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Oblique Derivative Problems for the Laplacian in Lipschitz Domains

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Introduction

The aim of this paper is to extend the results of Calderón [1] and Kenig-Pipher [12] on solutions to the oblique derivative problem to the case where the data is assumed to be BMO or Hölder continuous. Suppose D is a bounded Lipschitz domain in $\mathbb{R}^n$, $N(Q)$ is the unit normal at a point Q on the boundary of D and $V(Q)$ is a continuous bounded vector field defined on the boundary of D such that for some constant c, $\langle V, N \rangle \geqslant c > 0$. At each point $Q \in \partial D$, the cone $\Gamma(Q) \subseteq D$ is defined by

$$\Gamma(Q) = \{P \in D: |P - Q| \leqslant (1 + \alpha) \operatorname{dist}(P, \partial D)\} \quad \text{for some} \quad \alpha > 0.$$

For $v \in C^2(D)$, one can define the nontangential maximal function of v and the square function of v by

$$Nv(Q) = \sup_{X \in \Gamma(Q)} |v(X)|$$

$$S^2 v(Q) = \int_{\Gamma(Q)} d(X)^{2-n} |\nabla v(X)|^2 \, dX$$

where $d(X) = \operatorname{dist}(X, \partial D)$ is comparable to $|X - Q|$ for $X \in \Gamma(Q)$. Consider the boundary value problem

$$\text{(1.1)} \qquad \begin{cases} \Delta u = 0 & \text{in} \quad D \\ V \cdot \nabla u = g & \text{on} \quad \partial D \end{cases}$$

with $g \in L^p(\partial D)$. Because V is continuous and never tangent to the boundary of D, $V \cdot \nabla u$ may be regarded (locally) as a perturbation of $\partial u/\partial y$ and the following results are known.

Theorem A (Calderón [1]). *There exists a solution* $u(x)$ *to* (1.1) *with*

$$\|N(\nabla u)\|_{L^p} \leqslant \|g\|_{L^p} \quad \text{for} \quad 2 - \epsilon < p < 2 + \epsilon$$

where ϵ *depends on the domain and* g *is assumed to satisfy finitely many linear conditions.*

Theorem B (Kenig-Pipher [12]). *If* $g \in L^p(\partial D)$ *for* $2 < p < \infty$ *and satisfies finitely many linear conditions, then the solution* $u(x)$ *to* (1.1) *satisfies*

$$\|N(\nabla u)\|_{L^p} \lesssim \|g\|_{L^p}.$$

This last theorem indicates that the oblique derivative problem (1.1) behaves more like the Dirichlet problem than the Neumann problem on a Lipschitz domain. (Indeed, only where the domain is C^1 will (1.1) contain the Neumann problem as a special case.) In light of the results of Fefferman-Stein [6] for harmonic functions in $\mathbb{R}^n_+$ and the work of Fabes-Neri [7] on the Dirichlet problem with BMO data on Lipschitz domains, it is natural to ask for BMO solutions to the oblique derivative problem for data in BMO. This problem is addressed in section 2. Again in analogy with the Dirichlet problem it is of interest to consider behavior of solutions to this problem when the data g is assumed to be Hölder continuous and this result is formulated in section 3. In both cases, the method of proof closely follows that of [12], hence some aspects of the proof presented here are deliberately brief.

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2. The Boundary Value Problem (1.1) with $g \in \mathrm{BMO}(\partial D)$

If $g \in L^1(\partial D)$ then g belongs to $\mathrm{BMO}(\partial D)$ if there exists a constant C such that

$$\sup_{\Delta} \frac{1}{\sigma(\Delta)} \int_{\Delta} |g - g_{\Delta}| \, d\sigma < C, \tag{2.1}$$

where Δ is a surface ball contained in ∂D, $d\sigma$ is surface measure on ∂D and

$$g_{\Delta} = \frac{1}{\sigma(\Delta)} \int_{\Delta} g \, d\sigma.$$

The norm of g in this space is

$$\|g\|_* = \left|\int_{\partial D} g\, d\sigma\right| + \inf\{C: (2.1) \text{ holds for } C\}.$$

Equivalent definitions (and norms) are given by the L^p conditions for $1 < p < \infty$:

$$\sup_{\Delta}\left\{\frac{1}{\sigma(\Delta)}\int_{\Delta}|g - g_{\Delta}|^p\, d\sigma\right\}^{1/p} < \infty.$$

(See John-Nirenberg [10].) Our question is the following: when can (1.1) be solved with $g \in \mathrm{BMO}(\partial D)$ to yield a solution u with $\nabla u \in \mathrm{BMO}$, *i.e.*, all derivatives of u belong to BMO?

Consider first the following simple problem in the upper half plane. Let $V = (0, a(x))$ with $c \leqslant a \leqslant 1$ and suppose $g \in \mathrm{BMO}(\mathbb{R}, dx)$. Then $V \cdot \nabla u = g$ on $\mathbb{R}$ with u harmonic in $\mathbb{R}^2_+$ is the same as $\partial u/\partial y = a^{-1}g$ and one expects a to satisfy a further condition to insure that u_y belong to BMO, namely that a^{-1} be a BMO multiplier. Since $a \in L^\infty$ this is equivalent to demanding that a itself be a BMO multiplier (that is, for all $g \in \mathrm{BMO}$, $ag \in \mathrm{BMO}$). Necessary and sufficient conditions for a function to be a BMO multiplier were found by Stegenga [14] and we impose this additional condition on the components V_i of V. However, we require slightly more smoothness on V which is a VMO version of this multiplier condition (see Sarason [13] for the properties of VMO functions) and from now on each V_i will satisfy (2.2). Given $\epsilon > 0$, there exists $\delta > 0$ such that if the radius $r(\Delta)$ of $\Delta \subseteq \partial D$ is less than δ, then

$$\log\left(\frac{1}{r(\Delta)}\right)\int_{\Delta}|V_i - (V_i)_{\Delta}|\frac{d\sigma}{\sigma(\Delta)} < \epsilon. \tag{2.2}$$

Further comments about this VMO requirement will be made at the end of this section but it should be noted that continuity of V does not imply (2.2) (see S. Janson [8]). When D is a C^1 domain, the boundary value problem (1.1) could be taken to be the Neumann problem. In this case however it is possible to construct a C^1domain in $\mathbb{R}^2$ for which there exists a harmonic u with $\partial u/\partial n$ in BMO but $\nabla_T u$ (the tangential derivative) not exponentially integrable, and hence not in BMO. Construction of such a C^1 domain is essentially given in Kenig [11].

