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Compact Hypersurfaces with Constant Higher Order Mean Curvatures

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A fundamental question about hypersurfaces in the Euclidean space is to decide if the sphere is the only compact hypersurface (embedded or immersed) with constant higher order mean curvature H_r, for some $r = 1, \ldots, n$.

If the hypersurface M^n is star-shaped, Hsiung [3] solved afirmatively the problem for any r. In particular if the Gauss-Kronecker curvature H_n is constant, then M^n is a sphere, because in this case the Hadamard theorem asserts that M^n is convex. The convex case was studied previously by Liebmann [5] and Süss [9]. If the mean curvature H_1 is constant and M^n is embedded, Aleksandrov [1] proved that M^n is a sphere. In the immersed case Hsiang, Teng and Yu [4] and Wente [10] constructed non-spherical examples in higher dimension and in $\mathbb{R}^3$ respectively. If the scalar curvature H_2 is constant and the hypersurface is embedded we proved in [8] that it must be a sphere. In this paper we extend this result to higher order mean curvatures. In particular we prove that

«*The sphere is the only embedded compact hypersurface in the Euclidean space with H_r constant for some $r = 1, \ldots, n$.*»

In this paper we use as in [8] a method of Reilly [7]. Recently with S. Montiel [6] we obtained a different proof of the above theorem. Another proof has been published by N. Korevaar [11].

1. Preliminaries

Let $\psi: M^n \to \mathbb{R}^{n+1}$ be an orientable hypersurface immersed in the Euclidean space. Let N be an unit normal vector field on M, σ the second fundamental form of M with respect to N and k_i, $i = 1, \ldots, n$ the principal curvatures of M. For any $r = 1, \ldots, n$ we define the *mean curvature of order r*, H_r, by the identity

$$(1) \qquad (1 + tk_1) \cdots (1 + tk_n) = 1 + \binom{n}{1} H_1 t + \binom{n}{2} H_2 t^2 + \cdots + \binom{n}{n} H_n t^n$$

for any real number t. For instance, H_1 is simply the mean curvature $H_1 = H = (k_1 + \cdots + k_n)/n$. H_2 is, up a constant, the scalar curvature and $H_n = k_1 k_2 \ldots k_n$ is the Gauss-Kronecker curvature. From the Gauss equation we have that if r is even, then H_r is an intrinsic invariant of M. Note that for the unit sphere, and with respect to the unit inner normal, we have $H_r = 1$ for any r. By convenience we put $H_0 = 1$. These curvatures satisfy a basic relation in global hypersurface theory, which is stated in the following lemma.

Lemma (Minkowski Formulae [3]). *Let $\psi: M^n \to \mathbb{R}^{n+1}$ be a compact orientable hypersurface immersed in the Euclidean space. Then for any $r = 1, \ldots, n$ we have*

$$(2) \qquad \int_M H_{r-1}\, dA + \int_M H_r \langle \psi, N \rangle\, dA = 0.$$

Let Ω^{n+1} be a compact Riemannian manifold with smooth boundary $M^n = \partial\Omega$. Let dV and dA be the canonical measures on Ω and M respectively and V and A the volumen of Ω and the area of M. Given f in $C^\infty(\bar{\Omega})$ we denote $z = f_{|M}$ and $u = \partial f/\partial N$, where N is the inner unit normal on M. Reilly's formula [7] states that

$$(3) \qquad \int_\Omega [(\bar{\Delta} f)^2 - |\bar{\nabla}^2 f|^2 - \operatorname{Ric}(\bar{\nabla} f, \bar{\nabla} f)]\, dV = \int_M [-2(\Delta z)u + nHu^2 + \sigma(\nabla z, \nabla z)]\, dA,$$

$\bar{\nabla} f$, $\bar{\Delta} f$ and $\bar{\nabla}^2 f$ being the gradient, the Laplacian and the Hessian of f in Ω, Ric the Ricci cuvature of Ω, ∇z and Δz the gradient and the Laplacian of z in M, and σ and H the second fundamental form and the mean curvature of M with respect to N.

