Revista Matemática Iberoamericana
Vol. 3, n.^{os} 3 y 4, 1987

On the Spaces of Eisenstein Series of Hilbert Modular Groups

Toshitsune Miyake

Introduction

In the theory of automorphic forms, it is a basic result due to Hecke [1] that the space of elliptic modular forms of integral weight is the direct sum of the space of cusp forms and the space of Eisenstein series. This has been expected to be true in general and in fact it has been proved for Hilbert modular forms and for modular forms of half integral weight by several people, such as Kloosterman [2], Petersson [4], Pei [3], Shimizu [5], [6] and Shimura [9]. Especially Shimura [9] investigated automorphic eigen forms, which includes holomorphic automorphic forms, in great detail and proved that in most cases the orthogonal complements of (eigen) cusp forms in the spaces of the automorphic eigen forms of Hilbert modular groups are generated by Eisenstein series (and some other forms derived from Eisenstein series in certain special cases). There he omitted the case when the set of eigen values of differential operators is multiple in his sense. The purpose of the present paper is to prove that his result, which is the generalization of the classical fact mentioned in the beginning, is true without any restriction on eigenvalues.

Here we explain our result breifly restricting ourselves to only the case of integral weight, though our result includes also the case of half integral weight. Let F be a totally real algebraic number field, and $\mathbf{a}$ the set of all archimedian primes of F. Let H be the upper half plane. By $\mathbb{Z}^{\mathbf{a}}$, $\mathbb{C}^{\mathbf{a}}$ and $H^{\mathbf{a}}$, we understand copies the product of $\mathbf{a}$ of $\mathbb{Z}$, $\mathbb{C}$ and H, respectively. Then

$SL_2(F)$ acts on $H^{\mathbf{a}}$ in the usual way. For $\sigma \in \mathbb{Z}^{\mathbf{a}}$ and $v \in \mathbf{a}$, we define the differential operator L_v^σ on $H^{\mathbf{a}}$ by

$$L_v^\sigma = -4y_v^{2-\sigma_v}(\partial/\partial z_v)y_v^{\sigma_v}(\partial/\partial \bar{z}_v),$$

where $z = (z_v)$ is the variable on $H^{\mathbf{a}}$ and $y_v = \mathrm{Im}(z_v)$. Let $\lambda = (\lambda_v) \in \mathbb{C}^{\mathbf{a}}$. For each congruence subgroup Γ of $SL_2(F)$, we denote by $\mathcal{A}(\sigma, \lambda, \Gamma)$ the set of all C^∞-functions f on $H^{\mathbf{a}}$ satisfying the following conditions:

(1) $f(\gamma z) = \prod_{\gamma \in \mathbf{a}} (c_v z_v + d_v)^{\sigma_v} f(z)$ for every $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$;

(2) $L_v^\sigma f = \lambda_v f$ for every $v \in \mathbf{a}$;

(3) f is slowly increasing at every cusp.

We also denote by $\mathcal{S}(\sigma, \lambda, \Gamma)$ the space of cusp forms in $\mathcal{A}(\sigma, \lambda, \Gamma)$ which is defined by the condition at cusps as usual. Then our main result is that the orthogonal complement of $\mathcal{S}(\sigma, \lambda, \Gamma)$ in $\mathcal{A}(\sigma, \lambda, \Gamma)$ with respect to the Petersson inner product is generated by the special values of Eisenstein series with parameters (and some other functions derived from Eisenstein series in some special cases).

Recently Shimizu proved that it is also valid for automorphic eigen forms on GL_2 over any algebraic number field in the case of integral weight using representation theory ([6]).

1. Automorphic Eigen Forms

Let F be a totally real number field and $\mathbf{a}$ the set of all archimedian primes of F. For each set X, we denote by $X^{\mathbf{a}}$ the product of $\mathbf{a}$ copies of X or the set $\{(x_v)_v \mid v \in \mathbf{a}\}$. For each element x of $X^{\mathbf{a}}$, we denote by x_v the v-component of x. For two elements c and x of $\mathbb{C}^{\mathbf{a}}$, we put

$$c^x = \prod_v c_v^{x_v}$$

whenever each factor is well defined.

Let H be the upper half plane $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. For each $g \in SL_2(\mathbb{R})$ and $z \in H$, we put

$$g(z) = \frac{az+b}{cz+d}, \qquad j(g, z) = cz + d \quad \text{if} \quad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

By $z \mapsto g(z)$, $SL_2(\mathbb{R})$ acts on H and therefore $SL_2(\mathbb{R})^{\mathbf{a}}$ acts on $H^{\mathbf{a}}$. For $g \in SL_2(\mathbb{R})^{\mathbf{a}}$, $z = (z_v) \in H^{\mathbf{a}}$ and $\sigma = (\sigma_v) \in \mathbb{Z}^{\mathbf{a}}$, we put