The main result for $g \in \mathrm{BMO}(\partial D)$ will be formulated in terms of the Fefferman-Stein «sharp» function ([6]). For $f(Q)$ defined on ∂D, set

$$T^{\#,q}(f)(Q) = \sup_{\Delta \ni Q}\left\{\frac{1}{\sigma(\Delta)}\int_{\Delta}|f(Q) - f_{\Delta}|^q\, d\sigma\right\}^{1/q}$$

for $1 \leqslant q < \infty$. Then a function f belongs to BMO (∂D) if, for some q, $T^{\#,q}(f)$ belongs to L^∞, and if f belongs to L^p for $p > q$, $\|T^{\#,q}(f)\|_{L^p} \lesssim \|f\|_{L^p}$.

Theorem 2.3. *Suppose $V = (V_1, \ldots, V_n)$ is continuous, $(V, N) \geqslant c$ and V has property* (2.2), *and suppose $g \in \mathrm{BMO}(\partial D)$ and satisfies finitely many linear conditions. Let u be the harmonic solution to $V \cdot \nabla u = g$ on ∂D given by Theorem A. Then, given $\epsilon > 0$, there exists C_ϵ such that for a.e. $Q \in \partial D$,*

$$T^{\#,2}(\nabla u)(Q) < C \cdot \epsilon (M_\sigma[T^{\#,4}(\nabla u)]^2(Q))^{1/2} + C_\epsilon, \tag{2.4}$$

where M_σ is the Hardy-Littlewood maximal function on ∂D with respect to $d\sigma$.

The expression $T^{\#,2}(\nabla u)$ denotes $\sum_i T^{\#,2}(D_i u)$ and this sort of abbreviation will appear in other contexts later on. The constant C_ϵ in (2.4) depends on everything: the Lipschitz character of D, the modulus of continuity of V, the constant in (2.2), the diameter of D, the transversality constant c and the apertures of cones of square functions appearing in the proof of the theorem. The desired property of $|\nabla u|$ follows immediately.

Corollary 2.4. *All derivatives of u belong to* $\mathrm{BMO}(\partial D)$.

PROOF. By Stromberg ([16], Lemma 3.7), $\| T^{\#,4}(\nabla u)\|_{L^p} \leqslant C \| T^{\#,2}(\nabla u)\|_{L^p}$ with a constant independent of p for $p > 4$. By Theorem B, $\| T^{\#,2}(\nabla u)\|_p$ is finite for large p and hence if ϵ is sufficiently small, $\| T^{\#,2}(\nabla u)\|_{L^p} \leqslant C_\epsilon$. Letting p tend to ∞ gives $T^{\#,2}(\nabla u) \in L^\infty$. □

The proof of Theorem 2.3 consists of a series of lemmas. The first of these quantifies the intuition that $V \cdot \nabla u$ is locally a perturbation of $\partial u/\partial y$. We fix a regular family of cones (see Dahlberg [3]) $\Gamma(Q) \subset \bar{\Gamma}(Q) \subseteq \bar{\bar{\Gamma}}$ and $\bar{S}v$ will denote the square function taken with respect to $\bar{\Gamma}(Q)$. The cone $\Gamma_h(Q)$ (or $\bar{\Gamma}_h(Q)$) is the truncated cone $\Gamma(Q) \cap \{X \in D : |X - Q| < h\}$ and $S_h v$ (or $\bar{S}_h v$) denotes the square function with integration taken over $\Gamma_h(Q)$ (or $\bar{\Gamma}_h(Q)$). Let $Hu(X)$ denote the Hessian of u and set

$$S^2(\nabla u)(Q) = \int_{\Gamma(Q)} d(X)^{1-n} |Hu(X)|^2 \, dX.$$

Lemma 2.5. (Stein [15], p.213). *Suppose $S_h(\nabla u)(Q_0) < 0$ and choose coordinates so that B is a neighborhood of $Q_0 = (0, \ldots, 0)$ with $D \cap B = \{y : y > \Phi(X)\}$ for Φ Lipschitz. Then there exists a constant C such that*

$$S_h(\nabla u)(Q_0) \leqslant C\bar{S}_{\bar{h}}\left(\frac{\partial u}{\partial y}\right)(Q_0) + Ch^2 |\nabla u(0, h)|^2$$

for $\bar{h} > h$.

The following two lemmas will be needed to obtain $T^{\#}$ estimates from square function estimates, and vice versa.

Lemma 2.6. *If $\Delta = \Delta(Q_0, r_0)$ is a surface ball contained in ∂D, let*

$$T(\Delta) = \{X \in D: |X - Q_0| < r_0\}$$

be the Carleson region associated to Δ. Then, if $d(X) = \operatorname{dist}(X, \partial D)$,

$$\frac{1}{\sigma(\Delta)} \int_\Delta |f - f_\Delta|^2 \, d\sigma(Q) \lesssim \frac{1}{\sigma(\Delta)} \int_{T(\Delta)} d(X) |\nabla f|^2 \, dX$$

for f harmonic in D and $f \in L^2(\partial D)$.

PROOF. If Ω is a Lipschitz domain, let $S_\Omega f$ denote the square function of f with respect to the domain Ω. Then

$$\begin{aligned} \int_{T(\Delta)} d(X) |\nabla f|^2 \, dX &\geqslant \int_{T(\Delta)} \operatorname{dist}(X, \partial T(\Delta))^{2-n} |\nabla f|^2 \, dX \\ &\approx \int_{\partial T(\Delta)} S^2_{T(\Delta)}(f)(Q) \, d\sigma(Q) \\ &\geqslant \int_\Delta |f(Q) - f_\Delta|^2 \, d\sigma(Q) \end{aligned}$$

by Dahlberg's area integral theorem [3]. □

Lemma 2.7. *If $f \in L^2(\partial D)$ and $\Delta f = 0$ in D, $\Delta = \Delta(Q_0, r_0)$, then*

$$\left\{ \frac{1}{\sigma(\Delta)} \int_{T(\Delta)} d(X) |\nabla f|^2 \, dX \right\} \lesssim T^{\#,2}(f)(Q_0).$$

