Revista Matemática Iberoamericana
Vol. 3 n.° 2, 1987

Hardy Spaces and the Dirichlet Problem on Lipschitz Domains

Carlos E. Kenig and Jill Pipher

Introduction

Our concern in this paper is to describe a class of Hardy spaces $H^p(D)$ for $1 \leqslant p < 2$ on a Lipschitz domain $D \subset \mathbb{R}^n$ when $n \geqslant 3$, and a certain smooth counterpart of $H^p(D)$ on $\mathbb{R}^{n-1}$, by providing an atomic decomposition and a description of their duals. For a Lipschitz domain D,

$$H^p(D, d\sigma) = \{u: \Delta u = 0 \text{ in } D \text{ and } Nu(Q) \in L^p(\partial D, d\sigma)\}$$

where $Nu(Q) = \sup_{\Gamma(Q)} |u(x)|$ is the nontangential maximal function. When $p \geqslant 2$ H^p and L^p are essentially the same. When the dimension $n = 2$, $H^p(D)$ can be understood in terms of conformal mappings onto the upper half plane (Kenig [20]).

In 1979, B. Dahlberg overcame one major obstacle in providing the atomic decomposition of $H^1(D, d\sigma)$ in higher dimensions by showing that appropriately defined atoms belong to H^1. However, the pairing between BMO and H^1 was not established since, as we show, the most natural class of measures arising from the harmonic extensions of BMO functions do not satisfy the right Carleson measure condition.

At this point we would like to mention the work of Jerison-Kenig [18] and Dahlberg-Kenig [10] where the analogous theory on Lipschitz (and even NTA) domains was carried out for $H^p(D, d\omega)$ for harmonic functions ([18]) and systems of conjugate harmonic functions ([10]).

The paper is organized as follows. At the begining of section 1 we describe the notation to be used throughout. We then explain why Dahlberg's lemma (which was stated in [8], without proof) on the harmonic extension of BMO functions fails. In addition, we give an example which shows that there is no Carleson measure condition on the *harmonic* extension of a BMO function.

In section 2 we give (for completeness) the proof of Dahlberg's lemma on atoms ([8]). In order to exhibit the duality between H^1 and BMO, one requires that some extension of a BMO function be a Carleson measure. At the end of this section, we discuss which properties such an extension must satisfy, and give the motivation for the work which follows.

There are two approaches to obtain such an extension result. One approach is by duality, giving an atomic decomposition via a grand maximal function; the other approach is constructive, as in Varopoulos [25]. In section 3 we consider a related space, $H^p(w\,dx)$, a space of distributions on $\mathbb{R}^{n-1}$, where the weight $w(x)$ appears in the kernel used to define the maximal functions, and not as a weight on Lebesgue measure dx. For this space of distributions we prove that definitions in terms of grand maximal functions or vertical maximal functions are equivalent, give an atomic decomposition, a description of the dual space, and a Varopoulos type extension theorem for the dual. At the end of this section, a constructive proof of the extension theorem is given.

In section 4 we use the extension result and a localization argument to obtain the atomic decomposition for $H^1(D, d\sigma)$ and duality with $\mathrm{BMO}_\sigma(\omega)$. This is carried out first for starlike domains, and a separate argument gives the duality and decomposition for a general Lipschitz domain. Section 5 is devoted to the analogous results for $H^p(D, d\sigma)$, $1 < p < 2$. In this case the dual of H^p is characterized by a weighted «sharp» function, which arises from the defining condition of $\mathrm{BMO}_\sigma(\omega)$.

We would like to thank J. O. Stromberg for helpful conversations on weighted $H^p(\mathbb{R}^n)$ spaces and for preprints of his joint work with A. Torchinsky on this topic.

1. We begin by reviewing some basic facts about harmonic functions on a Lipschitz domain $D \subseteq \mathbb{R}^n$ and by setting up the notation to be used throughout. For $P \in D$, $d\omega^P$ is harmonic measure evaluated at P and $k(P, \cdot) = d\omega^P/d\sigma$, where d$\sigma$ is surface measure on ∂D. Then if $G(P, \cdot)$ is the Green's function with pole at P, we have

$$k(P, \cdot) = \frac{\partial}{\partial n} G(P, \cdot).$$

In D we fix an arbitrary P_0 and set $G(X) = G(P_0, X)$, $k(Q) = k(P_0, Q)$ and $d\omega = d\omega^{P_0}$. If $f(Q)$ is defined on ∂D, $u(P) = \int_{\partial D} f(Q) k(P, Q)\, d\sigma(Q)$ is the harmonic extension of f to D, where $d\sigma(Q)$ is surface measure on ∂D.

To each point $Q \in \partial D$ is associated a cone

$$\Gamma(Q) = \{P \in D: |P - Q| \leqslant c \operatorname{dist}(P, \partial D)\}$$

contained in D. For u harmonic in D, $Nu(Q) = \sup_{x \in \Gamma(Q)} |u(x)|$ is the non-tangential maximal function of u and

$$Su(Q) = \left\{ \int_{\Gamma(Q)} |\nabla u(X)|^2 \, d(X)^{-n+2} \, dX \right\}^{1/2}$$

is its square function.

Definition. $H^p(D, d\sigma) = \{u: \Delta u = 0$ *in* D, $u(P_0) = 0$ *and* $Nu \in L^p(\partial D, d\sigma)\}$, *and* $\|u\|_{H^p} = \|Nu\|_{L^p(d\sigma)}$.

By Dahlberg's theorem ([7]),

$$\|Nu\|_{L^p(d\sigma)} \approx \|Su\|_{L^p(d\sigma)} \quad \text{for all} \quad p > 0,$$

and so $H^p(D, d\sigma)$ could just as well be defined using square functions.

The normalizing condition $u(P_0) = 0$ says that if $u(P) = \int f(Q) \, d\omega^P(Q)$ for some function f then $\int_{\partial D} f \, d\omega = 0$. With this in mind we say that a harmonic function a on D is an *atom* if there exists a surface ball $\Delta \subset \partial D$ such that $a \equiv 0$ on $\partial D \setminus \Delta$, $\|a\|_\infty \leqslant \sigma(\Delta)^{-1}$ and $\int_\Delta a(Q) \, d\omega = a(P_0) = 0$. We will sometimes identify an atom a with its harmonic extension A.

Definitions.

(1) $H^1_{at}(\partial D, d\sigma) = \{f: f = \sum \lambda_k a_k$ *where the* a_k *are atoms and* $\sum |\lambda_k| < \infty\}$.
(2) $\mathrm{BMO}_\sigma(\omega) = \{g$: *there exists* $C < \infty$ *such that*

$$\sup_{\Delta \subseteq \partial D} \left\{ \int_\Delta |g - g_\Delta| \, d\omega / s(\Delta) + \int_{\partial D} |g(Q)| \, d\omega \leqslant C \right\},$$

where

$$g_\Delta = \frac{1}{\omega(\Delta)} \int_\Delta g(Q) \, d\omega.$$

(3) $$VMO_\sigma(\omega) = \left\{ g \in BMO_\sigma(\omega): \overline{\lim_{\sigma(\Delta) \to 0}} \int_\Delta |g - g_\Delta| \, d\omega / \sigma(\Delta) = 0 \right\}.$$

The space $H^1(D, d\sigma)$ is in fact a space of extensions of distributions, as not every element of H^1 is the Poisson integral of its pointwise boundary values. We should like to identify these boundary distributions with $H^1_{at}(\partial D, d\sigma)$ and thereby show its dual to be $\mathrm{BMO}_\sigma(\omega)$. At this point we take a closer look at $\mathrm{BMO}_\sigma(\omega)$. In particular, an alternative Carleson characterization of $\mathrm{BMO}_\sigma(\omega)$ will be considered, and rejected.

Suppose that $u(x)$ and $v(x)$ are the Poisson extensions of functions f and g on ∂D satisfying $u(P_0) = v(P_0) = 0$. Assume $u \in H^1(D, d\sigma)$. Then by Green's formula, together with the relationship between $G(x)$ and harmonic measure,

$$\int_{\partial D} f(Q)g(Q)\,d\omega = \int_{\partial D} f(Q)g(Q)k(Q)\,d\sigma = \int_D G(x)\Delta(u \cdot v)(x)\,dx$$
$$= 2\int_D G(x)\nabla u \cdot \nabla v\,dx.$$

In this way, the pairing between BMO and H^1 is typically established by bounding the solid integral over the domain. In our situation we have (with $d(x)$ abbreviating dist $(x, \partial D)$)

$$\int_D G(x)|\nabla u(x)|\,|\nabla v(x)|\,dx = \int_{\partial D}\int_{\Gamma(Q)} G(x)|\nabla u(x)|\,|\nabla v(x)|\,d(x)^{1-n}\,dx\,d\sigma(Q)$$
$$\leqslant \int_{\partial D}\left\{\int_{\Gamma(Q)} |\nabla u|^2 d(x)^{2-n}\,dx\right\}^{1/2} \cdot \left\{\int_{\Gamma(Q)} |\nabla v|^2 \frac{G^2}{d(x)} d(x)^{1-n}\,dx\right\}$$
$$= \int_{\partial D} Su(Q)Av(Q)\,d\sigma$$

with

$$A^2v(Q) \equiv \int_{\Gamma(Q)} |\nabla v|^2 \frac{G^2(x)}{d(x)} d(x)^{1-n}\,dx.$$

Observe that

$$\int_{\partial D} A^2v(Q)\,d\sigma = \int_D \frac{G^2(x)}{d(x)} \cdot |\nabla v|^2\,dx.$$

Then by mimicking the duality argument of Fefferman-Stein [13], it will follow that $\left|\int_{\partial D} fg\,d\omega\right|$ is finite if $|\nabla v|$ satisfies the Carleson condition

$$\iint_{T(\Delta)} |\nabla v|^2 \frac{G^2(X)}{d(X)}\,dX \leqslant c\sigma(\Delta) \tag{1.1}$$

where, for $\Delta = \Delta(Q_0, r_0)$, a surface ball centered at Q_0 with radius r_0, $T(\Delta) = \{X \in D\colon |X - Q_0| < r_0\}$.

This Carleson measure condition was introduced in Fabes-Kenig-Neri [11] for the analogous problem on C^1 domains and in the C^1 case was shown to be equivalent to the $\mathrm{BMO}_\sigma(\omega)$ boundary definition. However, for a Lipschitz domain in $\mathbb{R}^n$, $n \geqslant 3$, the Poisson kernel does not satisfy the appropriate decay and a $\mathrm{BMO}_\sigma(\omega)$ function can fail to satisfy (1.1).

Example 1. Let $D \subseteq \mathbb{R}^3$ be that part of the complement of the cone

$$\Gamma(m) = \{|x| \leqslant my\}$$

with vertex at the origin which is contained in the unit sphere S. Then there exists a function $h(x)$, harmonic in D, given by $h(x) = |x|^\alpha \varphi(x/|x|)$ where φ vanishes on the spherical cap cut out by $\Gamma(m)$, and $\varphi(0,0,1) = 1$. Thus $h(x)$ vanishes on $\partial D \cap \partial\Gamma(m)$ and moreover $\alpha = \alpha_m$ becomes arbitrarily small as $m \to \infty$, i.e., as the Lipschitz constant of D increases. (See Dahlberg [6] for a similar construction). Let $\Delta_r \subseteq \partial\Gamma(m)$ be a surface ball with radius $r << 1$ centered at the origin. Let D' be a subdomain of D containing

$$D \cap \{|x|^2 + y^2 \leqslant 1/2\} = D \cap S_{1/2}$$

such that

$$\partial D' \cap \partial D = \partial\Gamma(m) \cap S_{1/2}$$

but with $\partial D'$ smooth except at the origin. Then $h(x)$ satifies the $\mathrm{BMO}_\sigma(\omega)$ condition on $\partial D'$ since it is a continuous function which vanishes where the boundary of D' fails to be smooth.

Let us assume that D' does not contain the pole of $G_D(x)$, the Green's function for D. Then both $G(x)$ and $h(x)$ are harmonic in D' and vanish on $\partial D \cap \partial\Gamma(m)$. Hence by the comparison theorem ([6]), we may fix some z_0 away from the origin and obtain the estimate

$$\frac{G(x)}{h(x)} \approx \frac{G(z_0)}{h(z_0)} = c_0, \tag{1.2}$$

for $x \in D \cap S_{1/4}$. Set $T(\Delta_r)^+ = \{x \in T(\Delta_r) : d(x) > r/2\}$. The above estimate for $G(x)$ shows that

$$\int_{T(\Delta_r)^+} |\nabla h|^2 G^2(x)/d(x)\, dx \geqslant C_0 r^{2\alpha - 2} \int_{T(\Delta_r)^+} r^{2\alpha}/r\, dx = Cr^{4\alpha}.$$

Here the constant C depends only on the comparability constant in (1.2) and not on the radius r. Observe now that if the Carleson measure condition holds, the estimate

$$\int_{T(\Delta_r)} |\nabla h|^2 G^2/d\, dX \lesssim \sigma(\Delta_r)$$

implies that $r^{4\alpha} \leqslant Cr^2$. Letting $r \to 0$ forces $\alpha \geqslant 1/2$, but in dimension $n \geqslant 3$, α tends to zero.

This argument fails, as it must, for dimension $n = 2$. In the plane the Green's function for the complement of a cone can be computed explicitly by a conformal mapping onto the upper half plane and one finds that the restriction $\alpha > 1/2$ is satisfied.

There is, however, a more fundamental reason for the failure of the Carleson measure condition of Example 1. Namely, if $u \in H^1(D, d\sigma)$ and v is the harmonic extension of a $\mathrm{BMO}_\sigma(\omega)$ function, the integral $\int_D G(X)|\nabla u|\,|\nabla v|\, dX$ need not be absolutely convergent.

Example 2. Let $D, D' \subseteq \mathbb{R}^n$ and $h(x)$ be as in Example 1. Again let Δ_r be the surface ball on $\partial D \cap \partial\Gamma(m)$ with radius r centered at the origin. A plane perpendicular to the axis of the cone divides Δ_r into two pieces Δ_1 and Δ_2 with $\sigma(\Delta_1) \approx \sigma(\Delta_2) \approx r^{n-1}$. We construct an H^1 function with support on Δ_r by setting $u(x) = \omega^x(\Delta_1) - \beta\omega^x(\Delta_2)$ where $\beta = \omega(\Delta_1)/\omega(\Delta_2)$. By the doubling property of harmonic measure ([17]), β is bounded above and below by universal constants depending only on D and not, in particular, on the radius r. Hence $u(x)/r^{n-1}$ is an atom and Dahlberg's lemma ([18], see 2.3 of this paper) yields $\|u\|_{H^1} \leqslant Cr^{n-1}$, where $C = C(D)$.

We claim that there is no constant C such that

$$\int_{D'} |\nabla u(x)| G^2(x)/d(x)\, dx \leqslant Cr^{n-1}.$$

Because $|\nabla h(x)| \lesssim G(x)/d(x)$, this would imply that $\int_D G(x)|\nabla h(x)|\, |\nabla u(x)|\, dx$ has no bound in terms of $\|u\|_{H^1}$. Assume, to the contrary, that such a constant C_0 exists. Define $v(x) = u(x) + \beta$ and choose three points z_1, z_2, z_3 in D as follows. Let Q_1 (resp. Q_2) be a point in Δ_1 (resp. Δ_2) at a distance $r/2$ from the origin, the vertex of $\Gamma(m)$. If $Q \in \partial D$, let n_Q denote the unit normal at Q and set $z_1 = Q_1 + rn_{Q_1}$, $z_2 = Q_2 + rn_{Q_2}$ and let z_3 be the point on the vertical axis of $\Gamma(m)$ at a distance r from the origin. From now on, C will denote a constant which depends only on the domain D, not necessarily the same at each occurence.

Because $\omega^{z_1}(\Delta_1) \geqslant C$ (see [7]) and $v(x) = (1+\beta)\omega^x(\Delta_1)$ we have $v(z_1) \geqslant C$. Moreover, by Harnack's principle, $v(z_1) \approx v(z_2) \approx v(z_3)$. Hence

$$\begin{aligned} C \leqslant v(z_2) &= v(Q_2 + rn_{Q_2}) \\ &= v(Q_2 + \epsilon rn_{Q_2}) + \int_{\rho=\epsilon r}^{r} \frac{\partial}{\partial\rho} v(Q_2 + \rho n_{Q_2})\, d\rho \\ &\leqslant C\epsilon^{\gamma}(1+\beta) + \int_{\epsilon r}^{r} |\nabla v(Q_1 + \rho n_{Q_2})|\, d\rho. \end{aligned}$$

This last estimate follows from Hölder continuity since v is non-negative, bounded by $(1+\beta)$ and vanishes on Δ_1. If $\tilde{\Delta}_1$ is a surface ball with $\sigma(\tilde{\Delta}_1) \approx \sigma(\Delta_1)$ whose double is contained in Δ_1, the above estimate is valid with $Q \in \tilde{\Delta}_1$ replacing Q_2. Thus if we set

$$T_{\epsilon r}(\tilde{\Delta}_1) = \{X \in D : X = Q + \rho n_Q, Q \in \tilde{\Delta}_1, \epsilon r < \rho\},$$

integrating the above inequality on $Q \in \tilde{\Delta}_1$ yields

$$C \leqslant C'\epsilon^{\gamma}(1+\beta) + \int_{T_{\epsilon r}(\tilde{\Delta}_1)} |\nabla v(X)|\, dX/r^{n-1}$$

and so

$$Cr^{n-1} \leqslant C'\epsilon^{\gamma}(1+\beta)r^{n-1} + \int_{T_{\epsilon r}(\tilde{\Delta}_1)} |\nabla v(X)| G^2(X)/d(X) \cdot d(X)/G^2(X)\, dX$$
$$\leqslant C'\epsilon^{\gamma}(1+\beta)r^{n-1} + C(\epsilon)r/G^2(z_1) \int_{D'} |\nabla u(X)| G^2(X)/d(X)$$

where $C(\epsilon) \geqslant [G(x)/G(z_1)]^2$ depends only on ϵ and on D. Since $G(z_1) \approx G(z_3) \approx r^{\alpha}$, and, by our assumption that C_0 exists, we have

$$C \leqslant C'\epsilon^{\gamma}(1+\beta) + C(\epsilon)r^{1-2\alpha}.$$

Now fix a small α and then fix ϵ so that $C - C'\epsilon^{\gamma}(1+\beta) > 0$. The above inequality leads to contradiction as $r \to 0$.

2. In this section we give Dahlberg's proof that atoms belong to $H^1(D, d\sigma)$ and then discuss the two methods by which the atomic decomposition of H^1 and its duality with $\mathrm{BMO}_{\sigma}(d\omega)$ will be obtained. Recall that a harmonic function A is an atom if there is a surface ball $\Delta \subseteq \partial D$ such that $A = 0$ on $\partial D \setminus \Delta$, $\|A\|_{\infty} \leqslant \sigma(\Delta)^{-1}$ and A has mean value zero with respect to harmonic measure $d\omega = d\omega^{P_0}$.

Lemma 2.1 (Dahlberg [8]). *There is a constant C such that for all atoms A*

$$\int_{\partial D} NA(Q)\, d\sigma(Q) \leqslant C.$$

PROOF. Suppose that A is supported on $\Delta_r = \{Q: |Q - Q_0| < r\}$, where r is small. (If $r > c\sigma(\partial D)$, then $\int_{\partial D} NA(Q)\, d\sigma(Q)$ is bounded by $\|A\|_{L^2(\partial D)}\sigma(\partial D)^{1/2} \leqslant C$).

