REVISTA MATEMÁTICA IBEROAMERICANA
VOL. 4, N.° 1, 1988

The Boundedness of Calderón-Zygmund Operators on the Spaces $\dot{F}_p^{\alpha,q}$

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To our dear friend José-Luis Rubio de Francia

Introduction

Calderón-Zygmund operators are generalizations of the singular integral operators introduced by Calderón and Zygmund in the fifties [CZ]. These singular integrals are principal value convolutions of the form

$$Tf(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} K(x-y) f(y)\, dy = \text{p.v.}\, K * f(x),$$

where f belongs to some class of test functions. The kernel K is a function defined in $\mathbb{R}^n \setminus \{0\}$ and has the form:

$$K(x) = \Omega(x/|x|)|x|^{-n}$$

where Ω is usually assumed to satisfy some «smoothness» property (see [S] or [GC-RF]) on the unit sphere S^{n-1} as well as the important cancellation property

$$\int_{S^{n-1}} \Omega(x')\, d\sigma = 0.$$

This cancellation property may be rewritten, at least formally, as $T1 = 0$; *i.e.*, the action of T on the function identically equal to 1 (properly defined) is identically zero.

The L^2 boundedness of these operators can be proved by using Fourier transform arguments and, from this, the L^p boundedness for $1 < p < \infty$ is obtained by what are now considered to be standard real variable techniques. Although Fourier transform arguments are not applicable, the L^p boundedness results are still true for some more general principal value integrals

$$Tf(x) = \lim_{\epsilon \to 0} \int_{|x-y| > \epsilon} K(x,y) f(y)\, dy$$

that are not convolutions.

The general operators in question can be described as follows. Let $T: \mathfrak{D} \to \mathfrak{D}'$ be a continuous linear operator, where $\mathfrak{D} = \mathfrak{D}(\mathbb{R}^n)$ is the space of Schwartz test functions and $\mathfrak{D}' = \mathfrak{D}'(\mathbb{R}^n)$ is its dual. Let K denote the distributional kernel of T. That is, $K \in \mathfrak{D}'(\mathbb{R}^n \times \mathbb{R}^n)$ and satisfies:

$$\langle T\varphi, \psi \rangle = \langle K, \psi \otimes \varphi \rangle$$

for all φ and ψ in $\mathfrak{D}$. We assume that the restriction of K to the set $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}$ is a continuous function. Hence, if φ and ψ are in $\mathfrak{D}$ and $\operatorname{supp} \varphi \cap \operatorname{supp} \psi = \emptyset$, then

$$\langle T\varphi, \psi \rangle = \iint K(x,y) \varphi(y) \psi(x)\, dy\, dx.$$

Moreover, we suppose that K satisfies the «size and smoothness» conditions:

$$|K(x,y)| \leqslant C|x-y|^{-n}, \quad \text{and} \tag{0.1}$$

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leqslant C|x-x'|^{\epsilon}|x-y|^{-n-\epsilon} \tag{0.2}$$

whenever $2|x-x'| \leqslant |x-y|$, for some constant $C > 0$ and some ϵ in $(0,1]$.

These operators were introduced by Coifman and Meyer in [CM] where they showed that if T is bounded on L^2 then it is also bounded on L^p for $1 < p < \infty$.

The problem of charaterizing which operators with this type of kernel are bounded on L^2 was solved by David and Journé in 1985 with the «$T1$-Theorem». They found two necessary and sufficient conditions that T must satisfy. The first of these conditions is that both $T1$ and T^*1 (where T^* denotes the formal transpose of T) must be in BMO. In fact, David and Journé proved in [DJ] that the above condition can be reduced to the case $T1 = T^*1 = 0$; thus, a cancellation property appears again.

The operators studied by Calderón and Zygmund are invariant under the action of the group of translation and dilations of $\mathbb{R}^n$ (the «$ax + b$ group»). However, this is not always the case for this more general class of operators. It is then necessary and natural to impose some constraints on the behaviour of T under the action of this group. The second condition in the David and

Journé result is the so called Weak Boundedness Property (WBP) and it is intimately related to these constraints. We will describe it in detail in the next section.