PROOF. Let

$$f_1(X) = \int_{\Delta(Q_0, 2r_0)} (f(Q) - f_\Delta) \, d\omega^X(Q) \quad \text{and} \quad f_2(X) = f(X) - f_1(X)$$

so that

$$\nabla f = \nabla f_1 + \nabla f_2 .$$

Then

$$\begin{aligned} \frac{1}{\sigma(\Delta)} \int_{T(\Delta)} d(X) |\nabla f_1|^2 \, dX &\lesssim \frac{1}{\sigma(\Delta)} \int S^2 f_1(Q) \, d\sigma(Q) \\ &\leqslant \frac{1}{\sigma(\Delta)} \int_\Delta |f(Q) - f_\Delta|^2 \, d\sigma(Q) \\ &\lesssim (T^{\#,2}(f)(Q_0))^2. \end{aligned}$$

If $G(X)$ denotes the Green's function for D with pole at $P_0 \in D$ then since

$$f_2^*(X) \equiv \int_{c_{\hat{\Delta}}} |f(Q) - f_\Delta| \, d\omega^X(Q)$$

is a positive harmonic function which vanishes on Δ, the comparison theorem yields, for $X \in T(\Delta)$,

$$|\nabla f_2(X)| \lesssim |f_2^*(X)|/d(X) \lesssim G(X)/d(X) \cdot f_2^*(X_h)/G(X_h)$$

where $X_h \in T(\Delta)$ satisfies $\operatorname{dist}(X_h, \partial D) \geqslant ch$. By Hölder continuity of the Green's function, there is some $\alpha > 0$ such that $G(X)/G(X_h) \lesssim (d(X)/h)^\alpha$ (see [9]) and so

$$\frac{1}{\sigma(\Delta)} \int_\Delta d(X)|\nabla f_2(X)|^2\, dX \lesssim \left(\frac{1}{\sigma(\Delta)} \int_{T(\Delta)} d(X)^{2\alpha - 1} \cdot h^{-2\alpha}\, dX\right)(f_2^*(X_h))^2$$
$$\leqslant C(f_2^*(X_h))^2.$$

Estimates for $|f_2^*(X_h)|$ as in [5] or [7] show that

$$|f_2^*(X_h)| \leqslant CT^{\#,2}(\nabla u)(Q_0). \qquad \square$$

Fix a $Q_0 \in \partial D$. Then $\langle V(Q_0), N(Q_0)\rangle \geqslant c$ and since $V(Q)$ is continuous there is a neighborhood Δ of Q_0 such that $\langle V(Q), N(Q_0)\rangle \geqslant c/2$ for all $Q \in \Delta$. Given $\epsilon > 0$, choose a coordinate chart for ∂D near Q_0 with neighborhood $\Delta(Q_0, \delta)$ such that $V(Q_0)$ points in the direction of the y-axis and $|V(X) - V(Q)|^2 < \epsilon$ for all $X \in \bigcup_{Q \in \Delta(Q_0, \delta)} \bar{\Gamma}_\delta(Q)$, where $V_i(X)$ is the harmonic extension of $V_i(Q)$ to D. By Lemma 2.6, choose $h > 0$ so that

$$T^{\#,2}(\nabla u)(Q_0) \leqslant \frac{1}{\sigma(\Delta_h)} \int_{\Delta_h} S_h^2(\nabla u)(Q)\, d\sigma(Q)$$

and we can assume $h < \delta$. By Lemma 2.5 and the continuity of V, for all $Q \in \Delta_h$,

$$S_h^2(\nabla u)(Q) \leqslant \int_{\bar{\Gamma}_h(Q)} d(X)^{2-n} \sum_j \left|\sum_i V_i(Q) \frac{\partial^2 u}{\partial x_i \partial x_j}(X)\right|^2 dX + h^2|\nabla u(X_h)|^2$$

(2.8)

$$\leqslant C_\epsilon \int_{\bar{\Gamma}_h(Q)} d(X)^{2-n} \sum_j \left|\sum_i V_i(X) \frac{\partial^2 u}{\partial x_j \partial x_i}(X)\right|^2 dX + h^2|\nabla u(X_h)|^2.$$

Lemma 2.9. *If $X_h \in T(\Delta(Q_0, h))$ with* $\operatorname{dist}(X_h, \partial D) \geqslant ch$, *then*

$$|\nabla u(X_h)| \leqslant C \log(1/h) T^{\#,2}(\nabla u)(Q_0) + C.$$

PROOF. Let $\delta > h$ be fixed with $D \cap \{P: |P - Q_0| < 2\delta\} = \{(x, y): y > \Phi(x)\}$, a coordinate chart. Then

$$|\nabla u(x, \Phi(x) + h)| \lesssim |\nabla u(x, \Phi(x) + \delta)| + \int_h^\delta |Hu(x, \Phi(x) + r)|\, dr.$$

By Calderón's theorem,

$$|\nabla u(x, \Phi(x) + \delta)| \leqslant C_\delta \int N(\nabla u)\, d\sigma \lesssim C_\delta \|g\|_{L^2} \leqslant C_\delta \|g\|_* .$$

Let $X_r = (x, \Phi(x) + r)$ and $B_r = \{P \in D: |P - X_r| < r/2\}$. The mean value theorem gives

$$\int_h^\delta |Hu(X_r)|\, dr \leqslant \int_h^\delta \left(\frac{1}{r^n} \int_{B_r} |Hu(Z)|^2\, dZ \right)^{1/2} dr$$

$$\leqslant \log\left(\frac{1}{h}\right) \sup_r \left\{ \int_{T(\Delta(Q_0, r))} d(X) |Hu(X)|^2\, dX \right\}^{1/2}$$

$$\leqslant c \log\left(\frac{1}{h}\right) T^{\#,2}(\nabla u)(Q_0).$$

by Lemma 2.7. □

Set $v(x) = V(X) \cdot \nabla u(X)$. Inequality (2.8) together with Lemma 2.9 shows that, for $Q \in \Delta_h$,

$$S_h^2(\nabla u)(Q) \lesssim ch^2 \log^2\left(\frac{1}{h}\right) (T^{\#,2}(\nabla u)(Q_0))^2 + \bar{S}_h^2(v)(Q)$$

$$+ \int_{\bar{\Gamma}_h(Q)} d(X)^{2-n} \sum_{i,k} \left| \frac{\partial V_i}{\partial X_k}(X) \frac{\partial u}{\partial X_i}(X) \right|^2 dX + C_\delta .$$

Since $h < \delta$, the first term above will be smaller that $[T^{\#,2}(\nabla u)(Q_0)]^2/2$ for appropriate choice of ϵ and this proves

Lemma 2.10.

$$[T^{\#,2}(\nabla u)(Q_0)]^2 \lesssim C_\delta + \frac{1}{\sigma(\Delta_h)} \int_{\Delta_h} \bar{S}_h^2(v)(Q)\, d\sigma(Q)$$

$$+ \frac{1}{\sigma(\Delta_h)} \int_{T(\Delta_h)} d(X) |\nabla V|^2 |\nabla u|^2\, dX.$$

Lemma 2.11. *If V satisfies* (2.2), *then given $\epsilon > 0$, there exists $C = C(\epsilon)$ such that*

$$\left\{ \frac{1}{\sigma(\Delta_h)} \int_{T(\Delta_h)} d(X) |\nabla V(X)|^2 |\nabla u(X)|^2\, dX \right\}^{1/2} \leqslant C + C' \epsilon T^{\#,2}(\nabla u)(Q_0).$$

PROOF. Let $F(X) = \nabla u(X) - \nabla u(X_h)$ where $X_h \in T(\Delta_h)$ with $\operatorname{dist}(X_h, \partial D) \approx h$.