$$j_g(z)^\sigma = j(g, z)^\sigma = \prod_v j(g_v, z_v)^{\sigma_v}.$$

We shall define automorphic forms of integral weight and half integral weight. Put $u = (1, 1, \dots, 1) \in \mathbb{R}^{\mathbf{a}}$. A weight will be either an element of $\mathbb{Z}^{\mathbf{a}}$ (Case I, integral weight) or an element of $(1/2)u + \mathbb{Z}^{\mathbf{a}}$ (Case II, half integral weight). For each weight σ, we denote by $\mathcal{G}_\sigma$ the set of all pairs $(g, l(z))$ with an element $g \in SL_2(F)$ and a holomorphic function $l(z)$ on $H^{\mathbf{a}}$ such that

$$l(z)^2 = t \cdot j(g, z)^\sigma, \qquad t \in \mathbb{C}, \qquad |t| = 1.$$

The set $\mathcal{G}_\sigma$ is a group by the group law defined by

$$(g, l)(g', l') = (gg', l(g'(z))l'(z)).$$

We denote the projection of $\mathcal{G}_\sigma$ to $SL_2(F)$ by pr, or $pr((g, l)) = g$. For $\alpha = (g, l(z)) \in \mathcal{G}_\sigma$, we denote $l(z)$ also by $l_\alpha(z)$ and put $\alpha(z) = g(z)$ for $z \in H^{\mathbf{a}}$ and $j_\alpha^\sigma = j_g^\sigma$. For a function f on $H^{\mathbf{a}}$ and $\alpha \in \mathcal{G}_\sigma$, we define the function $f \| \alpha$ by

$$(f \| \alpha)(z) = l_\alpha(z)^{-1} f(\alpha(z)), \qquad z \in H^{\mathbf{a}}.$$

For $z = (z_v) \in H^{\mathbf{a}}$, we put $y_v = \operatorname{Im}(z_v)$ and $y = (y_v)$, and consider y as an $\mathbb{R}^{\mathbf{a}}$-valued function on $H^{\mathbf{a}}$. Then

$$y^p \| \alpha = l_\alpha^{-1} |j_\alpha|^{-2p} y^p \qquad (p \in \mathbb{R}^{\mathbf{a}}, \alpha \in \mathcal{G}_\sigma). \tag{1.1}$$

For $v \in \mathbf{a}$ and $\sigma \in \mathbb{R}^{\mathbf{a}}$, we define differential operators ϵ_v, δ_v^σ and L_v^σ operating on C^∞-functions f on $H^{\mathbf{a}}$ by

$$\epsilon_v f = -y_v^2 \, \partial f / \partial \bar{z}_v, \tag{1.2}$$

$$\delta_v^\sigma = -y_v^{\sigma v} \, \partial(y_v^{\sigma v} f)/\partial z_v, \tag{1.3}$$

$$L_v^\sigma = 4\, \delta_v^{\sigma'} \epsilon_v, \qquad \sigma'_v = \sigma_v - 2. \tag{1.4}$$

Let W be the subset of $SL_2(F)$ defined by [9, (1.10)] and h_g the holomorphic function on $H^{\mathbf{a}}$ for each $g \in W$ given in [9, Prop. 3.2]. For each weight σ, let Λ_σ be the injection

$$\Lambda_\sigma\colon\ SL_2(F) \to \mathcal{G}_\sigma \qquad \text{(Case I)},$$
$$\Lambda_\sigma\colon\ W \to \mathcal{G}_\sigma \qquad \text{(Case II)}$$

given by

$$\Lambda_\sigma(g) = \begin{cases} (g, j_g^\sigma), & g \in SL_2(F) \qquad \text{(Case I)}, \\ (g, h_g j_g^{\sigma - u/2}), & g \in W \qquad \text{(Case II)}. \end{cases}$$

Let $\mathfrak{g}$ be the maximal order of F, $\mathfrak{g}^\times$ the unit group of $\mathfrak{g}$ and $\mathfrak{d}$ the different of F. For each integral ideal $\mathfrak{c}$ of F, we put

$$\Gamma(\mathfrak{c}) = \{\alpha \in SL_2(F) \cap M_2(\mathfrak{g}) \mid \alpha - 1 \in \mathfrak{c} M_2(\mathfrak{g})\}.$$

We call a subgroup Δ of $\mathcal{G}_\sigma$ a congruence subgroup if it satisfies the following two conditions:

(1.5) Δ is isomorphic to a subgroup of $SL_2(F)$ by pr,

(1.6) Δ contains $\Lambda_\sigma(\Gamma(\mathfrak{c}))$ as a subgroup of finite index for some $\mathfrak{c}$ ($\mathfrak{c} \subset 8\mathfrak{d}^{-1}$ in Case II).

A real analytic function f on $H^{\mathbf{a}}$ is called an *automorphic eigen form* with respect to a congruence subgroup Δ of $\mathcal{G}_\sigma$, if it safisfies the following three conditions:

(1.7) $f \| \alpha = f$ for every $\alpha \in \Delta$;

(1.8) $L_v^\sigma f = \lambda_v f$ with $\lambda_v \in \mathbb{C}$ for every $v \in \mathbf{a}$;

(1.9) for every $\alpha \in \mathcal{G}_\sigma$, there exist positive numbers A, B and C (depending on f and α) such that

$$y^{\sigma/2}|(f \| \alpha)(x+iy)| \leqslant Ay^{cu} \quad \text{if} \quad y^u > B.$$

