

Structural Stability and Generic Properties of Planar Polynomial Vector Fields

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Dedicated to my father, Philip S. Shafer,
on his seventieth birthday
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In 1962 Peixoto [14] gave a complete characterization of the structurally stable C^1 vector fields on any compact, two-dimensional manifold without boundary, and showed that they form a dense, open set in the space of all C^1 vector fields with the uniform C^1 topology (see also [6], [7]). There later followed examples showing that, on any non-compact two-manifold, there exists an open set of vector fields, none of which is structurally stable ([10]; see also [16], [20]). Nevertheless, Kotus, Krych, and Nitecki [10] showed how to control behavior «at infinity» so as to guarantee stability of a vector field on any two-manifold under strong C^r perturbation, and gave a complete characterization of the structurally stable vector fields on \mathbb{R}^2 (see also [3]). In this paper we consider the set \mathfrak{P}_n of polynomial vector fields of degree $\leq n$ on \mathbb{R}^2 , and give sufficient conditions for structural stability of $X \in \mathfrak{P}_n$ with respect to perturbation in \mathfrak{P}_n (Theorem 3.2). Up to an added condition that asymptotically stable (or unstable) limit cycles be hyperbolic, these same conditions are also necessary (Theorem 3.3). We use the coefficient topology on \mathfrak{P}_n , and require that the equivalence homeomorphism lie in a pre-assigned compact-open neighborhood of $id_{\mathbb{R}^2}$. Briefly, in addition to the usual conditions that X be Morse-Smale, we have regularity conditions on the associated Poincaré vector field $\pi(X)$ along the equator S^1 of the Poincaré sphere S^2 , and conditions on certain distinguished orbits, so-called separatrices of

saddles-at-infinity, in their relation to the saddle points of X and to the critical points of $\pi(X)$ at infinity. The polynomial nature of the problem simplifies the dynamics to the point that we are able to prove existence of a dense, open set of structurally stable vector fields (Theorem 4.1). On the other hand, the analysis is complicated by the fact that the set of allowable perturbations is small, and every one of them is global and large at infinity. In particular, we have been unable to resolve the question, raised explicitly in [19] and implicitly in [1, §6.3], of whether a limit cycle of odd multiplicity must be hyperbolic in order to be locally structurally stable. (This question seems to have been overlooked in [21] and [23].)

We note that the results of this paper do not depend on the validity of Dulac's Theorem asserting that a polynomial vector field has at most finitely many limit cycles, a correct proof of which has not yet appeared in print. If a valid proof were given, then the statements and proofs of a number of results here would simplify greatly, and the similarities to the case of stability of smooth vector fields on compact manifolds increase.

In comparing this work with previous work on stability of polynomial vector fields ([19], [21], [23], for example), it should be noted that, while we exploit the Poincaré compactification of \mathbb{R}^2 (see §1), we do not require that the Poincaré vector fields $\pi(X)$ and $\pi(Y)$ be equivalent on S^2 in order for X and Y to be equivalent. This approach seems more natural, more closely mimics the compact case, and allows closer comparison with results of the general theory of stability of smooth vector fields on \mathbb{R}^2 . Of course, more complicated behavior is now consistent with structural stability.

Moreover, in our setting it is natural to allow smooth as well as polynomial perturbations of $X \in \mathfrak{P}_n$. Building on work of Kotus, Krych, and Nitecki [10], we fully characterize elements of \mathfrak{P}_n that are stable under small perturbation (with respect to the Whitney C^r topology) by C^r vector fields, $r \geq 1$ (Theorem 3.1). Again the equivalence homeomorphism must be close to the identity. Strong control at infinity leads to a particularly simple result: X is structurally stable if and only if it is «Morse-Smale», in the sense that saddles-at-infinity are included in the «no saddle connections» condition (and only hyperbolicity, not finitude, of critical elements is explicitly required). Structural stability is generic in this setting as well (Theorem 4.1).

Peixoto [13] also showed that on compact two-manifolds, any requirement that the equivalence homeomorphism be near the identity is redundant. This is not always the case on open two-manifolds (see pages 20-22 of [10]), but we are able to show that it holds for polynomials under general (C^r) perturbation (Theorem 5.1). For polynomial perturbation of polynomial vector fields, the effect of restriction of the equivalence homeomorphism depends on the degree of the vector fields involved; this question is discussed briefly in section 5, and will be treated in detail in a forthcoming paper with F. Dumortier [4].

This paper is organized as follows. Section 1 is devoted to background material, and Section 2 to the statements and proofs of a few results needed later. In Section 3 the structural stability theorems are stated and proved, and are used in Section 4 to show that stability is generic. Section 5 treats redundancy of restrictions on the equivalence homeomorphism.

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1. Background

Definitions and constructions in this section are generally confined to \mathbb{R}^2 . For general background and meaning of terms not defined here, consult Hartman [8] or Palis-deMelo [12]. For extensive discussion of various aspects of structural stability on open manifolds, see §2 of the memoir of Kotus, Krych, and Nitecki [10].

\mathfrak{F}_n will denote the set of vector fields on \mathbb{R}^2 of the form

$$X(x, y) = P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y},$$

where P and Q are polynomial functions in x and y of degree $\leq n$. Any such X is uniquely specified by the $(n + 1)(n + 2)$ coefficients of P and Q , hence may be identified with a unique point of $\mathbb{R}^{(n+1)(n+2)}$. The topology induced on \mathfrak{F}_n from the usual topology on $\mathbb{R}^{(n+1)(n+2)}$ by this identification is the *coefficient topology* on \mathfrak{F}_n . Further notation:

- \mathfrak{X}^r : the C^r vector fields on \mathbb{R}^2 , Whitney C^r topology (see [9])
- \mathfrak{Y}^r : the C^r vector fields on S^2 , uniform C^r topology
- H : the homeomorphisms of \mathbb{R}^2 , compact-open topology
- J : the homeomorphisms of S^2 , uniform C^0 topology

For $X \in \mathfrak{F}_n$ as written above, we let X^\perp denote the vector field

$$-Q(x, y) \frac{\partial}{\partial x} + P(x, y) \frac{\partial}{\partial y},$$

also in \mathfrak{F}_n .

Let \mathfrak{D} be \mathfrak{F}_n or \mathfrak{X}^r ($r \geq 1$), and $X \in \mathfrak{D}$. Then X generates a local flow on \mathbb{R}^2 , which will be denoted $\eta_X(t, p)$. X and another element X' of \mathfrak{D} are *topologically equivalent* if there exists $h \in H$ carrying orbits of the flow induced by X

onto orbits of the flow induced by X' , preserving sense but not necessarily parametrization; h is termed an *equivalence homeomorphism* between X and X' . X is *structurally stable (with respect to perturbation in \mathfrak{D})* if for any neighborhood M of $id_{\mathbb{R}^2}$ in H , there exists a neighborhood \mathfrak{K} of X in \mathfrak{D} , such that every X' in \mathfrak{K} is topologically equivalent to X by an equivalence homeomorphism h lying in M .

