi_{B(2^j r_k, y_k)} \right\|_{L^p_v} \quad \text{by (2)} \\
&\leqslant c \sum_j 2^{-j(n+\epsilon_0)} 2^{j\delta} \left\| \sum_k \lambda_k \chi_{B(r_k, y_k)} \right\|_{L^p_v} \quad \text{by (3)} \\
&= c \left\| \sum_k \lambda_k \chi_{B(r_k, y_k)} \right\|_{L^p_v}
\end{aligned}$$

since $\delta < n + \epsilon_0$. If $0 < q < 1$,

$$\begin{aligned}\|Tf\|_{L_u^q}^q &\leqslant c\sum_j 2^{-j(n+\epsilon_0)q}\Big\|\sum_k \lambda_k a(2^j r_k, y_k)\chi_{B(2^j r_k, y_k)}\Big\|_{L_u^q}^q \\ &\leqslant c\sum_j 2^{-j(n+\epsilon_0)q}\Big\|\sum_k \lambda_k \chi_{B(2^j r_k, y_k)}\Big\|_{L_v^p}^q \\ &\leqslant c\sum_j 2^{-j(n+\epsilon_0)q}2^{j\delta q}\Big\|\sum_k \lambda_k \chi_{B(r_k, y_k)}\Big\|_{L_v^p}^q \\ &= c\Big\|\sum_k \lambda_k \chi_{B(r_k, y_k)}\Big\|_{L_v^p}^q\end{aligned}$$

since $\delta < n + \epsilon_0$. This shows that if

$$f_N = \sum_1^N \lambda_k g_k, \qquad \lambda_k > 0,$$

and g_k is an atom associated with B_k, then

$$\|Tf_N\|_{L_u^q} \leqslant c\Big\|\sum_1^N \lambda_k \chi_{B_k}\Big\|_{L_v^p}.$$

Let $f \in \mathcal{S}_{0,0}$. Write $f = \sum_1^\infty \lambda_k g_k$ with $\lambda_k > 0$, g_k an atom with support B_k, and (see [10])

$$\Big\|\sum \lambda_k \chi_{B_k}\Big\|_{L_v^p} \leqslant c\|f\|_{H_v^p}.$$

If

$$f_N = \sum_1^N \lambda_k g_k, \qquad \|Tf_N\|_{L_u^q} \leqslant c\Big\|\sum_1^N \lambda_k \chi_{B_k}\Big\|_{L_v^p} \leqslant c\|f\|_{H_v^p}$$

and

$$\|Tf_N - Tf_M\|_{L_u^q} \leqslant c\Big\|\sum_M^N \lambda_k \chi_{B_k}\Big\|_{L_v^p} \to 0 \qquad (N > M \to \infty).$$

Thus, Tf_N converges in L_u^q to a function h with $\|h\|_{L_u^q} \leqslant c\|f\|_{H_v^p}$. Of course, the integral defining Tf converges since $f \in \mathcal{S}_{0,0}$, and we wish to show that $Tf = h$ a.e. It is enough to show that $Tf_N \to Tf$ pointwise. We know $f_N \to f$ pointwise and $f_N \leqslant cf^*$ (f^* = grand maximal function of f). Thus, by the Lebesgue dominated convergence theorem, it suffices to prove that

$$\int f^*(y)|K(x,y)|\,dy < \infty.$$

This is clear from the earlier estimates on K since $f^*(y) \leqslant c_L(1 + |y|)^{-L}$ for all L. This proves Theorem 1.

2. Proof of Theorem 2

It is enough by considering $|f|$ and $|K|$ to prove the result for

$$T_1 f(x) = \int f(y) K_1(x,y)\,dy, \qquad K_1(x,y) = \frac{a(|x-y|, x)}{|x-y|^n}.$$

We claim it is also possible to assume that K_1 satisfies (ii) with $\epsilon_0 = 1$. To see this, let $\phi(z)$ be a nonnegative smooth function supported in $|z| < 1$ with $\phi(0) = 1$. Let $\phi_t(z) = t^{-n}\phi(z/t)$, $t > 0$, and define

$$\tilde{K}_1(x,y) = \int K_1(x,z)\phi_{|x-y|/2}(y-z)\,dz.$$

It is easy to check that there exist constants c' and c with $0 < c' < c < \infty$ such that

$$c'K_1(x,y) \leqslant \tilde{K}_1(x,y) \leqslant cK_1(x,y) \quad \text{and} \quad |\nabla_y \tilde{K}_1(x,y)| \leqslant \frac{c}{|x-y|} K_1(x,y)$$

so that $\tilde{K}_1$ satisfies (ii) with $\epsilon_0 = 1$.

Since $v \in A_p$, (3) holds with $\delta = n$. Thus, by Theorem 1 and the fact that $H_v^p \equiv L_v^p$ for $v \in A_p$, we have $\|T_1 f\|_{L_u^q} \leqslant c\|f\|_{L_v^p}$ if $f \in \mathcal{S}_{0,0}$. For general $f \in L_v^p$, since $\mathcal{S}_{0,0}$ is dense in L_v^p, there exist $f_j \in \mathcal{S}_{0,0}$ with $f_j \to f$ in L_v^p. Thus $\{T_1 f_j\}$ converges in L_u^q to a function h with $\|h\|_{L_u^q} \leqslant c\|f\|_{L_v^p}$. It is enough to show that the integral defining $T_1 f$ converges a.e. and that $h = T_1 f$ a.e. We claim that if $N < \infty$ and $v \in A_p$, then

$$\int_{|x|<N} |T_1 f(x)|\,dx \leqslant c_{N,v}\|f\|_{L_v^p}. \tag{2.1}$$

This will imply (by replacing f by $|f|$) that $T_1 f$ converges absolutely a.e.; moreover, it implies that $T_1 f_{j_k} \to T_1 f$ a.e. in $|x| < N$ for some subsequence. Since $T_1 f_{j_k} \to h$ in L_u^q, a further subsequence converges pointwise a.e. to h, and so $h = T_1 f$ a.e.