For $\lambda = (\lambda_v) \in \mathbb{C}^{\mathbf{a}}$, we denote by $\mathcal{A}(\sigma, \lambda, \Delta)$ the set of all such f and by $\mathcal{A}(\sigma, \lambda)$ the union of $\mathcal{A}(\sigma, \lambda, \Delta)$ for all congruence subgroups Δ of $\mathcal{G}_\sigma$. We know $\mathcal{A}(\sigma, \lambda, \Delta)$ is finite dimensional and $\mathcal{A}(\sigma, \lambda)$ is stable under the action of $\alpha \in \mathcal{G}_\sigma$. If $f \in \mathcal{A}(\sigma, \lambda, \Delta)$, then it has a Fourier expansion of the form

$$f(x+iy) = b(y) + \sum_{0 \neq h \in \mathfrak{m}} b_h W(hy; \sigma, \lambda) e(hx)$$

with $\mathfrak{m}$ a lattice of F, $b_h \in \mathbb{C}$, $b(y)$ a function on $\mathbb{R}^{\mathbf{a}}$, $e(z) = \exp(2\pi i \Sigma z_v)$ for $z \in \mathbb{C}^{\mathbf{a}}$, and W the Whittaker function defined by [9, (2.19) and (2.20)]. We call $b(y)$ the constant term of f and call f a *cusp form* if the constant term of $f \| \alpha$ vanishes for every $\alpha \in \mathcal{G}_\sigma$. We denote the set of all cusp forms in $\mathcal{A}(\sigma, \lambda)$ by $\mathcal{S}(\sigma, \lambda)$ and put $\mathcal{S}(\sigma, \lambda, \Delta) = \mathcal{A}(\sigma, \lambda, \Delta) \cap \mathcal{S}(\sigma, \lambda)$.

For two continuous functions f and g satisfying (1.7), we put

$$\langle f, g \rangle = \mu(\Delta \backslash H^{\mathbf{a}})^{-1} \int_{\Delta \backslash H^{\mathbf{a}}} \bar{f} g y^\sigma \, d\mu(z)$$

where

$$d\mu(z) = y^{-2u} \prod_{v \in \mathbf{a}} dx_v \, dy_v.$$

This does not depend on the choice of Δ. We define subspaces $\mathcal{N}(\sigma, \lambda)$ and $\mathcal{N}(\sigma, \lambda, \Delta)$ of $\mathcal{A}(\sigma, \lambda)$ by

$$\mathcal{N}(\sigma, \lambda, \Delta) = \{ g \in \mathcal{A}(\sigma, \lambda, \Delta) \mid \langle f, g \rangle = 0 \quad \text{for all} \quad f \in \mathcal{S}(\sigma, \lambda, \Delta) \},$$

$$\mathcal{N}(\sigma, \lambda) = \{ g \in \mathcal{A}(\sigma, \lambda) \quad \mid \langle f, g \rangle = 0 \quad \text{for all} \quad f \in \mathcal{S}(\sigma, \lambda) \}.$$

Then $\mathcal{N}(\sigma, \lambda, \Delta) = \mathcal{N}(\sigma, \lambda) \cap \mathcal{A}(\sigma, \lambda, \Delta)$.

Let U be a subgroup of $\mathfrak{g}^\times$ of finite index. We call $\tau = (\tau_v) \in \mathbb{R}^{\mathbf{a}}$ *U-admissible* if

$$\Sigma \tau_v = 0 \quad \text{and} \quad |a|^{i\tau} = 1 \quad \text{for all} \quad a \in U. \tag{1.10}$$

We call τ ($\in \mathbb{R}^{\mathbf{a}}$) *admissible* if it is U-admissible for some U and denote by T_U the set of all U-admissible τ.

Hereafter we fix a weight σ and write $\mathcal{G} = \mathcal{G}_\sigma$ and $\Lambda = \Lambda_\sigma$. We call $\lambda \in \mathbb{C}^{\mathbf{a}}$ *critical* if $4\lambda_v = (1 - \sigma_v)^2$ for all $v \in \mathbf{a}$, and call λ *non-critical* if it is not critical.

Proposition 1.1. ([9, Prop. 3.1]). *The constant term $b(y)$ of an element $f \in \mathcal{A}(\sigma, \lambda)$ has one of the following forms.*

(1) *If λ is critical then*

$$b(y) = a_1 y^q + a_2 y^q \log y^u,$$

where $q = (q_v)$ and q_v is the multiple root of $X^2 - (1 - \sigma_v)X + \lambda_v = 0$.

(2) *If λ is non-critical then $b(y)$ is a linear combination of y^p with $p = (p_v) \in \mathbb{C}^{\mathbf{a}}$ satisfying*

(1.11) *p_v is a root of $\psi_v(X) = X^2 - (1 - \sigma_v)X + \lambda_v$,*

(1.12) *$p = su - (\sigma - i\tau)/2$ with $s \in \mathbb{C}$ and an admissible $\tau \in \mathbb{R}^{\mathbf{a}}$.*

When λ is non-critical, an element $p \in \mathbb{C}^{\mathbf{a}}$ is called an *exponent* attached to λ if it satisfies (1.11) and (1.12). We denote by $C(\sigma, \lambda)$ the set of all exponents attached to λ. For $p = (p_v) \in \mathbb{C}^{\mathbf{a}}$, we put $\bar{p} = (\bar{p}_v) \in \mathbb{C}^{\mathbf{a}}$. Then $C(\sigma, \bar{\lambda}) = \{\bar{p} \mid p \in C(\sigma, \lambda)\}$. We note if $C(\sigma, \lambda) = \phi$, then $\mathcal{A}(\sigma, \lambda) = \mathcal{S}(\sigma, \lambda)$. We call λ *simple* either if λ is critical or if λ is non-critical and $C(\sigma, \lambda)$ consists of exactly two elements. We also call λ *multiple* if $C(\sigma, \lambda)$ has more than two elements.