A critical point of X is *hyperbolic* if no eigenvalue of the linear part $dX(p)$ of X at p has real part zero. A closed orbit γ is *hyperbolic* if $\int_{\gamma} \text{div}(X)$ is non-zero. Choosing a sufficiently short line segment Σ through a point of γ , and a coordinate s on Σ with $s = 0$ corresponding to γ , the Poincaré first return map $f(s)$ is defined near zero, and both it and the difference map $d(s) = f(s) - s$, whose roots correspond to closed orbits of X near γ , are analytic. Thus closed orbits accumulate on γ only if a neighborhood of γ is composed of an *annular band* of closed orbits. The *multiplicity* of γ is the multiplicity of the zero of d at $s = 0$; γ is hyperbolic if and only if it has multiplicity 1.

A positive [negative] semi-orbit $o^+(p)$ [$o^-(p)$] of X is *bounded* if it is contained in a compact set, *escapes to infinity* if for every compact set K there exists $p' \in o^+(p)$ [$p' \in o^-(p)$] such that $o^+(p')$ [$o^-(p')$] is disjoint from K , and *oscillates* if it is neither bounded nor escapes to infinity.

A *saddle-at-infinity* (SAI) is a pair $(o^+(p), o^-(q))$ of semi-orbits, each escaping to infinity, such that there exist sequences $p_n \rightarrow p$ in \mathbb{R}^2 and $t_n \rightarrow \infty$ in \mathbb{R} such that $\eta(t_n, p_n) \rightarrow q$ in \mathbb{R}^2 . We identify this SAI with any formed using some $p' \in o^+(p)$ in the place of p or some $q' \in o^-(q)$ in the place of q , and further require that if for some sequence $\tilde{t}_i \rightarrow \infty$, $\eta(\tilde{t}_i, p_i) \rightarrow r$, then $r \in o^+(p) \cup o^-(q)$. Then $o^+(p)[o^-(q)]$ is termed the *stable [unstable] separatrix* of the SAI. The reader is cautioned that a SAI is a different object from a saddle (point) (of $\pi(X)$) on the line at infinity (see below).

A *saddle connection* is an orbit $o(p)$ such that $o^+(p)$ is a stable separatrix of a saddle, or of a SAI, while $o^-(p)$ is a separatrix of a saddle, or of a SAI. Note that, as this definition makes allowance for existence of SAIs, it is more general than the definition on a compact manifold. A *separatrix cycle* (elsewhere sometimes referred to as a *graph*) is a sequence $p_1, \sigma_1, p_2, \sigma_2, \dots, p_k, \sigma_k, p_{k+1}$ of orbits, each p_j a critical point, each σ_j a separatrix at p_j or at p_{j+1} and tending from p_j to p_{j+1} , and $p_{k+1} = p_1$. The definition is identical for vector fields on S^2 .

We let $W^+(X)$ [$W^-(X)$] denote the union of all orbits containing a stable [unstable] separatrix (of a saddle or SAI) of X . $\Omega(X)$ will denote the set of non-wandering points of X and $\text{Per}(X)$ the set of critical points and points on closed orbits. For $p \in \mathbb{R}^2$, $\alpha_X(p)$ and $\omega_X(p)$ denote the α - and ω -limit sets of p under X respectively.

For $X \in \mathfrak{B}_n$, an analytic vector field $\pi(X)$, the *Poincaré vector field corresponding to X* , is induced on S^2 as follows. Let

$$H^\pm(S^2) = \{(x, y, z) \in S^2 \subset \mathbb{R}^3 \mid \pm z > 0\},$$

identify \mathbb{R}^2 with $T_{(0,0,1)}S^2$, and let $f^\pm: T_{(0,0,1)}S^2 \rightarrow H^\pm(S^2)$ be central projection. Then $\pi(X)$ is the unique analytic extension of $z^{n-1}(f^\pm)_*X$ to all of S^2 , which is then termed the Poincaré sphere. We call \mathbb{R}^2 the *finite part of the plane (f.p.p.)*, and $S^1 \subset S^2$, to which $\pi(X)$ is tangent, (*the line at infinity*). Letting U_1 and U_2 be the hemispheres corresponding to $x > 0$ and $y > 0$ respectively, and choosing $\phi_i: U_i \rightarrow \mathbb{R}^2$, $i = 1, 2$, to be the inverses of the central projection from the vertical planes tangent to S^2 at $(1, 0, 0)$ and $(0, 1, 0)$ respectively, we have the following particularly simple coordinate representations of $\pi(X)$:

$$\text{(in } U_1) \quad D(s, t)t^n \left[Q\left(\frac{1}{t}, \frac{s}{t}\right) - sP\left(\frac{1}{t}, \frac{s}{t}\right), -tP\left(\frac{1}{t}, \frac{s}{t}\right) \right] \quad (1.1)$$

$$\text{(in } U_2) \quad D(s, t)t^n \left[P\left(\frac{s}{t}, \frac{1}{t}\right) - sQ\left(\frac{s}{t}, \frac{1}{t}\right), -tQ\left(\frac{s}{t}, \frac{1}{t}\right) \right] \quad (1.2)$$

where $D(s, t)$ is a positive function which is independent of X , where $\{(s, t) \mid t = 0\}$ corresponds to the equator $S^1 \subset S^2$, i.e., to infinity, with positive s -direction agreeing with the positive y -direction (in U_1) or the positive x -direction (in U_2) and where $\{(s, t) \mid t > 0\}$ corresponds to the f.p.p. Choosing V_i to be the open hemisphere opposite to U_i , $i = 1, 2$, and analogous coordinate mappings, we obtain the same expressions for $\pi(X)$ as (1.1) and (1.2), valid in V_1 and V_2 respectively, except for a multiplicative factor of $(-1)^{n-1}$. (The positive directions of the s and t axes disagree with the positive directions of the axes in \mathbb{R}^3 in these cases.) Complete details may be found in González [5]. We note in particular that, except for a positive scale factor, $\pi(X)$ appears in these coordinates as a polynomial vector field of degree $n + 1$, and that $S^1 \subset S^2$ is always invariant under $\pi(X)$.

Suppose that for some $r \in \mathbb{Z}^+ \cup \{0\}$, $\Pi \equiv \{\pi(X) \mid X \in \mathfrak{P}_n\}$ is given the subspace topology induced as a subset of \mathfrak{Y}^r . Then the coordinate expressions for $\pi(X)$ given above quickly lead to the fact that if \mathfrak{R} is a neighborhood of $\pi(X)$ in Π , then, $\pi^{-1}(\mathfrak{R})$ is a neighborhood of X in \mathfrak{P}_n (coefficient topology).