As in the argument for Lemma 2.7, split F into two components F_1 and F_2 where

$$F_1(X) = \int_{\Delta(Q_0, Mh)} F(Q)\, d\omega^X(Q) \quad \text{and} \quad F_2 = F - F_1$$

and M is a constant depending on the Lipschitz character of D. Because $d(X)|\nabla V(X)|^2\, dX$ is a Carleson measure and h is small,

$$\frac{1}{\sigma(\Delta_h)} \int_{T(\Delta_h)} d(X)|\nabla V(X)|^2 |F_1(X)|^2\, dX \leqslant \|V\|_{\mathrm{BMO}} \int_{\partial D} N^2(F_1) \frac{d\sigma}{\sigma(\Delta_h)}$$

$$< C'\epsilon \int_{\Delta_{Mh}} |\nabla u(Q) - \nabla u(X_h)|^2 \frac{d\sigma(Q)}{\sigma(\Delta_h)}$$

which is bounded by

$$C'\epsilon[T^{\#,2}(\nabla u)(Q_0)]^2$$

as

$$|\nabla u(X_h) - (\nabla u)_\Delta| \leqslant C T^{\#,2}(\nabla u)(Q_0)$$

(see [5]). To handle the term involving $|F_2(X)|$, one uses the usual estimates obtained by comparison with the Green's function and the fact that $|\nabla V(X)| \leqslant \epsilon/d(X)$ since $V \in \mathrm{VMO}$. Hence

$$\left\{ \frac{1}{\sigma(\Delta_h)} \int_{T(\Delta_h)} d(X)|\nabla V(X)|^2 |F_2(X)|^2\, dX \right\}^{1/2} \leqslant C\epsilon T^{\#,2}(\nabla u)(Q_0)$$

and it remains to bound

$$|\nabla u(X_h)|^2 \cdot \frac{1}{\sigma(\Delta_h)} \int_{T(\Delta_h)} d(X)|\nabla V(X)|^2\, dX.$$

By Lemma 2.9 and the multiplier condition (2.2) for V (which has an equivalent L^2 expression),

$$|\nabla u(X_h)|^2 \frac{1}{\sigma(\Delta_h)} \int_{T(\Delta_h)} d(X)|\nabla V(X)|^2\, dX$$

$$\lesssim C_\delta + \log^2\left(\frac{1}{h}\right) \frac{1}{\sigma(\Delta_h)} \int_{T(\Delta_h)} d(X)|\nabla V|^2\, dX [T^{\#,2}(\nabla u)(Q_0)]^2$$

$$\lesssim C_\delta + C\epsilon[T^{\#,2}(\nabla u)(Q_0)]^2. \quad \square$$

By Lemma 2.11 we may choose ϵ sufficiently small depending only on V, D so that

$$T^{\#,2}(\nabla u)(Q_0) \lesssim C(\epsilon) + \left\{ \frac{1}{\sigma(\Delta_h)} \int_{\Delta_h} \bar{S}_h^2(v)\, d\sigma(Q) \right\}^{1/2}. \tag{2.12}$$

From now on, we write

$$F(X) = \int_D G(X, Y)\, \Delta v(Y)\, dY$$

for the Green's potential of v. Then if $g(X)$ is the harmonic extension of g to D, we have $v(x) = -F(x) + g(x)$. Because $d(x)|\nabla g(x)|^2\, dx$ is a Carleson measure ([7]), the only term to be controlled involves $\bar{S}_h^2(F)$. The following good-λ inequality is a modification (and simplification) of Lemma 7 of [12] and details of the proof are provided only where they essentially differ from those of [12].

Lemma 2.13. *Let*

$$N_h(F)(Q) = \sup_{X \in \Gamma_h(Q)} |F(X)|$$

be the truncated nontangential maximal function of F. For γ sufficiently small, there exists constants C and η depending on D such that (if $d|\nabla F|$ abbreviates $d(x)|\nabla F(x)|$),

$$\sigma\{Q \in \Delta_h : S_h(F) > 2\lambda, N_h(F) \leqslant \gamma\lambda, N_h(d|\nabla F|) \leqslant \gamma\lambda, S_h(\nabla u) \cdot N_h(F) \leqslant (\gamma\lambda)^2\}$$
$$\leqslant C\gamma^\eta \sigma\{Q \in \Delta_h : S_h(F) > \lambda\}.$$

The inequality of Lemma 2.13 remains true for $v(x) = V(X) \cdot \nabla u(X)$ replacing $F(X)$ and consequently should be regarded as a good-λ inequality for the product of harmonic extensions of a BMO function and an L^2 *function.*

Corollary 2.13.

$$(T^{\#,2}(\nabla u)(Q_0))^2 \leqslant C + C \int_{\Delta_h} N_h^2(F) + N_h^2(d|\nabla F|) \frac{d\sigma}{\sigma(\Delta_h)}.$$

PROOF. Integrating the good-λ inequality and using (2.12) gives

$$(T^{\#,2}(\nabla u)(Q_0))^2 \leqslant C_\delta + \|g\|_*^2$$
$$+ C \int_{\Delta_h} N_h^2(F) + N_h^2(d|\nabla F|) \frac{d\sigma}{\sigma(\Delta_h)}$$
$$+ \int_{\Delta_h} S_h(\nabla u) N_h(F) \frac{d\sigma}{\sigma(\Delta_h)}.$$

The last term above is bounded by

$$\epsilon_0 \int_{\Delta_h} S_h^2(\nabla u) \frac{d\sigma}{\sigma(\Delta_h)} + C_{\epsilon_0} \int_{\Delta_h} N_h^2(F) \frac{d\sigma}{\sigma(\Delta_h)}$$

which in turn, by Lemma 2.7, is less than

$$\frac{1}{2}(T^{\#,2}(\nabla u)(Q_0))^2 + C \int_{\Delta_h} N_h^2(F) \frac{d\sigma}{\sigma(\Delta_h)}$$

for suitable choice of ϵ_0.