Several authors have obtained versions of the $T1$-Theorem in other spaces of functions or distributions (see for example [L], [MM], [M2]). In general, the results obtained are based on the same kind of conditions mentioned before. In addition, a higher degree of smoothness on K and a higher degree of cancellation on T (namely that T vanishes on polynomials up to certain order) is usually imposed. In [FHJW] Calderón-Zygmund operators acting on the Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}$ were studied for $\alpha \in [0, 1)$ and $1 \leqslant p, q \leqslant \infty$. In this paper we extend the study to the general case $\alpha \in \mathbb{R}$ and $0 < p, q \leqslant \infty$. Our main results are Theorems 3.1, 3.7 and 3.13.

One of the reasons for studying Calderón-Zygmund operators in the context of Triebel-Lizorkin spaces in that the results obtained can be translated to some other classical spaces. For example, we recover the result of David and Journé for L^2, and, using interpolation arguments, a criterion for the boundedness of Calderón-Zygmund operators on the Besov-Lipschitz spaces $\dot{B}_p^{\alpha,q}$. Also, for $\alpha > 0$ and $1 < p < \infty$, the inhomogeneous Triebel-Lizorkin space $F_p^{\alpha,2}$ coincides with the Sobolev space L_p^α and a criterion for this space can be easily obtained from the homogeneous case (see Corollary 3.33 below).

The main results in this paper were proved independently in [FW] and [T]. They were motivated by problems posed by B. Jawerth and E. Stein. Also, we would like to thank G. David for some helpful discussions.

We have included some technicals facts that can, essentially, be found elsewhere in the literature. However, we introduce some different notation that can be used to simplify the presentation. Hence, we restate some of these known facts here. The reader is referred to the expository article [FW] and to [T] for further details.

1. Some Preliminary Definitions

Generalized Calderón-Zygmund operators

Frazier and Jawerth have shown in [FJ1] and [FJ2] that the spaces $\dot{F}_p^{\alpha,q}$ can be decomposed in terms of some building blocks called «smooth atoms», and similarly into some more general building blocks, the «smooth molecules». Using this, it follows, in the tradition of [CW], [TW] and others, that to prove that an operator is bounded on these spaces, it is sufficient to prove that it maps «smooth atoms» into «smooth molecules». This was the strategy used in [FHJW] and it is also the one that we want to apply here.

We start by recalling the definitions of these atoms and molecules. Fix $\alpha \in \mathbb{R}$, and $0 < p, q \leqslant \infty$. Let $J = n/\min\{1, p, q\}$ and $L = \max\{[J - n - \alpha], [J - n]\}$

where, for $x \in \mathbb{R}$, $x_+ = \max\{0, x\}$, $[x]$ denotes the standard «greatest integer function» at x, and $x^* = x - [x]$.

Let Q be a dyadic cube of $\mathbb{R}^n$ with «lower left corner» at the point x_Q and with side length $\ell(Q)$. A smooth atom for $\dot{F}_p^{\alpha,q}$ associated to the cube Q, is a function $a_Q \in \mathfrak{D}$ satisfying

$$\text{Supp}\, a_Q \subset 3Q, \tag{1.1}$$

$$\int x^\gamma a_Q(x)\, dx = 0 \quad \text{if} \quad |\gamma| \leqslant L, \tag{1.2}$$

$$|D^\gamma a_Q(x)| \leqslant |Q|^{-1/2-|\gamma|/n} \quad \text{if} \quad |\gamma| \leqslant [\alpha]_+ + 1, \tag{1.3}$$

(where as usual, for a multi-index of non-negative integers $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$, we let $D^\gamma = (\partial^{\gamma_1}/\partial x_1^{\gamma_1})(\partial^{\gamma_2}/\partial x_2^{\gamma_2}) \cdots (\partial^{\gamma_n}/\partial x_n^{\gamma_n})$, $x^\gamma = x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_n^{\gamma_n}$ and $|\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_n$).