Lemma 1.2. *Assume λ is non-critical. For $p \in C(\sigma, \lambda)$, put $p' = u - \sigma - p$. Then $p' \in C(\sigma, \lambda)$ and $p' \neq p$. Furthermore $\bar{\lambda}$ is non-critical and $\overline{p'} = \bar{p}' \in C(\sigma, \bar{\lambda})$.*

PROOF. Since p_v is a root of $\psi_v(X) = 0$, $1 - \sigma_v - p_v$ is also a root of $\psi_v(X) = 0$. As $\psi_v(X) = 0$ has simple roots for at least one v, we have $p' \neq p$. Further since $p' = (1 - s)u - (\sigma + i\tau)/2$, p' satisfies also (1.12). The last statement is obvious. □

2. A Bilinear Relation of Coefficients of Constant Terms

The purpose of this section is to generalize [9, Theorem 6.1] to multiple λ.

Hereafter we assume λ is non-critical. Let Δ be a congruence subgroup of $\mathcal{G}$ and put $\Gamma = pr(\Delta)$. Put

$$P = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(F) \,\middle|\, c = 0 \right\}, \qquad \mathcal{P} = \{\alpha \in \mathcal{G} \mid pr(\alpha) \in P\}.$$

Then $\mathcal{P} \backslash \mathcal{G}/\Delta$ is a finite set. We call classes of $\mathcal{P} \backslash \mathcal{G}/\Delta$ cusp classes. Take a complete set of representatives X for $\mathcal{P} \backslash \mathcal{G}/\Delta$. For each $\xi \in X$, we put $Q_\xi = P \cap pr(\xi\Delta\xi^{-1})$. Let Θ be a subgroup of P of the form

$$\Theta = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \,\middle|\, a \in U_1, b \in \mathfrak{m} \right\},$$

with a fractional ideal $\mathfrak{m}$ of F and a subgroup U of $\mathfrak{g}^\times$ of finite index. Take U_1 and $\mathfrak{m}$ so that $\Lambda(\Theta) \subset \xi\Delta\xi^{-1}$ for all $\xi \in X$ and every a $(\in U_1)$ is totally positive. For $0 < r \in \mathbb{R}$, put

$$T_r = \{z \in H^{\mathbf{a}} \mid y^u > r\}, \qquad M_r = \{z \in H^{\mathbf{a}} \mid y^u = r\}.$$

Put $U = \{a^2 \mid a \in U_1\}$. Then $\Theta \backslash M_r$ is isomorphic to the product of $\mathbb{R}^{\mathbf{a}}/\mathfrak{m}$ and $\{y \in \mathbb{R}^{\mathbf{a}} \mid y^u = r, y >> 0\}/U$ up to the difference of orientations of $(-1)^{n(n-1)/2}$. For a fixed $v \in \mathbf{a}$, we put

$$\omega = y^{-2u} \prod_{v \in \mathbf{a}} dx_v \wedge dy_v,$$

$$\zeta_v = (i/2) y_v^{-2u} d\bar{z}_v \prod_{w \neq v} dx_w \wedge dy_w.$$

Since [9, Lemma 6.2] holds only when $\tau = 0$, its proper statement should be the following

Lemma 2.1. *For $s \in \mathbb{C}$ and $\tau \in T_U$, we have*

$$\int_{\Theta \backslash M_r} y^{su+\tau+v} \zeta_v = \begin{cases} (-i/2)\mu(\mathbb{R}^{\mathbf{a}}/\mathfrak{m}) R_U r^{s-1} & (\tau = 0) \\ 0 & (\tau \neq 0), \end{cases}$$

where $R_U = R_F[\mathfrak{g}^\times : U \cdot \{\pm 1\}]$ with the regulator R_F of F and $\mu(\mathbb{R}^{\mathbf{a}}/\mathfrak{m})$ is the volume of $\mathbb{R}^{\mathbf{a}}/\mathfrak{m}$.