SKETCH OF PROOF OF LEMMA 2.13. Observe that Lemma 7 of [12] was stated in terms of v, not F. However $\Delta F = \Delta v$ and the only important property of $v(x)$ used here is the fact that $|\Delta v| \leqslant |\nabla V|\,|Hu|$.

Let B_j be a Whitney cube of $\{S_h(F) > \lambda\}$ and let

$$F_j = B_j \cap \{S_h(F) > 2\lambda, N_\eta(F) \leqslant \gamma\lambda, N_\eta(d|\nabla F|) \leqslant \gamma\lambda, S_h(\nabla u) N_h(F) \leqslant (\gamma\lambda)^2\}.$$

Let Ω be the sawtooth region over F_j (see [4]) and fix an $X \in \Omega$ away from ∂D for the pole of $G_\Omega(X, Y)$, the Green's function for Ω. If $d\omega_\Omega$ is harmonic measure for Ω evaluated at X, then the estimate $\sigma(F_j) \leqslant C\gamma^n \sigma(B_j)$ follows from the estimate $\omega_\Omega(F_j) \leqslant \gamma^2$ (see [3]). At this point the proof in [12] carries over once one shows that

$$\int_\Omega G_\Omega(X, Y)|F(Y)|\,|\nabla V(Y)|\,|Hu(Y)|\,dY \leqslant (\gamma\lambda)^2.$$

The integral will be estimated first over the region $B_0(X)$, a ball of radius roughly $d(X) = \operatorname{dist}(X, \partial D)$ centered at the pole X of G_Ω. For $Y \in B_0(X)$,

$$|Hu(Y)| \lesssim \frac{1}{d(X)} S_h(\nabla u)(Q)$$

for any $Q \in F_j$ and $|F(Y)| \leqslant N_h(F)(Q)$, hence

$$\begin{aligned} &\int_{B_0(X)} G_\Omega(X, Y)|F(Y)|\,|\nabla V(Y)|\,|Hu(Y)|\,dY \\ &\qquad \lesssim N_h(F)(Q) S_h(\nabla u)(Q) \|V\|_\infty \int_{B_0(X)} G_\Omega(X, Y) d(X)^{-2}\,dX \\ &\qquad \leqslant C(\gamma\lambda)^2. \end{aligned}$$

For $Y \in \Omega B_0(X)$,

$$G_\Omega(X, Y) \lesssim G_D(P_0, Y)/\omega^{P_0}(\Delta_h)$$

where $G_D(P_0, \cdot)$ is the Green's function for D with pole at $P_0 \in D$. Hence

$$\int_{\Omega\setminus B_0(X)} G_\Omega(X, Y)|F|\,|\nabla V|\,|Hu|\,dY$$

$$\leqslant \frac{1}{\omega(\Delta_h)}\int_{\Delta_h\cap F_j}\int_{\Gamma h(Q)} d(Y)^{2-n}|F|\,|Hu|\,|\nabla V|\,dY\,d\omega(Q)$$

$$\leqslant \frac{1}{\omega(\Delta_h)}\int_{\Delta_h\cap F_j} N_h(F)S_h(\nabla u)S_h(V)\,d\omega\,(Q)$$

$$\lesssim (\gamma\lambda)^2\left[\frac{1}{\omega(\Delta_h)}\int_{\Delta_h} S_h^2(V)\,d\omega\,(Q)\right]^{1/2}$$

$$\lesssim C(\gamma\lambda)^2\left\{\frac{1}{\omega(\Delta_h)}\int_{T(\Delta_h)} G_D(P_0, Y)|\nabla V(Y)|^2\,dY\right\}^{1/2}$$

$$\leqslant C(\gamma\lambda)^2.$$

This last inequality follows from the fact that a function in BMO $(d\sigma)$ is also in

$$\mathrm{BMO}\,(d\omega) = \left\{g\in L^2(d\omega)\colon \sup\left(\frac{1}{\omega(\Delta)}\int_\Delta\left|g - \int_\Delta g\frac{d\omega}{\omega(\Delta)}\right|^2 d\omega\right)^{1/2} < \infty\right\}$$

and that $G(P_0, Y)|\nabla V(Y)|^2\,dY$ is a Carleson measure with respect to $d\omega$. (See Jerison-Kenig [9].) □

The following lemma provides a pointwise estimate for $N_h(F)(Q)$ which proves Theorem 2.3.

Lemma 2.14. *Given* $\epsilon > 0$ *and* $X\in\Gamma_\delta(Q)$, $\delta = \delta(\epsilon)$,

$$|F(X)| + \mathrm{dist}\,(X)|\nabla F(X)| \leqslant C\epsilon T^{\#,4}(\nabla u)(Q) + C\|g\|_* + C\epsilon S_h(\nabla u)(Q).$$

From the lemma one obtains

$$\int_{\Delta_h} N_h^2(F) + N_h^2(\mathrm{dist}\,|\nabla F|)\frac{d\sigma}{\sigma(\Delta_h)} \lesssim \|g\|_*^2 + C\epsilon M_\sigma(T^{\#,4}(\nabla u))^2(Q_0)$$

$$+ C\epsilon\frac{1}{\sigma(\Delta_h)}\int_{\Delta_h} S_h^2(\nabla u)\,d\sigma,$$

which by (2.13) yields (2.4), for sufficiently small ϵ.