We note finally that because \mathbb{R}^2 is not compact, the flow $\eta_X(t, p)$ generated by $X \in \mathfrak{P}_n$ need not be complete. But the vector field \tilde{X} formed from X by rescaling by $(1 + \|X\|)^{-1/2}$, say, is uniformly bounded, hence generates a complete flow with orbits identical as point-sets to those of X . Since equivalence homeomorphisms ignore parametrizations of orbits, we may safely ignore possible incompleteness of the flows involved, and treat all flows under discussion as if complete (as was already done in the definitions of this section).

2. Preliminary Propositions and Genericity Theorem

This section is devoted to the statement and proof of a few propositions that will be needed later. Several of them combine to show that a dense, open set of \mathfrak{F}_n consists of vector fields whose structure is particularly simple, both on \mathbb{R}^2 and when extended to S^2 (Theorem 2.9). This latter result is practically the same as a theorem of G. Tavares dos Santos [21]. See also the book by J. Sotomayor [18].

Proposition 2.1. *Suppose $X \in \mathfrak{F}_n$ is such that, for some $r \in \mathbb{Z}^+ \cup \{0\}$, for every neighborhood M of id_{S^2} in J there is a neighborhood \mathfrak{R} of $\pi(X)$ in \mathcal{Y} such that if $Y \in \mathfrak{R}$ is tangent to $S^1 \subset S^2$, then Y is topologically equivalent to X by some $h \in M$ satisfying $h(S^1) \subset S^1$. Then for any compact set $K \subset \mathbb{R}^2$ and any $\delta > 0$, there is a neighborhood \mathfrak{U} of X in \mathfrak{F}_n such that if $X_1 \in \mathfrak{U}$, then X_1 is topologically equivalent to X by some h that is C^0 - δ -close to $id_{\mathbb{R}^2}$ on K .*

PROOF. Given K and δ , choose a compact neighborhood $L \subset H^+(S^2)$ of an ϵ -neighborhood of $f^+(K)$, some $\epsilon > 0$. By uniform continuity of $(f^+)^{-1}|_L$, for some $\xi > 0$, $x, y \in L$ and $\text{dist}_{S^2}(x, y) < \xi$ imply $\text{dist}_{\mathbb{R}^2}((f^+)^{-1}(x), (f^+)^{-1}(y)) < \delta$. Choosing $\mu = \min(\epsilon, \xi)$, let M correspond to C^0 - μ -closeness to id_{S^2} , and let \mathfrak{R} be the neighborhood of $\pi(X)$ in \mathcal{Y} hypothesized to exist. We claim that $\mathfrak{U} \equiv \pi^{-1}(\mathfrak{R})$ is as required. It is a neighborhood of X , and for $X_1 \in \mathfrak{U}$, by hypothesis there exists $h \in M$ which satisfies $h(S^1) \subset S^1$ and is an equivalence between $\pi(X)$ and $\pi(X_1)$, which by choice of ϵ , ξ , and μ is easily seen to be uniformly δ -close to $id_{\mathbb{R}^2}$ on K . \square

Proposition 2.2. *For $X \in \mathfrak{F}_n$, any one of the following conditions implies the failure of the inclusion $Cl(W^+(X)) \cap Cl(W^-(X)) \subset \text{Per}(X)$:*

- (i) $\pi(X)$ has a separatrix cycle containing a point not in $S^1 \subset S^2$;
- (ii) X has an oscillating orbit;
- (iii) $\Omega(X) \not\subset \text{Per}(X)$;
- (iv) there is a point $p \in \mathbb{R}^2$ for which $\alpha_X(p)$ or $\omega_X(p)$ is not empty, nor precisely one critical point, nor precisely one closed orbit.

PROOF. Call the inclusion I . If (i) holds, I obviously fails. If (ii) holds, then for $\pi(X)$ on S^2 there is an arc in the f.p.p. that is composed of orbits of X and which joins critical points A and B of $\pi(X)$ on $S^1 \subset S^2$ ($A = B$ possible), such that every point in the arc is in $\alpha_X(p)$ [or in $\omega_X(p)$] (where $o(p)$ oscillates). Since $\alpha_{\pi(X)}(p)$ [$\omega_{\pi(X)}(p)$] is connected, a saddle connection of X (among separatrices of SAIs) is formed, so I fails. If (iv) holds, then either $\alpha_X(p)$ is compact, hence X has a separatrix cycle, so I fails, or $\alpha_X(p)$ is not compact, so $o^-(p)$ oscillates, so I fails by (ii).

Finally, we show that (iii) implies failure of I by showing the contrapositive. Hence assume I holds, and suppose $p \in \mathbb{R}^2 \setminus \text{Per}(X)$. Passing to $\pi(X)$ on S^2 , by the Poincaré-Bendixson Theorem the non-empty, connected set $\omega_{\pi(X)}(p)$ must be either a single equilibrium or a single closed orbit, else (i) holds, which would imply failure of I . But if $\omega_{\pi(X)}(p)$ is such, then either it is a sink for $\pi(X)$ in S^2 , which implies $p \notin \Omega(X)$, or it is a saddle for $\pi(X)$ containing p in a stable separatrix, which by I implies that $\alpha_{\pi(X)}(p)$ is a source for $\pi(X)$ in S^2 , which again implies $p \notin \Omega(X)$. \square

Proposition 2.3. *If $X \in \mathfrak{P}_n$ has only finitely many critical points, all critical points and closed orbits are hyperbolic, and there are no saddle connections (even when SAIs are taken into account), then $\Omega(X) = \text{Per}(X)$.*

PROOF. If X is as hypothesized and $p \notin \text{Per}(X)$, then $\omega_{\pi(X)}(p)$ [$\alpha_{\pi(X)}(p)$] is a nonempty, compact, connected subset of S^2 . By the Poincaré-Bendixson Theorem and the hypotheses, if it contains a separatrix, that separatrix is wholly contained in $S^1 \subset S^2$. In any event, it is composed of a single sink [source] (which could be a separatrix cycle) of $\pi(X)$, whose basin of attraction [region of repulsion] contains p . Thus $p \notin \Omega(X)$ \square

Proposition 2.4. *The set \mathfrak{G}_1 of all vector fields X in \mathfrak{P}_n such that $\pi(X)$ has only finitely many critical points, all hyperbolic, is open and dense in \mathfrak{P}_n .*

PROOF. G. Tavares [21] has shown that the set of X in \mathfrak{P}_n such that X has finitely many critical points, all hyperbolic, is open and dense. Examining the form of $\pi(X)$ in the charts on S^2 mentioned in section one, formulas (1.1) and (1.2), we see that the same arguments carry over to give a relatively open, relatively dense set with only finitely many singularities of $\pi(X)$ on $S^1 \subset S^2$, all hyperbolic. \square

This proposition is also an immediate consequence of a more general result, Proposition 4.1 of Chapter Two of [18].