We now prove (2.1). For $f \geqslant 0$

$$T_1 f(x) \leqslant \left(\int_{|y|<1} + \int_{|y|<2|x|} + \int_{\substack{|y|>1 \\ |y|>2|x|}} \right) f(y) K_1(x,y)\,dy = F_1 + F_2 + F_3.$$

Then

$$\int_{|x|<N} (F_1(x) + F_2(x))\,dx \leqslant \int_{|y|<2N+1} f(y) \left(\int_{|x-y|<3N+1} K_1(x,y)\,dx \right) dy.$$

Also, by (iv),

$$\int_{|x-y|<3N+1} K_1(x,y)\,dx \leqslant c_N \int_{|x-y|<3N+1} \frac{dx}{|x-y|^{n-\mu}} = c_{N,\mu},$$

and

$$\int_{|y|<2N+1} f(y)\,dy \leqslant \left(\int_{|y|<2N+1} v^{-1/(p-1)}\,dy\right)^{1/p'} \|f\|_{L^p_v}$$
$$= c_{N,v}\|f\|_{L^p_v}.$$

Finally, if $|y| > 2|x|$, then $K_1(x,y) \leqslant ca(|y|,0)/|y|^n$ and

$$\text{(2.2)}\quad \int_{|x|<N} F_3(x)\,dx \leqslant c_N \int_{|y|>1} f(y)\frac{a(|y|,0)}{|y|^n}\,dy$$
$$\leqslant c_N\|f\|_{L^p_v}\left(\int_{|y|>1} v(y)^{-1/(p-1)}\frac{a(|y|,0)^{p'}}{|y|^{np'}}\,dy\right)^{1/p'}.$$

Thus, (2.1) will follow if

$$A = \int_{|y|>1} v(y)^{-1/(p-1)}\frac{a(|y|,0)^{p'}}{|y|^{np'}}\,dy < \infty.$$

Let $B_0 = \{y: |y| < 1\}$. Cover $\{y: |y| > 1\}$ by balls $\{B_k\}_{k=1}^{\infty}$ with $|B_k|^{1/n} \approx \operatorname{dist}(B_k, 0)$ and $\sum_1^\infty \chi_{B_k} \leqslant c$. Let $\bar{B}_k = 10B_k$ and note that $\bar{B}_k \supset B_0$. Clearly,

$$A \approx \sum_1^\infty \frac{a(B_k)^{p'}}{|B_k|^{p'}}\int_{B_k} v^{-1/(p-1)}\,dy$$
$$\approx \sum_1^\infty a(B_k)^{p'} v(B_k)^{-1/(p-1)}$$

since $v \in A_p$. If $\lambda_k \geqslant 0$,

$$\sum_1^M \lambda_k a(\bar{B}_k) u(B_0)^{1/q} \leqslant \left\|\sum_1^M \lambda_k a(\bar{B}_k)\chi_{\bar{B}_k}\right\|_{L^q_u} \quad \text{since} \quad \bar{B}_k \supset B_0$$
$$\leqslant c\left\|\sum_1^M \lambda_k \chi_{\bar{B}_k}\right\|_{L^p_v} \quad \text{by (2)}$$
$$\leqslant c\left\|\sum_1^M \lambda_k \chi_{B_k}\right\|_{L^p_v} \quad \text{by (3)}$$
$$\leqslant c\sum_1^M \lambda_k^p v(B_k)$$

since the B_k's have bounded overlaps. Pick λ_k so that

$$\lambda_k a(\bar{B}_k) = \lambda_k^p v(B_k), \quad \textit{i.e.}, \quad \lambda_k = [a(\bar{B}_k)/v(B_k)]^{1/(p-1)}.$$

Then from above,

$$\left[\sum_{1}^{M}\lambda_k a(\bar{B}_k)\right]^{1-1/p} \leqslant cu(B_0)^{-1/q}.$$

Since the constant c is independent of M, we obtain

$$\sum_{1}^{\infty}\lambda_k a(\bar{B}_k) \leqslant cu(B_0)^{-p'/q}.$$

But the sum on the left equals

$$\sum_{1}^{\infty} a(\bar{B}_k)^{1+1/(p-1)}v(B_k)^{-1/(p-1)},$$

and since $1 + 1/(p-1) = p'$, it follows that $A \leqslant cu(B_0)^{-p'/q}$. This proves that A is finite and completes the proof of Theorem 2.

3. Proof of Theorem 3

The proof is similar to that of Theorem 2. We first note from [1] that $H_v^p \equiv L_v^p$ with equivalence of norms if $v(x)/|x|^p \in A_p$ and $v(x)/|x|^{np} \in A_p$: in fact, we can then write $v(x) = |x|^p w(x)$ with $w \in A_p$ and $w(x)/|x|^{(n-1)p} \in A_p$, which fulfills the requirements in [1]. We also note by Lemma 6.3 of [12] that if $w \in D_\infty$ and $\gamma \geqslant 0$, then $|x|^\gamma w \in D_\infty$.

Now suppose that $|x|^{-\epsilon p}v \in A_p$ for some ϵ with $0 < \epsilon \leqslant \epsilon_0$, $\epsilon_0 \leqslant 1$, and $|x|^{-np}v \in A_p$. We claim that $|x|^{-p}v \in A_p$. Consider first the case of a ball B which is small compared to its distance to 0. Then $|x|$ is essentially constant on B and, consequently,

$$\left(\frac{1}{|B|}\int_B |x|^{-p}v\,dx\right)\left(\frac{1}{|B|}\int_B [|x|^{-p}v]^{-1/(p-1)}\,dx\right)^{p-1} \tag{3.1}$$

's bounded by a fixed multiple of the A_p constant of $|x|^{-\epsilon p}v$ (or $|x|^{-np}v$). If, n the other hand, B is not small compared to its distance to 0, we may assume ' enlarging B by a fixed factor that $B = \{x\colon |x| < R\}$ for some R. Next, note ıt both $|x|^{-p}v$ and $(|x|^{-p}v)^{-1/(p-1)}$ are in D_∞ since they may be written, ›ectively, as

$$|x|^{(n-1)p}(|x|^{-np}v) = |x|^\gamma w, \qquad \gamma \geqslant 0, \qquad w \in A_p,$$

and

$$|x|^{(1-\epsilon)p'}(|x|^{-\epsilon p}v)^{-1/(p-1)} = |x|^\delta w', \qquad \delta \geqslant 0, \qquad w' \in A_{p'}.$$

Hence, the product in (3.1) is equivalent to a similar product with the domains of integration replaced by $R/2 < |x| < R$. This essentialy reduces the situation to the first case and proves the claim.