If M is a compact hypersurface embedded in $\mathbb{R}^{n+1}$, then M is the boundary of a compact domain $\Omega \subset \mathbb{R}^{n+1}$. So if x denotes the position vector in $\mathbb{R}^{n+1}$, then $\bar{\Delta}|x|^2 = 2(n+1)$, and from the divergence theorem we have

$$(4) \qquad (n+1)V + \int_M \langle \psi, N \rangle\, dA = 0,$$

ψ being the immersion of M in $\mathbb{R}^{n+1}$.

2. An inequality

For the next result we will follow closely the ideas of Reilly [7].

Theorem 1. *Let Ω^{n+1} be a compact Riemannian manifold with smooth boundary M and non-negative Ricci curvature. Let H be the mean curvature of M. If H is positive everywhere, then*

$$\int_M \frac{1}{H} dA \geqq (n+1)V. \tag{5}$$

The equality holds if and only if Ω is isometric to an Euclidean ball.

PROOF. Let f in $C^\infty(\bar{\Omega})$ be the solution of the Dirichlet problem

$$\begin{cases} \bar{\Delta} f = 1 \text{ in } \Omega, \\ z = 0 \quad \text{on} \quad M. \end{cases}$$

From the divergence theorem we have

$$V = \int_\Omega \bar{\Delta} f \, dV = -\int_M u \, dA. \tag{6}$$

Combining Schwarz inequality $(\bar{\Delta} f)^2 \leqslant (n+1)|\bar{\nabla}^2 f|^2$ with (3) we obtain

$$\frac{V}{n+1} \geqq \int_M H u^2 \, dA. \tag{7}$$

Finally, from (6), Schwarz inequality and (7) it follows that

$$\begin{aligned} V^2 &= \left(\int_M u \, dA \right)^2 = \left(\int_M (uH^{1/2}) H^{-1/2} \, dA \right)^2 \\ &\leqq \int_M u^2 H \, dA \int_M H^{-1} \, dA \\ &\leqq \frac{V}{n+1} \int_M \frac{1}{H} dA, \end{aligned}$$

and we have proved inequality (5).

If equality holds, then $\bar{\nabla}^2 f$ is proportional to the metric everywhere. As $\bar{\Delta} f = 1$, we conclude that

$$\bar{\nabla}^2 f = \frac{1}{n+1} \langle \, , \rangle. \tag{8}$$

Deriving covariantly we obtain $\bar{\nabla}^3 f = 0$ and from the Ricci-identity,

$$R(X, Y)\bar{\nabla} f = 0, \tag{9}$$

for any X, Y tangent vector to Ω, where R is the curvature of Ω. From the maximum principle f attains its minimum at some point x_0 in the interior of Ω. From (8) it follows that

$$\bar{\nabla} f = \frac{1}{n+1} r \frac{\partial}{\partial r},$$

where r is the distance to the point x_0, which combined with (9), Cartan's Theorem and the fact that f vanishes at the boundary of Ω implies that Ω is an Euclidean ball whose center is x_0, and f is given by

$$f(x) = [2(n+1)]^{-1}(|x - x_0|^2 - a^2)$$

in Ω, a being the radius of the ball.

3. Hypersurfaces with H_r Constant

In this section we prove the main result of this paper. Given $k = 1, \ldots, n$ we consider the function $\sigma_k \colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$\sigma_k(x_1, \ldots, x_n) = \text{elementary symmetric polynomial of degree } k \text{ in the variables } x_1, \ldots, x_n.$$

Thus $H_k = \sigma_k(k_1, \ldots, k_n)$. We denote by C_k the connected component of the set $\{x \in \mathbb{R}^n / \sigma_k(x) > 0\}$ which contains the vector $(1, \ldots, 1)$. From Gärding [2] we have that if $k < r$, then $C_k \supset C_r$. Moreover if $x \in C_r$ we have for $k \leqq r$ that $\sigma_k(x)^{(k-1)/k} \leqq \sigma_{k-1}(x)$. For $k \geqq 2$ equality holds if and only if x is proportional to $(1, \ldots, 1)$. Clearly if $x_i > 0$ for any i, then $(x_1, \ldots, x_n) \in C_k$.