Since $\|A\|_{L^2} \leqslant r^{1-n}$, by the L^2 theory for Lipschitz domains (Dahlberg [9]), we have

$$\int_{\Delta_{2r}} NA(Q)\, d\sigma(Q) \leqslant \sigma(\Delta_{2r})^{1/2} \|N(A)\|_{L^2} \leqslant C.$$

Thus it suffices to estimate $|A(X)|$ for $X \in D_{\rho} = \{P \in D: |P - Q_0| > \rho\}$ for $\rho > 2r$. Let $S_{\rho} = \{P \in D: |P - Q_0| = \rho\}$ and pick $Q_{\rho} \in S_{\rho}$ such that $\operatorname{dist}(Q_{\rho}, \partial D) = \max_{Q \in S_2} \operatorname{dist}(Q, \partial D)$. Extend A to all of $\mathbb{R}^n$ by putting $A = 0$ in $\mathbb{R}^n \setminus D$ and let $A^+ = \max(A, 0)$, $A^- = \max(-A, 0)$. Both A^+ and A^- are subharmonic in $\{X: |X - Q_0| > r\}$. If $d\tau$ denotes normalized surface measure on the unit sphere S, set

$$m_{\pm}(\rho) = \left(\int_Q |A^{\pm}(\rho\tau)|^2\, d\tau\right)^{1/2}.$$

By Huber [6] and Friedland and Hayman [14],

$$m_{\pm}(\rho) \leqslant \sqrt{2}\,(r/\rho)^{n-2} m_{\pm}(r) \exp\left(-\int_{2r}^{\rho} \alpha_{\pm}(t)\, dt/t\right) \tag{2.2}$$

where $\alpha_{\pm}(t)$ is the nonnegative root of the equation

$$\alpha(\alpha + n - 2) = \lambda(U^{\pm}(t)),\ U^{\pm}(t) = \{Q \in S: A^{\pm}(tQ) > 0\},$$

and

$$\lambda(U) = \inf\left\{\int |\nabla u|^2\, d\tau: u \in C_0^{\infty}(U), \int u^2\, d\tau = 1\right\}.$$

Set

$$h_\rho^{\pm}(X) = \int_{\partial D_\rho} A^{\pm}(Q)\, d\omega_{D_\rho}^X(Q)$$

where $d\omega_{D_\rho}^X$ is harmonic measure for the domain D_ρ and let $d\omega_{D_\rho}$ denote the harmonic measure for D_ρ evaluated at the point Q_ρ. Let $k_\rho = d\omega_{D_\rho}/d\sigma_\rho$ be the density of harmonic measure in surface measure on ∂D_ρ. Then

$$\begin{aligned}(2.3)\qquad h^{\pm}(Q_\rho) &\leqslant \left(\int_{\partial D_\rho} |A^{\pm}(Q)|^2\, d\sigma_\rho\right)^{1/2} \cdot \left(\int_{D_\rho} k_\rho^2\, d\sigma_\rho\right)^{1/2} \\ &\leqslant \left(\int_{\partial D_\rho} |A^{\pm}(Q)|^2\, d\sigma_\rho\right)^{1/2} \cdot \sigma(\partial D_\rho)^{-1/2} \cdot \int_{\partial D_\rho} k_\rho\, d\sigma_\rho \\ &\leqslant C\left(\int_{\partial D_\rho} |A^{\pm}(Q)|^2\, d\sigma_\rho\right)^{1/2} \rho^{1-n} = Cm_{\pm(\rho)},\end{aligned}$$

where the estimates above follow from the reverse Hölder condition on k_ρ (see Dahlberg [9]). By Harnack's inequality, $\max_{D_{2\rho}} |h_\rho^{\pm}| \leqslant Ch_\rho^{\pm}(Q_\rho)$, and by the comparison theorem ([16]),

$$\frac{h^+(Q_\rho)}{h^-(Q_\rho)} \approx \frac{h^+(P_0)}{h^-(P_0)}.$$

However, $A = h_\rho^+ - h_\rho^-$ and $A(P_0) = 0$, by assumption, so it follows that $h_\rho^+(Q_\rho) \approx h_\rho^-(Q_\rho)$. Hence, by (2.2) and (2.3)

$$\begin{aligned}(2.4)\quad \operatorname*{Max}_{D_{2\rho}} |A| &\leqslant C\{h_\rho^+(Q_\rho) \cdot h_\rho^-(Q_\rho)\}^{1/2} \\ &\leqslant Cr^{n-2}(m_+(r)m_-(r))^{1/2}\rho^{2-n} \exp\left(-\int_{2r}^{\rho} \frac{\alpha_+(t) + \alpha_-(t)}{2} \frac{dt}{t}\right).\end{aligned}$$

Again by the L^2 theory,

$$\begin{aligned} r^{n-1}m_{\pm}^2(r) &\leqslant \int_{\partial\Delta(Q_0, r)} |A^{\pm}(P)|^2\, d\sigma_r(P) \\ &\leqslant \int_{\Delta_r} N^2(A)(Q)\, d\sigma(Q) \\ &\leqslant C\int_{\Delta_r} |A(Q)|^2\, d\sigma(Q) \\ &\leqslant Cr^{1-n}.\end{aligned}$$

This gives the estimate

$$\operatorname*{Max}_{D_{2\rho}} |A| \leqslant Cr^{-1}\rho^{2-n} \exp\left(- \int_{2r}^{\rho} \frac{\alpha_+(t) + \alpha_-(t)}{2} \frac{dt}{t} \right).$$

Let

$$a_\pm(t) = \int_{U^\pm(t)} d\tau$$

an set $\varphi(x) = 2(1-x)$ for $1/4 \leqslant x \leqslant 1$,

$$\varphi(x) = \frac{1}{2} \log(4x)^{-1} + \frac{3}{2}$$

otherwise. Because φ is decreasing and convex, it follows that $\alpha_\pm(t) \geqslant \varphi_\pm(t)$. (See Friedland-Hayman [14]). Since D is Lipschitz and $A^+A^- = 0$ there exists a positive β such that $a_+(t) + a_-(t) \leqslant 1 - \beta$. Hence

$$\frac{1}{2}(\alpha_+(t) + \alpha_-(t)) \geqslant \varphi\left(\frac{1}{2}(a_+(t) + a_-(t)) \right) \geqslant 1 + \beta.$$

Substituting this estimate into (2.4) gives

$$\operatorname*{Max}_{D_{2\rho}} |A| \leqslant C\rho^{1-n-\beta} r^{\beta} \tag{2.5}$$

and therefore

$$NA(Q) \leqslant C \min\{ r^\beta |Q - Q_0|^{1-n-\beta}, r^{1-n} \}. \quad \square$$

Observe that the argument yields a strong pointwise estimate (2.5) on the harmonic function $A(x)$ away from the support of its boundary values. We will need this fact later.

Let us now assume that D is Lipschitz and starlike with respect to the origin. Let $d\omega$ denote $d\omega^0$.

Definition.

$$H^1(\partial D, d\sigma) = \left\{ f: f(Q) = \lim_{r \to 1} u(rQ), u \in H^1(D, \delta) \right\},$$

where the limit is taken in an appropriate sense to be made precise later on.

Definition. *For*

$$\mathrm{VMO}_\sigma(d\omega) = \left\{ g \in \mathrm{BMO}_\sigma(\omega): \lim_{\sigma(\Delta) \to 0} \frac{1}{\sigma(\Delta)} \int_\Delta |g - g_\Delta| \, d\omega = 0 \right\},$$

let $VMO^*_\sigma(d\omega)$ be its dual, i.e., the linear functionals acting continuously on $\mathrm{VMO}_\sigma(d\omega)$.

Definition. $H^1_{at}(\partial D, d\sigma) = \{f \in \mathrm{VMO}^*_\sigma(d\omega): f = \sum_j \lambda_j a_j$, *where the* a_j *are atoms,* $\sum_j |\lambda_j| < \infty$, *and the convergence takes place in* $\mathrm{VMO}^*_\sigma\ (d\omega)\}$.

Our goal in the next few sections is to establish the following.

Theorem 2.6.

$$H^1(\partial D, d\sigma) = \mathrm{VMO}^*_\sigma(d\omega) = H^1_{at}(\partial D, d\sigma)$$

and the dual of $H^1(\partial D, d\sigma)$ *is* $\mathrm{BMO}_\sigma(d\omega)$ *with pairing* $(f, g) = \int fg\, d\omega$, *on an appropiate dense subclass.*

There will be an analogous result for nonstarlike Lipschitz domains, which will be formulated at the end of section 4.

The main result we need is the following, establishing the pairing between $H^1(D, d\sigma)$ and $\mathrm{BMO}_\sigma(\omega)$.

Theorem 2.7. *If* $u \in \mathfrak{L}(\bar{D})$, *the space of functions on D Lipschitz on* $\bar{D}$, $\Delta u = 0$, $u(0) = 0$, *and* $f \in \mathrm{BMO}_\sigma(d\omega)$ *then*

$$\int_{\partial D} u(Q) f(Q)\, d\omega \leqslant \|N(u)\|_{L^1(d\sigma)} \|f\|_{\mathrm{BMO}}.$$

The idea behind the proof of Theorem 2.7 is to find some (non-harmonic) extension v of f to the domain D which satisfies a Carleson measure condition. For suppose $v(X)$ is some smooth extension of f to all of D, for which the following formal argument is justified. Let $G(x)$ be the Green's function with pole at 0.

$$\begin{aligned}\int_{\partial D} u(Q) f(Q)\, d\omega &= \int_D G(x)\Delta(u \cdot v)\, dx \\ &= 2\int_D G(x)\nabla u(x) \cdot \nabla v(x)\, dx + \int_D G(x) u(x) \Delta v(x)\, dx \\ &= 2\int_D G(x)\nabla u(x) \cdot \nabla v(x)\, dx - \int_D G(x)\nabla u(x) \cdot \nabla v(x)\, dx \\ &\quad - \int_D u(x)\nabla G(x) \cdot \nabla v(x)\, dx.\end{aligned}$$

If $d(x) = \mathrm{dist}\,(x, \partial D)$, then $|\nabla G(x)| \lesssim G(x)/d(x)$, when $x \notin K$ for K some region around the pole, and we have

$$\begin{aligned}\left|\int_{\partial D} u(Q) f(Q)\, d\omega(Q)\right| &\leqslant \int_D G(x)|\nabla u(x)|\,|\nabla v(x)|\, dx + \int_K u|\nabla G|\,|\nabla v|\, dx \\ &\quad + \int_D |u(x)|\,|\nabla v(x)| G(x)/d(x)\, dx.\end{aligned}$$

The first integral on the right-hand side is bounded by $\int_{\partial D} S(u)(Q)\, d\sigma(Q)$ if $|\nabla v|^2 G^2/d(x)$ is a Carleson measure. (See the argument at the beginning of §1).

The third integral is bounded by $\int_{\partial D} Nu(Q)\,d\sigma(Q)$ as long as $|\nabla v| G(X)/d(X)$ is a Carleson measure. The second integral also has this bound if $|v(x)|$ is bounded in K. Thus we seek an extension v of f which satisfies, for some constant C and all surface balls $\Delta \subseteq \partial D$,

$$\text{(2.8)}\quad \begin{array}{lll} \text{(i)} & \int_{T(\Delta)} |\nabla v(x)| G(x)/d(x)\,dx \leqslant C\sigma(\Delta) & \\ \text{(ii)} & |\nabla v(x)| \leqslant CG(x)^{-1}, & \text{for } x \text{ near } \partial D \\ \text{(iii)} & |v(x)| \leqslant C_K & \text{on a compact subset } K \text{ of } D. \end{array}$$

We shall find such an extension by a localization procedure which allows one to translate the problem to an analog on the upper half space $\mathbb{R}^n_+$. Roughly speaking, a $\mathrm{BMO}_\sigma(\omega)$ function f on ∂D can be cut off and projected onto $\mathbb{R}^{n-1}$ so that the resulting function lies in $\mathrm{BMO}(w\,dx)$, a weighted space of bounded mean oscillation. The counterpart of (2.8) on $\mathbb{R}^n_+$ will be obtained in two ways. The first approach is by duality. We introduce a space of distributions on $\mathbb{R}^{n-1}$, a «weighted» space of homogeneous type, which could be regarded as a smooth version of $H^1(D, d\sigma)$. The dual of this H^1 space will be $\mathrm{BMO}(w\,dx)$ and a representation of a $\mathrm{BMO}(w\,dx)$ in terms of an appropriate kernel and a Carleson measure is obtained. Obtaining such an extension constructively is the approach to the classical duality taken by Carleson [2], Varopoulos [25], and Jones [19]. The theory here becomes somewhat elaborate, although we think it is of interest in itself, and so an alternative constructive approach is developed. The constructive argument parallels that of Varopoulos [25] and will be given at the end of §3. Basic to both methods of proof is the following observation about harmonic measure.

Lemma 2.9. *There exists a constant C, an $\alpha > 0$, and a radius r_0, all depending only on the domain D, such that for every surface ball*

$$\Delta(Q_0, r) = \{Q \in \partial D \colon |Q - Q_0| < r\}$$

with $2r < r_0$,

$$\omega(\Delta(Q_0, 2^{-j}r)) \leqslant C 2^{-j(n-2+\alpha)} \omega(\Delta(Q_0, r))$$

PROOF. By Lemma 5.8 of [17], there exists r_0 and M such that whenever $2r < r_0$,

$$M^{-1} < \omega(\Delta(Q, r))/r^{n-2} G(X_r) < M$$

where X_r is a point in D whose distance to ∂D is approximately $|X_r - Q_0|$. Thus

$$\omega(\Delta(Q_0, 2^{-j}r))/\omega(\Delta(Q_0, r)) \leqslant C_M 2^{-j(n-2)} G(X_{2^{-j}r})/G(X_r).$$

The Green's function is harmonic in $\Delta(Q_0, 2r)$ and vanishes on the boundary of D so by Hölder continuity ([17]) we have

$$G(X_{2^{-j}r}) \leqslant C(|X_{2^{-j}r} - Q_0|/r)^{\alpha} G(X_r) \leqslant C2^{-j\alpha} G(X_r), \quad \text{for some} \quad \alpha > 0.$$

In the next section we define an H^1 space relative to a kernel with a weight satisfying the condition of Lemma 2.9. This weight should therefore be thought of as the projection of harmonic measure onto $\mathbb{R}^{n-1}$. The localization arguments and the translation of the problem in D to the situation which follows are given later.

3. Suppose w is a weight on $\mathbb{R}^{n-1}$ with the following properties:

(i) For some constant C an some $0 < \alpha < 1$,

$$w(Q)/w(2^j Q) \leqslant 2^{-j^{(n-2+\alpha)}}$$

for all cubes Q and their 2^j-fold enlargements $2^j Q$.

(3.1) (ii) $w(Q) \approx |Q|$ when the length $l(Q)$ of Q is larger than 1.

(iii) w satifies a reverse Hölder inequality with exponent two; that is

$$\left(\frac{1}{|Q|} \int_Q w^2 \, dx \right)^{1/2} \lesssim \frac{1}{|Q|} \int_Q w \, dx.$$

We remark that conditions (ii) and (iii) are stated in this seemingly strong way for convenience only. If we only ask that $w \in A_\infty$, i.e.,

$$w(E)/w(Q) \leqslant C(|E|/|Q|)^{\theta} \quad \text{for some } \theta,$$

then this would suffice instead of (ii) and (iii). (See Coifman-Fefferman [3] for the relevant properties of A_∞ weights).

Define BMO($w\,dx$) to be the space of functions $g \in L^1_{\text{loc}}(w\,dx)$ for which there exists a contant M such that

$$\sup_{Q:\,cube} \frac{1}{|Q|} \int_Q |g - g_Q| w \, dx < M$$

where $|Q|$ is the Lebesgue measure of Q and

$$g_Q = w(Q)^{-1} \int_Q g w \, dx.$$

Clearly if for each cube Q there exists some constant c_Q for which

$$\sup_Q \left\{ |Q|^{-1} \int_Q |g - c_Q| w \, dx \right\}$$

is finite, then g belongs to BMO($w\,dx$). Let VMO denote the closure of Lip_0, the compactly supported Lipschitz functions, in the BMO norm. This definition makes sense for if ψ belongs to Lip_0, $|Q|$ is small, take Q_0 to be the cube of length 1 containing Q. Then

$$\begin{aligned}\int_Q |\psi(x) - \psi(x_0)|\,w\,dx &\leq \|\psi\|_{\mathrm{Lip}} \int_Q |x - x_0|\,w\,dx \\ &\leq \|\psi\|_{\mathrm{Lip}}\, l(Q) w(Q) \\ &\leq \|\psi\|_{\mathrm{Lip}}\, l(Q) w(Q)/W(Q_0) w(Q_0) \\ &\leq \|\psi\|_{\mathrm{Lip}}\, l(Q)^{n-1+\alpha},\end{aligned}$$

by conditions (3.1) (i) and (ii). And when Q is large,

$$\int_Q |\psi(x)|\,w\,dx \leq C w(Q) \leq C|Q|.$$

Hence $\mathrm{Lip}_0 \subseteq$ BMO($w\,ds$). Fix, once and for all, a C^∞ bump function φ supported in the unit ball $B(0, 1)$ and $\varphi \equiv 1$ on $B(0, 1/2)$. The kernel function formed from φ, relative to the weight w, is

$$K(x, z, y) = \varphi(x - z/y)\left\{\int \varphi\left(\frac{x - z'}{y}\right) w(z')\,dz'\right\}^{-1}.$$

An f belonging to VMO*($w\,dx$), the dual of VMO($w\,dx$), will be called a distribution and the pairing $\langle f, \varphi\rangle$ will be denoted $\int f\varphi w\,dx$. For $f \in$ VMO*($w\,dx$), let

$$u(x, y) = \int_k (x, z, y) f(z) w(z)\,dz$$

be its «Poisson» extension to the upper half space.

Definition.

$$H^1(w\,dx) = \left\{f \in \mathrm{VMO}^*(w\,dx) : f^+(x) = \sup_{y>0} |u(x, y)| \in L^1(dx)\right\}.$$

The first goal is to give an atomic decomposition of $H^1(w\,dx)$ and for this purpose we need a definition of this space in terms of a grand maximal function. Let $\mathcal{S}$ be class of Schwartz functions and let

$$\mathfrak{A} = \left\{\psi \in \mathcal{S} : \int (1 + |x|)^N \Big(\sum_{|\alpha| \leq N} |D^\alpha \psi|^2\Big)\,dx \leq 1\right\},$$

for some large N. The pairing of an $f \in$ VMO*($w\,dx$) against $\psi \in \mathfrak{A}$ is well defined. To see this, fix a sequence of bump functions θ_j supported on $\{|x| \leq 2^{j+1}\}$, with $\theta_j \equiv 1$ for $|x| \leq 2^j$. Now if $\psi \in \mathfrak{A}$, $\theta_j\psi \in \mathrm{Lip}_0$ and we need $\|\theta_j\psi - \psi\|_{\mathrm{BMO}} \to 0$ as $j \to \infty$. Suppose $Q \subseteq \mathbb{R}^{n-1}$ is a cube with $l(Q) > 1$. We have

$$\begin{aligned}\int_Q |(1-\theta_j)\psi(x)|\,w\,dx &\leqslant \left(\int_{Q\cap\{|x|\geq 2^j\}} w^2(1+|x|)^{-N}\,dx\right)^{1/2}\\ &\quad\cdot\left(\int_{\{|x|\geq 2^j\}} |\psi|^2(1+|x|)^N\,dx\right)^{1/2}\\ &\leqslant 2^{-jN/2}\left(\int_Q w^2\frac{dx}{|Q|}\right)^{1/2}\cdot|Q|^{1/2}\\ &\leqslant 2^{-jN/2}|Q|\end{aligned}$$

by (3.1) (ii) and (iii). Alternatively, if $l(Q)$ is small, and x_0 is the center of Q,

$$\begin{aligned}\int_Q [(1-\theta_j)\psi(x)-(1-\theta_j)\psi(x_0)]\,w\,dx &\leqslant l(Q)\cdot\int_Q |\nabla((1-\theta_j)\psi)|\,w\,dx\\ &\leqslant C_j l(Q)w(Q)\leqslant C_j|Q|\end{aligned}$$

where $C_j \to 0$ as $j\to\infty$. The last inequality, valid for small cubes, is derived from an argument we have used before. For Q_0 is the cube of length 1 containing Q, condition (3.1) (i) yields

$$w(Q)\leqslant l(Q)^{n-2+\alpha}w(Q_0)$$

and again by (3.1) (ii),

$$l(Q)w(Q)\leqslant l(Q)^{n-1+\alpha}\leqslant l(Q)^{n-1}.$$

One can now define, for $\eta\in\mathfrak{A}$ and $B(x,t)=\{x'\in\mathbb{R}^{n-1}: |x-x'|<t\}$, the extension

$$f\eta(x,t)=\int f(z)\eta\left(\frac{x-z}{t}\right)w(z)\,dz\Big/ w(B(x,t)).$$

The grand maximal function of f is

$$f^*(x)=\sup_{\eta\in\mathfrak{A}}\ \sup_{|x-x'|<t}|f\eta(x',t)|.$$

Lemma 3.2. *There is a constant C such that*

$$\frac{1}{C}\|f^+\|_{L^1(dx)}\leqslant\|f^*\|_{L^1(dx)}\leqslant C\|f^+\|_{L^1(dx)}$$

PROOF. The argument is an adaptation of that of Fefferman-Stein [13], so only those details which indicate how the properties of the weight come into play are provided. Recall that φ is our fixed «nice» bump function. The non-tangential maximal function is

$$Nf(x)=\sup_{|x-x'|<t}|f_\varphi(x',t)|,$$

where $f_\varphi(x, t)$ is the $u(x, t)$ defined before. We shall first see that

$$\|Nf\|_{L^1(dx)} \approx \|f^*\|_{L^1(dx)}.$$

Assume $Nf \in L^1(dx)$ and let $\eta \in \mathfrak{A}$. A purely geometric argument (p. 185 of [13]) shows that the tangential maximal function

$$f^{**}(x) = \sup_{t,x'} |f_\varphi(x', t)| \left(\frac{t}{|x - x'| + t}\right)^N$$

belongs to $L^1(dx)$ for sufficiently large N. Suppose that for some $\psi \in \mathfrak{A}$, and $s < 1$, $\eta = \psi * \varphi_s$ where $\varphi_s(\cdot) = s^{-n+1}\varphi(\cdot/s)$. Then

$$\left|\int f(y)\eta(x - y/t)w(y)\,dy\right| = \left|\int f(y)\left\{\int \psi_t(x - z)s^{-n+1}\varphi\left(\frac{x-y}{st}\right)dx\right\}w(y)\,dy\right|$$

$$= \left|\int \psi_t(x - z)\left(\int f(y)s^{-n+1}\varphi\left(\frac{z-y}{st}\right)w(y)\,dy\right)dx\right|.$$

This interchange of integration can be justified in the same way one justifies the interchange in the ordinary convolution case and we obtain

$$|f\eta(x, t)| \leqslant \int |\psi_t(x - z)|s^{-n+1}\frac{w(B(z, st))}{w(B(x, t))}|f_\varphi(z, st)|\,dz$$

$$\leqslant \int |\psi_t(x - z)|s^{-n+1}\frac{w(B(z, st))}{w(B(x, t))}\left(\frac{st}{|x - z| + st}\right)^{-N} dz \cdot f^{**}(x)$$

$$= f^{**}(x)s^{-n+1}\sum_j \int_{|x-z|\approx 2^j t} t^{-n+1}\left|\psi\left(\frac{x-z}{t}\right)\right|\left(\frac{st}{2^j t + st}\right)^{-N}$$

$$\cdot \frac{w(B(z, st))}{w(B(x, t))}\,dz.$$

By (3.1) and the fact that $w \in A^\infty$,

$$\frac{w(B(z, st))}{w(B(x, t))} = \frac{w(B(z, st))}{w(B(z, 2^j t))} \cdot \frac{w(B(z, 2^j t))}{w(B(x, t))} \leqslant (s/2^j)^{n-2+\alpha} \cdot 2^{Mj}.$$

Thus

$$|f\eta(x, t)| \lesssim f^{**}(x)s^{-n+1} \cdot s^{n-2+\alpha}\sum_j 2^{Mj}\int_{|x-z|\sim 2^j t} t^{-n+1}\left|\psi\left(\frac{x-z}{t}\right)\right|(s/2^j)^{-N}\,dz$$

$$\lesssim f^{**}(x)s^{-1+\alpha}\sum_j 2^{Mj}s^{-N}2^{-jN}\left(t^{-n+1}\int_{|x-z|\sim 2^j t}\left|\psi\left(\frac{x-z}{t}\right)\right|dz\right)$$

$$\lesssim f^{**}(x)s^{-N-1+\alpha}, \quad \text{since} \quad \psi \in \mathfrak{A}.$$

The remainder of the argument for $\|f^*\|_{L^1(dx)} \leqslant \|Nf\|_{L^1(dx)}$ follows Fefferman-Stein verbatim, cutting up the support of $\hat{\eta}$ to express it as a sum of convolutions $\psi * \varphi_{s_j}$, with $\psi \in \mathfrak{A}$.