Let $f \in \mathfrak{A}(\sigma, \lambda, \Delta)$ and $g \in \mathfrak{A}(\sigma, \bar{\lambda}, \Delta)$. For each $\xi \in X$, we write

$$f \,\|\, \xi^{-1} = \Sigma a_{p,\xi} y^p + (\text{non-constant terms})$$

and

$$g \,\|\, \xi^{-1} = \Sigma b_{\bar{p},\xi} y^{\bar{p}} + (\text{non-constant terms}),$$

where p is taken over the elements of $C(\sigma, \lambda)$. Let $C_1 = C_1(\sigma, \lambda)$ be a subset of $C(\sigma, \lambda)$ such that

$$C(\sigma, \lambda) = C_1(\sigma, \lambda) \cup \{p' \mid p \in C_1(\sigma, \lambda)\} \qquad \text{(disjoint)}.$$

Then we can generalize [9, Theorem 6.1] to the following

Theorem 2.2. *We have*

$$\sum_{p \in C_1} (p_v - p'_v) \sum_{\xi \in X} \nu_\xi (a_{p,\xi} \bar{b}_{\bar{p}',\xi} - a_{p',\xi} \bar{b}_{\bar{p},\xi}) = 0,$$

where $\nu_\xi = [Q_\xi \cdot \{\pm 1\} : \Theta \cdot \{\pm 1\}]^{-1}$.

PROOF. Take a positive number r so that any two sets $\xi^{-1} Q_\xi \xi \setminus \xi^{-1}(T_r)$ $(\subset \Gamma \setminus H^{\mathbf{a}}, \xi \in X)$ have no intersection points. Let J be a union of small neighbourhoods of elliptic points on $\Gamma \setminus H^{\mathbf{a}}$, which are compact manifolds with boundary. Inducing a natural orientation into each set, we see that

$$\partial K = \sum_{\xi \in X} \xi^{-1} Q_\xi \xi \setminus \xi^{-1}(T_r) - \partial J.$$

Therefore for a Γ-invariant C^∞-form ϕ on $H^{\mathbf{a}}$ of codegree 1, we have

$$\int_K d\phi = \sum_{\xi \in X} \nu_\xi \int_{B_\xi} \phi \circ \xi^{-1} - \int_{\partial J} \phi,$$

where $B_\xi = \xi^{-1} \Theta \xi \setminus \xi^{-1}(M_r)$. We put $\phi = \bar{f}(\epsilon_v g) y^\sigma \zeta_v$ and $\phi' = \bar{g}(\epsilon_v f) y^\sigma \zeta_v$. Then

$$d\phi = \frac{1}{4} \bar{f} L_v^\sigma g y^\sigma \omega - \overline{(\epsilon_v f)}(\epsilon_v g) y^{\sigma'} \omega \qquad (\sigma' = \sigma - 2v)$$

and

$$d(\bar{\phi} - \phi') = \frac{1}{4}(f \overline{L_v^\sigma g} - L_v^\sigma f \bar{g}) y^\sigma \omega$$

$$= \frac{1}{4}(\lambda_v - \lambda_v) f \bar{g} y^\sigma \omega = 0.$$

This implies that

$$\int_{\partial J} (\bar{\phi} - \phi') = \sum_{\xi \in X} \nu_\xi \int_{B_\xi} (\bar{\phi} - \phi') \circ \xi^{-1}.$$

Now we have the expansions

$$\bar{\phi} \circ \xi^{-1} = \frac{i}{2} \sum_{p,q} q_v a_{p,\xi} \bar{b}_{\bar{q},\xi} y^{\bar{p} + \bar{q} + v + \sigma} \zeta_v + \cdots,$$

and

$$\phi' \circ \xi^{-1} = -\frac{i}{2}\sum_{p,q} p_v a_{p,\xi}\bar{b}_{\bar{q},\xi} y^{\bar{p}+\bar{q}+v+\sigma}\zeta_v + \cdots,$$

where p and q are taken over $C(\sigma, \lambda)$. Though the unwritten terms also contain terms which do not contain $e(hx)$, they tend to 0 in our later process of letting $r \to \infty$ since they decrease rapidly by [9, Prop. 2.1(2)]. Applying Lemma 2.1, we obtain

$$\int_{\partial J}(\bar{\phi} - \phi') = \mu(\mathbb{R}^{\mathfrak{a}}/\mathfrak{m})R_U \sum_{p \in C_1} (p_v - p'_v) \sum_{\xi \in X} \nu_\xi(a_{p,\xi}\bar{b}_{\bar{p}',\xi} - a_{p',\xi}\bar{b}_{\bar{p},\xi}) + \cdots,$$

where $C_1 = C_1(\sigma, \lambda)$. Since the unwritten terms tend to 0 when $r \to \infty$ as we mentioned above and $\int_{\partial J}(\bar{\phi} - \phi') \to 0$ when $J \to \phi$, we have the assertion. □

3. Eisenstein Series

Let ρ be an element of $\mathbb{C}^{\mathfrak{a}}$ such that

$$\rho = (\sigma - \tau)/2$$

with an admissible τ. A cusp class $\mathcal{P}\xi\Delta$ ($\xi \in X$) is called ρ-regular if

$$y^{-\rho} \,\|\, \alpha = y^{-\rho} \quad \text{for every} \quad \alpha \in \mathcal{P} \cap \xi\Delta\xi^{-1}.$$

We denote by $Y(\rho)$ the subset of X that represents all ρ-regular cusps. We also denote by $\varkappa(\rho)$ the number of elements of $Y(\rho)$. For each congruence subgroup Δ, we define its Eisenstein series by

$$E(z, s; \rho, \Delta) = \begin{cases} \sum_{\alpha \in \mathcal{P}\cap\Delta\backslash\Delta} y^{su-\rho} \,\|\, \alpha & \text{if } \mathcal{P}\Delta \text{ is } \rho\text{-regular}, \\ 0 & \text{otherwise.} \end{cases}$$