Let $M > J$ and $1 > \delta > \alpha^*$. A smooth (δ, M)-molecule is a function m_Q which satisfies

$$|m_Q(x)| \leqslant C|Q|^{-1/2}(1 + \ell(Q)^{-1}|x - x_Q|)^{-\max\{M, M-\alpha\}}, \tag{1.4}$$

$$\int x^\gamma m_Q(x)\, dx = 0 \quad \text{if} \quad |\gamma| \leqslant [J - n - \alpha], \tag{1.5}$$

$$|D^\gamma m_Q(x)| \leqslant C|Q|^{-1/2-|\gamma|/n}(1 + \ell(Q)^{-1}|x - x_Q|)^{-M} \quad \text{for} \quad |\gamma| \leqslant [\alpha], \tag{1.6}$$

and, for $|\gamma| = [\alpha]$,

$$|D^\gamma m_Q(x) - D^\gamma m_Q(x')| \leqslant C|Q|^{-1/2-(|\gamma|+\delta)/n}|x - x'|^\delta \sup_{|z| \leq |x-x'|} (1 + \ell(Q)^{-1}|z - (x - x_Q)|^{-M}, \tag{1.7}$$

where C is an absolute constant. (Our convention is that conditions (1.5), (1.6) or (1.7) are void if the term bounding $|\gamma|$ is negative.)

The degree of smoothness of the molecules increases with the parameter α; thus, in order to carry out our project, we will need to consider derivatives of the image of an atom by the Calderón-Zygmund operators that we want to study. This, in turn, will cause us to consider operators with sufficiently smooth kernels.

The usual «$|x - y|^{-n}$ size condition» of the kernels of Calderón-Zygmund operators, in general, is not satisfied by the kernels of the derivatives of these operators. In fact, the behaviour of those near the «diagonal» is worse, and, consequently, we are forced to consider a more general class of operators that we will now define. Let $T: \mathfrak{D} \to \mathfrak{D}'$ be a continuous linear operator and let K be its distributional kernel. For $m, l \in \mathbb{N}_0$ and $\epsilon \in (0, 1)$, we say that T is a (generalized) Calderón-Zygmund operator of «size m» and «smoothness $l + \epsilon$» if the restriction of K to $\Omega = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}$ is a function with continuous partial derivatives in the variables x up to order l which satisfy, for some constant $C > 0$:

$$|D_1^\gamma K(x,y)| \leqslant C|x-y|^{-n-m-|\gamma|} \quad \text{for} \quad |\gamma| \leqslant l \tag{1.8}$$

$$|D_1^\gamma K(x,y) - D_1^\gamma K(x',y)| \leqslant C|x-y|^{-n-m-l-\epsilon}|x-x'|^\epsilon \tag{1.9}$$

for $|\gamma| = l$ and $2|x-x'| < |x-y|$(where the subindex 1 stands for derivatives in the first variable, x). In such a case we will write $T \in \mathrm{CZO}\,(m, l+\epsilon)$ or $T \in \mathrm{CZO}\,(l+\epsilon)$ if $m = 0$. If β and γ are two multi-indices, we denote by $T_{\beta,\gamma}$ the continuous linear operator from $\mathfrak{D}$ to $\mathfrak{D}'$ whose distributional kernel is:

$$K_{\beta,\gamma} = (y-x)^\beta D_1^\gamma K, \tag{1.10}$$

where the derivatives are taken in the distributional sense. It is easy to check that if $T \in \mathrm{CZO}\,(m, l+\epsilon)$, $|\gamma| \leqslant l$ and $|\beta| \leqslant m$, then $T_{\beta,\gamma} \in \mathrm{CZO}\,(m - |\beta| + |\gamma|, l - |\gamma| + \epsilon)$ and that the restriction of $K_{\beta,\gamma}$ to Ω is the function:

$$K_{\beta,\gamma}(x,y) = (y-x)^\beta D_1^\gamma K(x,y).$$

We will write T_β if $|\gamma| = 0$ and $D^\gamma T$ if $|\beta| = 0$.

In contrast with (0.2), our definition of Calderón-Zygmund operators does not impose any smoothness in the $y-$ variables. Recall that the formal transpose of T, $T^*: \mathfrak{D} \to \mathfrak{D}'$, is defined by

$$\langle T^*\psi, \varphi \rangle = \langle T\varphi, \psi \rangle, \quad \text{for} \quad \varphi, \psi \in \mathfrak{D}, \tag{1.11}$$

and, hence, the kernel of T^* is given by $K^*(x,y) = K(y,x)$. We will require later that the kernel of T be smooth in both x and y; thus, both T and T^* will be generalized Calderón-Zygmund operators for appropriate numbers m and l.