Remark 2.5. The set of all vector fields in \mathfrak{P}_n which have only finitely many critical points, all hyperbolic, is dense, but not open. For example, let

$$H_a(x, y) = 2ax^3 - a^2x^2y - x^2 - 2axy + a^2y^2 + y,$$

$a \in \mathbb{R}$, and let $X_a \in \mathfrak{P}_2$ be the corresponding gradient vector field. Then X_0 is non-singular, but X_a has a non-hyperbolic critical point at

$$(x, y) = \left(\frac{1}{a}, \frac{1}{a^2} \right).$$

Proposition 2.6. *The set \mathcal{G}_2 of all vector fields X in $\mathcal{G}_1 \subset \mathfrak{F}_n$ such that $\pi(X)$ has no saddle connections, except in $S^1 \subset S^2$, is open and dense in \mathfrak{F}_n .*

PROOF. If $X_\mu(x, y) = X(x, y) + \mu X^\perp(x, y)$, then for non-zero μ near zero, X_μ is a rotation and scaling of X , hence breaks all saddle connections between saddle points of X . Tavares [21, Lemma 3.4] used Sotomayor's criterion to show that passage to X_μ breaks connections between saddles of $\pi(X)$ as well. Since X_μ is near X in \mathfrak{F}_n , the set in question is dense. It is open because the finite portion of a saddle separatrix varies continuously with $\pi(X)$ (see for example [1], Lemma 3 of §9.2). \square

Proposition 2.7. *The set \mathcal{G}_3 of all vector fields X in $\mathcal{G}_2 \subset \mathcal{G}_1 \subset \mathfrak{F}_n$ such that $\pi(X)$ has finitely many closed orbits, all hyperbolic, is open and dense in \mathfrak{F}_n .*

PROOF. Suppose $X \in \mathcal{G}_2$ has infinitely many closed orbits. They cannot accumulate on a critical point or a separatrix cycle, since X is in \mathcal{G}_2 , hence accumulate on a closed orbit γ of $\pi(X)$. Analyticity of $\pi(X)$, hence of a Poincaré first return map on a section through γ , implies that a neighborhood of γ is made up entirely of closed orbits of $\pi(X)$, hence that there is an annular band of closed orbits of X in \mathbb{R}^2 . Analyticity of X similarly implies that each boundary of the band is either a single critical point, a single closed orbit, or a separatrix cycle. (See [17] for a detailed discussion and proof.) While the outer boundary could be $S^1 \subset S^2$, the inner boundary must be a center or a separatrix cycle, both of which are impossible for X in \mathcal{G}_2 . Thus every X in \mathcal{G}_2 has at most finitely many closed paths.

Suppose now that $X \in \mathcal{G}_2$, and that X has some non-hyperbolic closed paths. Passing from X to X_a as in the proof of Proposition 2.6, for a non-zero a close enough to zero, any particular closed orbit of X of even multiplicity either disappears entirely or decomposes into two hyperbolic closed orbits, while any particular closed orbit of odd multiplicity persists and becomes hyperbolic (see for example [1], Theorems 72 and 73 of §32.4). If $S^1 \subset S^2$ was a closed orbit of $\pi(X)$ originally, it is easy to make a small adjustment in X to make it hyperbolic (see González [5] for example). Thus we have density of \mathcal{G}_3 . Openness is clear. \square

Remark 2.8. The set of vector fields in $\mathcal{G}_2 \subset \mathcal{G}_1 \subset \mathfrak{F}_n$ having only hyperbolic closed orbits is not itself open. For example, let

$$\begin{aligned} X_a(x, y) = & [a^4x^5 - x^4y - 2x^2y^3 - y^5 - 2a^2x^3 - 2x^2y - 2y^3 + x - y] \frac{\partial}{\partial x} \\ & + [x^5 + 3a^4x^4y + 2x^3y^2 + 3a^4x^2y^3 + xy^4 + a^4y^5 \\ & + 2x^3 - 4a^2x^2y + 2xy^2 - 2a^2y^3 + x + y] \frac{\partial}{\partial y}. \end{aligned}$$

In polar coordinates this becomes

$$X_a(r, \theta) = (ar - 1)^2(ar + 1)^2 r \frac{\partial}{\partial r} + [(r^2 + 1)^2 + a^2 r^2 \cos \theta \sin \theta (a^2 r^2 (1 + \cos \theta) - 2)] \frac{\partial}{\partial \theta},$$

so that for $|a| < 2/\sqrt{5}$, the coefficient of $\partial/\partial\theta$ is positive. Thus X_a has a hyperbolic unstable focus at $(0, 0)$ as its sole critical point, and a semi-stable cycle lying in the circle $x^2 + y^2 = 1/a^2$ as its sole closed orbit, if $a \neq 0$. The field X_0 is Morse-Smale, and $X_a \rightarrow X_0$ in the coefficient topology on \mathfrak{P}_5 as $a \rightarrow 0$.

Theorem 2.9. *There is a dense, open subset $\mathfrak{G} \subset \mathfrak{P}_n$, each of whose elements X has the following properties:*

- (i) $\pi(X)$ (hence X) has only finitely many critical points and closed orbits, all of them hyperbolic;
- (ii) $\pi(X)$ (hence X) has no saddle connections, except in $S^1 \subset S^2$; and
- (iii) $\Omega(X) = \text{Per}(X)$.

PROOF. Let \mathfrak{G} be the set \mathfrak{G}_3 of Proposition 2.7. Then \mathfrak{G} is a dense, open subset of \mathfrak{P}_n , all of whose elements satisfy conditions (i) and (ii). But then they satisfy (iii) as well, by Proposition 2.3. \square

3. Structural Stability Theorems

The first theorem of this section is an application of the characterization theorem of Kotus, Krych, and Nitecki [10] to the situation of smooth perturbation of polynomial vector fields. Recall Section 1 for definitions of terms.

Theorem 3.1. *$X \in \mathfrak{P}_n$ is structurally stable with respect to perturbation in \mathfrak{X} , $r \geq 1$ (Whitney C^r topology) if and only if*

- (1) X has only hyperbolic singularities and closed orbits (and there are only finitely many of the former);
- (2) X has no saddle connections (where separatrices of SAIs are taken into account); and
- (3) $\Omega(X) = \text{Per}(X)$.

PROOF. While the theorem can easily proved directly, for brevity we will simply show that the three conditions stated are equivalent to the conditions characterizing structural stability with respect to perturbation in \mathfrak{X} given in Theorems *A* and *B* of [10]:

- (i) there are no non-trivial minimal sets or oscillatory orbits;
- (ii) every critical point and closed orbit is hyperbolic; and
- (iii) $Cl(W^-(X)) \cap Cl(W^+(X)) \subset \text{Per}(X)$.