It is also not difficult to see that if w satisfies $w \in A_p \cap D_d$ and $\alpha > 0$, then $|x|^\alpha w \in A_r$ for any $r > \max\{p, d + \alpha/n\}$. Of course, if $w \in A_p$ then $w \in D_p$. It follows that if v is a weight which satisfies $|x|^{-\epsilon p} v \in A_p$, then $v \in A_r$ (and D_r) if $r > p(1 + \epsilon/n)$. Thus, if $|x|^{-\epsilon p} v \in A_p$, (3) holds for any $\delta > \epsilon + n$. The requirement in Theorem 1 that $\delta < n + \epsilon_0$ is then satisfied if $\epsilon < \epsilon_0$. In case $\epsilon = \epsilon_0$, this requirement is also satisfied since if $|x|^{-\epsilon_0 p} v \in A_p$, then $|x|^{-\epsilon p} v \in A_p$ for some $\epsilon < \epsilon_0$: this is a corollary of the fact that if a weight $w \in A_p$, then $|x|^\eta w \in A_p$ for some $\eta > 0$.

Combining facts and applying Theorem 1, we see that under the hypothesis of Theorem 3, $\|Tf\|_{L^q_u} \leqslant c\|f\|_{H^p_v} \approx c\|f\|_{L^p_v}$ if $f \in \mathcal{S}_{0,0}$. Moreover, $Tf = Sf$ for such f since $\int f = 0$. Hence, $\|Sf\|_{L^q_u} \leqslant c\|f\|_{L^p_v}$ if $f \in \mathcal{S}_{0,0}$. To show the same inequality holds for any $f \in L^p_v$, it is enough as in the proof of Theorem 2 (since $\mathcal{S}_{0,0}$ is dense in L^p_v for v satisfying the hypothesis of Theorem 3) to show that Sf converges a.e. if $f \in L^p_v$ and

$$\int_{\eta < |x| < N} |Sf(x)|\, dx \leqslant c_{\eta, N, v} \|f\|_{L^p_v}, \qquad 0 < \eta < N < \infty. \tag{3.2}$$

Inequality (3.2) is similar to (2.1) and serves the same purpose. We will actually prove a stronger version of (3.2), namely, its analogue for the operator

$$S_1 f(x) = \int |f(y)|\, |K(x,y) - K(x,0)|\, dy.$$

This will prove (3.2) and show that Sf converges absolutely a.e. in $\eta < |x| < N$, and so a.e. in $\mathbb{R}^n$.

Write

$$F_1(x) = \int_{|y| < |x|/2} + \int_{|x|/2 < |y| < 2|x|} + \int_{|y| > 2|x|} = F_1 + F_2 + F_3.$$

By (ii) and (iv),

$$\begin{aligned} F_1(x) &\leqslant c \int_{|y| < |x|/2} |f(y)| \left(\frac{|y|}{|x|}\right)^{\epsilon_0} \frac{a(|x|, 0)}{|x|^n}\, dy \\ &\leqslant c \frac{a(|x|, 0)}{|x|^{n+\epsilon_0}} \int_{|y| < N} |f(y)|\, |y|^{\epsilon_0}\, dy \quad \text{if} \quad |x| < N. \end{aligned}$$

We have

$$\begin{aligned} \int_{|y| < N} |f(y)|\, |y|^{\epsilon_0} &\leqslant \|f\|_{L^p_v} \Big(\int_{|y| < N} |y|^{\epsilon_0 p'} v(y)^{-p'/p}\, dy \Big)^{1/p'} \\ &= c_{N,v} \|f\|_{L^p_v}, \end{aligned}$$

since the fact that $v = |y|^{\epsilon p} w$, $w \in A_p$, $\epsilon \leqslant \epsilon_0$, implies that

$$\int_{|y|<N} |y|^{\epsilon_0 p'} v(y)^{-p'/p}\, dy = \int_{|y|<N} |y|^{(\epsilon_0-\epsilon)p'} w(y)^{-1/(p-1)}\, dy \leqslant N^{(\epsilon_0-\epsilon)p'} \int_{|y|<N} w(y)^{-1/(p-1)}\, dy.$$

It follows easily that

$$\int_{\eta<|x|<N} |F_1|\, dx \leqslant c_{\eta,N,v} \|f\|_{L_v^p}.$$

Now consider F_2. First note that f is integrable away from 0: for $\eta > 0$,

$$\int_{|y|>\eta} |f(y)|\, dy \leqslant \|f\|_{L_v^p} \left(\int_{|y|>\eta} v(y)^{-1/(p-1)}\, dy \right)^{1/p'} = c_{\eta,v} \|f\|_{L_v^p}$$

since the fact that $v = |y|^{np} w_1$, $w_1 \in A_p$, gives

$$\int_{|y|>\eta} v(y)^{-1/(p-1)}\, dy = \int_{|y|>\eta} \frac{w_1(y)^{-1/(p-1)}}{|y|^{np'}}\, dy,$$

which is finite since $w_1^{-1/(p-1)} \in A_{p'}$ (see, *e.g.*, (2.3) of [7] for the case $n = 1$). We have

$$F_2(x) \leqslant \int_{|x|/2<|y|<2|x|} |f(y)| \{ |K(x,y)| + |K(x,0)| \}\, dy.$$

Hence, by (i) and (iv),

$$\begin{aligned} \int_{\eta<|x|<N} |F_2(x)|\, dx &\leqslant c_N \int_{|y|>\eta/2} |f(y)| \left[\int_{|x-y|<3N} \frac{dx}{|x-y|^{n-\mu}} + \int_{\eta<|x|<N} \frac{dx}{|x|^n} \right] dy \\ &= c_{N,\eta} \int_{|y|>\eta/2} |f(y)|\, dy \\ &\leqslant c_{N,\eta,v} \|f\|_{L_v^p}. \end{aligned}$$

Finally, since $|K(x,y)| \leqslant c a(|y|,0)|y|^{-n}$ for $|y| > 2|x|$,

$$F_3(x) \leqslant c \int_{|y|>2|x|} |f(y)| \left[\frac{a(|y|,0)}{|y|^n} + \frac{a(|x|,0)}{|x|^n} \right] dy,$$

and

$$\int_{\eta<|x|<N} F_3(x)\,dx \leqslant c_N \int_{|y|>2\eta} |f(y)| \frac{a(|y|,0)}{|y|^n}\,dy + \int_{|y|>2\eta} |f(y)|\,dy \int_{\eta<|x|<N} \frac{a(|x|,0)}{|x|^n}\,dx.$$

The second term on the right is at most $c_{\eta,N,v}\|f\|_{L^p_v}$ as above. The first term is like the integral on the right in (2.2) and can be treated by the argument given there. This completes the proof of Theorem 3.