The action of generalized Calderón-Zygmund operators on functions of polynomial growth

Another characteristic of the atoms and molecules for the spaces $\dot{F}_p^{\alpha,q}$ is that they have a certain number of vanishing moments (properties (1.2) and (1.5)). Since we want that the Calderón-Zygmund operators preserve these vanishing moments, some cancellation property should be imposed on them. As we mention before, this is usually achieved by requiring that T (and T^*) vanishes on polynomials. However, since T is originally defined only on $\mathfrak{D}$, its action on polynomials has to be first defined. More generally, assume that $T \in \mathrm{CZO}\,(m, \epsilon)$. We want to extend T to the space:

$$\mathcal{O}^m = \{ f \in C^\infty(\mathbb{R}^n) : f(x) = O(|x|^m) \text{ as } x \to \infty \}.$$

Let $\{\varphi_j\}_{j=1}^\infty$ be a sequence of functions in $\mathfrak{D}$ satisfying the following properties:

$$\operatorname{supp}\varphi_j \subseteq B_{j+1} = \{x \in \mathbb{R}^n : |x| < j+1\}, \tag{1.12}$$

$$\varphi_j(x) = 1 \quad \text{for every} \quad x \in B_j, \quad \text{and} \tag{1.13}$$

$$\|\varphi_j\|_\infty \leqslant C \quad \text{for some} \quad C > 0, \quad \text{for every} \quad j. \tag{1.14}$$

Let $f \in \mathcal{O}^m$. By analogy with the definition of the action of convolution singular integrals on L^∞ (see [GC-RF, p. 202]), we define Tf by the equality

$$Tf = \lim_{j \to \infty} \left\{ T(\varphi_j f) - \int_{1 < |y| < j} K(0, y) f(y)\, dy \right\}. \tag{1.15}$$

This is justified by the following

Lemma 1.16. *Suppose $T \in \mathrm{CZO}(m, \epsilon)$, $f \in \mathcal{O}^m$ and $\{\varphi_j\}_j$ is as before. The limit in* (1.15) *exists in the weak*- topology of $\mathfrak{D}'$ and defines Tf as a distribution. In particular, if $g \in \mathfrak{D}_0$ where $\mathfrak{D}_0 = \left\{ \varphi \in \mathfrak{D} : \int \varphi\, dx = 0 \right\}$, then*

$$\langle Tf, g \rangle = \lim_{j \to \infty} \langle T\varphi_j f, g \rangle.$$

PROOF. Let

$$F_j = T\varphi_j f - \int_{1 < |y| < j} K(0, y) f(y)\, dy.$$

We want to show that

$$\lim_{j \to \infty} \langle F_j, g \rangle \quad \text{exists for any} \quad g \in \mathfrak{D}.$$

Fix $g \in \mathfrak{D}$ and let j_0 be large enough so that $\operatorname{supp} g \subseteq B_{j_0/2}$. For $j > j_0$, we can write

$$F_j = F_{j_0} + T(\varphi_j - \varphi_{j_0}) f - \int_{j_0 \le |y| < j} K(0, y) f(y)\, dy.$$

Then, since $(\varphi_j - \varphi_{j_0}) f$ and g have disjoint supports, we have

$$\begin{aligned}
\langle F_j, g \rangle &= \langle F_{j_0}, g \rangle + \iint K(x, y) [(\varphi_j - \varphi_{j_0}) f](y) g(x)\, dy\, dx \\
&\quad - \iint_{j_0 < |y| < j} K(0, y) f(y) g(x)\, dy\, dx \\
&= \langle F_{j_0}, g \rangle + \iint_{j_0 < |y| \le j_0 + 1} K(x, y) [1 - \varphi_{j_0} f](y) g(x)\, dy\, dx \\
&\quad - \iint_{j_0 < |y| \le j_0 + 1} K(0, y) f(y) g(x)\, dy\, dx \\
&\quad + \iint_{j_0 + 1 < |y| < j} [K(x, y) - K(0, y)] f(y) g(x)\, dy\, dx \\
&\quad + \iint_{j \le |y| < j+1} K(x, y) \varphi_j f(y) g(x)\, dy\, dx,
\end{aligned} \tag{1.17}$$

where all integrals are absolutely convergent.