To show that $\|Nf\|_{L^1(dx)} \leqslant \|f^+\|_{L^1(dx)}$ one uses a geometric argument of Burkholder-Gundy to prove

$$|Nf(x)| \leqslant \sup_{x \in Q} \left(\frac{1}{|Q|} \int_Q [f^+(z)]^r \, dz \right)^{1/r}, \qquad 0 < r < 1$$

and invoke the maximal theorem. Again, for details, we refer to [13].

The main step in the atomic decomposition for $H^1(w\,dx)$ is to provide an appropriate dense subclass of functions. Here we follow closely the method of Macías and Segovia [26].

Given a distribution $f \in H^1(w\,dx)$, let $\Omega_\lambda = \{x : f^*(x) > \lambda\}$. Associate to Ω_λ a partition of unity $\{\eta_j\}$ with $\sum_j \eta_j = \chi_{\Omega_\lambda}$, $\eta_j = 1$ on $B(x_j, r_j)$, supp $\eta_j \subseteq B(x_j, 2r_j)$, and $\|D^\alpha \eta_j\|_\infty \leqslant C^{r_j^{-|\alpha|}}$. Furthermore, $\bigcup B(x_j, r_j) = \Omega_\lambda$ and for some constant M, and some $c > 2$, no point of Ω_λ lies in more than M of the balls $B(x_j, cr_j)$. Define the mapping S_j on VMO$(w\,dx)$ by

$$S_j(\psi)(x) = \eta_j(x) \int [\psi(x) - \psi(z)] \eta_j(z) w(z) \, dz \cdot \left\{ \int \eta_j(z') w \, dz' \right\}^{-1}.$$

Lemma 3.3. *If* $\psi \in$ VMO, *then* $\sum_{j=1}^\infty S_j(\psi)$ *belongs to* VMO *and*

$$\left\| \sum_{j=1}^\infty S_j(\psi) \right\|_{\mathrm{BMO}} \leqslant C \|\psi\|_{\mathrm{BMO}}.$$

PROOF. Let $B_j = \operatorname{supp} \eta_j$ and $l(B_j) = \text{radius}(B_j)$. We first show that $\sum_{j=1}^\infty \eta_j(x)(\psi - \psi_{B_j})$ belongs to VMO, where $\psi_{B_j} = 1/w(B_j) \int_{B_j} \psi \, dx$. The difference between this sum and $\sum_j S_j(\psi)$ will be controlled later. Fix a cube $J \subseteq \mathbb{R}^{n-1}$ with center x_J. We need only consider those balls B_j which intersect J. These balls B_j are of two types: $l(J) < l(B_j)$ or $l(J) > l(B_j)$. Let $j \in \mathfrak{J}_1$ if $l(J) > l(B_j)$ and $j \in \mathfrak{J}_2$ if $l(J) \leqslant l(B_j)$. Set

$$c_J = \sum_{j \in \mathfrak{J}_2} \eta_j(x_J)(\psi - \psi_{B_j})_J.$$

Then

$$\int_J \left| \sum_{j=1}^\infty \eta_j(x)(\psi - \psi_{B_j}) - c_J \right| w \, dx \leqslant \int_J \left| \sum_{j \in \mathfrak{J}_1} \eta_j(x)(\psi - \psi_{B_j}) \right| w \, dx$$

$$+ \int_J \left| \sum_{j \in \mathfrak{J}_2} \eta_j(x)(\psi - \psi_J) + \sum_{j \in \mathfrak{J}_2} (\eta_j(x) - \eta_j(x_J)(\ \ )(\psi_J - \psi_{B_j}) \right| w \, dx.$$

The first term above is bounded by

$$\sum_{j\in\mathfrak{I}_1}\int_{B_j}|\psi-\psi_{B_j}|w\,dx\leqslant\|\psi\|_{\text{BMO}}\sum_{j\in\mathfrak{I}_1}|B_j|\leqslant C\int_{2J}\sum_j\chi_{B_j}(x)\,dx\leqslant C|J|.$$

We consider the two sums in the second term separately.

$$\int_J\left|\sum_{j\in\mathfrak{I}_2}\eta_j(x)(\psi(x)-\psi_J)\right|w\,dx\leqslant M\int_J|\psi(x)-\psi_J|w(x)\,dx\leqslant M|J|$$

since no point lies in more than M of the B_j's. In the second sum we add and subtract $\psi(x)$ from $\psi_J-\psi_{B_j}$ and it remains to bound

$$\int_J\left|\sum_{j\in\mathfrak{I}_1}(\eta_j(x)-\eta_j(x_J))(\psi-\psi_{B_j})\right|w\,dx.$$

The balls B_j which occur int this sum have $l(B_j)\geqslant l(J)$ and so at most M of them intersect J. Hence we have only to estimate a single term of the form

$$\int_J|\eta_j(x)-\eta_j(x_J)|\,|\psi(x)-\psi_{B_j}|w(x)\,dx \tag{3.4}$$

where $l(J)<l(B_j)$. Let J_n be the 2^n-fold enlargement of J and let n_0 be the smallest n for which $B_j\subseteq J_{n_0}$. Of course, for this n_0, $|J_{n_0}|$ is comparable to $|B_j|$. By the gradient estimate on η_j, (3.4) is less than

$$\frac{l(J)}{l(B_j)}\int_J\left|(\psi-\psi_{B_j})-(\psi-\psi_{B_j})_J+\sum_{l=1}^{n_0}[(\psi-\psi_{B_j})_{J_l}-(\psi-\psi_{B_j})_{J_{l-1}}]\right.$$
$$\left.+(\psi-\psi_{B_j})_{J_{n_0}}\right|w\,dx$$

Applying the BMO condition on ψ to the interval $J_l\cup J_{l-1}$ gives

$$|\psi_{J_l}-\psi_{J_{l-1}}|\leqslant|J_l|/w(J_l).$$

In particular,

$$\begin{aligned}\frac{l(J)}{l(B_j)}\int_J|\psi_{J_{n_0}}-\psi_{B_j}|w\,dx&\leqslant\|\psi\|_{\text{BMO}}\frac{l(J)}{l(B_j)}\frac{|B_j|}{w(B_j)}w(J)\\&\leqslant C|J|\frac{l(J)}{l(B_j)}\frac{|B_j|}{|J|}\left(\frac{l(J)}{l(B_j)}\right)^{n-2+\alpha}\\&\leqslant C|J|\left(\frac{l(J)}{l(B_j)}\right)^{\alpha}.\end{aligned}$$

A similar estimate is achieved on the sum:

$$\begin{aligned}\frac{l(J)}{l(B_j)}\int_J\left|\sum_{l=1}^{n_0}|\psi_{J_l}-\psi_{J_{l-1}}|\right. w\,dx &\leqslant \frac{l(J)}{l(B_j)}\int_J\sum_{l=1}^{n_0}\frac{|J_l|}{w(J_l)}w\,dx\\ &\leqslant \frac{l(J)}{l(B_j)}\sum_{l=1}^{n_0}|J_l|\frac{w(J)}{w(J_l)}\\ &\leqslant |J|\sum_{l=1}^{n_0}\frac{(2^l l(J))^{n-1}}{l(J)^{n-1}}\frac{l(J)}{l(B_j)}\left(\frac{l(J)}{2^l l(J)}\right)^{n-2+\alpha}\\ &= |J|\sum_{l=1}^{n_0}(2^l)^{1-\alpha}\frac{l(J)}{l(B_j)}\\ &\leqslant |J|\sum_{l=1}^{n_0}(2^l)^{1-\alpha}2^{-n_0}\\ &\leqslant C|J|.\end{aligned}$$

To control the difference between the sum $\sum_j \eta_j(\psi - \psi_{B_j})$ and $\sum_j S_j(\psi)$, note that each term is bounded by

$$\eta_j(x)\frac{1}{w(B_j)}\int_B|\psi-\psi_{B_j}|w\,dx$$

and, as in the previous argument it is possible to obtain a bound of $|J|$ on the BMO$(w\,dx)$ norm of the difference.

We have shown that $\sum_j S_j(\psi) \in \mathrm{BMO}(w\,dx)$, and it remains to verify that it belongs to VMO$(w\,dx)$ when $\psi \in \mathrm{VMO}(w\,dx)$. Suppose that $\psi \in \mathrm{Lip}_0$. Then

$$\begin{aligned}\sum_j S_j(\psi)(x)-\sum_j S_j(\psi)(x_0) &\leqslant \sum_j[\eta_j(x)-\eta(x_0)]\int[\psi(x)-\psi(z)]\frac{\eta_j w\,dz}{\int\eta_j w\,dz'}\\ &\quad+\sum_j\eta_j(x_0)\int[\psi(x)-\psi(x_0)]\frac{\eta_j w\,dz}{\int\eta_j w\,dz'}\\ &\leqslant \sum_j[\eta_j(x)-\eta_j(x_0)]\int|\psi(x)-\psi(z)|\frac{\eta_j w\,dz}{\int\eta_j w\,dz'}\\ &\quad+C|x-x_0|.\end{aligned}$$

By considering, separately the cases $l(B_j) < |x - x_0|$ and $l(B_j) > |x - x_0|$ in the sum above, and using $|\psi(x) - \psi(z)| \leqslant l(B_j)$ we can see that $\sum_j S_j(\psi)$ is also Lip_0. If $\psi \in \mathrm{VMO}(w\,dx)$, let θ be a Lip_0 function such that $\|\psi - \theta\|_{\mathrm{BMO}} < \epsilon$. Then

$$\sum_j S_j(\psi) = \sum_j S_j(\psi - \theta) + \sum_j S_j(\theta).$$

The second term is Lip_0 and the first is BMO with norm bounded by $\|\psi - \theta\|_{\mathrm{BMO}}$. □

Lemma 3.5. *If* $\psi \in \mathrm{VMO}$, *set*

$$\psi_t(x) = \int \psi(y) K(x, y, t) w(y)\, dy$$

where

$$K(x, y, t) = \varphi(x - y/t)\Big\{\int \varphi(x - z'/t) w(z')\, dz'\Big\}^{-1}.$$

Then ψ_t *converges to* ψ *in* $\mathrm{VMO}(w\,dx)$ *as* $t \to 0$.

PROOF. We first show that $\|\psi_t\|_{\mathrm{BMO}} \leqslant C\|\psi\|_{\mathrm{BMO}}$ for $t < 1$. Let Q be a cube in $\mathbb{R}^{n-1}$. If $t < l(Q)$,

$$\begin{aligned}\int_Q |\psi_t(x) - \psi_Q| w\, dx &\leqslant \int_Q \int_{y \in B(x,t)} |\psi(y) - \psi_Q| K(x, y, t) w(y)\, dy\, w(x)\, dx \\ &\leqslant \int_{y \in 2Q} |\psi(y) - \psi_Q| \int_{x \in B(y,t)} \frac{\varphi(x - y/t)}{w(B(x,t))} w(x)\, dx\, w(y)\, dy \\ &\leqslant C\|\psi\|_{\mathrm{BMO}} |Q|.\end{aligned}$$

If instead $t > l(Q)$, let Q_t be the cube of length $t \cdot l(Q)$ containing Q. Then

$$\begin{aligned}&\int_Q |\psi_t(x) - \psi_t(x_0)| w(x)\, dx \\ &\quad = \int_Q \left| \int (\psi(y) - \psi_{Q_t})((K(x, y, t) - K(x_0, y, t)) w(y)\, dy \right| w(x)\, dx \\ &\quad \leqslant \int_Q \int_{y \in Q_t} |\psi(y) - \psi_{Q_t}|\, |x - x_0| \Big/ t \frac{w\, dy}{w(Q_t)} w(x)\, dx \\ &\quad \lesssim \frac{l(Q)}{t} \|\psi\|_{\mathrm{BMO}} \frac{|Q_t|}{w(Q_t)} w(Q) \\ &\quad \lesssim \|\psi\|_{\mathrm{BMO}} (tl(Q))^{n-1} \left(\frac{l(Q)}{tl(Q))}\right)^{n-2+\alpha} \frac{l(Q)}{t} \\ &\quad \lesssim \|\psi\|_{\mathrm{BMO}} l(Q)^{n-\alpha} (l(Q)/t)^{\alpha} \\ &\quad \leqslant \|\psi\|_{\mathrm{BMO}} l(Q)^{n-1}\end{aligned}$$

since $l(Q)$ is also less than 1.

We now need to see that if $\psi \in \text{Lip}_0$, then $\psi_t \in \text{VMO}(w\,dx)$ and $\|\psi_t - \psi\|_{\text{BMO}} \to 0$ as $t \to 0$. Clearly $\psi_t \in \text{Lip}_0$ if ψ is Lip_0, so we now estimate $\|\psi_t - \psi\|_{\text{BMO}}$. In fact, for $0 < \beta < 1$, we have $\|\psi_t - \psi\|_{\text{Lip}(\beta)} \leqslant t^{1-\beta}$. Consider $|(\psi_t - \psi)(x) - (\psi_t - \psi)(x_0)|$. If $t < |x - x_0|$,

$$|\psi_t(x) - \psi(x)| \leqslant \int |\psi(y) - \psi(x)| K(x, y, t) w(y)\,dy$$
$$\leqslant t \leqslant t^{1-\beta}|x - x_0|^{\beta}.$$

If $t > |x - x_0|$,

$$|(\psi_t - \psi)(x) - (\psi_t - \psi)(x_0)| \leqslant \int |\psi(y) - \psi(x)|\,|K(x, y, t) - K(x_0, y, t)|\,w(y)\,dy$$
$$+ |\psi(x) - \psi(x_0)| \int K(x_0, y, t) w(y)\,dy$$
$$\leqslant |x - x_0|$$
$$< t^{1-\beta}|x - x_0|^{\beta}.$$

If we choose $\beta = 1 - \alpha$, and $l(Q)$ is small, then

$$\int_Q |(\psi_t - \psi)(x) - (\psi_t - \psi)(x_0)|\,w(x)\,dx \leqslant t^{\alpha}\|\psi\|_{\text{Lip}} \int_Q |x - x_0|^{1-\alpha} w\,dx \leqslant t^{\alpha}|Q|,$$

which tends to zero with t. When $|Q|$ is large, $|\psi_t - \psi| \lesssim t$ and the same bound is achieved.

Finally, if $\psi \in \text{VMO}$, write $\psi = \theta + \eta$ where $\theta \in \text{Lip}_0$ and $\|\eta\|_{\text{BMO}} < \epsilon$. Then $\|\psi_t - \psi\|_{\text{BMO}} \leqslant \|\theta_t - \theta\|_{\text{BMO}} + C\|\epsilon\| \leqslant C\epsilon$, if t is small.

Lemma 3.6. (Compare Macías-Segovia [22], Lemma 3.2). *Suppose f is a distribution in $H^1(w\,dx)$ and let $\Omega = \{f^* > \lambda\}$. Let $\{\eta_j\}$ be the partition of unity associated to Ω described above. Define the distribution b_j on* VMO$(w\,dx)$ *by setting*

$$\langle b_j, \psi \rangle = \langle f, S_j(\psi) \rangle$$

for all $\psi \in$ VMO$(w\,dx)$. *Then*

$$Nb_j(x_0) \leqslant C\lambda(r_j/|x_0 - x_j|)^{n-1+\alpha}\chi_{cB(x_j, cr_j)}(x_0) + Cf^*(x_0)\chi_{B(x_j, cr_j)}(x_0) \tag{3.7}$$

the series $\sum_j b_j$ converges weakly in VMO*$(w\,dx)$ *to a distribution b satisfying*

$$\int b^*(x)\,dx \leqslant C \int_{\Omega} f^*(x)\,dx, \tag{3.8}$$

and if $g = f - b$,

$$Ng(x_0) \leqslant C\lambda \sum_j \left(\frac{r_j}{|x_0 - x_j| + r_j}\right)^{n-1+2} + Cf^*(x_0)\chi_{c_\Omega}(x_0). \tag{3.9}$$

PROOF. We shall prove (3.7) and (3.8); the proof of (3.9) requires no new ideas and the argument is essentially provided in Macías-Segovia [22].

PROOF OF (3.7). To estimate $Nb_j(x_0)$, let us first assume that $x_0 \notin B(x_j, cr_j)$ for some constant c, which does not depend on j. Then

$$\left| \int b_j(y)\varphi(x_0 - y/t)w(y)\,dy \cdot w(B(x_0, t))^{-1} \right| = w(B(x_0, t))^{-1} \cdot \left| \int f(y) S_j(\varphi_{x_0,t})(y) w(y)\,dy \right|$$

where $\varphi_{x_0,t}(\bullet) = \varphi(x_0 - \bullet/t)$. For the above expression to be nonzero, we must have $t > c|x_0 - x_j|$. Therefore,

$$\begin{aligned} |S_j(\varphi_{x_0,t})(x)| &= \left| \eta_j(x)\left\{\eta_j(z')w(z')\,dz'\right\}^{-1} \cdot \int (\varphi_{x_0,t}(x) - \varphi_{x_0,t}(z))\eta_j(z)w(z)\,dz \right| \\ &\leqslant \int_{B(x;2r_j)} (|x - x|/t)w(z)\,dz \cdot \left\{\int \eta_j(z')w(z')\,dz'\right\}^{-1} \\ &\leqslant cr_j/t, \end{aligned}$$

and

$$|D^\alpha(S_j(\varphi_{x_0,t}))| \leqslant \sum_{\beta \leq \alpha} c_{\alpha,\beta}(1/r_j)^{|\alpha| - |\beta|} t^{-|\beta|} \leqslant c(r_j/t) r_j^{-|\alpha|}.$$

Now pick $y_j \in {}^c\Omega$ with $|x_j - y_j| \approx r_j$. The function $S_j(\varphi_{x_0,t})$ has support in $B(x_j, 2r_j) \subseteq B(y_j, cr_j)$ so we can write $S_j(\varphi_{x_0,t})(y) = \theta(y_j - y/r_j)$. The above computations show that $Ct/r_j \cdot \theta(z)$ belongs to $\mathfrak{A}$. This gives the estimate

$$\left| \left\langle f, \frac{t}{r_j} S_j(\varphi_{x_0,t}) \right\rangle \right| w(B(y_j, cr_j))^{-1} \lesssim f^*(y_j)$$

and so

$$|\langle b_j, \varphi_{x_0,t}\rangle| \cdot w(B(x_0, t))^{-1} \lesssim \frac{w(B(y_j, cr_j))}{w(B(x_0, t))} \frac{r_j}{t} f^*(y_j) \lesssim \lambda \left(\frac{r_j}{t}\right)^{n-2+\alpha} \frac{r_j}{t},$$

using (3.1) (i) and the fact that $y_j \in {}^c\Omega$. Because $|x_0 - x_j| \lesssim t$, the above is bounded by $c\lambda(r_j/|x_0 - x_j|)^{n-1+\alpha}$, which is the first summand in (3.7).

Let us assume now that $x_0 \in B(x_j, cr_j)$, and consider two cases. If $t > r_j$, then $\operatorname{supp} S_j(\varphi_{x_0,t}) \subseteq \operatorname{supp} \eta_j$. Moreover,

$$\|S_j(\varphi_j(\varphi_{x_0,t})\|_\infty \leqslant r_j/t \leqslant 1 \quad \text{and} \quad \|D^\alpha S_j(\varphi_{x_0,t})\|_\infty \leqslant r_j^{-|\alpha|}.$$

Reasoning as before we conclude that

$$|\langle b_j, \varphi_{x_0,t}\rangle|/w(B(x_0, t)) \lesssim f^*(x_0).$$

On the other hand, if $t < r_j$, split $S_j(\varphi_{x_0,t})$ in two parts as

$$S_j(\varphi_{x_0,t})(y) = \eta_j(y)\varphi_{x_0,t}(y) - \eta_j(y)\Big\{\int \eta_j(z')w(z')\,dz'\Big\}^{-1}\int \eta_j(z)\varphi_{x_0,t}(z)w\,dz$$
$$= h_1(y) - h_2(y).$$

The function $h_1(y)$ has support in $B(x_0, 2t)$, and satisfies $\|h_1\|_\infty \leqslant 1$ and $\|D^\alpha h_1\|_\infty \leqslant t^{-|\alpha|}$. Consequently,

$$|\langle f, h_1\rangle|\, w(B(x_0,t))^{-1} \lesssim f^*(x_0).$$

The function h_2 has support in $B(x_j, 2r_j)$ and satisfies

$$\|h_2\|_\infty \leqslant w(B(x_0,t))/w(B(x_j,2r_j))$$

and

$$\|D^\alpha h\|_\infty \leqslant w(B(x_0,t))/w(B(x_j,wr_j))\cdot r_j^{-|\alpha|}.$$

Since $|x_0 - x_j| < r_j$,

$$|\langle f, h_2\rangle|\, w(B(x_0,t))^{-1} \leqslant \left|\left\langle f, \frac{w(B(x_0,r_j))}{w(B(x_0,t))}h_2\right\rangle\right| w(B(x_0,r_j)) \leqslant f^*(x_0).$$

Altogether, this gives (3.7).