4. Proof of Theorem 4

We first show that (2) is necessary for $\|Mf\|_{L^{p'}_v} \leqslant c\|f\|_{L^{q'}_u}$ if $1<p,\ q<\infty$ and

$$Mf(x) = \sup_{B \ni x} \frac{a(B)}{v(B)} \int_B |f|u.$$

The proof is by duality. For $\lambda_k > 0$,

$$\left\|\sum \lambda_k a(B_k)\chi_{B_k}\right\|_{L^q_u} = \sup_{\|g\|_{L^{q'}_u}=1} \int \left(\sum \lambda_k a(B_k)\chi_{B_k}\right) gu.$$

The integral on the right equals

$$\begin{aligned}\sum \lambda_k a(B_k) \int_{B_k} gu &= \sum \lambda_k v(B_k) \left\{\frac{a(B_k)}{v(B_k)} \int_{B_k} gu\right\}\\ &\leqslant \sum \lambda_k v(B_k) \inf_{B_k} M(g)\\ &\leqslant \sum \lambda_k \int_{B_k} M(g)v\\ &= \int \left(\sum \lambda_k \chi_{B_k}\right) M(g)v.\end{aligned}$$

By Hölder's inequality and hypothesis, this is at most

$$\left\|\sum \lambda_k \chi_{B_k}\right\|_{L^p_v} \|Mg\|_{L^{p'}_v} \leqslant \left\|\sum \lambda_k \chi_{B_k}\right\|_{L^p_v} c\|g\|_{L^{q'}_u} = c\left\|\sum \lambda_k \chi_{B_k}\right\|_{L^p_v}.$$

Taking the supremum over g, we obtain (2). Note that the proof works without any hypothesis on a, u, and v except $a \geqslant 0$ and $v(B) > 0$.

For the converse, we also assume (iii) and $v \in D_\infty$. If I is a cube, let $a(I) = a(B)$ where B is the smallest ball containing I. It is then easy to see that (2) for balls is equivalent to (2) for cubes. For example, if I_k is a cube and B_k is the smallest ball containing I_k, and if (2) holds for balls, then

$$\begin{aligned}\Big\|\sum \lambda_k a(I_k)\chi_{I_k}\Big\|_{L^q_u} &\leqslant \Big\|\sum \lambda_k a(B_k)\chi_{B_k}\Big\|_{L^q_u} \qquad (\lambda_k \geqslant 0)\\ &\leqslant c\Big\|\sum \lambda_k \chi_{B_k}\Big\|_{L^p_v}\\ &\leqslant c\Big\|\sum \lambda_k \chi_{I_k}\Big\|_{L^p_v}\end{aligned}$$

by (3), since $B_k \subset \alpha I_k$, $\alpha = \alpha_n$. A similar argument shows that (2) for cubes implies (2) for balls. Similarly, since $v \in D_\infty$ and (iii) holds, Mf is equivalent to its analogue defined by using cubes rather than balls.

Let G be a fixed dyadic grid of cubes. For $t \in \mathbb{R}^n$, let ${}^tG$ be the grid obtained by shifting G by t. Define

$${}^tMf(x) = \sup_{\substack{I \ni x\\ I \in {}^tG}} \frac{a(I)}{v(I)} \int_I |f|u.$$

We need the following analogue of Lemma 2 of [9].

Lemma 4.1. *Let a satisfy* (iii) *and $v \in D_\infty$. Then*

$$\sup_{\substack{I \ni x\\ |I| \leq r^n}} \frac{a(I)}{v(I)} \int_I |f|u \leqslant c\frac{1}{r^n}\int_{|t|<r} {}^tMf(x)\,dt, \qquad 0 < r < \infty,$$

'th c independent of r and x.

ᵓ. Fix $x = x_0$ and a cube I_0 containing x_0 with edgelength $h_0 \leqslant r$. Con- ose dyadic I in $B_{2r}(x_0)$ with edgelength $2h_0$. The number of such I is '. If I is such a cube and x_I is the center of I, then shifting I by $+\,\eta$ for any η with $|\eta| < h_0/2$ gives a cube ${}^tI$ containing I_0 with ince different I's lead to essentially disjoint sets of t's, the set E of all t is $\approx (r^n/|I_0|)(h_0/2)^n \approx r^n$. If $t \in E$ and ${}^tI$ is the then by (iii) and the doubling property of v,

$$\begin{aligned}\frac{a(I_0)}{v(I_0)}\int_{I_0}|f|u &\leqslant c\frac{a({}^tI)}{v({}^tI)}\int_{{}^tI}|f|u\\ &\leqslant c\,{}^tMf(x_0).\end{aligned}$$

Hence,

$$\frac{a(I_0)}{v(I_0)}\int_{I_0}|f|u \leqslant c\frac{1}{r^n}\int_{|t|<r} {}^tMf(x_0)\,dt,$$

and the lemma follows.

Denote the expression on the left in Lemma 4.1 by $M_r f(x)$ and apply Minkowski's integral inequality to obtain

$$\|M_r f\|_{L_v^{p'}} \leqslant c \frac{1}{r^n} \int_{|t|<r} \|{}^t Mf\|_{L_v^{p'}} \, dt$$

with c independent of r. Hence, if we prove that

$$\|{}^t Mf\|_{L_v^{p'}} \leqslant c \|f\|_{L_u^{q'}} \tag{4.2}$$

with c independent of t, then we will obtain $\|M_r f\|_{L_v^{p'}} \leqslant c\|f\|_{L_u^{q'}}$ with c independent of r, and Theorem 4 will follow by letting $r \to \infty$.