Hence suppose X satisfies (1), (2), and (3). Since no flow on \mathbb{R}^2 has a non-trivial minimal set, the first half of (i) holds automatically. Proposition 2.2(ii), finitude of the set of separatrices, and (2) combine to exclude oscillatory semi-orbits, so (i) holds. Condition (ii) follows from (1), and (iii) from (2) and finitude of the number of separatrices.

Conversely, suppose conditions (i), (ii), and (iii) hold. First, since by (ii) all critical points of X are hyperbolic, they are isolated, hence by Bézout's Theorem [22] there are at most n^2 of them. This and (ii) imply that (1) holds. Certainly (iii) implies (2). Truth of (3) follows from Proposition 2.2(iii). \square

Turning to polynomial perturbations of polynomial vector fields, we have the following sufficient conditions for structural stability. These same conditions will prove necessary for stability, with the possible exception noted in Theorem 3.3.

Theorem 3.2. $X \in \mathfrak{P}_n$ is structurally stable with respect to perturbation in \mathfrak{P}_n (coefficient topology) if

- (1) X has finitely many critical points and closed orbits, all of them hyperbolic;
- (2) X has no saddle connections (where separatrices of SAIs are taken into account);
- (3) if $S^1 \subset S^2$ is a closed orbit of $\pi(X)$, it is hyperbolic, or
- (3') if p is a critical point of $\pi(X)$ on $S^1 \subset S^2$, then it is hyperbolic, or $d\pi(X)(p)$ has a non-zero eigenvalue with corresponding eigenvector not in $T_p S^1 \subset T_p S^2$; and
- (4a) no separatrix $o^+(p)$ [$o^-(p)$] of a SAI tends under $\pi(X)$ to a saddle-node on $S^1 \subset S^2$ in forward [reverse] time, and
- (4b) no separatrix $o^+(p)$ [$o^-(p)$] common to two distinct SAIs tends under $\pi(X)$ to a non-hyperbolic saddle on $S^1 \subset S^2$ in forward [reverse] time.

PROOF. Suppose X satisfies the hypotheses of the theorem and $\pi(X)$ has $S^1 \subset S^2$ a closed orbit. Then $\pi(X)$ is Morse-Smale, hence structurally stable in \mathfrak{Y}^r , $r \geq 1$. Moreover, the equivalence homeomorphism can be required to preserve $S^1 \subset S^2$ and to be close to id_{S^2} . Hence by Proposition 2.1, X is structurally stable.

If X satisfies the hypotheses and $S^1 \subset S^2$ is composed entirely of critical points, then condition (3') implies that on a neighborhood of S^1 , $\pi(X)(x, y, z) = zY(x, y, z)$, where $Y(x, y, 0)$ is non-zero and not tangent to S^1 . Thus for all

$R \in \mathbb{R}^+$ sufficiently large, X points everywhere outward from (or inward to) $B_R(0, 0)$ all along $\partial B_R(0, 0)$. Moreover X has no SAIs. If a neighborhood N of $id_{\mathbb{R}^2}$ in the compact-open topology is specified, then there is a compact set K and a number $\epsilon > 0$ such that if h is ϵ -close to $id_{\mathbb{R}^2}$ on K , then $h \in N$. By hypothesis (2) and Proposition 2.2(iii), $\Omega(X) \subset \text{Per}(X)$, hence we may choose $R \in \mathbb{R}^+$ so large that $\Omega(X) \cup K \subset \text{Int}(B_R(0, 0))$, and non-tangency of X to $\partial B_R(0, 0)$ is true. Since $X|_{Cl(B_R(0, 0))}$ is Morse-Smale, there is a neighborhood \mathfrak{M} of X in $\mathfrak{X}(Cl(B_R(0, 0)))$ (uniform C^r topology) so that $Y \in \mathfrak{M}$ implies Y equivalent to X by h uniformly C^0 - ϵ -close to $id_{\mathbb{R}^2}$ on $Cl(B_R(0, 0))$, and Y is nowhere tangent to $\partial B_R(0, 0)$. By compactness of $B_R(0, 0)$, clearly there is a neighborhood \mathfrak{D} of X in \mathfrak{B}_n such that $Y \in \mathfrak{D}$ implies $Y \in \mathfrak{M}$. Given $Y \in \mathfrak{M}$, the corresponding h extends to a homeomorphism of \mathbb{R}^2 (possibly far from $id_{\mathbb{R}^2}$ off $B_R(0, 0)$) as usual: for $p \in \mathbb{R}^2 \setminus Cl(B_R(0, 0))$, there exist unique $\tau(p) \in \mathbb{R}$, $\tilde{p} \in \partial B_R(0, 0)$, such that $\eta_X(\tau(p), \tilde{p}) = p$; define $h(p) \equiv \eta_Y(-\tau(p), h(\tilde{p}))$. Since X has no SAIs, there are no separatrices other than those of the saddles of X , and these all lie in $B_R(0, 0)$, hence h preserves all distinguished orbits, and \mathfrak{D} is the required neighborhood of X in \mathfrak{B}_n .

Since in the coordinate charts discussed in Section 1 $\pi(X)$ is a scaled polynomial vector field, the only case remaining is X satisfying conditions (1) through (4) and $\pi(X)$ having a finite non-zero number of critical points on $S^1 \subset S^2$. Let a neighborhood N of $id_{\mathbb{R}^2}$ be specified, and let compact set K and $\epsilon > 0$ be such that if h is any homeomorphism that is C^0 - ϵ -close to id on K , then $h \in N$.

Let C_X denote $\text{Per}(X)$ together with all points lying in separatrices that limit on elements of $\text{Per}(X)$ in both directions. Choose $R \in \mathbb{R}^+$ so large that the ϵ -neighborhood of the compact set $C_X \cup K$ lies in $\text{Int}(B_R(0, 0))$. About each saddle point q of X choose a closed neighborhood N_q , bounded by a quadrilateral whose sides are line segments transverse to X , which is contained in an $\epsilon/4$ -neighborhood of q , and which contains no critical point besides q , and no entire closed orbit. About each critical point q [closed orbit γ] which is a source or a sink choose a closed neighborhood N_q [N_γ], bounded by a circle [pair of simple closed curves] transverse to X , similarly.