The series is convergent for $\mathrm{Re}(s) > 1$ and can be continued as a meromorphic function in s to the whole s-plane. If $\Delta \supset \Delta'$, we see that

$$[\mathcal{P} \cap \Delta : \mathcal{P} \cap \Delta']E(z, s; \rho, \Delta) = \sum_{\gamma \in \Delta\backslash\Delta'} E(z, s; \rho, \Delta') \,\|\, \gamma. \tag{3.1}$$

For each cusp class $\mathcal{P}\xi\Delta$, we put

$$E_\xi(z, s; \rho, \Delta) = E(z, s; \rho, \xi\Delta\xi^{-1}) \,\|\, \xi.$$

Then we see

$$E_\xi(z, s; \rho, \Delta) \,\|\, \alpha = E_\xi(z, s; \rho, \Delta) \quad \text{for every} \quad \alpha \in \Delta. \tag{3.2}$$

We denote by $\mathcal{E}[\rho, \Delta]$ the complex vector space generated by the functions $E_\xi(z, s; \rho, \Delta)$ for all $\xi \in X$. Using (3.1), we can prove that if $\Delta \supset \Delta'$, then $\mathcal{E}[\rho, \Delta] \subset \mathcal{E}[\rho, \Delta']$ and

$$(3.3) \qquad \mathcal{E}[\rho, \Delta] = \{g = g(z, s) \in \mathcal{E}[\rho, \Delta'] \mid g \parallel \alpha = g \text{ for every } \alpha \in \Delta\}.$$

Now by [9, Prop. 5.2], we have for $\xi, \eta \in Y(\rho)$,

$$(3.4) \qquad E_\xi \parallel \eta^{-1} = \delta_{\xi\eta} y^{su-\rho} + f_{\xi\eta}(s) y^{u-su-\bar{\rho}} + \sum_{0 \neq h \in \mathfrak{n}} g_{\xi\eta}(h, s, y) e(hx)$$

where $f_{\xi\eta}$ and $g_{\xi\eta}$ are meromorphic functions in s, $\delta_{\xi\eta}$ is the Kronecker's delta and $\mathfrak{n}$ is a lattice in F.

For $s_0 \in \mathbb{C}$, we denote by $\mathcal{E}[s_0, \rho, \Delta]$ the subspace of $\mathcal{E}[\rho, \Delta]$ consisting of all functions $g(z, s)$ that are holomorphic at s_0. We put

$$\mathcal{E}(s_0, \rho, \Delta) = \{g(z, s_0) \mid g \in \mathcal{E}[s_0, \rho, \Delta]\}.$$

Then by [9, Prop. 7.1],

$$\mathcal{E}(s_0, \rho, \Delta) \subset \mathcal{A}(\sigma, \lambda, \Delta)$$

with $\lambda = (\lambda_v)$, $\lambda_v = (s_0 - \rho_v)(1 - s_0 - \bar{\rho}_v)$.

The following lemma is stated in [9, Prop. 7.2] under the assumption that λ is simple, but the assertion holds also for multiple λ without any changes of the proof.

Lemma 3.1. (1) $\dim \mathcal{E}[\rho, \Delta] = \varkappa(\rho)$.

(2) *The map* $g(z, s) \to g(z, s_0)$ *gives an isomorphism of* $\mathcal{E}[s_0, \rho, \Delta]$ *onto* $\mathcal{E}(s_0, \rho, \Delta)$.

Conversely for a fixed non-critical λ, we express $p \in C(\sigma, \lambda)$ as

$$p = s_p u - (\sigma - i\tau_p)/2$$

with $s_p \in \mathbb{C}$ and an admissible τ_p. We put $\rho_p = (\sigma - i\tau_p)/2$ and also set $Y(p) = Y(\rho_p)$ and $\varkappa(p) = \varkappa(\rho_p)$. We note

$$p' = (1 - s_p)u - \rho_p \quad \text{and} \quad \varkappa(p') = \varkappa(p)$$

by [9, Prop. 7.5].

Theorem 3.2. *Suppose* λ *is non-critical,* $\mathcal{E}[\rho_p, \Delta] = \mathcal{E}[s_p, \rho_p, \Delta]$ *and* $\mathcal{E}[\bar{\rho}_p, \Delta] = \mathcal{E}[\bar{s}_p, \bar{\rho}_p, \Delta]$ *for any* $p \in C_1(\sigma, \lambda)$. *Then*

$$\mathfrak{N}(\sigma, \lambda, \Delta) = \bigoplus_{p \in C_1} \mathcal{E}(s_p, \rho_p, \Delta) \qquad (C_1 = C_1(\sigma, \lambda)).$$

PROOF. It is easy to see that the right-hand side is a direct sum and is contained in $\mathfrak{N}(\sigma, \lambda, \Delta)$ by [9, Prop. 7.1]. Therefore we have

$$\dim(\mathfrak{N}(\sigma, \lambda, \Delta)) \geqslant \sum_{p \in C_1} \varkappa(p).$$