PROOF OF (3.8). Let

$$b^k = \sum_{j\le k} b_j(x).$$

The above estimates imply that $\{b^k\}$ is a Cauchy sequence in $H_1(w\,dx)$, since

$$\sum_{j=N+1}^{M}\int Nb_j(x)\,dx \leqslant \sum_{j=N+1}^{M} c\lambda\int_{{}^cB(x_j,cr_j)} (r_j/|x-x_j|)^{n-1+\alpha}\,dx$$
$$+\sum_{j=N+1}^{M}\int_{B(x_j,cr_j)} f^*(x)\,dx$$
$$\leqslant c\lambda\sum_{j=N+1}^{M}\sum_{l>1}\int_{|x-x_j|\sim 2^l r_j}(r_j/2^l r_j)^{n-1+\alpha}\,dx$$
$$+\sum_{j=N+1}^{M}\int_{B(x_j,cr_j)} f^*(x)\,dx$$
$$\leqslant c\lambda\sum_{j=N+1}^{M}\sum 2^{-l\alpha}r_j^{n-1} + \sum_{j=N+1}^{M}\int_{B(x_j,cr_j)} f^*(x)\,dx$$
$$\leqslant c\sum_{j=N+1}^{M}\int_{B(x_j,cr_j)} f^*(x)\,dx$$

as $f^* > \lambda$ on $B(x_j, r_j)$. By the finite overlap property of $\{B(x_j, cr_j)\}$ the above sum will be small once N and M are large.

We now want to see that $\{b^k\}$ converges (weakly) to a distribution in VMO*($w\,dx$). If $\psi \in \mathrm{VMO}(w\,dx)$,

$$|\langle b^k, \psi\rangle| = \left|\left\langle f, \sum_{j=1}^{k} S_j(\psi)\right\rangle\right| \leqslant \|f\|_{\mathrm{VMO}^*}\left\|\sum_{j=1}^{k} S_j(\psi)\right\|_{\mathrm{BMO}} \leqslant C\|f\|\;\|\psi\|_{\mathrm{BMO}},$$

by lemma 3.3. Since the $\{b^k\}$ are uniformly bounded in VMO*($w\,dx$), some subsequence $\{b^{k_j}\}$ has a weak limit, b, in VMO*($w\,dx$). Because the b^k are Cauchy in $H^1(w\,dx)$, the argument to follow shows that b is the H^1 limit of $\{b^{k_j}\}$, and hence of the entire sequence $\{b^k\}$. To compute the $H^1(w\,dx)$ norm of b, we test against φ and by weak convergence in VMO*($w\,dx$),

$$Nb(x) \leqslant \overline{\lim_{k_j\to\infty}}\, Nb^{k_j} \lesssim \lambda \sum_{j=1}^{\infty} (r_j/|x - x_j|)^{n-1+\alpha}\chi_{{}^cB(x_j, cr_j)}(x) + cf^*(x)\chi_{{}^cB(x_j, cr_j)}(x)$$

so that

$$\int b^*\,dx \leqslant C\int_\Omega f^*\,dx. \quad \square$$

When $f \in \mathrm{VMO}^*(w\,dx)$, set

$$\tilde{f}(y,t) = \langle f, K(\cdot, y, t)\rangle = \int f(x)\varphi(x-y)/t)\left\{\int \varphi(x - z'/t)w(z')\,dz'\right\}^{-1} w(x)\,dx.$$

Then one can show that $\langle \tilde{f}(\cdot, t), \psi\rangle = \langle f, \psi_t\rangle$ for all ψ and since $\psi_t \to \psi$ in VMO($w\,dx$) as $t \to 0$, by Lemma 3.4, $\tilde{f}(y,t)$ will converge to $f(y)$ in the ditribution sense as $t \to 0$.

Lemma 3.10. *If f is a distribution in* VMO*($w\,dx$) *which satisfies*

$$\int (f^*(x))^2\,dx < \infty,$$

then there exists a function $F \in L^2(dx)$ such that, for all $\psi \in \mathrm{Lip}_0$,

$$\langle f, \psi\rangle = \int F(x)\psi(x)w(x)\,dx.$$

PROOF. Notice that $\tilde{f}(y,t)$ is not quite the same as $f_\varphi(y,t)$, however

$$\int \varphi(x - z'/t)w(z')\,dz' \approx w(B(y,t)) \quad \text{for} \quad |x-y| < t.$$

This implies that $|\tilde{f}(y,t)| \lesssim f^*(y)$ for all t and so $\tilde{f}(y,t)$ is uniformly in $L^2(dy)$, by the same argument used in the proof of Lemma 3.2. By passing to a subsequence we get an $L^2(dx)$ function $F(x)$ such that $\tilde{f}(\cdot, t)$ converges weakly to $F(\cdot)$ as $t \to 0$. We also know that $\tilde{f}(\cdot, t) \to f$ when tested against Lip_0 functions. Let $\psi \in \mathrm{Lip}_0$. Then

$$\langle f, \psi \rangle = \lim_{t} \langle \tilde{f}(\cdot, t), \psi \rangle = \lim_{t \to 0} \int \tilde{f}(y, t)\psi(y)w(y)\,dy.$$

The reverse Hölder condition of exponent two tells us that $\psi_w \in L^2(dy)$, and we have shown that $\tilde{f}(\cdot, t) \in L^2(dy)$. Hence

$$\lim_{t} \langle \tilde{f}(\cdot, t), \psi \rangle = \int F(y)\psi(y)w(y)\,dy$$

which proves the lemma. $\square$

Lemma 3.11. *Suppose f is a distribution in* VMO*$(w\,dx)$ *with* $\int f^*(x)\,dx < \infty$. *Then, given $\epsilon > 0$, there exists a function $\tilde{g} \in L^2(dx)$ such that*

(i) *$\tilde{g}$ has a unique extension, g, to* VMO*$(w\,dx)$
(ii) $\int (f - g)^* w\,dx < \epsilon$.

PROOF. Choose $\lambda > \int f^*(x)\,dx$ such that

$$\int_{\Omega} f^*(x)\,dx < \epsilon, \quad \text{for} \quad \Omega = \{f^* > \lambda\}.$$

By Lemma (3.6), $f = g + b$, where

$$\int b^*(x)\,dx \leqslant c\int_{\Omega} f^*(x)\,dx$$

and

$$g^*(x) \leqslant c\lambda \sum_{j} (r_j/|x - x_j| + r_j)^{n-1+\alpha} + cf^*(x)\chi_{c\Omega}(x).$$

Therefore,

$$\begin{aligned}
\int (g^*(x))^2\,dx &\leqslant c\lambda^2 \int \Big\{\sum_{j} (r_j/|x - x_j| + r_j)^{n-1+\alpha}\Big\}^2 dx + \int_{c\Omega} (f^*(x))^2\,dx \\
&\leqslant c\lambda^2|\Omega| + \lambda \int_{\Omega} f^*(x)\,dx \\
&\leqslant c\lambda \int f^*(x)\,dx
\end{aligned}$$

By lemma (3.10), there exists an $L^2(dx)$ function $\tilde{g}$ such that $\langle g, \psi \rangle = \langle \tilde{g}, \psi \rangle$ for all $\psi \in \mathrm{Lip}_0$. And

$$\int (f - g)^*\,dx = \int b^*(x)\,dx < \epsilon. \quad \square$$

Definition. *A function $a(x)$ is an atom if the support of $a(x)$ is contained in some ball $B(x_0, r)$, $\|a\|_\infty \leqslant r^{-n+1}$ and $\int a(x)w(x)\,dx = 0$.*

Lemma 3.12. *There exists a constant C such that for all atoms $a(x)$,*

$$\int a^*(x)\,dx \leqslant C.$$

PROOF. Let $a(x)$ be an atom supported in $B(x_0, r) = B_r$. Let

$$M_w(h)(x) = \sup_{Q \ni x} \left\{ \int_Q |h(y)| w(y)\,dy / w(Q) \right\}$$

be the Hardy-Littlewood maximal function formed from the measure $w\,dx$. Consider

$$|a_\varphi(x, t)| \leqslant \left| \int a(y)\varphi(x - y)/t)w(y)\,dy \cdot w(B(x, t)) \right|$$

$$\leqslant \frac{1}{w(B(x, t))} \int_{B(x,t)} |a(y)| w(y)\,dy,$$

which shows that

$$\sup_t |a_\varphi(x, t)| \leqslant M_w a(x).$$

Now

$$\begin{aligned} \int_{B(x_0, 2r)} Na(x)\,dx &\lesssim |B_r|^{1/2} \left(\int (M_w a)^2(x)\,dx \right)^{1/2} \\ &= |B_r|^{1/2} \left(\int (M_w a)^2(x) w^{-1}(x) w(x)\,dx \right) \\ &\lesssim |B_r|^{1/2} \left(\int a^2\,dx \right)^{1/2} \\ &\leqslant |B_r| \cdot \|a\|_\infty \leqslant C. \end{aligned}$$

The last few inequalities follow from the fact that $w^{-1} \in A_2(w\,dx)$, (Muckenhoupt [23]) and the properties of atoms. It remains to estimate $Na(x)$ when $x \in {}^cB(x_0, 2r)$. By the mean value property of atoms,

$$\begin{aligned} |a_\varphi(x, t)| = &\left| \left\{ \int \varphi(x - z/t) w\,dz \right\}^{-1} \right. \\ &\left. \cdot \int_{y \in B_r} a(y)[\varphi(x - y/t) - \varphi(x - x_0/t)] w(y)\,dy \right| \\ \leqslant &\; w(B(x, t))^{-1} \cdot \|a\|_\infty \int_{B_r} \frac{|y - x_0|}{t} w(y)\,dy \end{aligned}$$

Note that if $x \in {}^cB(x_0, 2r)$, then t must be larger than $c|x - x_0|$ in order that $\operatorname{supp}(x - \bullet/t) \cap \operatorname{supp} a$ be nonempty. So $|a_\varphi(x, t)|$ is bounded by

$$|B_r|^{-1} \cdot \frac{r}{|x - x_0|} \frac{w(B_r)}{w(B(x, t))} \leqslant |B_r|^{-1} \left(\frac{r}{|x - x_0|} \right)^{n - 1 + \alpha}$$

by (4.1) (i). This implies that

$$\begin{aligned} \int_{{}^cB(x_0, 2r)} Na(x)\, dx &\leqslant \sum_{l > 2} \int_{\{x:\, |x - x_0| \approx 2^l r\}} |B_r|^{-1} \cdot 2^{-l(n-1+\alpha)}\, dx \\ &\leqslant \sum_l |B_r|^{-1} \approx 2^{-l(n-1+\alpha)} \cdot (2^l r)^{n-1} \\ &\leqslant C. \quad \square \end{aligned}$$

Definition.

$$H^1_{at} = \left\{ f = \sum_k \lambda_k a_k \text{: the } a_k \text{ are atoms and } \|f\|_{H^1_{at}} = \sum_k |\lambda_k| \text{ is finite} \right\}$$

where $f \in \mathrm{VMO}^*(w\,dx)$ *and the convergence is in* $\mathrm{VMO}^*(w\,dx)$.

By Lemma 3.12, H^1_{at} is contained in $H^1(w\,dx)$. Standard arguments show that the dual of H^1_{at} is $\mathrm{BMO}(w\,dx)$.

Lemma 3.13. H^1_{at} *is the dual of* $\mathrm{VMO}(w\,dx)$.

PROOF. We refer to Coifman-Weiss [5], p. 638, for a proof of

$$\mathrm{VMO}(\mathbb{R}^n, dx) = H^1_{at}(\mathbb{R}^{n-1})$$

which can be modified to yield our lemma. $\square$

Theorem 3.14. *Let f be a distribution in* $\mathrm{VMO}^*(w\,dx)$ *with* $\int f^*(x)\,dx < \infty$. *Then there exists a sequence of numbers* $\{\lambda_k\}$ *and atoms* $\{a_k\}$ *such that* $f = \sum \lambda_k a_k$ *in the sense that*

$$\langle f, \psi \rangle = \left\{ \sum_k \lambda_k a_k, \psi \right\}$$

for all $\psi \in \mathrm{Lip}_0$ *and the norms* $\|f\|_{H^1(w\,dx)}$ *and* $\sum_k |\lambda_k|$ *are equivalent.*

We shall confine ourselves to a few remarks about the proof of Theorem 4.14; the technical details of the construction are standard from the information we have at hand, and the reader is referred to Latter [21], Stromberg-Torchinsky [24], or Macías-Segovia [22]. We observe the following.

(1) The density Lemma (3.11) provides a class of $L^1_{\mathrm{loc}}(w\,dx)$ functions on which the construction of the atoms can take place. This forms, essentially, the heart of the proof.

(2) We already know that elements of $H^1(w\,dx)$ have an atomic decomposition $H^1(w\,dx)$ as a subspace of $\mathrm{VMO}^*(w\,dx) = H^1_{at}$. The point of course is the equivalence of the two norms. Moreover, by Lemma (4.13) and continuity, the relationship $\langle f, \psi\rangle = \langle \Sigma\lambda_k a_k, \psi\rangle$ is true for all $\psi \in \mathrm{VMO}(w\,dx)$.

(3) Theorem (3.14) shows that $H^1(w\,dx)$ is complete in the norm $\int f^*(x)\,dx$. It is equivalent to this fact since lemma 4.12 guarantees that $\|f\|_{H^1(w\,dx)}$ is bounded by $\|f\|_{H^1_{at}}$ and so the comparability of these two norms is equivalent to the completeness in $\|\bullet\|_{H^1(w\,dx)}$ norm.

(4) Finite sums of atoms are dense in $H^1(w\,dx)$.

(5) If one uses the approach in Stromberg-Torchinsky [24], an atomic decomposition for $H^1(w\,dx)$ can be obtained with arbitrarily large vanishing moments on the atoms. That is, whenever $p(x)$ is a polynomial of degree less than N, the atoms $a(x)$ will satisfy

$$\int a(x)p(x)w(x)\,dx = 0.$$

Corollary 3.15. *The dual of* $H^1(w\,dx)$ *is* $\mathrm{BMO}(w\,dx)$, *with pairing*

$$\langle f, \psi\rangle = \int f\psi w\,dx,$$

for f a finite sum of atoms.

Theorem 3.16. *Let* $\psi \in \mathrm{BMO}(w\,dx)$ *with compact support. Set*

$$\varphi_s(\bullet) = s^{-n+1}\varphi(\bullet/s).$$

Then $\psi = b_0 + b$ *where* $b_0(x)w(x) \in L^\infty(dx)$ *and b has an extension $h(x,s)$ to the upper half plane (in the sense that $h(x,s) \to b$ weakly in $L^1(w\,dx)$) which satisfies*

(1) $|\nabla h(x,s)|\,w * \varphi_s(x)$ *is a Carleson measure, and*

(2) $|\nabla h(x,s)|\,w * \varphi_s(x) \cdot s \leqslant C$.

Remark. As in P. Jones [19], one can obtain such an extension for all $\mathrm{BMO}(w\,dx)$ functions, once it is known for compactly supported functions.

PROOF. Recall that for $u \in H^1(w\,dx)$,

$$u(x,y) = \int \varphi(x - z/y)f(z)w(z)\,dz \cdot \left\{\int \varphi\left(\frac{x-z'}{y}\right)w(z')\,dz'\right\}^{-1} \tag{3.17}$$

We claim that if λ is a linear functional on $H^1(w\,dx)$ and $u \in H^1$

$$\lambda(u) = \int u(x) g_\infty(x)\,dx + \sum_{n=-\infty}^{n=\infty} \int u(x, y_n) g_n(x)\,dx$$

where $y_n \to -\infty$ as $n \to \infty$, $y_n \to +\infty$ as $n \to -\infty$, $|y_n - y_{n+1}| < \min\{\delta, y_n^2\}$ and

$$\|g\|_\infty + \left\| \sum_{n=-\infty}^{\infty} |g_n| \right\|_\infty \leqslant \|\lambda\|.$$

(We assume here, by the density lemma, that in (3.17), $f \in L^1 \cap L^2(dx)$). The argument for this is due to C. Fefferman; it depends on the fact that $u^*(x)$, the vertical maximal function defines an equivalent norm on $H^1(w\,dx)$. Our source for this is Garnett [15].

Fix a ball B, and let $\psi \in \mathrm{BMO}(w\,dx)$ with support in B. Then if f is an atom supported in B,

$$\begin{aligned}\int f(x)\psi(x)w(x)\,dx &= \int f(x) g_\infty(x)\,dx + \sum_{-\infty}^{\infty} \int u(x, y_n) g_n(x)\,dx \\ &= \int f g_\infty\,dx + \lim_{N\to\infty} \int f(z) w(z) \sum_{-N}^{N} \int \varphi\left(\frac{x-z}{y_n}\right) \\ &\quad \cdot \left\{ \int \varphi\left(\frac{x-z'}{y_n}\right) w\,dz' \right\}^{-1} g_n(x)\,dx\end{aligned}$$

Let $b_0 = g_\infty / w \in L^1_{\mathrm{loc}}(dw)$ and set $b = \psi - b_0$. Then b belongs to $\mathrm{BMO}(w\,dx)$. Set

$$h_N(z) = \sum_{-N}^{N} \int \varphi\left(\frac{x-z}{y_n}\right) \left\{ \int \varphi\left(\frac{x-z'}{y_n}\right) w(z')\,dz' \right\}^{-1} g_n(x)\,dx$$

so that

$$\int b f w\,dz = \lim_{N\to\infty} \int f(z) h_N(z) w(z)\,dz.$$

The function $h_N(z)$ is well defined, as

$$|h_N(z)| \leqslant \sum_{-N}^{N} |B(z, y_n)| / w(B(z, y_n)).$$

Now test h_N against an atom a:

$$\begin{aligned}\int a(z)h_N(z)w(z)\,dz &= \sum_{-N}^{N}\int a(x,y_n)g_n(x)\,dx\\ &\leqslant \int a^+(x)\sum_{-N}^{N}|g_n(x)|\,dx\\ &\leqslant C\|a\|_{H^1}\leqslant C.\end{aligned}$$

So $\|h_N\|_{\mathrm{BMO}}$ is bounded by a constant independent of N.

We now claim that BMO $\subseteq L^{1+\epsilon}_{\mathrm{loc}}(dw)$ for some $\epsilon > 0$. By the A_2 property of w^{-1}, $\psi^2 w^2 \in L^1_{\mathrm{loc}}(dx)$ for any $\psi \in \mathrm{BMO}(w\,dx)$. But $w \in A_\infty(dx)$ so there exists a small $\delta > 0$ such that $w^{-\delta} \in L^1_{\mathrm{loc}}(dx)$. (See Coifman-Fefferman [3]). In particular, $h_N \in L^{1+\epsilon}_{\mathrm{loc}}(w\,dx)$ and so on B, there exists constant c_N and a constant A for which $\int |h_N - c_N|^{1+\epsilon}\,dw \leqslant A$, all N. Thus there is a weak limit for $\{h_N - c_N\}$; call this h. We have

$$\int fh\,w\,dx = \int fb\,w\,dx$$

for all $f \in L^\infty$ with support in B, and $\int f w\,dx = 0$. Hence $b = h + c$ for some constant C. As in Varopoulos [25], set

$$h_N(z,s) = \sum_{-N}^{N}\theta(s/y_n)\int \varphi\left(\frac{x-z}{y_n}\right)\left\{\int \varphi\left(\frac{x-z'}{y_n}\right)w(z')\,dz'\right\}^{-1} g_n(x)\,dx$$

where $\theta \in C^\infty$ satisfies $\theta(t) \equiv 1$ for $0 \leqslant t \leqslant 1/2$, $\theta(t) = 0$ for $t > 1$. Then

$$|\nabla h_N(z,s)| \leqslant \int_{y=s}^{\infty}\int_{\{|x-z|<y\}}\{yw(B(z,y))\}^{-1}\sum_{j=-N}^{N}g_j(x)\,dS_j(x,y) \tag{3.17}$$

where $dS_j(x,y) = dx$ on $y = y_j$. If we set

$$d\sigma(x,y) = \sum_{j=-N}^{N} g_j(x)\,dS_j(x,y)$$

then $d\sigma(x,y)$ is a Carleson measure. In fact

$$\int_{x\in Q}\int_0^\infty d\sigma(x,y) \leqslant C|Q|,$$

which we will refer to as the vertical Carleson measure estimate. Hence, using (3.1) (i)

$$\begin{aligned}|\nabla h_n(z,s)| &\leqslant \sum_{l>0}\int_{y\approx 2^l s}\int_{|x-z|<2^l s}(yw(B(z,s)))^{-1}\frac{w(B(z,s))}{w(B(z,y))}\,d\sigma(x,y)\\ &\leqslant \sum_{l>0}\int_{y\approx 2^l s}\int_{|x-z|<2^l s}(2^l s)^{-1}w(B(z,s))^{-1}(2^{-l})^{n-2+\alpha}\,d\sigma(x,y)\end{aligned}$$

$$|\nabla h_n(z,s)| \leqslant \sum_{l>0} (2^{-l})^{n-1+\alpha}\{sw(B(z,s))\}^{-1}\sigma(B(z,2^l s)\times[0,2^l s])$$
$$\leqslant \sum_{l>0} 2^{-l\alpha}\cdot s^{n-2}/w(B(z,s))$$

because $d\sigma$ is Carleson. This proves (2) for $h_N(x,s)$. Now consider the Carleson property of $|\nabla h_N| w * \varphi_s(\cdot)/s^{n-1}$. Fix a cube $Q \subseteq \mathbb{R}^{n-1}$ and let $S(Q) = Q \times [0, l(Q)]$. We split the integral in (3.17) in two parts, estimating each separately.

$$(3.18)\quad \int_{(z,s)\in S(Q)} \int_{y=s}^{2l(Q)} \int_{x\in B(z,y)} \frac{|w*\varphi_s(y)|}{yw(B(z,y))}\, d\sigma(x,y)\, dz\, ds$$
$$\leqslant C\int_{(x,y)\in S(4Q)} \int_{z\in B(x,y)} \int_{s=0}^{y} \frac{w(B(z,s))}{w(B(z,y))} y^{-1}s^{-n+1}\, dz\, ds\, d\sigma(x,y)$$
$$\leqslant C\int_{(x,y)\in S(4Q)} \int_{z\in B(x,y)} \int_{s=0}^{y} (s/y)^{n-2+\alpha} y^{-1}s^{-n+1}\, dz\, ds\, d\sigma(x,y)$$
$$\leqslant C\iint_{S(4Q)} d\sigma(x,y) \leqslant C|Q|.$$

$$(3.19)\quad \int_{(z,s)\in S(Q)} \sum_{k>0} \int_{y\approx s^k l(Q)} \int_{x\in 2^k Q} y^{-1}\frac{w(B(z,s)}{w(B(z,y)} s^{-n+1}\, d\sigma(x,y)\, dz\, ds$$
$$\leqslant \int_{(z,s)\in S(Q)} \sum_{k>0} \int_{y\approx 2^k l(Q)} \int_{x\in 2^k Q} 2^{-k} l(Q)^{-1}[s/2^k l(Q)]^{n-2+\alpha} s^{-n+1}\, d\sigma(x,y)\, dz\, ds$$
$$\leqslant \sum_{k>0} \int_{y\approx 2^k l(Q)} \int_{x\in 2^k Q} (2^{-k})^{n-1+\alpha}|Q|\, l(Q)^{-(n-1+\alpha)} \int_{s=0}^{l(Q)} s^{-1+\alpha}\, ds\, d\sigma(x,y)$$
$$\leqslant \sum_{k>0} 2^{-k\alpha}|Q| = C|Q|.$$

Combining (3.18) and (3.19) gives (3.16) (1) for $|\nabla h_N(x,s)|$ with a Carleson measure constant independent of N.