Each separatrix of a SAI, and each separatrix of a saddle point that escapes to infinity, tends under $\pi(X)$ to a unique critical point of $\pi(X)$ on $S^1 \subset S^2$. By assumption (3'), that critical point is a node, topological saddle, or saddle-node, in a neighborhood of which $\pi(X)$, expressed in local coordinates (1.1) or (1.2), is nowhere horizontal besides along the s -axis, corresponding to S^1 . Thus for R sufficiently large, the separatrix in question crosses $\partial B_R(0, 0)$ precisely once, without tangency, or, in the case of a separatrix of a SAI that escapes to infinity in both directions, exactly twice. Fix such a number R . On such a separatrix, choose a point p_1 outside $Cl(B_R(0, 0))$ and a point p_2 in the interior of the isolating neighborhood of the critical element on which it limits, or, in

the case of a separatrix of a SAI tending to infinity in both directions, outside $Cl(B_R(0, 0))$ and near the second critical point on $S^1 \subset S^2$ to which the separatrix tends. Choose points p_1, p_2 similarly for each separatrix joining critical elements of X in \mathbb{R}^2 . For each of the finitely many compact orbit segments $[p_1, p_2]$ so obtained, there exists a number $\delta > 0$ such that if at every point p on the orbit segment a perpendicular segment Σ_p of length 2δ (centered at p) is erected, then $\Sigma_p \cap \Sigma_q = \emptyset$ for $p \neq q$. Choose the minimum of all such numbers (one for each separatrix) and $\epsilon/4$, and for each orbit segment $[p_1, p_2]$ form the set $N[p_1, p_2]$ composed of all the transverse segments Σ_p for $p \in [p_1, p_2]$. Shrinking δ if necessary we can insure that the neighborhoods of orbit segments are pairwise disjoint. Let \mathfrak{F} denote the closed set which is the union of all the neighborhoods $N[p_1, p_2]$ of orbit segments, the isolating neighborhoods of critical elements of X , and the complement of $Int(B_R(0, 0))$.

Now consider a perturbation of X to a sufficiently close element Y of \mathfrak{P}_n . By well-known theorems, hypotheses (1) and (2), and the fact that choosing Y close enough to X in \mathfrak{P}_n makes Y arbitrarily C^r -close to X on pre-assigned compact sets, the critical elements and separatrices of saddles limiting on them of Y properly persist and lie in \mathfrak{F} . We now show that if σ is a separatrix of a saddle point of X escaping to infinity, then the corresponding separatrix σ' of Y also escapes to infinity, never leaving \mathfrak{F} ; that if σ is a separatrix of a SAI of X , there is a unique corresponding separatrix of a SAI of Y , lying in \mathfrak{F} (and that Y has no additional separatrices of SAIs); and that if the semi-orbit opposite to that in σ forming the SAI escapes to infinity, then so does the corresponding opposite semi-orbit of σ' .

First let σ be an unstable separatrix of a saddle q of X which escapes to infinity, with orbit segment (p_1, p_2) , $p_2 \in N_q$ and $p_1 \notin B_R(0, 0)$, and tending to critical point r_0 of $\pi(X)$ in $S^1 \subset S^2$. If r_0 is a node, then for Y close enough to X , $\pi(Y)$ is so close to $\pi(X)$ that σ' tends to a critical point r'_0 of $\pi(Y)$ arbitrarily close to r_0 , $\sigma' \subset \mathfrak{F}$, and for all $p \in [p_1, p_2]$, σ' crosses Σ_p precisely once. If r_0 is a saddle, then by hypothesis (2), moving along S^1 in either direction from r_0 , we encounter finitely many (possibly none) saddle-nodes, r_1, r_2, \dots, r_l , all contracting onto S^1 , followed by a stable node r_{l+1} . Under sufficiently small perturbation, σ' clearly remains in \mathfrak{F} and tends to a critical point arbitrarily near one of r_0, r_1, \dots, r_{l+1} , and crosses Σ_p precisely once, for each $p \in [p_1, p_2]$. If r_0 is a saddle-node, then by hypothesis (4a), proceeding in the direction induced on S^1 by the flow of $\pi(X)$ near r_0 , we must again have a sequence of saddle-nodes, each contracting onto S^1 , followed by a stable node, and reach the same conclusion. Note that in all these cases there is a subinterval $(a, b) \subset \Sigma_{p_1}$ containing p_1 in its interior, every point of which escapes to infinity under both X and Y . The same arguments handle the case of a stable separatrix of X that escapes to infinity in backward time.

Now let $\sigma = o^+(p)$ be a stable separatrix of a SAI of X with orbit segment $[p_1, p_2]$ and tending to $r_0 \in S^1 \subset S^2$. By hypothesis (4a), r_0 is a saddle of $\pi(X)$, and of the two critical points adjacent to r_0 on S^1 , at least one is a saddle point of $\pi(X)$. If both adjacent critical points, call them r_1 and r_2 , are saddles, then by hypothesis (4b) r_0 is hyperbolic, so for Y close enough to X , $\pi(Y)$ has a unique hyperbolic saddle point r'_0 near r_0 with stable separatrix σ' near σ and meeting each transverse segment in $N[p_1, p_2]$ precisely once. There are also unique closest critical points r'_1, r'_2 to r'_0 , near r_1 and r_2 , and each with a unique unstable separatrix, so each SAI of which σ' is now a stable separatrix persists. If precisely one of the critical points of $\pi(X)$ adjacent to r_0 on S^1 , call it r_1 , is a saddle point, then proceeding in the opposite direction along S^1 away from r_0 , by hypothesis (4a) we encounter finitely many saddle-nodes (possibly none), all contracting onto S^1 , followed by a stable node. Thus there is a subinterval $(a, b) \subset \Sigma_{p_1}$, containing p in its interior, such that every point of $(a, p_1]$ escapes to infinity, but no point of (p_1, b) does. Under small enough perturbation, Y has a unique pair of critical points r'_0 near r_0 and r'_1 near r_1 having no critical points between them along the arc of S^1 under consideration; r'_0 has a unique stable separatrix σ' , and r'_1 has a unique unstable separatrix, so the SAI persists, and its stable separatrix σ' is appropriately near σ for Y close enough to X . If σ' meets Σ_{p_1} at p'_1 , then every point on $(a, p'_1]$ escapes to infinity, but no point of (p'_1, b) does. The same arguments handle the case of an unstable separatrix of a SAI.

The persistence of escape to infinity of the opposite semi-orbit from a semi-orbit forming a separatrix of a SAI has exactly the same proof as persistence of escape to infinity of separatrices of saddles.