For each $p \in C_1$, let $Y'(p)$ be the set of all $\xi \in X$ such that $\mathcal{P}\xi\Delta$ is $\bar{\rho}_p$-regular. Then the number of elements of $Y'(p)$ is $\varkappa(p)$ by [3, Prop. 7.5]. For $f \in \mathfrak{A}(\sigma, \lambda, \Delta)$ and $\xi \in X$, write

$$f \parallel \xi^{-1} = \sum_{p \in C_1} (a_{p,\xi} y^p + a_{p',\xi} y^{p'}) + (\text{non-constant terms}).$$

Then the map $\Psi: f \to ((a_{p,\xi})_{p \in C_1, \xi \in Y(p)}, (a_{p',\xi})_{p \in C_1, \xi \in Y'(p)})$ gives an injection of $\mathfrak{A}(\sigma, \lambda, \Delta)/\mathcal{S}(\sigma, \lambda, \Delta)$ into $\mathbb{C}^{2\mu}(\mu = \sum_{p \in C_1} \varkappa(p))$. For each $p \in C_1$, take $v \in \mathbf{a}$ so that $p_v \neq p'_v$. Let $g \in \mathcal{E}(\bar{s}_p, \bar{\rho}_p, \Delta)$ and $\xi \in Y'(p)$. Denote the Fourier expansion of $g \parallel \xi^{-1}$ by

$$g \parallel \xi^{-1} = b_{\bar{p},\xi} y^{\bar{p}} + b_{\bar{p}',\xi} y^{\bar{p}'} + (\text{non-constant terms}).$$

The using Theorem 2.2, we have a linear relation

$$\sum_{\xi \in Y'(p)} \nu_\xi (a_{p,\xi} \bar{b}_{\bar{p}',\xi} - a_{p',\xi} \bar{b}_{\bar{p},\xi}) = 0$$

among $(a_{p,\xi}, a_{p',\xi})$ for each g. Since these linear relations are independent if p's are different, we have at least μ independent linear relations. This implies the dimension of the image of Ψ is at most μ and therefore is equal to μ. □

If λ is non-critical, by [9, Remark 7.4 (1), (2)], we can take the set $C_1(\sigma, \lambda)$ and s_p for each $p \in C_1(\sigma, \lambda)$ so that they satisfy the conditions of Theorem 3.2, except for the case when $\lambda = 0$, $\sigma = 0$ ($p = u$, $p' = 0$, $s_p = 1$) in Case I and the case when $\sigma_v - 1/2$ is either an even non-negative integer or an odd negative integer for every $v \in \mathbf{a}$ ($p = 3/4 - (1/2)\sigma$, $p' = 1/4 - (1/2)\sigma$, $s_p = 3/4$) in Case II. To discuss these cases, we denote by $\mathcal{E}^*[s_0, \rho, \Delta]$ the set of elements of $\mathcal{E}[\rho, \Delta]$ that have at most a simple pole at s, and by $\mathcal{E}^*(s_0, \rho, \Delta)$ the set of residues of elements in $\mathcal{E}^*[s_0, \rho, \Delta]$. The following theorem is a generalization of [9, Theorem 7.9], which can be proved similarly as [9, Theorem 7.9] together with the modification used in Theorem 3.2.

Theorem 3.3. *Suppose λ is real and non-critical. Suppose also for all $p \in C_1(\sigma, \lambda)$, $\mathcal{E}[\rho_p, \Delta] = \mathcal{E}^*[s_p, \rho_p, \Delta]$ and a cusp class of Δ is ρ_p-regular if and only if it is $\bar{\rho}_p$-regular. Then $\mathfrak{N}(\sigma, \lambda, \Delta)$ has dimension $\sum_{p \in C_1} \varkappa(p)$ and is the direct sum*

$$\bigoplus_{p \in C_1} (\mathcal{E}(s_p, \rho_p, \Delta) \oplus \mathcal{E}^*(s_p, \rho_p, \Delta)), \qquad (C_1 = C_1(\sigma, \lambda)).$$

Using Theorem 3.2 and Theorem 3.3, we obtain the following

Theorem 3.4. *If* λ *is non-critical, then* $\mathfrak{N}(\sigma, \lambda, \Delta)$ *is generated by special values and the residues of Eisenstein series and has dimension* $\sum_{p \in C_1} \varkappa(p)$ $(C_1 = C_1(\sigma, \lambda))$.

PROOF. This is a direct result of Theorem 3.2 and Theorem 3.3 together with [9, Remark 7.10]. The only thing we would like to mention here is that in Theorem 3.3, we have assumed a cusp class of Δ is ρ_p-regular if and only if it is $\bar{\rho}_p$-regular for all $p \in C_1$. To avoid this restriction, take a subgroup Δ' of Δ of finite index so small that any cusp class of Δ' is ρ_p-regular and also $\bar{\rho}_p$-regular for all $p \in C_1$. Then by (3.3), we have the result. □