To prove (4.2), fix t and f and let

$$E_k = \{x : 2^k < {}^t Mf(x) \leqslant 2^{k+1}\},$$

$k = 0, \pm 1, \pm 2, \ldots$ The E_k are disjoint. By considering maximal cubes, it follows from the dyadic nature of ${}^t G$ that we may write $E_k = \bigcup_j I_{j,k}$, $I_{j,k} \in {}^t G$, $I_{j,k}$ disjoint in j (and so in j and k), and

$$2^k < \frac{a(I_{j,k})}{v(I_{j,k})} \int_{I_{j,k}} |f| u \leqslant 2^{k+1}.$$

Therefore,

$$\begin{aligned} 2^{kp'} v(E_k) &= \sum_j 2^{kp'} v(I_{j,k}) \\ &\approx \sum_j \left(\frac{a(I_{j,k})}{v(I_{j,k})} \int_{I_{j,k}} |f| u \right)^{p'} v(I_{j,k}) \\ &= \sum_j \left(\frac{a(I_{j,k})}{v(I_{j,k})^{1/p}} \int_{I_{j,k}} |f| u \right)^{p'}. \end{aligned}$$

Since

$$\|{}^t Mf\|_{L_v^{p'}} \approx \Big(\sum_j 2^{kp'} v(E_k) \Big)^{1/p'},$$

we obtain

$$\begin{aligned} \|{}^t Mf\|_{L_v^{p'}} &\approx \left[\sum_{j,k} \left(\frac{a(I_{j,k})}{v(I_{j,k})^{1/p}} \int_{I_{j,k}} |f| u \right)^{p'} \right]^{1/p'} \\ &\approx \sup \sum_{j,k} b_{j,k} \frac{a(I_{j,k})}{v(I_{j,k})^{1/p}} \int_{I_{j,k}} |f| u, \end{aligned}$$

where the supremum is taken over all sequences $\{b_{j,k}\}$ with $b_{j,k} \geqslant 0$ and

$$\|\{b_{j,k}\}\|_{l^p} = \Big(\sum_{j,k} b_{j,k}^p\Big)^{1/p} = 1.$$

Now

$$\begin{aligned}\sum_{j,k} b_{j,k}\frac{a(I_{j,k})}{v(I_{j,k})^{1/p}}\int_{I_{j,k}}|f|u &= \int |f|\Big\{\sum_{j,k} b_{j,k}\frac{a(I_{j,k})}{v(I_{j,k})^{1/p}}\chi_{I_{j,k}}\Big\}u \\ &\leqslant \|f\|_{L_u^{q'}}\Big\|\sum_{j,k} b_{j,k}\frac{a(I_{j,k})}{v(I_{j,k})^{1/p}}\chi_{I_{j,k}}\Big\|_{L_u^q} \\ &\leqslant c\|f\|_{L_u^{q'}}\Big\|\sum_{j,k} b_{j,k}\frac{1}{v(I_{j,k})^{1/p}}\chi_{I_{j,k}}\Big\|_{L_v^p}\end{aligned}$$

by hypothesis, with c independent of t and f. We shall use duality to show that

$$\Big\|\sum_{j,k} b_{j,k}\frac{1}{v(I_{j,k})^{1/p}}\chi_{I_{j,k}}\Big\|_{L_v^p} \leqslant c_{p,v}. \tag{4.3}$$

For $g \in L_v^{p'}$, $g \geqslant 0$,

$$\begin{aligned}\int \sum_{j,k} b_{j,k}\frac{1}{v(I_{j,k})^{1/p}}\chi_{I_{j,k}} gv &= \sum_{j,k} b_{j,k}\frac{1}{v(I_{j,k})^{1/p}}\int_{I_{j,k}} gv \\ &\leqslant \|\{b_{j,k}\}\|_{l^p}\Big(\sum_{j,k}\Big[\frac{1}{v(I_{j,k})}\int_{I_{j,k}} gv\Big]^{p'} v(I_{j,k})\Big)^{1/p'} \\ &\leqslant \Big(\sum_{j,k}\int_{I_{j,k}} H_v(g)^{p'}v\Big)^{1/p'} \\ &\leqslant \|H_v(g)\|_{L_v^{p'}},\end{aligned}$$

where $H_v(g)$ denotes the maximal function of Hardy-Littlewood type defined by

$$H_v(g)(x) = \sup_{I \ni x}\frac{1}{v(I)}\int_I |g|v\,dy.$$

Since $v \in D_\infty$, $\|H_v(g)\|_{L_v^{p'}} \leqslant c\|g\|_{L_v^{p'}}$, and it follows that (4.3) holds. Collecting estimates, we obtain (4.2). This completes the proof of Theorem 4.

5. Proof of Theorem 5

It is enough to prove the analogous result for cubes instead of balls, *i.e.*, to show that under the hypothesis of Theorem 5,

$$\left\| \sum \lambda_k a(I_k)\chi_{I_k} \right\|_{L^q_u} \leqslant c \left\| \sum \lambda_k \chi_{I_k} \right\|_{L^p_v} \tag{5.1}$$

if $\lambda_k \geqslant 0$ and the I_k are cubes. In fact, it is enough to prove this for dyadic cubes as we now show. If I is any cube, there are 2^n dyadic cubes J_i with $|J_i| \approx |I|$ and $I \subset UJ_i$. By (iii) and (iv), $a(I) \leqslant ca(J_i)$ with c independent of I and J_i. Find such a covering for each I_k and denote the dyadic cubes by $J_{k,i}$. Then

$$\sum \lambda_k a(I_k)\chi_{I_k} \leqslant \sum_k \lambda_k a(I_k) \sum_i \chi_{J_{k,i}}$$
$$\leqslant c \sum_i \left[\sum_k \lambda_k a(J_{k,i})\chi_{J_{k,i}} \right].$$

Thus

$$\left\| \sum \lambda_k a(I_k)\chi_{I_k} \right\|_{L^q_u} \leqslant c \sum_i \left\| \sum_k \lambda_k a(J_{k,i})\chi_{J_{k,i}} \right\|_{L^q_u}$$
$$\leqslant c \sum_i \left\| \sum_k \lambda_k \chi_{J_{k,i}} \right\|_{L^p_v},$$

assuming that (5.1) holds for dyadic cubes. However, it is easy to see from (3) that

$$\left\| \sum_k \lambda_k \chi_{J_{k,i}} \right\|_{L^p_v} \leqslant c \left\| \sum_k \lambda_k \chi_{I_k} \right\|_{L^p_v}$$

with c independent of i. Thus, since the number of i's is finite, we obtain by combining inequalities that

$$\left\| \sum \lambda_k a(I_k)\chi_{I_k} \right\|_{L^q_u} \leqslant c \left\| \sum \lambda_k \chi_{I_k} \right\|_{L^p_v},$$

as desired.