By (3.16) (2), when $s > 0$, $|\nabla h_N(\cdot, s)|$ is uniformly bounded on compact sets. A similar result holds for higher order derivatives. Thus there exists a sequence of constans $\{a_N\}$ such that $h_N(x,s) + a_N \to h(x,s)$ uniformly on compact sets as $N \to \infty$, and also $\nabla h_N(x,s) \to \nabla h(x,s)$. This $h(x,s)$ satisfies conditions (1) and (2) of the theorem.

For fixed N, the expression

$$\sum_{-N}^{N} \theta(s/y_n)\varphi\left(\frac{x-y}{y_n}\right) g_n(x) \left\{ \int \varphi\left(\frac{x-z'}{y_n}\right) w(z')\, dz' \right\}^{-1}$$

is bounded by some constant c_N and converges pointwise as $s \to 0$. So by dominated convergence, $h_N(x, s) \to h_N(x)$ as $s \to 0$ in $L^1_{\text{loc}}(w\,dx)$.

For fixed $s > 0$,

$$\int h(x, s) f(x) w(x)\, dx = \lim_{N \to \infty} \int h_N(x, s) f(x) w(x)\, dx$$

for all $f \in L^\infty(B)$ with $\int f w\, dx = 0$. Hence if f is an atom,

$$\int h(x) f(x) w(x)\, dx - \int h(x, s) f(x) w(x)\, dx = \lim_{N \to \infty} \int [h_N(x) - h_N(x, s)] f(x) w(x)\, dx$$

and

$$|h_N(x) - h_N(x, s)| \leqslant \sum_{-\infty}^{\infty} |g_n(x)| u^+(x)[1 - \theta(s/y_n)]$$

which converges to zero as $s \to 0$. So for all atoms f,

$$\lim_{s \to 0} \int h(x, s) f(x) w(x)\, dx = \int h(x) f(x) w(x)\, dx = \int b(x) f(x) w(x)\, dx. \tag{3.20}$$

Now let f be an arbitrary L^∞ function supported in B. Set $g = (f - (f)_B)\chi_B$. By (3.20) with g in place of f, we get

$$\int f b w\, dx - (f)_B \int_B b w\, dx = \lim_{s \to 0} \int g(x) h(x, s) w(x)\, dx$$

which implies

$$b(x) = \lim_{s \to 0} \left[h(x, s) - \int_B h(x, s) w(x) \frac{dx}{w(B)} \right] + (b)_B,$$

where the limit is taken weakly in $L^1(w\,dx)$. Thus $(b - (b)_B)\chi_B$ has an extension $h(x, s) - C(s)$ to the upper half plane where

$$C(s) = \lim_{s \to 0} \int_B h(x, s) w(x)\, dx / w(B).$$

Using the vertical Carleson measure estimate, it is not hard to check that $C(s)$ verifies (1) and (2). □

We now describe an alternative, constructive mean of obtaining the extension theorem for BMO($w\,dx$) functions. We mimic the approach of Varopoulos [25]. Let us assume that $f \in \mathrm{BMO}(w\,dx)$ has support in Q_0 the unit cube) with $(f)_{Q_0} = 0$. Then we claim the following.

Lemma 3.21. *If $f \in \mathrm{BMO}(Q_0, w\,dx)$ with $\|f\|_{\mathrm{BMO}} \leqslant 1$, there exists a sequence of dyadic cubes $\{I_j\}$ and constants $\{\alpha_{I_j}\}$ such that*

(1) $\left|f(x) - \sum \alpha_{I_j} \chi_{I_j}(x)\right| \leqslant \frac{3}{2} w(x)^{-1}$.

(2) $|\alpha_{I_j}| \leqslant |I_j|/w(I_j)$.

(3) $\sum_{I_j \subseteq I_0} |I_j| \leqslant C|I_0|$, *for all dyadic* I_0.

PROOF. The proof is obtained in the same way as its counterpart in $\mathrm{BMO}(dx)$, the argument in that situation is due to Garnett ([15]. p.) and closely follows the stopping time procedure used to prove the John-Nirenberg theorem. We observe that one only needs to show that if f belongs to $\mathrm{BMO}(w\,dx)$, then for any two consecutive dyadic intervals I, I', the difference $|(f)_I - (f)_{I'}|$ is less than $C|I|/w(I)$ where $I = I' \cup I''$. Moreover, this condition characterizes the difference between $\mathrm{BMO}(w\,dx)$ and dyadic $\mathrm{BMO}(w\,dx)$. □

The function $f_0(x) = \sum \alpha_I \chi_I(x)$ can be extended to the upper half space in a discrte way by setting

$$F(x, y) = \sum_I \alpha_I \chi_{\tilde{I}}(x, y),$$

where $\tilde{I} = I \times [0, L(I)]$.

Lemma 3.22. *In the distribution sense, $|\nabla F(x, y)|\, w * \varphi_y(x)/y^{n-1}$ is a Carleson measure on $\mathbb{R}^n_+$.* (Comparare Varopoulos [15], p. 226).

PROOF. We give the proof in $\mathbb{R}^2_+$, for the mesure $|\nabla F|\, w * \varphi_y(x)/y^{n-1}$. Note that $\partial\chi_{\tilde{I}}/\partial x$, with $I = [a, b]$, is Lebesgue measure on $\{x = a, 0 \leqslant y \leqslant l(I)\}$ and $\{x = b, 0 \leqslant y \leqslant l(I)\}$ and that $\partial\chi_{\tilde{I}}/\partial y$ is Lebesgue measure on $\{x \in I, y = l(I)\}$. Consider first $\sum \alpha_I (\partial\chi_{\tilde{I}}(x, y)/\partial y) w * \varphi_y(x)/y^{n-1}$. Then for $(x, y) \in \operatorname{supp} \partial\chi_{\tilde{I}}/\partial y$, $w * \varphi_y(x)/y^{n-1}$ is comparable to $w(I)/|I|$. Let $\beta_I = \alpha_I w(I)/|I|$. By (3.21) (2), $|\beta_I| \leqslant 1$ so we must show that $\sum_I \beta_I \partial\chi_{\tilde{I}}/\partial y$ is Carleson when the intervals $\{I\}$ have the packing property (3.2) (3). But this is precisely Varopoulos's result.

Now consider $\sum \alpha_I \partial\chi_{\tilde{I}}(x, y)/\partial x) w * \varphi_y(x)/y^{n-1}$. Let I_0 be a dyadic interval of the form $[p/2^m, p + 1/2^m]$ for $n, p \geqslant 0$. Let I_1 and I_2 be its adjacent dyadic intervals and set

$$\mu_i = \sum_{I \subseteq I_i} \alpha_I \partial\chi_{\tilde{I}}/\partial x, \qquad i = 0, 1, 2.$$

Then

$$\int_{x\in I_0}\int_{y=0}^{l(I_0)} w*\varphi_y(x)/y^{n-1}\,d\mu_i(x,y)=\int_{x\in I_0}\int_{y=0}^{l(I_0)}\sum_{I\subseteq I_0}\alpha_I w(B(a,y))/y^{n-1}\,dS_I(x,y)$$

where $dS_I(x,y)=dy$ for $\{x=a, 0<y<l(I)\}$ and $\{x=a+l(I), 0<y\leqslant l(I)\}$. By (3.2) (2) and our basic estimate for $w(B)/w(2^kB)$, the above is bounded by

$$\sum_{I\subseteq I_0}\int_{x\in I_0}\int_{y=0}^{l(I)}\frac{|I|}{w(I)}w(B(a,y))/y^{n-1}\,dS_I(x,y)$$

$$\leqslant \sum_{I\subseteq I_0}\int_{x\in I_0}\int_{y=0}^{l(I)}[l(I)/y]^{n-1}[y/l(I)]^{n-2+\alpha}\,dS_I(x,y)$$

$$\leqslant 2\sum_{I\subseteq I_0}\int_{y=0}^{l(I)}(l(I)/y)^{1-\alpha}\,dy$$

$$=2\sum_{I\subseteq I_0}l(I)\leqslant C|I_0|,$$

which gives the desired estimate for the measure μ_i. The rest of the argument of [25] goes through with the help of the following estimates:

$$|(f)_I-(f)_{2I}|\leqslant |I|/w(I)$$

and

$$\sum_{I\subseteq I_0}\alpha_I w(I)\leqslant |I_0|.\quad\square$$

The discrete version of our extension given by Lemma 3.22 can be smoothed to obtain a continuous one by setting $\bar{F}(x,y)=F*\varphi_y(x,y)$ for some smooth bump function φ. The estimate for $|\nabla_x\bar{F}(x,y)|$, both the pointwise bound and the Carleson measure condition, is not hard. The main difficulty lies in the estimate for $|\nabla_y\bar{F}(x,y)|$. This can be overcome however by writing $\partial\varphi_y(\cdot)/\partial y$ as $\nabla_x\eta_y(x,y)$ for some other smooth bump function η.

4. We shall prove in this section, the main results for $H^1(D,d\sigma)$, $D\subset\mathbb{R}^n$ and Lipschitz. The fundamental tool will be the extension Theorem 3.16. Let us begin by localizing to a part of the boundary of D which is the graph of a Lipschitz function.

There exists some $\delta>0$ and a finite convering of $\{x\colon \operatorname{dist}(x,\partial D)\leqslant\delta\}$ by balls $B_j=B(Q_j,r_j)$ such that

$$B(Q_j,4r_j)\cap D=B(Q_j,4r_j)\cap\{(x,y)\colon y>\Phi_j(x)\}$$

where each Φ_j is Lipschitz. Let $\{\psi_j\}$ be a finite partition of unity for $\{x: \text{dist}(x, \partial D) \leqslant \delta\}$ subordinate to $\{B_j\}$ with $\psi_j \in C_0^\infty$. Let w_j be defined on $\mathbb{R}^{n-1}$ by $w_j(x) = k(x, \Phi_j(x))$ for $(x, \theta_j(x)) \in \partial D \cap B(Q_j, 4r_j)$, where $k = d\omega/d\sigma$.

Lemma 4.1. *The measure $w_j(x)\,dx$ on $\mathbb{R}^{n-1}$ obtained by extending w_j (the projection of harmonic measure) by reflection across the face of a cube*

$$I_j \subseteq \{x: (x, \Phi_j(x) \in \partial D \cap B(Q_j, r_j)\}$$

satisfies conditions (3.1) (i), (ii), *and* (iii).

PROOF. Let us assume that $w(x)$ is the projection of harmonic measure onto the unit cube Q_0 and then extended by repeated reflection across the faces of Q_0 to all of $\mathbb{R}^{n-1}$. Thus $\mathbb{R}^{n-1} = \bigcup Q_l$, each Q_l has unit size and $w(Q_l) = w(Q_k)$ all l, k; and when the cubes are large, conditions (i)-(iii) are obviously satisfied.

Let us consider the first condition (3.1) (i): $w(Q)/w(2^jQ) \leqslant C2^{-j(n-2+\alpha)}$ for all cubes Q. When both Q and 2^jQ are contained in some Q_l, this is clear. If the length of 2^jQ is large, let j_0 be the largest integer such that $l(2^{j_0}Q) < 1$. Then $w(Q)/w(2^jQ) = w(Q)/w(2^{j_0}Q) \cdot w(2^{j_0}(Q)/w(2^jQ)$ and by our previous remark it is enough to show (3.1) (i) under the assumption that $l(2^jQ)$ is small. We shall reduce this situation to the case where both cubes are contained in some Q_l.

We claim that there exists a Q' and Q_j', both contained in some Q_l, with $w(Q')/w(Q_j')$ comparable to $w(Q)/w(2^jQ)$ and $w(Q') \leqslant 2^{-j(n-2+\alpha)}w(Q_j')$. To see this choose a Q_l such that $|Q \cap Q_l| > |Q|/2$. Then

$$w(Q) = \sum_k w(Q \cap Q_k) \leqslant 2^{n-1} w(Q \cap Q_l),$$

since w was extended by reflection. Inside $Q \cap Q_l$ there is a cube Q' with $|Q'| \approx |Q \cap Q_l|$ and since $w \in A_\infty(Q_l, dx)$ we also have $w(Q') \approx w(Q \cap Q_l)$. The same argument gives a cube Q_j', $Q' \subset Q_j' \subset Q_l$ with $w(2^jQ) = w(Q_j')$ and $l(Q_j') \approx 2^j l(Q')$.

The argument for the reverse Hölder condition of exponent two consists of the same case by case analysis and will be omitted.

Lemma 4.2. *If $f \in \mathrm{BMO}_\sigma(d\omega)$ and $\theta \in C_0^\infty(\mathbb{R}^n)$ with $|\nabla\theta| \leqslant c$, then $f\theta \in \mathrm{BMO}_\sigma(d\omega)$ with $\|f\theta\|_{\mathrm{BMO}} \leqslant c\|f\|_{\mathrm{BMO}}$.*

PROOF. Let $\Delta = \Delta(Q_0, r)$ be a small surface ball contained in ∂D and let $c_\Delta = \theta(Q_0)(f)_\Delta$ where Q_0 is the center of Δ. Then

$$\begin{aligned}\int_\Delta |f\theta - c_\Delta|\, d\omega &\leqslant \int_\Delta |f(Q)[\theta(Q) - \theta(Q_0)]\, d\omega + \int_\Delta \theta(Q_0)[f - (f)_\Delta]\, d\omega \\ &\leqslant \sigma(\Delta)\|\theta\|_\infty \|f\|_{\mathrm{BMO}} + \int_\Delta |f(Q)|\,|Q - Q_0|\, d\omega \\ &\leqslant c\|f\|_{\mathrm{BMO}} \cdot \sigma(\Delta) + r \cdot \int_\Delta |f|\, dw.\end{aligned}$$

Let $2^j\Delta$ denote the surface ball contained in ∂D with $\sigma(2^j\Delta) = (2^j)^{n-1}\sigma(\Delta)$ and let $m = \inf\{j: \sigma(2^j\Delta) > 1/2\,\sigma(\partial D)\}$. Then

$$\int_\Delta |f|\, d\omega \leqslant \int_\Delta |f - f_\Delta|\, d\omega + \sum_{j=1}^m \int_\Delta |(f)_{2^{j-1}\Delta} - (f)_{2^j\Delta}|\, d\omega + \int_\Delta |(f)_{2^m\Delta}|\, d\omega \tag{4.3}$$

Applying the $\mathrm{BMO}_\sigma(d\omega)$ condition to the interval $2^{j-1}\Delta$ we see that

$$|(f)_{2^{j-1}\Delta} - (f)_{2^j\Delta}| \leqslant \sigma(2^j\Delta)/\omega(2^j\Delta)$$

Therefore

$$\begin{aligned}\sum_{j=1}^m \int_\Delta |(f)_{2^{j-1}\Delta} - (f)_{2^j\Delta}|\, d\omega &\leqslant \omega(\Delta) \cdot \sum_{j=1}^m \sigma(2^j\Delta)/\omega(2^j\Delta) \\ &\leqslant \sum_{j=1}^m (2^{-j})^{n-2+\alpha} \cdot \operatorname{diam}(\partial D) \cdot (2^j r)^{n-2} \\ &\leqslant Cr^{n-2}.\end{aligned}$$

Also

$$|(f)_{2^m\Delta}| \leqslant C\int_{\partial D} |f|\, dw \leqslant C\|f\|_{\mathrm{BMO}}.$$

Altogether then (4.3) is bounded by $\|f\|_{\mathrm{BMO}}\{\sigma(\Delta) + r^{n-2} + \omega(\Delta)\}$, and

$$\begin{aligned}r \cdot \int_\Delta |f|\, d\omega &\leqslant \|f\|_{\mathrm{BMO}}\{\operatorname{diam}(\partial D) \cdot \sigma(\Delta) + r^{n-1} + r \cdot r^{n-2+\alpha}\} \\ &\leqslant c\|f\|_{\mathrm{BMO}} r^{n-1}. \quad \square\end{aligned}$$

Lemma 4.4. *Assume $f \in \mathrm{BMO}_\sigma(d\omega)$. Then there exists an f_0 with $f_0 k \in L^\infty$ ($k = d\omega/d\sigma$) such that if $f_1 = f - f_0$, there is an extension F of f_1 in the sense that*

$$\int_{\partial D} u(Q) f_1(Q)\, d\omega = \int_D G(x)\nabla u(x) \cdot \nabla F(x)\, dx - \int_D u(x)\nabla G(x) \cdot \nabla F(x)\, dx$$

for all $u \in \mathfrak{L}(\bar{D})$ and $|\nabla F(x)|G(x)/d(x)$ is a Carleson measure on D. Moreover, $|\nabla F(x)| \lesssim G^{-1}(x)$ near ∂D and $|\nabla F(x)| + |F(x)|$ is bounded in a compact subset of D. (Compare Lemma 2.3 of Fabes-Kenig [11]).

PROOF. Let $\{\psi_j\}$ be the partition of unity associated to D subordinate to the covering $\{B_j\}$ and let $f_{\psi_j}(x) = \psi_j(x, \Phi_j(x)) \cdot f(x, \Phi_j(x))$. Let $w_j(x)\,dx$ be the measure of Lemma 4.2. By Lemma 4.2, $f_{\psi_j}(x)$ belongs to $\mathrm{BMO}(\mathbb{R}^{n-1}, w_j\,dx)$. Let $\bar{B}_j$ be the support of $f_{\psi_j}(x)$. Theorem 3.16 tells us that $f_{\psi_j} - g_j$ (where $g_j w_j \in L^\infty$) has an extension F_{ψ_j} to the upper half space such that $|\nabla F_{\psi_j}(x,s)|\,w_j * \varphi_s(x)/s^{n-1}$ is a Carleson measure and $|\nabla F_{\psi_j}(x,s)|(w_j * \varphi_s(x)/s^{n-2})^{-1}$ is bounded.

Pick $\theta_j \in C_0^\infty(B_j)$, identically 1 in a neighborhood of the support of ψ_j. Set $F_j(X) = \theta_j(X) \cdot F_{\psi_j}(x, y - \theta_j(x))$ for $X \in B_j \cap D$, $X = (x,y)$ and $y > \theta_j(x)$; put $F_j \equiv 0$ outside $B_j \cap D$. We will first show that

$$|\nabla(\theta_j(x, t + \Phi_j(x)) \cdot F_{\psi_j}(x,t)|\,w_j(B(x,t))/t^{n-1}$$

is a Carleson measure on $\mathbb{R}^{n-1}_+(dx\,dt)$ for all cubes Q with $l(Q) = r$, $r \leqslant r_0$, a number which depends only on the domain D. When the gradient falls on $F_{\psi_j}(x,t)$ our estimate is known. When the gradient falls on $\theta_j((x, t + \Phi_j(x))$, consider the integral

$$\begin{aligned}(4.5)\quad &\int_0^r \int_{Q_r} |F_{\psi_j}(x,t)|\,w_j(B(x,t))/t^{n-1}\,dx\,dt \\ &\qquad = \int_0^r \int_{Q_r} |F_{\psi_j}(x,t) - F_{\psi_j}(x,R)|\,w_j(B(x,t))/t^{n-1}\,dx\,dt \\ &\qquad\quad + \int_0^r \int_{Q_r} |F_{\psi_j}(x,R)|\,w_j(B(x,t))/t^{n-1}\,dx\,dt,\end{aligned}$$

where R is the radius of B_j, the support of f_{ψ_j}. The first integral is bounded by

$$\begin{aligned}&\int_0^r \int_{Q_r} \int_{s=t}^R |\nabla F_{\psi_j}(x,s)|\,ds\,w_j(B(x,t))/t^{n-1}\,dx\,dt \\ &\quad \leqslant \int_0^r \int_{Q_r} \int_{s=t}^R |\nabla F_{\psi_j}(x,s)|\,w_j(B(x,s))/s^{n-1} \cdot (t/s)^{n-2+\alpha} \cdot (s/t)^{n-1}\,ds\,dx\,dt \\ &\quad \leqslant R \int_{X \in Q_r} \int_{s=0}^\infty |\nabla F_{\psi_j}(x,s)|\,w_j(B(x,s))/s^{n-1}\,dx\,dt \\ &\quad \leqslant CR|Q_r|\end{aligned}$$

since the Carleson measure property is satisfied over all vertical lines for our extension. But R is just a constant which depends only on δ in the covering of $\{X: \operatorname{dist}(X, \partial D) < \delta\}$.