PROOF OF (2.14). Let us consider only the term $|F(X)|$; the estimate for $d(X)|\nabla F(X)|$ is exactly the same. If $B_0(X)$ is the ball centered at X with radius comparable to $d(X)$, and $G(X, \bullet)$ is the Green's function for the domain D,

$$|F(X)| \leqslant \int_D G(X, Y)|\nabla V(Y)|\,|Hu(Y)|\,dY = \int_{B_0(X)} + \int_{D\setminus B_0(X)}.$$

As in Lemma 2.13, using $|Hu(Y)| \leqslant d(X)^{-1}S_h(\nabla_h(Q))$,

$$\int_{B_0(X)} G(X, Y)|\nabla V(Y)|\,|Hu(Y)|\,dY$$

$$\leqslant S_h(\nabla u)(Q)\left\{\int_{B_0(X)} G(X, Y)|\nabla V(Y)|^2\,dY\right\}^{1/2}\left\{\frac{1}{d(X)}\int_{B_0(X)} G(X, Y)\,dY\right\}^{1/2},$$

For $Y \in B_0(X)$, $G(X, Y) \approx |X - Y|^{2-n}$, so that $G(X, Y)$ is comparable to $G_{\tilde{B}_0}(X, Y)$, the Green's function for $\tilde{B}_0 = B_0(X, 2d(X))$. Hence the third term in the product above is finite and

$$\left\{\int_{B_0(X)} G(X, Y)|\nabla V(Y)|^2\,dY\right\}^{1/2} \leqslant \left\{\int_{\partial\tilde{B}_0} |V(P) - V(X)|^2\,d\omega_{\tilde{B}_0}(P)\right\}^{1/2}$$

which is less than ϵ since V is continuous and $h < \delta(\epsilon)$.

To bound that part of the Green's potential over $D \setminus B_0(X)$ we introduce the regions Ω_j. For $j = 0, \ldots, N$, set

$$\Omega_j = \{Y \in D : |Y - Q| \leqslant 2^{j-1}d(X)\}, \quad \text{and} \quad R_j = \Omega_{j+1} \setminus \Omega_j,$$

where $2^N d(X) = \delta$. Thus the regions Ω_j form a nested sequence of domains which fills out $\Omega_\delta = T\Delta(Q, \delta))$. We assume that δ has been chosen so that

$$\frac{1}{\omega(\Delta)}\int_{T(\Delta)} G_D(Y)|\nabla V(Y)|^2\,dY < \epsilon$$

whenever radius $(\Delta) < \delta$. Choose a sequence of points $\{X_j\}$ such that $X_j \in \Omega_j$ and $d(X_j) \approx 2^j d(X)$. Then if $Y \in R_j$, the comparison theorem yields (see [2])

$$G(X, Y)/G_{\Omega_{j+1}}(X_{j+1}, Y) \lesssim G(X, X_j)/G_{\Omega_{j+1}}(X_{j+1}, X_j).$$

By Hölder continuity of G, there is some $\alpha > 0$ such that

$$G(X, X_j) \lesssim 2^{-j\alpha}G(X_{j+1}, X_j)$$

and altogether, for $Y \in \Omega_j$,

$$G(X, Y) \leqslant 2^{-j\alpha}G_{\Omega_{j+1}}(X_{j+1}, Y) \leqslant \frac{2^{-j\alpha}}{\omega(\Delta_{j+1})}G_D(P_0, Y)$$

where $\Delta_{j+1} = \partial D \cap \Omega_{j+1}$. It follows that

$$\int_{\Omega_\delta \setminus B_0(X)} G(X, Y)|\nabla V(Y)|\,|Hu(Y)|\,dY$$

$$\lesssim \sum_{j=0}^{N} 2^{-j\alpha} \frac{1}{\omega(\Delta_{j+1})} \int_{T(\Delta_{j+1})} G_D(P_0, Y)|\nabla V(Y)|\,|Hu(Y)|\,dY$$

$$\leqslant \sum_{j\geq 0} 2^{-j\alpha} \left\{ \frac{1}{\omega(\Delta_{j+1})} \int_{T(\Delta_{j+1})} G(P_0, Y)|\nabla V|^2\,dY \right\}^{1/2}$$

$$\left\{ \frac{1}{\omega(\Delta_{j+1})} \int_{T(\Delta_{j+1})} G(P_0, Y)|Hu|^2\,dY \right\}^{1/2}$$

$$\leqslant \sum_j 2^{-j\alpha} \epsilon \sup_{\Delta \ni Q} \left\{ \frac{1}{\omega(\Delta)} \int_\Delta |\nabla u - C_\Delta|^2\,d\omega \right\}^{1/2}$$

$$\leqslant \sum_j 2^{-j\alpha} \epsilon T^{\#,4}(\nabla u)(Q) \leqslant C\epsilon T^{\#,4}(\nabla u)(Q).$$

The last inequality follows from the fact that $d\omega/d\sigma$ satisfies a reverse Hölder inequality of exponent two (Dahlber, [2]).

Let $D_\delta = \{Y \in D\colon \operatorname{dist}(Y, \partial D) \leqslant \delta\}$. We have estimated that part of the Green's potential over the region $D_\delta \cap \Omega_\delta$. The remainder, $D_\delta \setminus \Omega_\delta$, consists of a union of regions $\Omega_\delta^{(k)}$, which are, roughly speaking, translates of $\Omega_\delta^{(k)}$ by a factor of $2^k\delta$. The estimates on each of these are similar to the above, and similar to those of [12], Lemma 10, so the details are omitted.

Consider now the region $D \setminus D_\delta$. Away from the pole X of $G(X, Y)$ and at a distance at least δ from ∂D, one simply uses the pointwise estimates $|G(X, Y)| \leqslant C_\delta$, $|\nabla V(Y)| \leqslant C_\delta$ and $|Hu(Y)| \leqslant C_\delta |\nabla u(Y)|$ to obtain

$$\int_{D \setminus D_\delta} G(X, Y)|\nabla V|\,|Hu|\,dY \leqslant C(\delta) \sup_{Y \in D \setminus D_\delta} |\nabla u(Y)| \leqslant C \left\{ \int_{\partial D} N^2(\nabla u)\,d\sigma \right\}^{1/2}$$

$$\leqslant C\|g\|_*. \quad \square$$

3. The boundary value problem (1.1) with Hölder continuous data

The Green's function $G(P_0, \cdot)$ for D is Hölder continuous of some order γ_0 where γ_0 depends on the Lipschitz character of D. Thus, it can be shown that if $g(Q)$ on ∂D is Hölder continuous of order $\gamma < \gamma_0$, then the solution of the Dirichlet problem with boundary values $g(Q)$ satisfies $|\nabla g(x)| \lesssim d(x)^{\gamma - 1}$. Solutions to the boundary value problem (1.1) will also have this property when the vector field $\vec{V}$ is smooth enough. The proof requires the information established in §2, namely that the solution is known to be of bounded mean oscillation.