4. Remarks on Multiple λ

In [9, Remark 5.5], Shimura gave an example of multiple λ, when the field F is a quadratic field. We explain it in a slightly more general situation, because it seems to be the only case when multiple λ appears. Let F be a totally real number field of degree $2n$ containing a quadratic field L as a subfield. Denote by $\{v, w\}$ the set of archimedian primes of L and by $\mathbf{a} = \{v_1, \ldots, v_n, w_1, \ldots, w_n\}$ the set of archimedian primes of F of which $v_1, \ldots, v_n$ are lying over v and $w_1, \ldots, w_n$ are over w. Let U_1 be a subgroup of the unit group of L of finite index and take θ so that $T_{U_1} = \mathbb{Z}\theta$. Let U be a subgroup of the unit group of F of finite index such that $\{N_{F/L}(\epsilon) \mid \epsilon \in U\} \subset U_1$. Let τ be an element of $\mathbb{R}^{\mathbf{a}}$ such that

$$\tau_{v_i} = \theta_v, \qquad \tau_{w_i} = \theta_w \quad \text{for any} \quad i \ (1 \leqslant i \leqslant n).$$

Then $|\epsilon|^{i\tau} = |N_{F/L}(\epsilon)|^{i\theta} = 1$ for any $\epsilon \in U$. Put for integers m, n $(m \neq n)$,

$$p = \{(1 + 2ni\theta_v)u - (\sigma - 2mi\tau)\}/2,$$
$$q = \{(1 + 2mi\theta_v)u - (\sigma - 2ni\tau)\}/2.$$

Then p, $\bar{p}$, q, $\bar{q}$ are all distinct and are exponents belonging to $C(\sigma, \lambda)$ with $\lambda = (\lambda_{v_1}, \ldots, \lambda_{v_n}, \lambda_{w_1}, \ldots, \lambda_{w_n})$ given by

$$4\lambda_{v_i} = (1 - \sigma_{v_i})^2 + (m + n)^2\theta_v^2, \qquad 4\lambda_{w_i} = (1 - \sigma_{w_i})^2 + (m - n)^2\theta_w^2.$$

Therefore λ is multiple.

Now we return to the general situation and assume λ is multiple. Then by [9, Prop. 3.2]

(4.1) λ is real and $X^2 - (1 - \sigma_v)X + \lambda_v = 0$ has either a multiple root or two simple roots which are complex conjugate.

Therefore if p is an exponent attached to λ, then $p' = \bar{p}$ and for any other exponent $q \in C(\sigma, \lambda)$, we see that q_v is either p_v or $\bar{p}_v$ for all $v \in \mathbf{a}$. The following proposition suggests that even for multiple λ, $C(\sigma, \lambda)$ cannot contain so many exponents.

Proposition 4.1. *Let F be a totally real number field of degree $(n \geqslant 3)$. Assume λ is multiple. Let $p = (p_1, \ldots, p_n)$ $(p_i = p_{v_i}$ for $v_i \in \mathbf{a})$ be an exponent attached to λ. Assume $\bar{p}_i \neq p_i$ and put $q = (p_1, \ldots, \bar{p}_i, \ldots, p_n)$. Then q is not an exponent attached to λ.*

PROOF. We may assume $i = 1$ by changing the indices. Assume q is also an exponent attached to λ and put

$$p = su - (\sigma - i\tau)/2, \qquad q = s'u - (\sigma - i\tau')/2 \qquad (s, s' \in \mathbb{C})$$

with admissible τ and τ'. Since $\mathrm{Re}(s) = \mathrm{Re}(s') = 1/2$ by (1.11) and (4.1), we can write

$$s = 1/2 + it, \qquad s' = 1/2 + it'$$

with $t, t' \in \mathbb{R}$. Then we see

$$t' + \tau_1' = -(t + \tau_1), \qquad t' + \tau_j' = t + \tau_j \qquad (2 \leqslant j \leqslant n).$$

Therefore $\tau_j - \tau_j' = t' - t$ for all j $(2 \leqslant j \leqslant n)$. Putting $a = t - t'$, we obtain

$$\tau_1 - \tau_1' = -\left(\sum_{j=2}^{n} \tau - \sum_{j=2}^{n} \tau'\right) = (n-1)a.$$

This implies

$$\tau - \tau' = -au + (na, 0, \ldots, 0).$$

Take a subgroup of the unit group of F of finite index such that τ, τ' are U-admissible. Then we see for any $\epsilon \in U$,

$$1 = |\epsilon|^{i(\tau - \tau')} = |\epsilon_i|^{ina}.$$

Since the rank of U is $n - 1$, we see $a = 0$ if $n \geqslant 3$. This implies $t = t'$, $\tau = \tau'$ and therefore $p_1 = \bar{p}_1$, which is a contradiction. □

It is an interesting problem to determine whether all multiple λ can be obtained as Shimura's example mentioned in the beginning of this section or not, though it is a problem solely on the structure of the unit group of number fields.

Corollary 4.2. *If* $[F:\mathbb{Q}]=3$, *then there exists no multiple* λ *for any weight* σ.

PROOF. Assume λ is a multiple eigenvalue and let p be an exponent attached to λ. Then $\bar{p}$ is also an exponent. If q is an exponent attached to λ, then q is obtained by changing p or $\bar{p}$ at only one prime v. But this is not allowed by Proposition 4.1. □

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Toshitsune Miyake
Department of Mathematics
Hokkaido University
Sapporo 060, Japan