We claim that $|F_{\psi_j}(x,R)|$ is bounded by a constant which depends only on R and hence only on the domain D. Recall from the proof of (3.16) that $F_{\psi_j}(x,R)$ has the form $h(x,R) - (1/w(B))\int_B h(y,R)w\,dy$. Hence

$$|F_{\psi_j}(x,R)| \leqslant \sup_{t \in B} |\nabla h(t,R)|\,|B|,$$

which is at most $(w(B)/R^{n-1}) \cdot |B| \leqslant w(B)$. Therefore, the second integral in (4.5) hast the bound

$$C_D \int_0^r \int_{Q_r} \frac{w_j(B(x,t))}{w(B(x,r))} \frac{w(B(x,r))}{t^{n-1}} \, dx \, dt$$

$$\begin{aligned} &\leqslant C_D \int_0^r \int_{Q_r} (t/r)^{n-2+\alpha} t^{-n+1} \, dx \, dt \cdot w(Q_r) \\ &\leqslant C_D w(Q_r) r^{-n+2} \cdot |W_r| \\ &\leqslant C_D r w(Q_r) \\ &\leqslant C_D r^{\alpha} |Q_r| \\ &\leqslant R^{\alpha} C_D |Q_r|. \end{aligned}$$

The pointwise gradient estimate holds for $\theta_j(x, t + \Phi_j(x)) \nabla F_{\psi_j}(x, t)$ and when the differentiation falls on θ_j, we have

$$\begin{aligned} |\nabla \theta_j(x, t + \Phi_j(x)) \cdot F_{\psi_j}(x, t)| &\leqslant |F_{\psi_j}(x, t) - F_{\psi_j}(x, R)| + |F_{\psi_j}(x, R)| \\ &\leqslant \int_{s=t}^{R} |\nabla F_{\psi_j}(x, s)| \, ds + C_R \\ &\leqslant \int_{s=t}^{R} \frac{s^{n-2}}{w(B(x,s)} \frac{w(B(x,t))}{t^{n-2}} \, ds \frac{t^{n-2}}{w(B(x,t))} + C_R \\ &\leqslant \frac{t^{n-2}}{w(B(x,t))} \cdot \left\{ \int_{s=t}^{R} \frac{s^{n-2}}{t^{n-2}} \left(\frac{t}{s}\right)^{n-2+\alpha} ds + C'_R \right\} \\ &\leqslant \frac{C t^{n-2}}{w(B(x,t))} \end{aligned}$$

This shows all the bounds for the extension. We now show that F is an extension in the required sense.

Consider

$$\begin{aligned} \int_{\partial D} u(Q) f(Q) \, d\omega &= \sum_j \int_{\partial D} u \psi_j \theta_j f \, d\omega \\ &= \sum_j \int u(x, \Phi_j(x)) \theta_j(x, \Phi_j(x)) [f \psi_j(x) - g_j(x) + g_j(x)] w_j(x) \, dx. \end{aligned}$$

Let $h_j(x) = f_{\psi_j}(x) - g_j(x)$. Then $f_0 = \sum_j g_j(x, \Phi_j(x))$ satisfies $f_0 k \in L^\infty(d\sigma)$. We look at a single term

$$\int u(x, \Phi_j(x)) \theta_j(x, \Phi_j(x)) h_j(x) w_j(x) \, dx.$$

Let F_{ψ_j} be the extension of h_j. Then the above is equal to

$$\lim_{s\to 0}\int u(x,\Phi_j(x)+s)\theta_j(x,\Phi_j(x)+s)F_{\psi_j}(x,s)w_j(x)\,dx$$

$$= \lim_{s\to 0}\int_{\partial D} u(x,\Phi_j(x)+s))F_j(x,\Phi_j(x)+s)\,d\omega.$$

Now for fixed $s>0$, $F_j(x,\Phi_j(x)+s)$ can be approximated by C^2 functions and since s is fixed, $F_j\in \mathcal{L}(\bar{D})$. Then Green's theorem can be applied and the formal reasoning of the argument following the statement of Theorem 2.7 is justified. Thus we have established

Theorem 2.7. *If $u\in\mathcal{L}(\bar{D})$, $\Delta u=0$ and $u(P_0)=0$, then*

$$\left|\int_{\partial D} u(Q)f(Q)\,dw\right| \leqslant C\|Nu\|_{L^1(d\sigma)}\|f\|_{\mathrm{BMO}}.$$

Lemma 4.6. *Let $\mathcal{L}(\partial D)$ denote the functions which are Lipschitz on ∂D. $\mathcal{L}(\partial D)$ is dense in* $\mathrm{VMO}_\sigma(w)$.

PROOF. Clearly $\mathcal{L}(\partial D)$ is contained in $\mathrm{VMO}_\sigma(w)$. By Lemma 4.2, if $\psi\in \mathrm{VMO}_\sigma(w)$ then $\theta\psi\in\mathrm{VMO}_\sigma(w)$, where $\theta\in C_0^\infty$ and supported on B where

$$B\cap\partial D=\{(x,\Phi(x)):\Phi \text{ Lipschitz on } \mathbb{R}^{n-1}\}.$$

Then if $\bar{\psi}(x)=(\theta\psi)(x,\Phi(x))$ we need only show that $\bar{\psi}\in\mathrm{VMO}(w\,dx)$ (see section 3) to establish the density. Write $\bar{\psi}=(\bar{\psi}-\bar{\psi}_t)+\bar{\psi}_t$. Clearly $\bar{\psi}_t$ is in Lip_0. Thus we only need to see that $\bar{\psi}-\bar{\psi}_t$ has small BMO norm when t is sufficiently small. We follow an argument in Garrett, p. 272. Let

$$M_\delta(\bar{\psi})=\sup_{l(J)<\delta}\frac{1}{|J|}\int_J|\bar{\psi}-\bar{\psi}_J|w\,dx.$$

We know that $M_\delta(\bar{\psi})\to 0$ as $\delta\to 0$. Fix a cube $Q\subseteq\mathbb{R}^{n-1}$ and a δ so that $M_\delta(\bar{\psi})<\epsilon$.

If $l(Q)>\delta$, express $Q=\bigcup Q_j$ with $l(Q_j)<\delta$. Let $h=\sum_j\bar{\psi}_{Q_j}\chi_{Q_j}(x)$. We estimate the BMO norms of $(\bar{\psi}-h)$, $(\bar{\psi}_t-h_t)$ and h_t-h. First,

$$\int_Q|\bar{\psi}-h|w\,dx\leqslant\sum_j\int_{Q_j}|\bar{\psi}-\bar{\psi}_{Q_j}|w\,dx\leqslant M_\delta(\bar{\psi})\sum_j|Q_j|<\epsilon|Q|.$$

For h_t-h we have

$$(h_t-h)(x)=\int K(x,y,t)[h(y)-h(x)]w(y)\,dy$$

and if $t < l(Q_j)$, $|h(y) - h(x)| \lesssim |\bar{\psi}_{Q_j} - \bar{\psi}_{Q'_j}|$ where Q'_j is adjacent to Q_j. By applying the BMO condition to $Q_j \cup Q'_j$ one sees that

$$|\bar{\psi}_{Q_j} - \bar{\psi}_{Q'_j}| \leqslant CM_\delta(\bar{\psi})|Q_j|/w(Q_j).$$

Hence

$$\sum_J \int_{Q_j} |h_t - h| w\,dx \leqslant CM_\delta(\bar{\psi}) \sum_J \frac{|Q_j|}{w(Q_j)} \cdot w(Q_j) \subset \epsilon|Q|.$$

Similarly, one can show that

$$\int_Q |\bar{\psi}_t - h_t| w\,dx \leqslant \epsilon|Q|. \quad \square$$

Lemma 4.7. $H'_{at}(\partial D, d\sigma) = \mathrm{VMO}^*_\sigma(w)$.

PROOF. The proof in Coifman-Weiss [5] can be modified to work in our situation, once we know Lemma 4.6.

Lemma 4.8. $K(x, Q) = d\omega^x/d\omega$ *belongs to* $\mathrm{VMO}_\sigma(\omega))$.

PROOF. Fix $x \in D$. If A is any atom,

$$|A(x)| = \left| \int A(Q)K(x, Q)\,dw(Q) \right| \leqslant \int_{\partial D} NA(Q)\,d\sigma(Q) \leqslant C,$$

which means $K(x, \cdot)$ belongs to $\mathrm{BMO}_\sigma(\omega)$. Then if $\Delta = \Delta(Q_0, r_0)$ is so small that dist $(x, Q_0) > 2r_0$, the pointwise estimate (2.5) for the harmonic extension of atoms, i.e.,

$$|A(x)| \leqslant r_0^\beta |x - Q_0|^{1-n-\beta}$$

implies that $K(x, \cdot)$ belongs to $\mathrm{VMO}_\sigma(\omega)$. $\square$

We now assume that D is starlike with respect to the origin. At this point we define

$$H^1(\partial D, d\sigma) = \left\{ f = \lim_{r \to 1} u(rQ), u \in H^1(D, d\sigma) \right\}$$

where the limit is taken in $\mathrm{VMO}^*_\sigma(\omega)$. The arguments to follow will show that this limit exists and there is uniqueness in the sense that $f = 0$ implies $u \equiv 0$.

Lemma 4.9. *Let* $f \in H^1_{at}(\partial D, d\sigma)$. *Set* $u(x) = \int f(Q)K(x, Q)\,d\omega(Q)$. *Then* $u \in H^1(D, d\sigma)$ *and* $\lim_{r \to 1} u(rQ) = f(Q)$ *in* $\mathrm{VMO}^*_\sigma(\omega)$.

PROOF. First, $u(x)$ is well-defined since $K(\bullet, Q) \in VMO_\sigma(\omega)$, by Lemma 4.8. Let

$$f = \sum_j \lambda_j a_j$$

where the a_j are atoms, and $\sum |\lambda_j| < \infty$. Then

$$Nu(Q) \leqslant \sum_j |\lambda_j| Na_j(Q),$$

so $u \in H^1(D, d\sigma)$.

Given $\epsilon > 0$, let N be large enough so that

$$f = \sum_{j=1}^{N} \lambda_j a_j + R_N$$

where $\|N(R_N)\|_{L^1(d\sigma)} < \epsilon$. Then

$$u(rQ) = \sum_{j=1}^{N} \lambda_j a_j(rQ) + R_N(rQ).$$

If $g \in \mathrm{VMO}_\sigma(\omega)$, then by Theorem 2.7,

$$\left| \int \left[u(rQ) - \sum_{j=1}^{N} \lambda_j a_j(rQ) \right] g(Q)\, d\omega \right| \leqslant \|N(R_N)\|_{L^1(d\sigma)} \|g\|_{\mathrm{BMO}} \leqslant \epsilon.$$

Moreover, by Theorem 2.7,

$$\lim_{r \to 1} \int \sum_j \lambda_j a_j(rQ) g(Q)\, d\omega = \int \sum_{j=1}^{N} \lambda_j a_j(q) g(Q)\, d\omega.$$

Thus $\lim_{r \to 1} u(rQ) = f(Q)$ in $\mathrm{VMO}_\sigma^*(w)$. □

At this point we can give the proof of

Theorem 2.6. *$H^1(\partial D, d\sigma) = H^1_{at}(\partial D, d\sigma)$, with comparable norms.*

PROOF. Lemma 4.9 implies that $H^1_{at}(\partial D, d\sigma)$ is continuously imbedded in $H^1(D, d\sigma)$. If $u \in H^1(D, d\sigma)$, then Theorem 2.7 shows that $\{u(rQ)\}$ is bounded in $\mathrm{VMO}_\sigma^*(\omega)$ and so there exists a subsequence $\{r_{j_k}\}$, $r_{j_k} \to 1$ as $k \to \infty$, such that $u(r_{j_k}) \to f \in \mathrm{VMO}_\sigma^*(\omega)$ as $k \to \infty$. This limiting distribution is unique, for by Theorem 2.7, if $r_j \to 1$ and $r_l \to 1$,

$$\left| \int [u(r_j Q) - u(r_l Q)] g(Q) \right| d\omega \leqslant C \|N(u(r_j, \bullet) - u(r_l, \bullet))\|_{L^1} \|g\|_{\mathrm{BMO}}$$

and the above tends to zero as $r_j, r_l \to 1$. By uniqueness and Lemma 4.9 this shows that every $u \in H^1(D, d\sigma)$ can be written as

$$u(x) = \int f(Q)K(x, Q)\, dw(Q)$$

where $f(Q) = \lim_{r\to 1} u(rQ)$ belongs to $H^1_{at}(\partial D, d\sigma)$. $\square$

We now consider the case of a general Lipschitz domain D, not assumed to be starlike. In this situation we have the following result.

Theorem 4.10. *If $\Delta u = 0$ and $Nu(Q) \in L^1(\partial D, d\sigma)$, there exists a sequence of constants $\{\lambda_j\}$ and atoms $\{a_j\}$ such that*

$$u(x) = \sum_j \lambda_j \int a_j(Q)K(x, Q)\, d\omega(Q)$$

for $x \in D$; with $\|N(u)\|_{L^1(d\sigma)} \approx \sum_j |\lambda_j|$. Moreover, the above sum of atoms determines $u(x)$ uniquely.

Let us fix our domain D and a u in $H^1(D, d\sigma)$, and set up some notation. There exists a finite collection of starlike Lipschitz domains $D_i \subseteq D$ such that $\bigcup D_i$ covers a neighborhood of ∂D in D, with $N_{D_i}(u) \in L^1(\partial D_i\, d\sigma_i)$ where $N_{D_i} u$ is the nontangential maximal of u relative to the domain D_i. Let $u_i = u|_{\partial D_i}$, so that u_i belongs to $H^1(D_i, d\sigma_i)$. The domains D_i will have the additonial property that there exists subdomains $\tilde{D}_i$ of D_i, with the same starcenter as D_i and with $\bigcup \tilde{D}_i$ covering a neighborhood of ∂D within D, such that $2\tilde{\Delta}_i = \Delta_i$, where $\Delta_i = \partial D_i \cap \partial D$ and $\tilde{\Delta}_i = \partial \tilde{D}_i \cap \partial D$. Fix the pole of the Green's function for D at $P \in D$ and let dw denote $d\omega_D^P$. Define $\mathrm{VMO}_\sigma(w)$ as the closure of $\mathcal{L}(\partial D)$ under the norm

$$\int_{\partial D} |\psi|\, d\omega + \sup_{\Delta \subseteq \partial D}\ \inf_{\psi_\Delta \text{contant}} \left\{ \int_\Delta |\psi - \psi_\Delta|\, d\omega/\sigma(\Delta) \right\}.$$

Its dual $\mathrm{VMO}_\sigma^*(d\omega)$ can be identified with $H^1_{at}(\partial D, d\sigma)$. Let $d\omega_i$ be harmonic measure for D_i evaluated at the starcenter.

Lemma 4.11. *There exists distributions f_i in $\mathrm{VMO}_\sigma^*(\partial D_i, d\omega_i)$ with the property that if $\psi \in \mathrm{VMO}_\sigma(\partial D, d\omega)$ and is supported in a compact subset of Δ_i, then (if we call the origin the starcenter of D_i),*

$$\langle f_i, \psi \rangle_{(\partial D_i, d\omega_i)} = \lim_{r\to 1} \int_{\partial D_i} u(rQ)\psi(Q)\, d\omega_i. \tag{4.12}$$

Moreover, if the above limit is zero for all such $\psi \in \mathrm{VMO}_\sigma(\partial D, d\omega)$, and all i, then $u \equiv 0$.

PROOF. If $K \subset\subset \Delta_i$, then $d\omega_i/d\omega$ and $d\omega/d\omega_i$ are bounded on K. Hence if ψ is supported on K, its $\mathrm{VMO}_{\sigma_i}(\partial D_i, d\omega_i)$ and $\mathrm{VMO}_\sigma(\partial D, d\omega)$ norms are comparable, with constants depending only on K. Since $u_i \in H^1(D_i, d\sigma_i)$, Theorem 2.6 for starlike domains gives distributions $f_i \in \mathrm{VMO}_\sigma(\partial D_i, d\omega_i)$ satisfying (4.12). To establish the uniqueness of the $\{f_i\}$, we use an argument of Dahlberg-Kenig [10]. If the expressións in (4.12) are zero, we will see that $N_{D_i}(u_i)$ belongs to $L^2(\partial\tilde{\Delta}_i, d\sigma)$ and that the nontangential limit of u_i is zero on $\tilde{\Delta}_i$. By the L^2 uniqueness in the Dirichlet problem (Dahlberg [9]) this would imply that u is identically zero.

With this in mind, let $\psi \in \mathrm{Lip}(\partial D_i)$ be supported in a compact subset of $\partial D_i \setminus \Delta_i$. Let 0 be the starcenter of D_i and assume that $\psi_i(0) = u(0)$, where $\psi_i(x) = \int_{\partial D_i} \psi(Q)\, d\omega_i^x(Q)$ is the harmonic extension of ψ to D_i. Set $v_i = u_i - \psi_i$. Then v_i belongs to $H^1(\partial D_i\, d\sigma_i)$, $v_i(0) = 0$ and its boundary distribution g_i is supported in $\partial D_i \setminus \Delta_i$. Then v_i has an atomic decomposition, $v_i = \sum_j \lambda_j a_j^i$, where we may assume that each a_j^i has support in $\partial D_i \setminus K_i$ where $\tilde{\Delta}_i \subset K_i \subset\subset \Delta_i$. By the pointwise estimate (2.5) on atoms, $N_{D_i}(v_i) \in L^2(\tilde{\Delta}_i)$. Let h_i be the nontangential limit of v_i on $\tilde{\Delta}_i$, an L^2 function on a neighborhood of $\tilde{\Delta}_i$. Then for all $\theta \in \mathrm{Lip}(\tilde{\Delta}_i)$, by dominated convergence we have

$$\int_{\tilde{\Delta}_i} h_i \theta\, d\omega_i = \lim_{r\to 1} \int_{\tilde{\Delta}_i} v_i(rQ)\theta(Q)\, d\omega_i = \lim_{r\to 1} \int_{\partial D_i} v_i(rQ)\theta(Q)\, d\omega_i = 0.$$

Hence h_i is zero almost everywhere $\tilde{\Delta}_i$, but therefore u_i has zero nontangential limit on Δ_i. □

Lemma 4.13. $L^2(\partial D, d\sigma) \subset H^1_{at}(\partial D, d\sigma)$.

PROOF. Let $\psi \in \mathrm{BMO}_\sigma(d\omega)$. Then $\int_{\partial D} f\psi\, d\omega = \int_{\partial D} f\psi k\, d\sigma$, but $\psi k \in L^2(d\sigma)$ since

$$\sup_\Delta \left\{ \int_\Delta |\psi - \psi_\Delta|^2 k^2\, d\sigma/\sigma(\Delta) \right\}^{1/2} \quad \text{and} \quad \sup_\Delta \left\{ \int_\Delta |\psi - \psi_\Delta|\, d\omega/\sigma(\Delta) \right\}^{1/2}$$

define equivalent $\mathrm{BMO}_\sigma(\omega)$ norms.

Lemma 4.14. *Let $f \in H^1_{at}(\partial D, d\sigma)$ with $f = \sum_j \lambda_j a_j$. Suppose there exists a $g \in L^2(\partial D, d\sigma)$, with $g \equiv f$ on $\Delta \subseteq \partial D$ in the sense that*

$$\int_{\partial D} f\psi\, d\omega = \int_{\partial D} g\psi\, d\omega$$

for all $\psi \in \mathrm{VMO}_\sigma(\omega)$ with $\operatorname{supp}\psi \subseteq \Delta$. Then for any $\Delta' \subset\subset \Delta$, if

$$u = \sum_j \lambda_j \int a_j(Q) k(x, Q)\, d\omega,$$

$N(u)$ belongs to $L^2(\Delta', d\sigma)$.