For the rest of the proof, we will assume that the cubes $\{I_k\}$ are dyadic. Fix $\{\lambda_k\}$, $\lambda_k \geqslant 0$, and let

$$M_\epsilon(x) = \sup_k \lambda_k a(I_k) u(I_k)^\epsilon \chi_{I_k}(x). \tag{5.2}$$

We claim that

$$\sum \lambda_k a(I_k)\chi_{I_k}(x) \leqslant c[M_\epsilon(x) M_{-\epsilon}(x)]^{1/2}, \qquad \epsilon > 0, \tag{5.3}$$

with c depending only on ϵ, n, and u. Fix x. Note that there is at most one I_k of a given size which contains x since the cubes are dyadic. We may then assume that those I_k containing x are ordered in size, *i.e.*, that $I_k \subset I_j$ if $k < j$. For $k_0 = k_0(x)$ to be chosen and $\epsilon > 0$, write

$$\sum \lambda_k a(I_k)\chi_{I_k}(x) = \sum_{k \le k_0} \{\lambda_k a(I_k)u(I_k)^{-\epsilon}\chi_{I_k}(x)\}u(I_k)^{\epsilon}$$

$$+ \sum_{k > k_0} \{\lambda_k a(I_k)u(I_k)^{\epsilon}\chi_{I_k}(x)\}u(I_k)^{-\epsilon}$$

$$\leqslant M_{-\epsilon}(x) \sum_{k \le k_0} u(I_k)^{\epsilon} + M_{\epsilon}(x) \sum_{k > k_0} u(I_k)^{-\epsilon}.$$

The I_k's in these sums are dyadic cubes containing x, and they are ordered in size. There may not be cubes containing x of every (dyadic) size, but if there are missing sizes we just add cubes of those sizes, thereby increasing the sums on the right side above. We do not add any cubes to the collection used to define M_ϵ and $M_{-\epsilon}$. It follows easily from the doubling condition on u and the dyadic nature of the cubes that for any k_0

$$\sum_{k \le k_0} u(I_k)^{\epsilon} \leqslant cu(I_{k_0})^{\epsilon}$$

and

$$\sum_{k \le k_0} u(I_k)^{-\epsilon} \leqslant cu(I_{k_0})^{-\epsilon},$$

with c depending only on n, ϵ and u. Thus,

$$\sum \lambda_k a(I_k)\chi_{I_k}(x) \leqslant c[M_{-\epsilon}(x)u(I_{k_0})^{\epsilon} + M_{\epsilon}(x)u(I_{k_0})^{-\epsilon}].$$

Pick k_0 so that

$$u(I_{k_0})^{\epsilon} \approx \{M_{\epsilon}(x)/M_{-\epsilon}(x)\}^{1/2},$$

and (5.3) follows immediately.

By (5.3)

$$\left\| \sum \lambda_k a(I_k)\chi_{I_k} \right\|_{L_u^q} \leqslant c\left(\int M_{\epsilon}^{q/2} M_{-\epsilon}^{q/2} u \, dx\right)^{1/q}.$$

For small $\epsilon > 0$ to be chosen, let q_1 and q_2 be defined by

$$\frac{1}{q_1} = \frac{1}{q} - \epsilon, \qquad \frac{1}{q_2} = \frac{1}{q} + \epsilon.$$

Then $1/q_1 + 1/q_2 = 2/q$, so that $2q_1/q$ and $2q_2/q$ are conjugate indices. By Hölder's inequality

$$\left(\int M_{\epsilon}^{q/2} M_{-\epsilon}^{q/2} u \, dx\right)^{1/q} \leqslant \|M_{\epsilon}\|_{L_u^{q_1}}^{1/2} \|M_{-\epsilon}\|_{L_u^{q_2}}^{1/2}.$$

The hypothesis (1) that $a(I)u(I)^{1/q} \leqslant cv(I)^{1/p}$ may be rewritten as both

$$a(I)u(I)^{1/q_1 + \epsilon} \leqslant cv(I)^{1/p} \quad \text{and} \quad a(I)u(I)^{1/q_2 - \epsilon} \leqslant cv(I)^{1/p}.$$

Also, since $q > p$, we have $q_1, q_2 \geqslant p$ for small $\epsilon > 0$. To complete the proof, we need the following lemma.

Lemma 5.4. *Let* $u \in D_\infty$ *and* $a(I)$ satisfy (iii) *and* (iv). *Let* M_ϵ *be defined by* (5.2) *for a collection of dyadic cubes* $\{I_k\}$, *and assume that*

$$a(I)u(I)^{1/q+\epsilon} \leqslant cv(I)^{1/p}$$

for all cubes, $1 < p \leqslant q < \infty$. *There is a number* $\eta > 0$ *depending on u and a so that if* $\epsilon > -\eta$ *and* $v \in A_p$,

$$\|M_\epsilon\|_{L^q_u} \leqslant c\Big\|\sum \lambda_k \chi_{I_k}\Big\|_{L^p_v}.$$

As we shall see, the value of η can be taken to be $\mu/n\sigma$ where μ is the parameter in (iv) and $u \in D_\sigma$.