Now, the first three terms in (1.17) do not depend on j, and, using (1.8), they are easily seen to be bounded by $P(g)$ where P is some continuous seminorm in $\mathcal{S}(\mathbb{R}^n)$ which may depend on the support of g.

By (1.9) and the dominated convergence theorem, the fourth term tends to

$$\iint_{j_0+1<|y|} [K(x,y) - K(0,y)] f(y) g(x)\, dy\, dx \quad \text{as} \quad j \to \infty,$$

and it is bounded by $C\|g\|_{L^1}$ where C is a constant depending on f, the support of g and the constant in (1.8) and (1.9).

Finally, the fifth term is bounded by

$$C \int_{\operatorname{supp} g} \int_{j - j_0/2 < |y| < j+1+j_0/2} \frac{|y|^m}{|y|^{n+m}} |g(x)|\, dy\, dx$$

$$\leqslant C\|g\|_{L^1} \log \frac{j+1+j_0/2}{j - j_0/2} \to 0 \quad \text{as} \quad j \to \infty. \quad \square$$

Observe that if $f \in \mathfrak{D}$, the above definition of Tf and the original action of T on f differ by a constant. Hence, Tf should be viewed as a distribution on $\mathfrak{D}_0$.

It is easy to see that definition of Tf does not depend on the choice of the sequence $\{\varphi_j\}_{j=1}^{\infty}$ and that, for $m = 0$, it is equivalent to the one given in [DJ].

If $T \in \mathrm{CZO}(m, \epsilon)$ and $g \in \mathfrak{D}_0$, we can use Lemma 1.16 to extend the action of T^*g to $\mathcal{O}^m$ by

$$\langle T^*g, f\rangle \equiv \lim_{j\to\infty} \langle T^*g, \varphi_j f\rangle = \lim_{j\to\infty} \langle T\varphi_j f, g\rangle = \langle Tf, g\rangle,$$

which is consistent with (1.11).

Note also that for $w \in \operatorname{supp} g$ and

$$y \in G = \{z \in \mathbb{R}^n: \operatorname{dist}(z, \operatorname{supp} g) > 2 \operatorname{diam}(\operatorname{supp} g)\},$$

$$\begin{aligned} \left|\int K(x,y) g(x)\, dx\right| &= \left|\int [K(x,y) - K(w,y)] g(x)\, dx\right| \\ &\leqslant \int_{\operatorname{supp} g} C|x-w|^{\epsilon} |y-w|^{-n-m-\epsilon} \|g\|_{\infty}\, dx \\ &\leqslant C[\operatorname{diam}(\operatorname{supp} g)]^{n+\epsilon} \|g\|_{\infty} |y-w|^{-n-m-\epsilon}, \end{aligned} \tag{1.18}$$

where we used that g has integral zero. But, if $\operatorname{supp} f \subseteq G$, the function $f(y) \int K(x,y) g(x)\, dx$ is an integrable function of y, since $f \in \mathcal{O}^m$, and, therefore,

$$\begin{aligned} \langle T^*g, f\rangle &= \lim_{j\to\infty} \iint K(x,y) \varphi_j f(y) g(x)\, dy\, dx \\ &= \lim_{j\to\infty} \int \varphi_j f(y) \int_{\operatorname{supp} g} K(x,y) g(x)\, dx\, dy \\ &= \int f(y) \int K(x,y) g(x)\, dx\, dy. \end{aligned}$$

Hence, we see that «far away» from the support of g, the distribution T^*g agrees with the continuous function $\int K(x,y)g(x)\,dx$.