PROOF. We can assume that there exists a starlike Ω with $\partial\Omega \cap \partial D = \Delta$ by taking Δ small, and that $u|_\Omega \in H^1(\Omega, d\sigma)$. Set $v(x) = \int_\Delta g(Q)k(x, Q)\, d\omega$. The proof of Lemma 4.11 on the domain Ω shows that $N(u - v) \in L^2(\Delta', d\sigma)$. $\square$

Lemma 4.15. *Let a be an atom on ∂D_i with respect to $d\omega_i$ supported in $K \subset\subset \Delta_i$. Then there exists a constant $C = C(K)$ such that*

$$\|a\|_{H^1_{at}(\partial D, d\sigma)} \leqslant C.$$

PROOF. Clearly,

$$\|a\|_{H^1_{at}(\partial D, d\sigma)} \leqslant C(K) \sup \left\{ \int_{\partial D} a\psi\, d\omega : \psi \in \mathrm{VMO}_\sigma(\omega), \right.$$
$$\left. \|c\|_{\mathrm{BMO}} \leqslant 1, \text{ and } \operatorname{supp}\psi \subseteq K \right\}.$$

Fix such a ψ in $\mathrm{VMO}_\sigma(\omega)$. By the construction in Lemma 4.4, ψ has an extension F such that $|\nabla F| G(x)/d(x)$ is a Carleson measure and $F \equiv 0$ outside D_i. We can also assume that $F = 0$ at the poles of both G and G_i (the Green's function for D_i). Let $u_i(x) = \int a(Q)\, d\omega_i^x$ be the harmonic extension of a to D_i. Let θ be a C_1^∞ function with $\theta \equiv 1$ on $\operatorname{supp} F$ and $\operatorname{supp}\theta \subseteq \overline{D_i \cap D}$. Set $v_i = \theta u_i$. Then, by the argument used in the proof of Lemma 4.4 the following formal calculation can be justified.

$$\begin{aligned}
\left| \int_{\partial D} a\psi\, d\omega \right| &= \left| \int_D \Delta(v_i F) G(x)\, dx \right| \\
&= \left| \int_D (\Delta v_i) F G\, dx + \int_D v_i \Delta(F) G\, dx + 2\int_D G \nabla v_i \cdot \nabla F\, dx \right| \\
&\quad \left| \int_D G \nabla v_i \cdot \nabla F\, dx - \int_D v_i \nabla F \cdot \nabla G\, dx \right| \\
&\leqslant C\left(\int_{D_i} G_i |\nabla F|\, |\nabla u_i|\, dx + \int_{D_i} |\nabla F| (G_i/d_i) |u_i|\, dx \right)
\end{aligned}$$

which is bounded by $\|Na\|_{L^1(\partial D_i, d\sigma)}$, as before. $\square$

PROOF OF THEOREM 4.10. Let $\{f_i\}$ be the distributions obtained in Lemma 4.11 for $u \in H^1(D, d\sigma)$. Fix i and let $\eta_i \in C_0^\infty$ satisfy $\eta_i \equiv 1$ on a neighborhood of $\tilde{\Delta}_i$ and $\operatorname{supp}\eta_i \subset\subset \Delta_i$. Set $g_i = f_i \eta_i$. Let $\psi_i \in \operatorname{Lip}(\partial D_i)$ be compactly supported in Δ_i such that

$$g_i(0) = \int_{\partial\Omega_i} \psi_i(Q)\, d\omega_i^0.$$

By Lemma 4.15, g_i has an atomic decomposition on $(D, d\omega)$,

$$\|g_i\|_{H^1_{at}(\partial D, d\sigma)} \leqslant C \quad \text{and} \quad g_i = \psi_i + \sum_j \lambda_j^i a_j^i$$

where the a_j^i are atoms on D compactly supported in a neighborhood of Δ_i.

Now consider

$$v_i(x) = u(x) - \int_{\partial D} g_i(Q)K(x, Q)\, d\omega.$$

By Lemma 4.14, $N(v_i) \in L^2(K_i)$, $\tilde{\Delta}_i \subset K_i \subset \Delta_i$. On D_l, $l \neq i$, $w_{i,l} = v_{i,l} = v_i|_{D_l}$ belongs to $H^1(\partial D_l, d\sigma)$. Let $f_{i,l}$ denote the boundary value distribution of v_i on ∂D_l. As before, multiply $f_{i,l}$ by a cut off function η_l and set $g_{i,l} = \eta_l f_{i,l}$. Again, $\|g_{i,l}\|_{H^1(\partial D, dw)} \leqslant C$ and

$$g_{i,l} = \psi_l + \sum_k \lambda_k^l a_k^l$$

with a_k^l, and atom on $(\partial D, d\omega)$, compactly supported in a neighborhood of $\hat{\Delta}_l$. Set

$$u_{i,l}(x) = v_i(x) - \int_{\delta D} g_{i,l}(Q)K(x, Q)\, d\omega.$$

We have $N(v_{i,l}) \in L^2(\tilde{\Delta}_l)$, but we claim that $N(v_{i,l})$ is in L^2 on a neighborhood of $\tilde{\Delta}_i \cup \tilde{\Delta}_l$. By the pointwise estimate (2.5) on atoms, the harmonic extension of $g_{i,l}$ is in L^2 away from $\tilde{\Delta}_l$, and v_i is in $L^2(\tilde{\Delta}_i)$. It remains to consider the behavior of $N(v_{i,l})$ on the intersection of a neighborhood of $\tilde{\Delta}_i$ with a neighborhood of $\tilde{\Delta}_l$. At such a point, however, the boundary values of v_i and $g_{i,l}$ are in L^2 and so in this case, the claim follows by Lemma 4.14. Proceeding in this manner, we find $v_1(x), \ldots, v_N(x)$ such that $u(x) - \sum_i v_i(x)$ has non-tangential maximal function in $L^2(\partial D)$, $v_i(x) = \int_{\partial D} g_i(Q)K(x)Q)\, d\omega$ and with each $g_i \in H^1_{at}(\partial D, d\sigma)$. By Lemma 4.13 this proves the theorem. □

5. The results for the $H^p(D, d\sigma)$ spaces, $1 < p < 2$. In this section we discuss the results for the $H^p(D, d\sigma)$ spaces, $1 < p < 2$. As before, we introduce a related space on $\mathbb{R}^{n-1}(dx)$ and obtain our results in this setting first. In what follows we shall use the notation of section 3. Our weight $w(x)$ satisfies conditions 3.1 (i)-(iii). Note the difference in the normalization in our definition of atoms below.

Definitions.

(1) *An atom $a(x)$ on $\mathbb{R}^{n-1}$ has support in a cube Q and satisfies $\|a\|_\infty \leqslant 1$ and*

$$\int_p a(x)w(x)\, dx = 0.$$

(2) $L^{\#,q}_{dx}(w\, dx) = \{g \in L^1_{\text{loc}}(w\, dx) : M^{\#}_w(g) \in L^q(dx)\}$ *where* $q > 1$ *and*

$$M^{\#}_w g(x) = \sup_{Q \ni x} \left\{ \frac{1}{|Q|} \int_Q |g - g_Q| w(x)\, dx \right\}.$$

(3) $\mathcal{L}^{\#,q}_{dx}(w\,dx)$ = *closure of the* Lip_0 *functions in* $L^{\#,q}_{dx}(w\,dx)$.

(*We observe that if* $\psi \in \mathrm{Lip}_0$, $M^{\#}_w\psi(x) \leqslant C$ *and if* x *is far from the support of* ψ, $M^{\#}_w\psi(x) \leqslant C/|x|^{n-1}$ *so that* Lip_0 *is contained in* $L^{\#,q}_{dx}(w\,dx)$).

(4) $H^p(\mathbb{R}^{n-1}, w\,dx) = \{f \in (\mathcal{L}^{\#,p'}_{dx}, w\,dx))^*: Nf \in L^p(dx)\}$ *where* $(\ )^*$ *denotes the dual space and* $1/p + 1/p' = 1$.

(5) $H^p_{at}(w\,dx) = \{f: f = \sum \lambda_k a_k$ *where the* a_k *are atoms supported in balls* B_k, $\|\sum \lambda_k \chi_{B_k}\|_{L^p(dx)} < \infty$ *and the convergence takes place in* $(\mathcal{L}^{\#,p'}_{dx}(w\,dx))^*\}$.

Lemma 5.1. *A «distribution» f belongs to $H^p(w\,dx)$ if and only if either f^+ or f^* belongs to $L^p(dx)$ and all maximal functions have comparable norms.*

PROOF. To argue as before one needs only check that if $\psi \in \mathfrak{A}$ then the pairing $\langle f, \psi\rangle$ is well-defined. To check this one must first see that $M^{\#}_w(\psi) \in L^{p'}(dx)$ (for $p' > 2$) and that $\theta_j\psi \to \psi$ in $L^{\#,p'}_{dx}(w\,dx)$ where θ_j is smooth bump function supported on $\{|x| \leqslant 2^j\}$. □

Lemma 5.2. *The functions in $L^2(dx) \cap H^p(w\,dx)$ are dense in $H^p(w\,dx)$.*

The proof of Lemma 5.2 proceeds exactly as the proof of Lemma 3.11 once we have the following facts. (See Lemmas 3.3-3.6 for the notation appearing below).

Proposition 5.2.1. *If $\psi \in \mathcal{L}^{\#,q}_{dx}(w\,dx)$ then so is $\sum_{j=1}^{\infty} S_j(\psi)$ and*

$$\left\| \sum_{j=1}^{\infty} S_j(\psi) \right\|_{L^{\#,q}_{dx}(w\,dx)} \lesssim \|\psi\|_{L^{\#,q}_{dx}(w\,dx)}.$$

PROOF. Our proof of Lemma 3.3 shows that in fact

$$M^{\#}_w\left(\sum_{j=1}^{\infty} S_j(\psi)\right) \leqslant CM(M^{\#}_w\psi)$$

where m is the Hardy-Littlewood maximal function. □

Proposition 5.2.2. *If $\psi \in \mathcal{L}^{\#,q}_{dx}(w\,dx)$ and $\psi_t(x) = \int \psi(y)K(x,y,t)w(y)\,dy$, then $\psi_t \to \psi$ in $L^{\#,q}_{dx}(w\,dx)$ as $t \to 0$.*

PROOF. If $\psi \in L^{\#,q}_{dx}(w\,dx)$, then we claim that

$$\|\psi_t\|_{L^{\#,q'}_{dx}(w\,dx)} \lesssim \|\psi\|_{L^{\#,q}_{dx}(w\,dx)}.$$

The proof of Lemma 3.5 shows that $M_w^{\#}(\psi_t)(x_0) \lesssim CM_w^{\#}\psi(x_0)$ so this is immediate. Then if $\psi \in \mathrm{Lip}_0$, we need $\psi_t \in \mathfrak{L}_{dx}^{\#,q}(w\,dx)$ and $\|M_w^{\#}(\psi_t - \psi)\|_{L^q(dx)} \to 0$ as $t \to 0$. But again, in the proof of Lemma 3.5 we found that

$$\|\psi_t - \psi\|_{\mathrm{Lip}(\beta)} \lesssim t^{1-\beta}, \quad \text{for} \quad 1 > \beta > 0.$$

Hence, if $t < 1$, and x_Q is the center of Q,

$$\begin{aligned} M_w^{\#}(\psi_t - \psi)(x_0) &\leqslant \sup_{Q \ni x_0} \frac{1}{|Q|} \int_Q |(\psi_t - \psi)(x) - (\psi_t - \psi)(x_Q)|\, w\,dx \\ &\leqslant \sup_{Q \ni x_0} \frac{1}{|Q|} \int_Q t^{1-\beta} |x - x_Q|^{\beta} w\,dx \\ &\lesssim t^{1-\alpha} \quad \text{if} \quad l(Q) < t < 1. \end{aligned}$$

If $l(Q) > t$

$$\begin{aligned} \frac{1}{|Q|} \int_Q |(\psi_t - \psi)(x)|\, w\,dx &\leqslant \frac{1}{|Q|} \int_Q \int |\psi(y) - \psi(x)| K(x, y, t) w(y)\,dy\, w(x)\,dx \\ &\leqslant t \leqslant t^{1-\alpha} \end{aligned}$$

The estimate $M_w^{\#}(\psi_t - \psi)(x) \leqslant t^{1-\alpha}$ can be used for $x_0 \in \operatorname{supp}(\psi_t - \psi)$. If $(\psi_t - \psi)$ has support, say, in $B(0, 1)$ and $|x_0| > 2$,

$$\begin{aligned} \sup_{Q \ni x_0} \frac{1}{|Q|} \int_Q |(\psi_t - \psi)(x)|\, w(x)\,dx &\leqslant \frac{1}{B(0, |x_0|)} \int_{B(0,1)} \|\psi\|_{\mathrm{Lip}_0} t \\ &\quad \cdot \int K(x, y, t) w(y)\,dy\, w(x)\,dx \\ &\leqslant \frac{ct}{|x_0|^{n-1}} \end{aligned}$$

and then $\|M_w^{\#}(\psi_t - \psi)\|_{L^q({}^cB(0,1), dx)} \lesssim t$. □

Theorem 5.3. *If $f \in H^p(w\,dx)$, then there exists a sequence of positive constants $\{\lambda_k\}$ and a sequence of atoms a_k, supported in balls B_k such that the sum $\sum_k \lambda_k a_k$ converges to f in $(\mathfrak{L}_{dx}^{\#,p'}(w\,dx))^*$ and in $H^p(w\,dx)$ norm, with*

$$\Big\|\sum_k \lambda_k \chi_{B_k}\Big\|_{L^p(dx)} \leqslant \|f\|_{H^p(w\,dx)}.$$

Moreover, if $\{a_k\}$ is a sequence of atoms such that $\Big\|\sum_k \lambda_k \chi_{B_k}\Big\|_{L^p(dx)} < \infty$, then

$$\sum_k \lambda_k a_k \in H^p(w\,dx)$$

and

$$\left\| \sum_k \lambda_k a_k \right\|_{H^p(w\,dx)} \leqslant C \left\| \sum_k \lambda_k \chi_{B_k} \right\|_{L^p(dx)}.$$

PROOF. Both the statement of the teorem and the ideas in its proof follow Stromberg-Torchinsky [24]. Again, the decomposition of an $f \in H^p(w\,dx)$ is fairly standard, using the ideas of Later [21], and we refer to Stromberg-Torchinsky [24] for the proof in this case. We turn to the proof of the second half of the theorem.

Let

$$f(x) = \sum_{k=1}^{N} \lambda_k a_k$$

be a finite linear combination of atoms. Consider

$$\begin{aligned} Na_k(x) &= \sup_t |(a_k)_\varphi(x, t)| \\ &= \sup_t \left| \int a_k(y)\varphi(x - y/t)w(y)\,dy \cdot \left\{ \int \varphi(x - y'/t)w(y')\,dy' \right\}^{-1} \right| \end{aligned}$$

when $x \in B_k = \operatorname{supp} a_k$,

$$\sup_t |(a_k)_\varphi(x, t)| \leqslant C$$

when $x \in 2^j B_k \setminus 2^{j-1} B_k$, $j > 1$, and r_k is the radius of B_k,

$$\begin{aligned} |(a_k)_\varphi(x, t)| &\leqslant \left| \int a_k(y) \left[\varphi\left(\frac{x-y}{t}\right) - \varphi\left(\frac{x-y}{t}\right) \right] \frac{w(y)\,dy}{w(B(x, t)} \right| \\ &\leqslant \frac{C}{w(B(x, t))} \int_{B_k} \frac{|y - y_k|}{t} w(y)\,dy \\ &\leqslant \frac{r_k}{t} \frac{w(B_k)}{w(B(x, t)}. \end{aligned}$$

But in order that $(a_k)_\varphi(x, t)$ be nonzero when $x \in 2^j B_k \setminus 2^{j-1} B_k$, we must have $t > c2^j r_k$, hence the above is bounded by

$$c \frac{r_k}{2^j r_k} \frac{w(B_k)}{w(2^j B_k)} \leqslant C \left(\frac{r_k}{2^j r_k} \right)^{n-1+\alpha} = c2^{-j(n-1+\alpha)}.$$

Let $a_{k,0} = \chi_{B_k}(x)$ and $a_{k,j}(x) = \chi_{2^j B_k}(x)$. Then we have shown that

$$Na_k(x) \leqslant ca_{k,0}(x) + \sum_{j=1}^{\infty} 2^{-j(n-1+\alpha)} a_{k,j}(x).$$

Therefore

$$\left\| N\left(\sum_{k=1}^{N} \lambda_k a_k\right)\right\|_{L^p(dx)} \leqslant \left\|\sum_{k=1}^{N} \lambda_k a_{k,0}\right\|_{L^p(dx)} + \sum_{j=1}^{\infty} 2^{-j(n-1+\alpha)}\left\|\sum_{k=1}^{N} \lambda_k a_{k,j}\right\|_{L^p(dx)}$$

$$\leqslant \left\|\sum_{k=1}^{N} \lambda_k \chi_{B_k}\right\|_{L^p(dx)}$$

$$+ \sum_{j=1}^{\infty} 2^{-j(n-1+\alpha)} 2^{j((n-1)/p)}\left\|\sum_{k=1}^{N} \lambda_k \chi_{B_k}\right\|_{L^p(dx)}$$

$$\leqslant C\left\|\sum_{k=1}^{N} \lambda_k \chi_{B_k}\right\|_{L^p(dx)},$$

where the next-to-last inequality follows by a change of variables in the integral. □

Corollary. *The dual of $H^p(w\,dx)$ is $L^{\#,p'}(w\,dx)$, with $1/p + 1/p' = 1$ and pairing $\langle f, g\rangle = \int f(x)g(x)w(x)\,dx$, for f a finite linear combination of atoms.*

PROOF. Suppose $f = \sum \lambda_k a_k$ is a sum of atoms with

$$\left\|\sum \lambda_k \chi_{B_k}\right\|_{L^p(dx)} \approx \|f\|_{H^p(w\,dx)}.$$

If $g \in L^{\#,p'}_{dx}(w\,dx)$, we have

$$\int f(x)g(x)w\,dx = \int \sum_k \lambda_k a_k g(x) w\,dx$$

$$\leqslant \sum_k \lambda_k \int_{B_k} |g - g_{B_k}| w\,dx$$

$$\leqslant \sum_k \lambda_k |B_k| \inf_{x\in B_k} M_w^{\#} g(x)$$

$$\leqslant \left\|\sum \lambda_k \chi_{B_k}\right\|_{L^p} \|M_w^{\#} g\|_{L^{p'}(dx)}.$$

If Λ is a linear functional on $H^p(w\,dx)$, one can show that Λ is given by a $g \in L^{\#,p'}_{dx}(w\,dx)$, with pairing $\langle f, g\rangle = \int fgw\,dx$, as in Coifman-Weiss [5]. It can also be shown that H^p_{at} is the dual of $\mathcal{L}^{\#,p'}_{dx}(w\,dx)$.

Lemma 5.5. *If Λ is continuous linear functional on $H^p(w\,dx)$, there exists $\{y_n\}$ with $y_n \to 0$ as $n \to +\infty$ and $y_n \to \infty$ and $n \to -\infty$ and functions $g_\infty(x)$, $\{g_n(x)\}_{n=-\infty}^{n=\infty}$ such that for all $f \in L^2 \cap L^1(dx)$,*

$$\Delta(f) = \int f(x) g_\infty(x)\,dx + \sum_{n=-\infty}^{\infty} \int u(x, y_n) g_n(x)\,dx$$

where

$$u(x, y_n) = \int f(z)\varphi\left(\frac{x-z}{y_n}\right)w(z)\,dz \cdot \left\{\int \varphi\left(\frac{x-z'}{y_n}\right)w(z')\,dz'\right\}^{-1}$$

and

$$|g_\infty(x)| + \sum_{n=-\infty}^{\infty} |g_n(x)| \in L^{p'}(dx).$$

PROOF. Again, we refer to Garnett [15] for C. Fefferman's argument in the case $p = 1$, which may be easily modified to give the above characterization when $p > 1$.

Our strategy for proving the atomic decomposition for $H^p(D, d\sigma)$ and the duality result is the same as that for the $H^1(D, d\sigma)$ situation. To carry this out we need a Varopoulos-type extension theorem for $L^{\#,p}_{dx}(w\,dx)$ functions. We shall formulate our result within the framework of the theory of tent spaces. In what follows the functions and measures are defined on $\mathbb{R}^n_+$ and $\Gamma(x)$ denotes a cone with vertex at x.

Definitions.

(1) $T^p_\infty = \{f: A_\infty(f)(x) \in L^p(dx)\}$ *where* $A_\infty f(x) = \sup_{\Gamma(x)} |f(x', y)|$.

(2) $T^p_q = \{f: A_q f(x) \in L^p(dx)\}$, $p < \infty$, $q < \infty$, *where*

$$A_q f(x) = \left\{\int_{\Gamma(x)} |f(x', y)|^q\, dx'\, dy/y^{n-1}\right\}^{1/q}.$$

(3) $\tau^p_1 = \{\mu: A_1(\mu) \in L^p\}$ *for* $p < \infty$, *where* $d\mu$ *is a measure on* $\mathbb{R}^n_+$ *and*

$$A_1(\mu) = \int_{\Gamma(x)} y^{-n+1}\, d\mu(x', y).$$

(4) $\tau^\infty_1 = \{\mu: C_1(\mu) \in L^\infty\}$ *where*

$$C_1(\mu)(x) = \sup_{Q \ni x}\left\{\frac{1}{|Q|}\int_{Q\times[0, l(Q)]} d\mu(x', y)\right\}.$$

Theorem 5.6. (Coifman-Meyer-Stein [4] and Alvarez-Milman [1]).

(i) $(T^p_\infty)^* = \tau^{p'}_1$, $\quad \dfrac{1}{p} + \dfrac{1}{p'} = 1.$

(ii) $(T^p_2)^* = T^{p'}_2$, $\quad \dfrac{1}{p} + \dfrac{1}{p'} = 1.$

Theorem 5.7. *Suppose $g \in L^{\#,p}_{dx}(w\,dx)$. Then*

$$g(x) = g_0(x) + g_1(x),$$

where $g_0 \in L^p(w\,dx)$ and $g_1(x)$ has an extension $g(x,s)$ to the upper half space $\mathbb{R}^n_+$ in the sense that $g(x,s) \to g(x)$ weakly in $L^1(w\,dx)$ with

(i) $$|\nabla g(x,s)|\,w * \varphi_s(x) \in \tau_1^p$$

and

(ii) $$|\nabla g(x,s)|s \cdot w * \varphi_s(x) \in T_2^p.$$

Proof. We argue as in Theorem 3.16, omitting those details which are merely repetitious.