Theorem 2. *Let M^n be a compact hypersurface embedded in the Euclidean space $\mathbb{R}^{n+1}$. If H_r is constant for some $r = 1, \ldots, n$, then M^n is a sphere.*

Proof. As M has an elliptic point, H_r must be a positive constant. As the principal curvatures are continuous functions we have that $(k_1, \ldots, k_n) \in C_r$ at any point. Hence $(k_1, \ldots, k_n) \in C_k$ for k smaller than r. In particular $H_k > 0$ for $k < r$. Moreover

$$H_k^{(k-1)/k} \leqq H_{k-1} \qquad k = 1, \ldots, r, \tag{10}$$

As consequence

$$H_r^{1/r} \leqq H \quad \text{in} \quad M. \tag{11}$$

Now we use Minkowski formulae and (3):

$$\begin{aligned} 0 &= \int_M H_{r-1}\, dA + \int_M H_r \langle \psi, N \rangle\, dA \\ &= \int_M H_{r-1}\, dA + H_r \int_M \langle \psi, N \rangle\, dA \\ &= \int_M H_{r-1}\, dA - (n+1) H_r V. \end{aligned}$$

Combining this equality with (10) we have

$$(n+1) H_r V = \int_M H_{r-1}\, dA \geqslant A H_r^{(r-1)/r}$$

and so

$$H_r^{1/r} \geqq \frac{A}{(n+1)V}. \tag{12}$$

On the other hand from (5) and (11) we obtain

$$(n+1)V \leqq \int_M \frac{dA}{H} \leqq A H_r^{-1/r},$$

which is

$$H_r^{1/r} \leqq \frac{A}{(n+1)V}, \tag{13}$$

and the equality holds if and only if M is a sphere. The theorem follows from (12) and (13).

4. An Extensión of the Aleksandrov Theorem

First we observe that a compact hypersurface embedded in the Euclidean space is a critical point of the isoperimetric functional if and only if it has constant mean curvature.

Theorem 3. *Let $\bar{M}^{n+1}$ be a Riemannian manifold with non-negative Ricci curvature, and let Ω be a compact domain in $\bar{M}$ with smooth boundary. If Ω is a critical point of the isoperimetric functional*

$$\Omega \to \frac{A(\partial\Omega)^{n+1}}{V(\Omega)^n},$$

then Ω is isometric to an Euclidean ball.

PROOF. Given a smooth function f on $\partial\Omega$, we consider the normal variation of $\partial\Omega$ defined by $\psi_t: \partial\Omega \to \bar{M}$, $\psi_t(p) = \text{Exp}_p(-tf(p)N(p))$, where Exp is the exponential map of $\bar{M}$. ψ_t determine a variation of Ω, Ω_t for $|t| < \epsilon$. We put $V(t) = V(\Omega_t)$ and $A(t) = A(\partial\Omega_t)$. The first variation formulae of the functionals above are given by

$$A'(0) = n \int_{\partial\Omega} fH\,dA,$$
$$V'(0) = \int_{\partial\Omega} f\,dA.$$

By hypothesis we have

$$\frac{d}{dt}\bigg|_{t=0} \frac{A(t)^{n+1}}{V(t)^n} = 0,$$

or equivalently

$$\int_{\partial\Omega} f[(n+1)VH - A]\,dA = 0, \quad \text{for any} \quad f.$$

Then $H = A/(n+1)V$ and we have equality in (5). Therefore Ω is isometric to an Euclidean ball.

Remark. Let $\psi: M^n \to \mathbb{R}^{n+1}$ be an *immersed* compact hypersurface. Suppose that M is the boundary of a certain manifold Ω^{n+1}, and that the immersion ψ extends to an immersion of Ω, $\bar{\psi}: \Omega^{n+1} \to \mathbb{R}^{n+1}$. It is easy to see that Aleksandrov proof [1] can be adapted to this situation: If M^n has constant mean curvature, it must be a sphere.

U. Pinkall pointed out to me that Reilly's method can also be used in this case. In fact, taking on Ω the pull-back of the Euclidean metric, inequality (5) remains true and the same holds for identity (3). On the other hand, Minkowski formulae hold for any immersed hypersurface. So theorem 2 extends to the above type of immersed hypersurfaces.

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