By hypotheses (3') and (4a), SAIs of X are in one-to-one correspondence with saddle connections of $\pi(X)$ that are subarcs of $S^1 \subset S^2$, hence there are finitely many of them, and clearly none are created under sufficiently small perturbation of X . Since therefore X has finitely many critical elements, saddle separatrices, and SAI separatrices, by the discussion of the previous paragraphs, there is a neighborhood \mathfrak{M} of X in \mathfrak{B}_n such that if $Y \in \mathfrak{M}$, then separatrices and critical elements of Y lie in \mathfrak{S} , and by a long but straightforward procedure, we can construct a homeomorphism $\hat{h}_Y: \mathfrak{S} \rightarrow \mathfrak{S}$ which (a) is $id_{\mathbb{R}^2}$ on $\partial\mathfrak{S}$, (b) is $C^{\epsilon-1/2}$ -close to id on $B_R(0, 0)$, and (c) carries critical elements and separatrices of Y back onto the corresponding objects of X . We can now finish the proof in several ways. On the one hand, we can apply the classical techniques of M. C. Peixoto and M. M. Peixoto [15] to create a homeomorphism of each canonical region of $\pi(X)|H^+(S^2)$ to the corresponding canonical region of $\pi(Y)|H^+(S^2)$. There are several new types of canonical regions, but their techniques carry over, inducing an equivalence homeomorphism of X and Y in \mathbb{R}^2 , which as in [15] can be made ϵ -close to $id_{\mathbb{R}^2}$ on the compact set K , simply by choosing Y close enough to X . On the other hand,

imitating the procedure in Kotus, Krych, and Nitecki [10], we can increase \mathfrak{S} in a natural way so as to include one orbit from each canonical region of X , hence of Y , in \mathbb{R}^2 , and can extend \hat{h}_Y to a homeomorphism $h_Y: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by the identity off \mathfrak{S} , and define a C^0 -flow on \mathbb{R}^2 by $\mu(t, p) = h_Y(\eta_Y(t, h^{-1}(p)))$. Obviously h_Y^{-1} carries μ -orbits back onto η_Y -orbits, and is $C^{0-\epsilon}/2$ -close to $id_{\mathbb{R}^2}$ on $B_R(0, 0)$. But μ has exactly the same critical elements and separatrices as η_X , and exactly the same orbits as η_Y off \mathfrak{S} . Hence for Y close enough to X , as outlined in [10] we can use techniques of Neumann [11] to construct a homeomorphism h of \mathbb{R}^2 that carries η_X -orbits to μ -orbits, and is $C^{0-\epsilon}$ -close to id on K , and carries η_X -orbits onto η_Y -orbits, as required. \square

We now turn to the necessity of the conditions of Theorem 3.2 for structural stability. We have the following partial result.

Theorem 3.3. *If $X \in \mathfrak{P}_n$ is structurally stable with respect to perturbation in \mathfrak{P}_n , and has no non-hyperbolic limit cycles of odd multiplicity, then X satisfies conditions (1) through (4) of Theorem 3.2.*

PROOF. Let $X \in \mathfrak{P}_n$ be structurally stable with respect to perturbation in \mathfrak{P}_n . Since X has any topological property satisfied by a dense subset of \mathfrak{P}_n , it follows from Theorem 2.9 that X has only finitely many critical points, each one a node, focus, or topological saddle, finitely many closed orbits, none semi-stable, and no saddle connections (including when SAIs are taken into account), and that $\Omega(X) = \text{Per}(X)$. Condition (2) of Theorem 3.2 thus follows immediately. The additional hypothesis on limit cycles of odd multiplicity implies that all cycles are hyperbolic.

To establish hyperbolicity of critical points, choose disjoint compact neighborhoods N_1, N_2, \dots, N_k and M_1, M_2, \dots, M_l of the closed orbits $\gamma_1, \gamma_2, \dots, \gamma_k$ and critical points p_1, p_2, \dots, p_l , isolating them from one another. Suppose X has at least one non-hyperbolic critical point, say p_1 . If $\det dX(p_1) > 0$ and $\text{Tr } dX(p_1) = 0$, then p_1 is a focus, since it is not a center. But then for any neighborhood \mathfrak{R} of X there is a rotation and scaling $X_\mu = X + \mu X^\perp$ of X that is in \mathfrak{R} , but has a closed orbit wholly contained in M_1 (cf. [1, Remark 3 of §10.3]). Then X_μ has at least $k + 1$ closed orbits, hence is not equivalent to X , a contradiction. Thus all non-hyperbolic critical points of X are nodes and saddles, and the M_i can be chosen so that the ∂M_i are circles and quadrilaterals to which X is transverse, except at the corners of the latter. It follows that there is a neighborhood $\mathfrak{R}'' \subset \mathfrak{R}$ each of whose elements has a least one critical point in each of the M_i . For we simply choose $\mathfrak{R}' \subset \mathfrak{R}$ so that $Y \in \mathfrak{R}'$ is also transverse to ∂M_i in the case of each non-hyperbolic node, and apply the Poincaré-Bendixson Theorem. Then choose $\mathfrak{R}'' \subset \mathfrak{R}'$ so that $Y \in \mathfrak{R}''$ is nowhere opposed to X on ∂M_i in the case of any

non-hyperbolic saddle, so that the index of Y with respect to ∂M_i is the same as that of X , namely (by Bendixson's formula) -1 , hence ∂M_i surrounds a critical point of Y .

But now if $\det dX(p_1) = 0$, we use the well-known fact that for any neighborhood \mathfrak{R} of X , there is a $Y \in \mathfrak{R}$ having at least two critical points in M_1 , so Y has at least $l + 1$ critical points, a contradiction. Thus X satisfies condition (1) of Theorem 3.2.

Condition (2) was mentioned already. To establish (3) for X suppose first that $S^1 \subset S^2$ is a closed orbit of $\pi(X)$, but non-hyperbolic, say (by condition (1)) asymptotically stable in $H^+(S^2)$. Then n is odd, and letting

$$m = \frac{n + 1}{2} - 1 \in \mathbb{Z},$$

the vector field

$$Y(x, y) = [P(x, y) - \alpha(n + 1)x(x^2 + y^2)^m] \frac{\partial}{\partial x} + [Q(x, y) - \alpha(n + 1)y(x^2 + y^2)^m] \frac{\partial}{\partial y}$$

is in \mathfrak{B}_n . It is readily verified that if $\alpha > 0$ is sufficiently small, $S^1 \subset S^2$ is a hyperbolic, asymptotically unstable limit cycle for $\pi(X)$, so that by Poincaré's Theorem there is a closed orbit of Y outside any pre-assigned compact subset of \mathbb{R}^2 . Since the closed orbits of X are all hyperbolic, they all persist under perturbation to Y , so Y is not equivalent to X , having one more closed orbit than X does, a contradiction. If condition (3') fails for X at $p \in S^1 \subset S^2$, it is not difficult to find an arbitrarily small change in the coefficients of X to produce a vector field Y for which $\pi(Y)$ has a critical point in the f.p.p. arbitrarily close to p , so that Y has at least one more critical point than X , contradicting stability of X . Since condition (3') holds for X , it follows from Theorem 65, §21 of [2] that every critical point of $\pi(X)$ on $S^1 \subset S^2$ is topologically a node, saddle, or saddle-node.