By Lemma 5.5 since g determines a continuous linear functional on $H^{p'}(w\,dx)$ we have

$$\int g(x)f(x)w(x)\,dx = \int f(x)g_\infty(x)\,dx + \lim_{N\to\infty}\sum_{-N}^{N}\int u(x,y_j)g_j(x)\,dx$$

for all

$$f = \sum_{k=1}^{M}\lambda_k a_k,$$

a finite linear combination of atoms, and where

$$|g_\infty(x)| + \sum_{-\infty}^{\infty}|g_n(x)| \in L^p(dx).$$

Set $g_0(x) = g_\infty(x)/w(x)$ and get

$$h_N(z) = \sum_{j=-N}^{N}\int \varphi\left(\frac{x-z}{y_j}\right)\left\{\int \varphi\left(\frac{x-z'}{y_j}\right)w\,dz'\right\}^{-1} g_j(x)\,dx.$$

For each N, h_N belongs to $L^{\#,p}_{dx}(w\,dx)$ with norm bounded by a constant which is independent of N. For $\theta \in C^\infty$ with $\theta(t) \equiv 1$ when $0 < t < 1/2$ and $\theta(t) \equiv 0$ when $t > 1$, define

$$h_N(z,s) = \sum_{-N}^{N}\theta(s/y_j)\int \varphi\left(\frac{x-z}{y_j}\right)\left\{\int \varphi\left(\frac{y-z'}{y_j}\right)w\,dz'\right\}^{-1} g_j(x)\,dx.$$

If we assume that g has compact support then, as in our argument for (3.16), there are constants c_N such taht $\{h_N - c_N\}$ has a weak limit in $L^p(w\,dx)$, call it $h(x)$, and $g_1(x) = h_1(x) + c$ for some constant c. Then we need only prove the estimates (i) and (ii), uniformly in N, for $|\nabla h_N(z,s)|$.

Let us first check condition (i). We write

$$h_N(z, s) = \int_x \int_{y=s/2} K(x, y, z)\, d\sigma(x, y)$$

where

$$K(x, y, z) = \varphi\left(\frac{x-z}{y}\right)\left\{\int \varphi\left(\frac{x-z'}{y}\right) w(z')\, dz'\right\}^{-1},$$

$$d\sigma(x, y) = \sum_{j=-N}^{N} g_j(x)\, d\sigma_j(x, y) \quad \text{and} \quad d\sigma_j(x, y) = dx \quad \text{on} \quad y = y_j.$$

By the properties of $\{g_j\}$, a computation shows that $d\sigma \in \tau_1^p$, with

$$\|d\sigma\|_{\tau_1^p} \leqslant \|g\|_{L_{dx}^{\#,p}(w\,dx)}.$$

We must show then that $|\nabla h_N(z, s)|\, w * \varphi_s(z)$ belongs to τ_1^p whenever $d\sigma$ belongs to τ_1^p, with comparable norms. We have already (Theorem 3.16) argued for this in the case $p = \infty$. By interpolation (see Álvarez-Milman [1]) it suffices to check this in the case $p = 1$. Assume then that $d\sigma \in \tau_1^1$, i.e., that

$$\int_{\mathbb{R}^{n-1}} \int_{\Gamma(x)} y^{-n+1}\, d\sigma(x, y) = \int_0^\infty \int_{\mathbb{R}^{n-1}} d\sigma(x, y) < \infty.$$

Then, since

$$|\nabla h_N(z, s)| \leqslant \int_{y=s}^{\infty} \int_{\{|x-z|<y\}} [y w(B(x, y)]^{-1}\, d\sigma(x, y),$$

we have

$$\begin{aligned}
&\int_{s=0}^{\infty} \int_{\mathbb{R}^{n-1}} w * \varphi_s(x) |\nabla h_N(z, s)|\, dz\, ds \\
&\leqslant \int_{y=0}^{\infty} \int_{\mathbb{R}^{n-1}} \int_{s=0}^{y} \int_{\{|x-z|<y\}} w(B(x, s))/w(B(x, y)) s^{-n+1} y^{-1}\, dz\, dx\, d\sigma(x, y) \\
&\leqslant \int_{y=0}^{\infty} \int_{\mathbb{R}^{n-1}} \int_{s=0}^{\infty} (s/y)^{n-2+\alpha} y^{n-2} s^{-n+1}\, ds\, d\sigma(x, y) \\
&\leqslant C \int_{y=0}^{\infty} \int_{\mathbb{R}^{n-1}} d\sigma(x, y) < \infty
\end{aligned}$$

where the second inequality used the basic estimate 3.1 (i) on the measure $w\,dx$.

We turn now to condition (ii). We want to show that

$$|\nabla h_N(z, s)| \cdot s(w * \varphi_s(x)) \in T_2^p$$

under the condition that $d\sigma \in \tau_1^p$. Recall that

$$T_2^\infty = \left\{f\colon \sup_{Q \ni x,\, cube} \left\{\frac{1}{|Q|} \int_{Q \times [0, l(Q)]} |f(x', y)|^2\, dx'\, dy/y\right\}^{1/2} \in L^\infty\right\}$$

We have shown (3.16) that the above condition on $|\nabla h_N(z,s)|$ holds in the case $p=\infty$. By interpolation (Coifman-Meyer-Stein [4]) it suffices to show that this statement holds for $p=1$. If $d\sigma \in \tau_1^1$, $|\nabla h_N(z,s)|w*\varphi_s(z) \in \tau_1^1$ and therefore

$$\int_{\Gamma(x)} |\nabla h_N(z,s)|w*\varphi_s(z)\cdot s\,dz\,ds/s^n$$

is in $L^1(dx)$. To show that

$$\left\{\int_{\Gamma(x)} |\nabla h_N(z,s)|^2|w*\varphi_s(x)\cdot s|^2\,dz\frac{ds}{s^n}\right\}^{1/2}$$

belongs to $L^1(dx)$, we estimate

$$\sup_{\Gamma(x)} |\nabla h_N(z,s)|s\cdot w*\varphi_s(z),$$

which is less than

$$\begin{aligned}\sup_{\Gamma(x)} s^{-n+1}w(B(z,s)) &\int_{y=s}^{\infty}\int_{\{|x-z|<y\}} \{yw(B(x,y))\}^{-1}\,d\varphi(x,y)\\ &\leqslant \sup_{\Gamma(x)} s^{-n+2}\int_{y=s}^{\infty}\int_{\{|x-z|<y\}} y^{-1}(s/y)^{n-2+\alpha}\,d\sigma(x,y)\\ &\leqslant \sup_{\Gamma(x)}\int_0^{\infty}\int_{\{|x-z|<y\}} y^{-n+1}\,d\sigma(x,y)\end{aligned}$$

which belongs to $L^1(dx)$ since $d\sigma\in\tau_1^1$. □

Having obtained the main result for $H^p(w\,dx)$, we now give the description of $H^p(D,d\sigma)$ and duality with

$$L_\sigma^{\#,p'}(\partial D,d\omega) = \left\{g\in L^1(d\omega)\colon \sup_{\Delta\ni Q}\left\{1/\sigma(\Delta)\int_\Delta |g-g_\Delta|\,d\omega\right\} + \int_{\partial D}|g|\,d\omega \in L^{p'}(\partial D,d\sigma)\right\}.$$

We will use the same localization procedure and notation as in the beginning of section 4.

Lemma 5.8. *Let* $\theta\in C_0^\infty(\mathbb{R}^n)$ *with* $|\nabla\theta|\leqslant c$. *Then if* $g\in L_\sigma^{\#,p'}(\partial D,d\omega)$, $g\theta\in L_\sigma^{\#,p'}(\partial D,d\omega)$ *also.*

PROOF. The proof of Lemma 4.2 shows that $M_w^{\#}(\psi g)(x_0)\leqslant cg^{\#}(x_0)+c$. □

An $H^p(\partial D, d\sigma)$ atom is function A, harmonic in D, with $\|A\|_\infty \leqslant 1$ and with boundary values $A(Q)$ supported in a surface ball $\Delta \subseteq \partial D$ and satisfying $\int_\Delta A(Q)\, d\omega(Q) = 0$. We have the following analog of Lemma 4.6.

Lemma 5.2.4. *$\mathcal{L}(\partial D)$, the space of Lipschitz functions on ∂D, is dense in $L_\sigma^{\#,p'}(\partial D, d\omega)$.*

PROOF. Our argument for Lemma 4.6 (the density of $\mathcal{L}(\partial D)$ in $\mathrm{VMO}_\sigma(\omega)$) will work in this context if we know that a compactly supported ψ belonging to $L_{dx}^{\#,p'}(\mathbb{R}^{n-1}, w\,dx)$ satisfies a «small oscillation» condition. That is, we want to see that

$$\sup_{\substack{l(Q)<\delta \\ Q\ni x_0}} \frac{1}{|Q|}\int_Q |\psi - \psi_Q|\, w\, dx = M_\delta(x_0)$$

satisfies: $\|M_\delta(x_0)\|_{L^{p'}(dx)} \to 0$ as $\delta \to 0$. But this is just a consequence of the dominated convergence theorem together with the fact that $w(Q)/|Q| \to w(x_0)$ as $l \to 0$, which is finite almost everywhere.

One can then show

Lemma 5.2.5. $H_{at}^p = (L_\sigma^{\#,p'}(\partial D, dw))^*$ *(although this information is not necessary for the duality argument).*

Lemma 5.9. *If $\sum_{k=1}^\infty \lambda_k A_k$ is an infinite linear combination of atoms, the λ_k are positive and $\Delta_k = \operatorname{supp} A_k$, then*

$$\begin{aligned}\Big\|\sum \lambda_k A_k\Big\|_{H^p(D,d\sigma)} &\equiv \Big\{\int_{\partial D} N\Big(\sum \lambda_k A_k\Big)^p d\sigma\Big\}^{1/p}\\ &\leqslant \Big\|\sum \lambda_k \chi_{\Delta_k}\Big\|_{L^p(d\sigma)}.\end{aligned}$$

PROOF. Since $K(x, Q) \in \mathrm{VMO}_\sigma(\omega)$, the harmonic extension of this infinite linear combination of atoms makes sense by duality. Now let $B_j = B(x_j, r_j)$ be the finite covering of $\{\operatorname{dist}(x, \partial D) < \delta\}$ for $\delta = \delta(D)$ with $B(x_j, 4r_j) \cap \partial D = \{(x, y) : y = \Phi_j(x)\}$. We can assume that for each k, $\sigma(\Delta_k) < \delta$ so that all atoms have support contained in one of these coordinate charts. Let m_k be the largest m such that $2^m\Delta_k$ $(= \Delta_k(x_k, 2^m r_k))$ is contained in a ball of radius no more than δ. By Dahlberg's pointwise estimate (2.5) on atoms, there is a $\beta > 0$ so that

$$NA_k(Q) \lesssim \chi_{\Delta_k}(Q) + \sum_{l=1}^{m_k} 2^{-l(n-1+\beta)}\chi_{2^l\Delta_k}(Q) + 2^{-(m_k+1)(n-1+\beta)}\chi_{\partial D}(Q).$$

Hence

$$\left\{\int_{\partial D} N^p\Big(\sum_k \lambda_k A_k\Big)\,d\sigma\right)^{1/p} \leqslant \left\{\int_{\partial D}\Big(\sum_k \lambda_k \chi_{\Delta_k}(Q)\Big)^p\right\}^{1/p}$$

$$+\left\{\int_{\partial D}\Big(\sum_k \lambda_k \sum_{l=1}^{m_k} 2^{-l(n-1+\beta)}\chi_{2^l\Delta_k}(Q)\Big)^p\,d\sigma(Q)\right\}^{1/p} + \sum_k \lambda_k 2^{-(m_k+1)(n-1+\beta)}.$$

Let $\epsilon_{k,l} = 1$ if $l \leqslant m_k$ and $\epsilon_{m,l} = 0$ otherwise. The second term in the sum is bounded by

$$\sum_{l=1}^{\infty} 2^{-l(n-1+\beta)}\left\{\int_{\partial D}\Big(\sum_k \epsilon_{k,l}\lambda_k\chi_{2^l\Delta_k}(Q)\Big)^p\,d\sigma\right\}^{1/p}$$

with by a change of variable, valid since each $2^l\Delta_k$ is contained in a coordinate chart, is less than

$$C_D\sum_{l=1}^{\infty} 2^{-l(n-1+\beta)}2^{l(n-1)/p}\Big\|\sum \lambda_k\chi_{\Delta_k}\Big\|_{L^p(d\sigma)}.$$

The third term in the sum is bounded by

$$C(\delta)\cdot\sum_k \lambda_k 2^{-(m_k+1)(n-1+\beta)}\int_{\partial D}\chi_{2^{m_k}\Delta_k}(Q)\,d\sigma$$

$$\leqslant C(\delta)\sum_k \lambda_k 2^{-(m_k+1)(n-1+\beta)}2^{m_k(n-1)/p}\int_{\partial D}\chi_{\Delta_k}(Q)\,d\sigma$$

$$\leqslant C(\delta)\sigma(\partial D)^{1/p'}\Big\|\sum_k \lambda_k\chi_{\Delta_k}\Big\|_{L^p(d\sigma)}. \quad \square$$

Lemma 5.10. *If $u \in \mathcal{L}(\bar{D})$ with $\Delta u = 0$ and $u(p_0) = 0$ then*

$$\left|\int_{\partial D} u(Q)f(Q)\,d\omega(Q)\right| \leqslant C\|N(u)\|_{L^p(d\sigma)}\|f\|_{L^{\#,p}_{\sigma}(\partial D,\,d\omega)}$$

for all $f \in L^{\#,p'}_{\sigma}(\partial D, d\omega)$.

PROOF. By Green's theorem, (see the argument following Theorem 2.7 in section 2),

$$\left|\int_{\partial D} u(Q)f(Q)\,d\omega(Q)\right| \leqslant \int_D G(x)|\nabla u|\,|\nabla v|\,dx + \int_D \nabla G\cdot\nabla v u\,dx$$

where $d(x) = \operatorname{dist}(x, \partial D)$ and v is some smooth extension of $f(Q)$ to D such that this formal argument is justified. Suppose that $|v| + |\nabla v| \leqslant C$ in $K \subset\subset D$ and that

$$(5.11)\quad \begin{array}{ll} \text{(i)} & |\nabla v(x)|G(x)/d(x) \in \tau_1^{p'} \\ \text{(ii)} & |\nabla v(x)|G(x) \in T_2^{p'} \end{array}$$

where τ_1^p, $T_2^{p'}$ have the obvious definitions on a Lipschitz domain D. Let $B(p_0)$ be a ball containing the pole of $G(x)$. Then

$$\begin{aligned} &\int_{DB(p_0)} G(x)|\nabla u|\,|\nabla v|\,|\nabla v|\,dx \\ &\qquad \leqslant \int_{\partial D}\int_{\Gamma(Q)} G(x)|\nabla u|\,|\nabla v|\frac{dx}{d(x)^{n-1}}\,d\sigma(Q) \\ &\qquad \leqslant \int_{\partial D} Su(Q)\left\{\int_{\Gamma(Q)} |\nabla v|^2 G^2(x) d(x)^{-n}\,dx\right\}^{1/2} d\sigma(Q) \\ &\qquad \leqslant \|Su\|_{L^p(d\sigma)}\,\||\nabla v|G\|_{T_2^{p'}(D)} \\ &\qquad \leqslant C\|Nu\|_{L^p(d\sigma)}\,\|f\|_{L^{\#,p}(\partial D, d\sigma)}. \end{aligned}$$

Using $|\nabla G(x)| \leqslant G(x)/d(x)$ away from p_0,

$$\begin{aligned} \int_{D\setminus B(p_0)} G/d\,|u|\,|\nabla v|\,dx &\leqslant \int_{\partial D}\int_{\Gamma(Q)} G/d\,|u|\,|\nabla v|\,d(x)^{-n+1}\,dx\,d\sigma(Q) \\ &\leqslant \int_{\partial D} Nu(Q)\int_{\Gamma(Q)} G/d\,|\nabla v|\,d(x)^{-n+1}\,dx \\ &\leqslant \|N(u)\|_{L^p(d\sigma)}\,\|G/d\,|\nabla v|\,\|_{\tau_1^{p'}} \\ &\lesssim \|Nu\|_{L^p(d\sigma)}\,\|f\|_{L^{\#,p'}}. \end{aligned}$$

Since G is integrable on $B(p_0)$ and $|v| + |\nabla v| \leqslant C$ here, the integral over $G(p_0)$ is handled as before. □

Thus it remains to find such an extension. Because we have the result on $\mathbb{R}_+^n$ (Theorem 5.7) and we can localize to a coordinate chart of ∂D (Lemma 5.8), the argument is just a variant of that given in Lemma 4.4 and we omit the details.

Assume now that D is starshaped with respect to the origin.

Definition.

(i) $H^p(\partial D, d\sigma) = \{f: f(Q) = \lim_{r\to 1} u(rQ),\ u \in H^p(D, d\sigma)$, *with convergence in* $(L_\sigma^{\#,p'}(\partial D, d\omega))^*\}$.

(ii) $H_{at}^p(\partial D, d\sigma) = \{f: f = \sum \lambda_k A_k$ *where the* A_k *are* H^p *atoms and* $\|\sum \lambda_k \chi_{\Delta_k}\|_{L^p(d\sigma)} \leqslant \infty\}$,

and the convergence takes place in $(L_\sigma^{\#,p'}(\partial D, dw))^*$.

Theorem 5.11. *$H^p_{at}(\partial D, d\sigma) = H^p(D, d\sigma)$ and $(H^p(D, d\sigma))^* = L_\sigma^{\#, p'}(\partial D, dw)$.*

PROOF. We refer the reader to the proof of Theorem 2.6 given in section 4. □

Finally, as before, we have the following theorem for domains, D, not assumed to be starlike.

Theorem 5.12. *If $\Delta u = 0$ and $Nu \in L^p(\partial D, d\sigma)$, $p \leqslant 2$ there exists a sequence of constants $\{\lambda_j\}$ and atoms $\{A_j\}$ such that*

$$u(x) = \sum_j \lambda_j \int A_j(Q) K(x, Q)\, dw(Q)$$

for $x \in S$, with $\|Nu\|_{L^p(d\sigma)} \approx \|\sum \lambda_j \chi_{\Delta_j}\|_{L^p(d\sigma)}$. Moreover, given $\{\lambda_j\}$ and $\{A_j\}$ sequences of constants and atoms satisfying $\|\sum \lambda_j \chi_{\Delta_j}\|_{L^p(d\sigma)} \leqslant \infty$, there exists a harmonic $u(x) = \sum \lambda_j A_j(x)$ with $Nu \in L^p(d\sigma)$. These boundary values determine $u(x)$ uniquely in the sense that the limiting distributions on the starlike subdomains of D are zero if $u \equiv 0$ on ∂D.

References

[1] Alvarez, J. and Milman, M. Spaces of Carleson measures; duality and interpolation, preprint.

[2] Carleson, L. Two remarks on H^1 and BMO, *Adv. in Math.* **22** (1976), 269-277.

[3] Coifman, R. R. and Fefferman, C. Weighted norm inequalities for maximal functions and singular integrals, *Studia Math.* **51** (1974), 241-250.

[4] Coifman, R. R., Meyer, Y., Stein, E. M. Some new functions spaces and their applications to harmonic analysis, *J. Funct. Anal.* **62** (1985), 304-335.

[5] Coifman, R. R. and Weiss, G. Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.* **83** (1977), 569-645.

[6] Dahlberg, B. E. J. On estimates of harmonic measure, *Arch. Rat. Mech. Anal.* **65** (1977), 272-288.

[7] Dahlberg, B. E. J. Weighted norm inequalities for the Lusin area integral and the nontangential maximal function for functions harmonic in a Lipschitz domain, *Studia Math.* **67** (1980), 297-314.

[8] Dahlberg, B. E. J. A note in H^1 and BMO, Proceedings of the Plëyel Conference, 1979, 23-30.

[9] Dahlberg, B. E. J. On the Poisson integral for Lipschitz and C^1 domains, *Studia Math.*

[10] Dahlberg, B. E. J., and Kenig, C. E. Hardy spaces and the Neumann problem in L^p for Laplace's equation in Lipschitz domains, to appear in *Annals of Math.*

[11] Fabes, E. B. and Kenig, C. E. On the Hardy spaces of a C^1 domain, *Ark. Mat.* **19** (1981), 1-22.

[12] Fabes, E. B., Kenig, C. E., Neri, U. Carleson measures, H^1 duality, and weighted BMO in non-smooth domains, *Indiana, J. Math.* **30** (4) (1981), 547-581.

[13] Fefferman, C. and Stein, E. M. H^p spaces of several variables, *Acta Math.* **129** (1972), 137-193.
[14] Friedland S. and Hayman, W. K. Eigenvalue inequalities for the Dirichlet problem on spheres and the growth of subharmonic functions, *Comm. Math. Helv.* **51** (1976), 133-161.
[15] Garnett, J. B. Bounded Analytic Functions, *Academic Press*, 1981.
[16] Huber, A. «Uber Wachsturmseigenschaften gewissen Klassen von subharmonischen Funcktionen», *Comm. Math. Helv.* **26** (1952), 81-116.
[17] Jerison, D. S. and Kenig, C. E. Boundary value problems on Lipschitz domains, *MAA Studies*, Vol. 23, 1-68.
[18] Jerison, D. S. and Kenig, C. E. Boundary behavior of harmonic functions in nontangentially accessible domains, *Adv. in Math.* **46** (1), (1982), 80-147.
[19] Jones P. W. Extension theorems for BMO, *Indiana J. Math.* **29** (1980), 41-66.
[20] Kenig, C. E. Weighted Hardy spaces on Lipschitz domains, *Amer. J. Math.* **102** (1980), 129-163.
[21] Latter, R. A characterization of $H^p(\mathbb{R}^n)$ in terms of atoms, *Studia Math.* **62** (1978), 93-101.
[22] Macías, R. and Segovia, C. A decomposition into atoms of distributions on spaces of homogeneous type, *Adv. in Math.* **33** (1979), 171-209.
[23] Muckenhoupt, B. Weighted norm inequalities for the Hardy maximal function, *Trans. Amer. Math. Soc.* **165** (1972), 207-226.
[24] Stromberg, J. O. and Torchinsky, A. Book to appear. (See also *Bull. Amer. Math. Soc.* **3**, No. 3 (1980)).
[25] Varapoulos, N. Th. BMO functions and the $\bar{\partial}$-equation, *Pacific J. Math.* **71** (1977), 221-273.

Carlos E. Kenig* and Jill Pipher**
Department of Mathematics
University of Chicago
Chicago, IL 60637

* Supported in part by the NSF and the J. S. Guggenheim Memorial Foundation.
** Supported in part by the NSF.