Theorem 3.1. *If the components $V_i(Q)$ of $\vec{V}(Q)$ and the boundary data $g(Q)$ are all Hölder continuous of order $\gamma < \gamma_0$, γ_0 as above, then if $u(x)$ is the solution to* (1.1) *given by Calderón's Theorem, there exists a constant C depending on D, $\|g\|_\gamma$ and $\|V\|_\gamma$ such that $|Hu(X)| \leqslant Cd(X)^{\gamma-1}$.*

Fix such a $\gamma < \gamma_0$. It will first be shown that ∇u is Hölder continuous of some order $\alpha < \gamma$; this information will then be used to modify some estimates and obtain the desired result. The results of section 2 guarantee that $\nabla u \in \mathrm{BMO}\,(d\sigma)$. Hence $T^{\#,2}(\nabla u)(Q)$ is bounded by $\|g\|_\infty$. Fix $Q_0 \in \partial D$ and set

$$H(r) = \left\{\frac{1}{\sigma(\Delta(Q_0, r))} \int_{T(\Delta_r)} d(X)|Hu(X)|^2\, dX\right\}^{1/2}.$$

A perturbation argument like that of section 2 will be used to prove

Theorem 3.2. *Given $\alpha < \gamma$ and $\epsilon > 0$, there exists a constant C_ϵ such that*

$$H(r) < \epsilon H(2r) + C_\epsilon r^\alpha. \tag{3.3}$$

Because $H(s)$ is finite for all $s > 0$, inequality (3.3) gives Hölder continuity of ∇u by repeated iteration of itself and by choosing $\epsilon \leqslant 2^{-\gamma-1}$. The notation and terminology of section 2 will be used in the subsequent lemmas. Fix a small $\epsilon > 0$, a $Q_0 \in \partial D$ and assume $r < \epsilon$. Recall that $v(x) = V(X) \cdot \nabla u(X)$, where $V(X)$ is the harmonic extension of $V(Q)$, and that Δ_r denotes the ball of radius r centered at Q_0.

Lemma 3.4. *If $X_r \in T(\Delta_r)$ with* $\operatorname{dist}(X_r, \partial D) \geqslant cr$, *then*

$$H(r) \leqslant C\left[r|\nabla u(X_r)| + \left\{\frac{1}{\sigma(\Delta_r)} \int_{\Delta_r} \tilde{S}^2_{2r}(v)(Q)\, d\sigma(Q)\right\}^{1/2} + \left\{\frac{1}{\sigma(\Delta_r)} \int_{T(\Delta_r)} d(X)|\nabla V(X)|^2 |\nabla u(X)|^2\, dX\right\}^{1/2}\right].$$

PROOF. The proof is identical to the argument leading up to Lemma 2.10, given that $T^{\#,2}(\nabla u)(Q_0) \leqslant \|g\|_\infty$. □

Lemma 3.5. *For any $\alpha < \gamma$,*

$$\frac{1}{\sigma(\Delta_r)} \int_{T(\Delta_r)} d(X)|\nabla V(X)|^2 |\nabla u(X)|^2\, dX \leqslant Cr^{2\alpha}.$$

PROOF. The proof of Lemma 2.11 and the fact that $\nabla u \in \mathrm{BMO}\,(d\sigma)$ yields the estimate, for $X \in T(\Delta_r)$,

$$|\nabla u(X)| \leqslant C \log\left(\frac{1}{r}\right). \tag{3.6}$$

Since $|\nabla V(X)| \leqslant d(X)^{\gamma-1}$ we have

$$\frac{1}{\sigma(\Delta_r)} \int_{T(\Delta_r)} d(X)|\nabla V(X)|^2 |\nabla u(X)|^2 \, dX \leqslant C \log^2\left(\frac{1}{r}\right) r^{2\gamma},$$

which proves the lemma for any $\alpha < \gamma$. □

It should be observed that this is the only point in the proof where the expected order of Hölder continuity is not achieved. The reason is that the bound (3.6) for $|\nabla u(X)|$ is not sharp. However, once some order of continuity of ∇u is established, the bound (3.6) can be replaced by

$$|\nabla u(X)| \leqslant C \tag{3.6$'$}$$

and the inequality of Lemma 3.5 can be replaced by

$$\frac{1}{\sigma(\Delta_r)} \int_{T(\Delta_r)} d(X)|\nabla V(X)|^2 |\nabla u(X)|^2 \, dX \leqslant C r^{2\gamma}. \tag{3.5$'$}$$

which yields the right order of Hölder continuity.

Moreover, (3.6) and Lemmas 3.4 and 3.5 now show that

$$H(r) \leqslant C r^{\alpha} + \left\{ \frac{1}{\sigma(\Delta_r)} \int_{\Delta_r} \bar{S}^2_{2r}(v)(Q) \, d\sigma(Q) \right\}^{1/2} \tag{3.7}$$

Write

$$v(x) = g(x) - F(x)$$

where

$$F(x) = \int_D G(X, Y) \Delta v(Y) \, dY$$

and since

$$g(x) = \int_{\partial D} g(Q) \, d\omega^X(Q)$$

satisfies $|\nabla g(x)| \leqslant d(X)^{\gamma-1}$ it suffices to consider the term involving $\bar{S}^2_{2r}(F)$. The good-λ inequality (2.13) yields

(3.8)
$$H(r) \leqslant C r^{\alpha} + \left\{ \frac{C_\epsilon}{\sigma(\Delta_r)} \int_{\Delta_r} N^2_r(F) \, d\sigma(Q) \right\}^{1/2} + \epsilon \left\{ \frac{1}{\sigma(\Delta_r)} \int_{\Delta_r} S^2_{2r}(\nabla u)(Q) \, d\sigma(Q) \right\}^{1/2}$$

and the last term in the summand above is bounded by $\epsilon H(2r)$. The next lemma will be used to handle the second term above; it is stated in a more general form.

Lemma 3.9. *Let γ_0 be the order of Hölder continuity of the Green's function $G(P_0, \cdot)$ of D. Suppose V and U are harmonic in D with $|\nabla V(X)| \leqslant d(X)^{\gamma-1}$ for $\gamma < \gamma_0$ and $U|_{\partial D}$ in* BMO $(d\sigma)$. *Set $v(x) = V(X)U(X)$, and*

$$F(X) = \int_D G(X, Y)\,\Delta v(Y)\,dY.$$

Then if $d(X) \leqslant r$, $|F(X)| \leqslant Cr^\gamma$.