We have used the «size» of T to define its action on $\mathcal{O}^m$. A similar argument can be given which uses the «smoothness» of T. More precisely, let

$$\mathfrak{D}_l = \left\{\varphi \in \mathfrak{D}\colon \int x^\gamma \varphi(x)\,dx = 0 \quad \forall |\gamma| \leqslant l\right\}$$

and let $\mathfrak{D}_l'$ be the dual space of $\mathfrak{D}_l$ with respect to the topology inherited from $\mathfrak{D}$. Then we have

Lemma 1.19. *Let $T \in \mathrm{CZO}\,(l+\epsilon)$, $f \in \mathcal{O}^l$ and let $\{\varphi_j\}_{j=1}^\infty$ be as before. Then the limit*

$$Tf = \lim_{j\to\infty}\left\{T\varphi_j f - \sum_{|\gamma|\leq l}\int_{1<|y|<j}\frac{D_1^\gamma K(0,y)}{\gamma!}f(y)\,dy\,x^\gamma\right\}$$

exists in the weak-topology of $\mathfrak{D}_{l-1}'$. In particular, if $g \in \mathfrak{D}_l$, then*

$$\langle Tf, g\rangle = \lim_{j\to\infty}\langle T\varphi_j f, g\rangle .$$

The proof of the lemma is similar to that of Lemma 1.16 and we leave it for the reader. Again, to avoid ambiguities, Tf should be regarded as an element of $\mathfrak{D}_l'$.

If $T\colon \mathfrak{D} \to \mathfrak{D}'$ is linear and continuous, then it is easy to check that

$$(D^\gamma T)\varphi = D^\gamma(T\varphi) \tag{1.20}$$

in the sense of distributions. Namely, let $T \in \mathrm{CZO}\,(l+\epsilon)$ and $f \in \mathcal{O}^l$. If $g \in \mathfrak{D}$, then $D^\gamma g \in \mathfrak{D}_{l-1}$ for $|\gamma| = l$ and hence we may consider

$$\langle D^\gamma(Tf), g\rangle \equiv (-1)^{|\gamma|}\langle Tf, D^\gamma g\rangle$$

$$= (-1)^{|\gamma|}\lim_{j\to\infty}\left\{\langle T\varphi_j f, D^\gamma g\rangle - \iint_{1<|y|<j} D_1^\gamma K(0,y)f(y)\,dy\,\frac{x^\gamma}{\gamma!}D^\gamma g(x)\,dx\right\}$$

$$= \lim_{j\to\infty}\left\{\langle D^\gamma T\varphi_j f, g\rangle - \iint_{1<|y|<j} D_1^\gamma K(0,y)f(y)\,dy\,g(x)\,dx\right\},$$

as an easy integration by parts shows. However, since $T \in \mathrm{CZO}\,(l+\epsilon)$, $D^\gamma T \in \mathrm{CZO}\,(l,\epsilon)$ and, from the last expression and Lemma 1.19, we have

$$\langle D^\gamma(Tf), g\rangle = \langle (D^\gamma T)f, g\rangle$$

which is consistent with (1.20). As in the remarks following Lemma 1.16, we may define the action of T^*g on $\mathcal{O}^l$ for $g \in \mathfrak{D}_l$ and verify that

$$\langle T^*g, f\rangle = \langle Tf, g\rangle.$$

Also, as in (1.18), this time using that $g \in \mathfrak{D}_l$, we have for $w \in \operatorname{supp} g$ and y «far away» from $\operatorname{supp} g$

$$(1.21)\quad \left|\int K(x,y)g(x)\,dx\right| = \left|\int\left[K(x,y) - \sum_{|\gamma|\le l}\frac{D_1^\gamma K(w,y)}{\gamma!}(x-w)^\gamma\right]g(x)\,dx\right|$$
$$\leqslant \int_{\operatorname{supp} g} C|x-w|^{l+\epsilon}|y-w|^{-n-l-\epsilon}\|g\|_\infty\,dx$$
$$\leqslant C[\operatorname{diam}(\operatorname{supp} g)]^{n+l+\epsilon}\|g\|_\infty|y-w|^{-n-l-\epsilon},$$

and, therefore, as before, away from the support of g, the distribution T^*g agrees with the continuous fuction $\int K(x,y)g(x)\,dx$.