Before proving the lemma, we note that Theorem 5 follows by combining it with the facts above, since then for small $\epsilon > 0$

$$\|M_\epsilon\|^{1/2}_{L^{q_1}_u}\|M_{-\epsilon}\|^{1/2}_{L^{q_2}_u} \leqslant c\Big\|\sum \lambda_k \chi_{I_k}\Big\|^{1/2+1/2}_{L^p_v} = c\Big\|\sum \lambda_k \chi_{I_k}\Big\|_{L^p_v}.$$

PROOF OF LEMMA 5.4. Write

$$\|M_\epsilon\|^q_{L^q_u} = \int \sup_k \{\lambda_k a(I_k)u(I_k)^\epsilon \chi_{I_k}\}^q u\,dx$$
$$\leqslant \sum \lambda_k^q a(I_k)^q u(I_k)^{\epsilon q+1}.$$

Let

$$g(x) = \sum \lambda_k \chi_{I_k}(x).$$

For $j = 0, \pm 1, \pm 2, \ldots$, let $\mathfrak{I}_j = \{I_k : |I_k| = 2^{jn}\}$ and define

$$g_j(x) = \left(\frac{1}{|I_k|}\int_{I_k} g\right)\chi_{I_k}(x), \qquad I_k \in \mathfrak{I}_j.$$

Then

$$\sum \lambda_k^q a(I_k)^q u(I_k)^{\epsilon q+1} = \sum_j \sum_{I_k \in \mathfrak{I}_j} \lambda_k^q a(I_k)^q u(I_k)^{\epsilon q+1}$$
$$\leqslant \sum_j \sum_{I_k \in \mathfrak{I}_j}\left(\frac{1}{|I_k|}\int_{I_k} g_j^q\right)a(I_k)^q u(I_k)^{\epsilon q+1}$$
$$(5.5) \qquad = \sum_j \int g_j(x)^q\Big\{2^{-jn}\sum_{I_k \in \mathfrak{I}_j} a(I_k)^q u(I_k)^{\epsilon q+1}\chi_{I_k}(x)\Big\}\,dx.$$

Think of $g_j(x)$ as a function on $\mathbb{R}^n \times \{2^j\}$, and think of

$$\mathrm{dm}\,(x, j) = 2^{-jn} \sum_{I_k \in \mathfrak{J}_j} a(I_k)^q u(I_k)^{\epsilon q+1} \chi_{I_k}(x)\, dx$$

as a measure. We claim that dm is a (q, p)-Carleson measure with respect to $v(x)\, dx$ *i.e.*, that if J is a cube in $\mathbb{R}^n$ with edgelength h, then

$$\sum_{2^j \leq h} \int_J \left\{ 2^{-jn} \sum_{I_k \in \mathfrak{J}_j} a(I_k)^q u(I_k)^{\epsilon q+1} \chi_{I_k}(x) \right\} dx \leqslant c v(J)^{q/p} \tag{5.6}$$

with c independent of J. Since $v \in D_\infty$, we may assume that J is dyadic. The I_k's in a given $\mathfrak{J}_j$ are disjoint, and those above which intersect J have edgelength $2^j \leqslant h$, *i.e.*, are smaller than J, and so are contained in J. Hence, after performing the integration with respect to x, we see that the left side of (5.6) equals

$$\sum_{2^j \leq h} \sum_{\substack{|I_k| = 2^{jn} \\ I_k \subset J}} a(I_k)^q u(I_k)^{\epsilon q+1}. \tag{5.7}$$

From (iv)

$$a(I_k) \leqslant c \left(\frac{|I_k|}{|J|} \right)^{\mu/n} a(J) = c \left(\frac{2^j}{h} \right)^{\mu} a(J).$$

Thus, (5.7) is at most

$$c \frac{a(J)^q}{h^{\mu q}} \sum_{2^j \leq h} 2^{j\mu q} \sum_{\substack{I_k \in \mathfrak{J}_j \\ I_k \subset J}} u(I_k)^{\epsilon q+1}.$$

In case $\epsilon \geqslant 0$, we have

$$u(I_k)^{\epsilon q+1} \leqslant u(J)^{\epsilon q} u(I_k) \quad \text{if} \quad I_k \subset J,$$

and therefore the last estimate is bounded by

$$c \frac{a(J)^q}{h^{\mu q}} \sum_{2^j \leq h} 2^{j\mu q} u(J)^{\epsilon q} \sum_{\substack{I_k \in \mathfrak{J}_j \\ I_k \subset J}} u(I_k) \leqslant c \frac{a(J)^q}{h^{\mu q}} h^{\mu q} u(J)^{\epsilon q} u(J)$$
$$= c a(J)^q u(J)^{\epsilon q+1}$$

since the cubes in each $\mathfrak{J}_j$ are disjoint. By hypothesis,

$$a(J)^q u(J)^{\epsilon q+1} \leqslant c v(J)^{q/p},$$

and (5.6) follows in this case. If instead $\epsilon < 0$, then since $u \in D_\sigma$

$$u(J) \leqslant c(|J|/|I_k|)^\sigma u(I_k) \quad \text{if} \quad I_k \subset J,$$

and we see the estimate above is at most

$$c\frac{a(J)^q}{h^{\mu q}}\sum_{2^j\le h}2^{j\mu q}\sum_{\substack{I_k\in\mathfrak{I}_j\\ I_k\subset J}}[(|I_k|/|J|)^\sigma u(J)]^{\epsilon q}u(I_k)$$

$$= c\frac{a(J)^q}{h^{\mu q+n\sigma\epsilon q}}\sum_{2^j\le h}2^{j(\mu q+n\sigma\epsilon q)}u(J)^{\epsilon q}\sum_{\substack{I_k\in\mathfrak{I}_j\\ I_k\subset J}}u(I_k)$$

$$= ca(J)^q u(J)^{\epsilon q+1},$$

provided $\mu q + n\sigma\epsilon q > 0$, *i.e.*, $\epsilon > -\mu/n\sigma$. Thus, (5.6) again follows.

Using (5.6) and Carleson's theorem, we see that (5.5) is at most $c\|g^*\|_{L^p_v}$, where

$$g^*(x) = \sup\{g_j(y): (y,j) \text{ satisfies } |x-y| < 2^j\}.$$

Fix x and suppose $|x-y| < 2^j$. By definition of g_j, if $y \in I_k \in \mathfrak{I}_j$,

$$g_j(y) = \frac{1}{|I_k|}\int_{I_k} g \leqslant \frac{c}{|I|}\int_I g$$

where I is any interval containing x of edgelength $\approx 2^j$. If there is no such I_k for y, then $g_j(y) = 0$. Thus, g^* is majorized by a multiple of the Hardy-Littlewood maximal function of g. Therefore, $\|g^*\|_{L^p_v} \leqslant c\|g\|_{L^p_v}$ by [8] since $v \in A_p$. Combining estimates, we obtain Lemma 5.4; this completes the proof of Theorem 5.