Finally, we must establish (4a) and (4b). First suppose that X is structurally stable but that (4a) fails, say $(o^+(r_0), o^-(s_0))$ a SAI such that under $\pi(X)$, $r_0 \rightarrow r_1 \in S^1$ and $s_0 \rightarrow s_1 \in S^1$ (reverse time), and r_1 is a saddle-node of $\pi(X)$. Let r_2 be the first critical point of $\pi(X)$ encountered in moving away from r_1 along S^1 in the direction opposite to that of s_1 , and let $\rho = \text{dist}_{S^1}(r_2, s_1)$. Rotating the coordinate system in \mathbb{R}^2 so that r_1 is at the end of the x -axis, it is not difficult to check, using chart (1.1), that there is an arbitrarily small change in the coefficients of X that removes the critical point of $\pi(X)$ at r_0 , so that either the SAI is destroyed completely, or has separatrices tending to points of $S^1 \subset S^2$ that are at least $\rho/2 > 0$ apart. Thus choosing $K = Cl(B_R(0, 0))$

of sufficiently large radius R , and $\epsilon = 1$, the new vector field Y obtained from X by the perturbation is either not equivalent to X at all, or else not by a homeomorphism that is C^0 - ϵ -close to $id_{\mathbb{R}^2}$ on K , contradicting stability of X .

If (4b) fails, we can always split the offending critical point in $S^1 \subset S^2$ into three or more by a small perturbation, similarly violating structural stability of X . \square

Let us say that a closed orbit γ of $X \in \mathfrak{D}$ is *structurally unstable in \mathfrak{D}* if for any neighborhoods N of γ in \mathbb{R}^2 and \mathfrak{X} of X in \mathfrak{D} , there exists $X_1 \in \mathfrak{X}$ which has other than exactly one closed orbit entirely contained in N . It is well known that if $\mathfrak{D} = \mathfrak{X}^r$, then γ is structurally unstable if and only if it is non-hyperbolic. Thus when $\mathfrak{D} = \mathfrak{F}_n$, hyperbolicity implies stability. The converse, however, has so far eluded proof:

Question 3.4. (Cf. [1 §6.3], [19]). *Is a non-hyperbolic limit cycle γ of $X \in \mathfrak{F}_n$ necessarily structurally unstable in \mathfrak{F}_n ?*

If γ has even multiplicity, a simple geometric argument shows that there is an arbitrarily small rotation of X producing a vector field Y having no closed orbit in N , so the answer to the question is «yes» in this case. An affirmative answer in general would mean that the conditions listed in Theorem 3.2 fully characterize structural stability in \mathfrak{F}_n .

4. Genericity of stability

Results of the previous sections yield the following genericity result.

Theorem 4.1. *There is a dense, open subset \mathfrak{G} of \mathfrak{F}_n , every element of which is structurally stable with respect to perturbation in either \mathfrak{X}^r , $r \geq 1$ (Whitney C^r topology), or in \mathfrak{F}_n (coefficient topology).*

PROOF. Let \mathfrak{G} be the set described in Theorem 2.9, which is dense and open in \mathfrak{F}_n . Structural stability in \mathfrak{X}^r and \mathfrak{F}_n follow from Theorems 3.1 and 3.2 respectively. \square

5. Equivalence Homeomorphisms

This section examines the role of restrictions on the equivalence homeomorphism in the definition of structural stability. In the case of arbitrary smooth perturbation, we are able to duplicate Peixoto's theorem from the compact case.

Theorem 5.1. *In the definition of structural stability of $X \in \mathfrak{P}_n$ with respect to perturbation in \mathfrak{X}^r , $r \geq 1$, the requirement that the equivalence homeomorphisms lie in a pre-assigned neighborhood of $id_{\mathbb{R}^2}$ is superfluous. That is, Theorem 3.1 is also valid in the case that no restriction is put on h .*

PROOF. It is obvious that if $X \in \mathfrak{P}_n$ is structurally stable when h is restricted to being close to $id_{\mathbb{R}^2}$, then X is structurally stable when h is unrestricted. To prove the converse, we suppose that $X \in \mathfrak{P}_n$ is structurally stable in the setting of arbitrary equivalence homeomorphisms, and demonstrate that X satisfies conditions (1) through (3) of Theorem 3.1, hence by the sufficiency statement of that theorem is structurally stable in the original sense.

To begin with, we claim that X has no semi-stable limit cycles. For if it did, then given any neighborhood \mathfrak{R} of X in \mathfrak{X}^r , we could find a positive C^∞ function $\epsilon(x, y): \mathbb{R}^2 \rightarrow \mathbb{R}$ so small that $Y(x, y) = X(x, y) + \epsilon(x, y)X^\perp(x, y)$ would lie in \mathfrak{R} . Yet such Y has only hyperbolic limit cycles ([1], Theorem 71, §32), contradicting stability of X . But then all the limit cycles of X are in fact hyperbolic, since the fact that we have smooth perturbations at our disposal means that if X were to have a non-hyperbolic limit cycle γ , then by an arbitrarily small C^r perturbation of X supported in a neighborhood of γ , a semi-stable limit cycle can be made to bifurcate off from γ (see for example [10], Corollary 8.7(ii)).

If X had more than n^2 critical points, then there would be an algebraic curve of critical points of X [22]. In such a case there exists a $Y \in \mathfrak{P}_n$ which is arbitrarily close to X in \mathfrak{P}_n and which has finitely many critical points. Then letting $f(x, y)$ be a C^∞ bump function which is identically 1 on $B_R(0, 0)$ and vanishes off $B_{2R}(0, 0)$, for R large enough and Y close enough to X , $f(x, y)Y(x, y)$ is close to X in \mathfrak{X}^r , but is not equivalent to X , a contradiction. It is well known that an arbitrarily C^r -small perturbation of X supported on a neighborhood of each of its critical points will yield a vector field with only hyperbolic critical points, hence each of the finitely many critical points of X is a node, focus, or topological saddle. At none of them can $\det dX$ vanish, else by a C^r perturbation the critical point in question can be split into several, contradicting stability of X . Thus any non-hyperbolic critical point p of X is a weak focus. Regardless of its multiplicity there is a C^r perturbation of X which causes an odd number of limit cycles to bifurcate from p , so that either the stability of p changes, or the new vector field near X has a semi-stable limit cycle near p , in either case contradicting the stability of X . In sum, X has finitely many critical points, all hyperbolic. Thus (1) of Theorem 3.1 holds.

The requirement of condition (2) now makes sense, and it is clear that it must actually hold for X , since saddle connections are easily destroyed by local smooth perturbation.

Finally, (3) follows from Proposition 2.3. \square

In the case of polynomial perturbation of $X \in \mathfrak{P}_n$, stability of X depends on whether or not h is restricted to be close to $id_{\mathbb{R}^2}$, at least for n large, the reason being that separatrices of SAIs can make a sudden jump as X is changed. On the other hand, if closeness to id is overly restricted, say uniformly C^0 -close rather than close in the compact-open topology, then as in the case of general smooth X ([10], Proposition 2.10), any X with an orbit escaping to infinity in both time-directions will be structurally unstable. A detailed analysis of the effects of restriction of h in the setting of polynomial perturbations will be given in [4].

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