PROOF. Fix X with $d(X) \leqslant r$ and observe that $|\nabla v(Y)| \leqslant |\nabla V(Y)|\,|\nabla U(Y)|$. Let $B_0(X) = \{ Y \in D: |Y - X| < \frac{1}{2}d(X) \}$ be the ball centered at X containing the pole of $G(X, Y)$. Then

$$\int_{B_0(X)} G(X, Y)|\nabla V|\,|\nabla U|\,dY \leqslant \frac{r^\gamma}{d(X)} \int_{B_0(X)} G(X, Y)\left[\frac{1}{d(X)}\,\|U\|_{\mathrm{BMO}}\right] dX$$
$$\leqslant Cr^\gamma.$$

Choose N such that $2^N r = 1$, and for each $j = 1, \ldots, N$ introduce the regions Ω_j as in the proof of Lemma 2.14, and let Δ_j denote $\partial D \cap \Omega_j$. As before, one applies the estimates for $G(X, Y)$ in the annular regions $\Omega_j \setminus \Omega_{j+1}$ to obtain

$$\int_{\Omega\delta} G(X, Y)|\nabla V|\,|\nabla U|\,dY$$
$$\leqslant \sum_{j=1}^{N} 2^{-j\gamma_0} \frac{1}{\omega(\Delta_j)} \int_{\Omega_j} G(P_0, Y)|\nabla V(Y)|\,|\nabla U(Y)|\,dY$$
$$\leqslant \sum_{j=1}^{N} 2^{-j\gamma_0} \left\{\frac{1}{\omega(\Delta_j)} \int_{\Omega_j} G(P_0, Y)|\nabla V|^2\,dY\right\}^{1/2} \left\{\frac{1}{\omega(\Delta_j)} \int_{\Omega_j} G(P_0, Y)|\nabla U|^2\,dY\right\}^{1/2}$$
$$\leqslant \sum_{j=1}^{N} 2^{-j\gamma_0} \left\{\frac{1}{\omega(\Delta_j)} \int_{\Delta_j} S^2_{2^j r}(V)\,d\omega\right\}^{1/2}$$

where the last inequality follows from the fact that $U \in \mathrm{BMO}\,(d\omega)$. When $|\nabla V(Y)| \leqslant d(Y)^{\gamma-1}$, a pointwise estimate on the truncated square function of V follows; namely $S_{2^j r}(V)(Q) \lesssim (2^j r)^\gamma$. For $\gamma < \gamma_0$, the above inequality can be summed, obtaining.

$$\int_{\Omega_j} G(X, Y)|\nabla V|\,|\nabla A U|\,dY \lesssim Cr^\gamma.$$

Let $D_\delta = \{X \in D: d(X) < \delta\}$ and then $D_\delta = \bigcup_k \Omega_\delta^{(k)}$ where $\Omega_\delta^{(k)}$ is a region like Ω_δ at a distance roughly $2^k\delta$ from Ω_δ. One achieves similar estimates for the

Green's potential when the region of integration is $\Omega_\delta^{(k)}$ and these in turn may be summed on k (see for example [12]). The term involving integration over the remaining region D_δ is easily handled by simple pointwise estimates:

$$\begin{aligned}\int_{D_\delta} G(X, Y)|\nabla V|\,|\nabla U|\,dY &\leqslant C_\delta d(X)^{\gamma_0} \sup_{Z\in D_\delta} |\nabla u(Z)| \\ &\leqslant Cr^\gamma \|N(\nabla u)\|_{L^2} \\ &\leqslant Cr^\gamma \|g\|_{\mathrm{BMO}}. \quad \square\end{aligned}$$

Lemma 3.9 is applied to estimate $N_r(F)(Q)$ at any $Q \in \Delta_r$ and together with (3.8) shows that $H(r) \leqslant Cr^\alpha + \epsilon H(2r)$, *i.e.*, that $H(r) \leqslant Cr^\alpha$ for all r. This estimate implies an immediate improvement of itself (see the argument following Lemma 3.5) and so $H(r) \leqslant Cr^\gamma$ for any $\gamma < \gamma_0$. From the fact that $H(r) \leqslant Cr^\gamma$ it follows, by the mean value property, that $|Hu(X)| \lesssim d(X)^{\gamma-1}$, which implies that u is of class $C^{1,\gamma}$.

References

[1] Calderón, A. P. Boundary value problems for the Laplace equation in Lipschitzian domains, *Recent Progress in Fourier Analysis*, North-Holland Mathematics Studies, **111**(1983), 33-48.

[2] Dahlberg, B. E. J. On estimates of harmonic measure, *Arch. Rational Mech. Anal.* **65**(1977), 272-288.

[3] Dahlberg, B. E. J. Weighted norm inequalities for the Lusin area integral and the nontangential maximal function for functions harmonic in a Lipschitz domain, *Studia Math.* **64**(1980), 297-314.

[3] Dahlberg, B. E. J., Jerison, D. S. and Kenig, C. E. Area integral estimates for elliptic differential operators with non-smooth coefficients, *Arkiv. Mat.*, **22**(1984), 97-108.

[5] Fabes, E. B., Kenig, C. E. and Neri, U. Carleson measures, H^1 duality, and weighted BMO in non-smooth domains, *Indiana J. Math.* **30**(1981), 547-581.

[6] Fefferman, C. and Stein, E. H^p spaces of several variables, *Acta Math.* **129**(1972), 137-193.

[7] Fabes, E. B. and Neri, U. Dirichlet problem in Lipschitz domains with BMO data, *Proc. Amer. Math. Soc.* **78**(1980), 33-39.

[8] Janson, S. On functions with conditions on the mean oscillation, *Arkiv Mat.* **14**(1976), 189-196.

[9] Jerison, D. S. and Kenig, C. E. Boundary behavior of harmonic functions in non-tangentially accessible domains, *Adv. in Math.* **46**(1) (1982), 80-147.

[10] John, F. and Nirenberg, L. On functions of bounded mean oscillation, *Comm. Pure Applied Math.* **14**(1961), 415-426.

[11] Kenig, C. E. Weighted Hardy spaces in Lipschitz domains, *Amer. J. Math.* **102**(1980), 129-163.

[12] Kenig, C. E. and Pipher, J. The oblique derivative problem on Lipschitz domains with L^p data, to appear, *Amer. J. Math.*

[13] Sarason, D. Functions of vanishing mean oscillation, *Trans. Amer. Math. Soc.* **207**(1975), 391-405.
[14] Stegeniga, D. A. Bounded Toeplitz operators on H^1 and applications of the duality between H^1 and the functions of bounded mean oscillation, *Amer. J. Math.* **98**(1976), 573-589.
[15] Stein, E. M. *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.
[16] Strömberg, J. O. Bounded mean oscillation with Orlicz norms and duality of Hardy spaces, *Indiana J. Math.*, **28**(1979), 511-544.

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