6. Proof of Theorem 6

The proof is similar to that of Theorem 5. As there, it is enough to prove the result for dyadic cubes instead of balls. Fix x and order those I_k containing x according to size as before. Let $g(x) = \sum \lambda_k \chi_{I_k}(x)$ and

$$A_\epsilon(x) = \sup_k\{\lambda_k a(I_k)^{1+\epsilon}\chi_{I_k}(x)\}, \qquad \epsilon > 0.$$

For k_0 and ϵ to be chosen, write

$$\sum \lambda_k a(I_k)\chi_{I_k}(x) = \sum_{k\le k_0} + \sum_{k>k_0}$$

$$\leqslant \Big[\sup_{k\le k_0} a(I_k)\chi_{I_k}(x)\Big]g(x) + A_\epsilon(x)\sum_{k>k_0} a(I_k)^{-\epsilon}\chi_{I_k}(x).$$

By adding cubes if necessary, we may assume that $\sup_{k\le k_0} a(I_k)\chi_{I_k}(x)$ and $\sum_{k>k_0} a(I_k)^{-\epsilon}\chi_{I_k}(x)$ are taken over cubes of every dyadic size containing x.

We do not add any cubes to the collection used to define g or A_ϵ. Thus, by (iii) and (iv),

$$\sum \lambda_k a(I_k)\chi_{I_k}(x) \leqslant ca(I_{k_0})g(x) + ca(I_{k_0})^{-\epsilon}A_\epsilon(x).$$

Pick k_0 so that $a(I_{k_0}) \approx (A_\epsilon(x)/g(x))^{1/(1+\epsilon)}$. Then

$$\sum \lambda_k a(I_k)\chi_{I_k}(x) \leqslant cg(x)^{1/r'}A_\epsilon(x)^{1/r}, \qquad r = 1 + \epsilon.$$

By Hölder's inequality,

$$\left\| \sum \lambda_k a(I_k)\chi_{I_k} \right\|_{L^p_u} \leqslant c\left(\int g^p v \right)^{1/pr'} \left(\int A^p_\epsilon \left(\frac{u}{v} \right)^r v \right)^{1/pr}. \tag{6.1}$$

If we show that

$$\int A^p_\epsilon \left(\frac{u}{v} \right)^r v \leqslant c \int g^p v, \qquad r = 1 + \epsilon, \tag{6.2}$$

then the right side of (6.1) is at most $c\|g\|_{L^p_v}$, and Theorem 6 follows.

To prove (6.2), note that since

$$A_\epsilon(x)^p \leqslant \sum \lambda^p_k a(I_k)^{pr}\chi_{I_k}(x), \qquad r = 1 + \epsilon,$$

the integral on the left is bounded by

$$\sum \lambda^p_k a(I_k)^{pr} \int_{I_k} \left(\frac{u}{v} \right)^r v.$$

Using the same notation as in the proof of Theorem 5, we see this equals

$$\sum_j \sum_{I_k \in \mathfrak{Z}_j} \left(\frac{1}{|I_k|} \int_{I_k} g^p_j \right) a(I_k)^{pr} \int_{I_k} \left(\frac{u}{v} \right)^r v$$

$$= \sum_j \int g_j(x)^p \left\{ \sum_{I_k \in \mathfrak{Z}_j} a(I_k)^{pr} \left(\frac{1}{|I_k|} \int_{I_k} \left(\frac{u}{v} \right)^r v \right) \chi_{I_k}(x) \right\}.$$

Thus, (6.2) will be as before if we show that the expression in curly brackets is a (p, p) Carleson measure with respect to $v(x)\,dx$. If J is a cube in $\mathbb{R}^n$ and $|J| = h^n$, we must show that

$$\sum_{2^j \leq h} \int_J \left\{ \sum_{I_k \in \mathfrak{Z}_j} a(I_k)^{pr} \left(\frac{1}{|I_k|} \int_{I_k} \left(\frac{u}{v} \right)^r v \right) \chi_{I_k}(x) \right\} dx \leqslant cv(J). \tag{6.3}$$

Arguing as before with J dyadic, we see this amounts to proving that

$$\sum_{2^j \leq h} \sum_{\substack{I_k \in \mathfrak{Z}_j \\ I_k \subset J}} a(I_k)^{pr} \int_{I_k} \left(\frac{u}{v} \right)^r v \leqslant cv(J).$$

Since $a(I_k) \leqslant c(|I_k|/|J|)^{\mu/n} a(J)$ if $I_k \subset J$, the sum on the left is majorized by

$$ca(J)^{pr} \sum_{2^j \leq h} \left(\frac{2^j}{h}\right)^{\mu pr} \sum_{\substack{I_k \in \mathfrak{I}_j \\ I_k \subset J}} \int_{I_k} \left(\frac{u}{v}\right)^r v \leqslant ca(J)^{pr} \sum_{2^j \leq h} \left(\frac{2^j}{h}\right)^{\mu pr} \int_J \left(\frac{u}{v}\right)^r v,$$

since the cubes in $\mathfrak{I}_j$ are disjoint and have edgelength 2^j. The last expression equals

$$ca(J)^{pr} \int_{I_k} \left(\frac{u}{v}\right)^r v. \tag{6.4}$$

Since (1)′ holds, the following lemma shows that if r is chosen near 1 (*i.e.*, ϵ is chosen small), then (6.4) is at most $cv(J)$, and the proof of (6.3) is complete.

Lemma 6.4. *If* $v \in A_\infty$, *the following two conditions are equivalent*:

(*a*) *condition* (1)′, *i.e., there exists* $s > 1$ *such that*

$$a(I)^p \left(\frac{1}{|I|} \int_I u^s\right)^{1/s} \leqslant c \frac{v(I)}{|I|}$$

for all cubes I;

(*b*) *there exists* $r > 1$ *such that*

$$a(I)^p \left(\frac{1}{v(I)} \int_I \left(\frac{u}{v}\right)^r v\right)^{1/r} \leqslant c$$

for all cubes I.

The proof is essentially the same as that given in [2] for the case $a(I) = |I|^{1/n}$ and we shall not repeat the details.

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