We want to point out that if $g \in \mathfrak{D}_l$, more general sequences than $\{\varphi_j\}_{j=1}^\infty$ can be used to compute $\langle Tf, g\rangle$. In fact, let $\{\psi_j\}_{j=1}^\infty \subseteq \mathfrak{D}$ satisfy the condition (extending (1.13))

(1.13)′ for each compact set $B \subset \mathbb{R}^n$ there exists $N > 0$ such that $\psi_j(x) = 1$ for all $x \in B$ and $j > N$,

as well as

$$(1.14)'\qquad \|\psi_j\|_\infty \leqslant C \quad \text{for all} \quad j.$$

Then it is easy to show, using (1.16) (see [T]) that, as in (1.16),

$$\langle Tf, g\rangle = \lim_{j\to\infty}\langle T\psi_j f, g\rangle.$$

Moreover, the following result due to Y. Meyer holds ([M2], *cf.* also [FW]).

Lemma 1.22. *Let $T \in \mathrm{CZO}(l+\epsilon)$ and let $f \in \mathcal{O}^l$. For $\varphi \in \mathfrak{D}$ let $\varphi^t(x) = \varphi(x/t)$, $t > 0$. Then for all $g \in \mathfrak{D}_l$*

$$\lim_{t\to\infty}\langle T\varphi^t f, g\rangle = \varphi(0)\langle Tf, g\rangle.$$

The Weak Boundedness Property

In [DJ] the following concept of Weak Bondedness Property was introduced.

For $\eta \in \mathfrak{D}$, $z \in \mathbb{R}^n$ and $t > 0$, let

$$\eta^{z,t}(x) = \eta\left(\frac{x-z}{t}\right).$$

A linear and continuous operator $T: \mathfrak{D} \to \mathfrak{D}'$ is said to satisfy the Weak Boundedness Property if for every bounded subset $\mathcal{B}$ of $\mathfrak{D}$ there exists a positive constant $C = C(\mathcal{B})$ such that for all $\varphi, \psi \in \mathcal{B}$, all $z \in \mathbb{R}^n$ and $t > 0$,

$$|\langle T\varphi^{z,t}, \psi^{z,t}\rangle| \leqslant Ct^n. \tag{1.23}$$

If K is the kernel of T, then (1.23) may be rewritten as

$$|\langle K, \psi^{z,t} \otimes \varphi^{z,t}\rangle| \leqslant Ct^n. \tag{1.24}$$

Moreover, since the linear span of $\mathfrak{D}(\mathbb{R}^n) \otimes \mathfrak{D}(\mathbb{R}^n)$ is dense in $\mathfrak{D}(\mathbb{R}^n \times \mathbb{R}^n)$, it follows that the above is equivalent to the condition

$$|\langle K, f^{z,t}\rangle| \leqslant Ct^n,$$

for every function f in a bounded subset $\tilde{\mathcal{B}}$ of $\mathfrak{D}(\mathbb{R}^n \times \mathbb{R}^n)$, all of whose elements
have support in, say, a cube centered on the diagonal (where, $f^{z,t}(x, y)$ is of course $f((x - z)/t, (y - z)/t)$).

If we denote by $T^{z,t}$ the operator from $\mathfrak{D}$ to $\mathfrak{D}'$ whose kernel, $K^{z,t}$, is defined by

$$\langle K^{z,t}, f\rangle = \frac{1}{t^n}\langle K, f^{z,t}\rangle,$$

then the Weak Boudedness Property says that for all $f \in \tilde{\mathcal{B}}$

$$|\langle K^{z,t}, f\rangle| \leqslant C.$$

That is, the family of operators $\{T^{z,t}\}_{z \in \mathbb{R}^n, t > 0}$ is in some sense uniformly bounded. These are the constraints imposed on the behavior of T under the action of the «$ax + b$ group» that we mentioned in the introduction.

In order to include in our discussion operators of «size m» we now extend the definition of the Weak Bondedness Property in the following way.

A linear and continuous operator $T: \mathfrak{D} \to \mathfrak{D}'$ is said to satisfy a Weak Boundedness Property of order m if for every bounded subset $\mathcal{B}$ of $\mathfrak{D}(\mathbb{R}^n \times \mathbb{R}^n)$ whose elements have supports in a fixed cube centered on the diagonal, there exists a positive constant $C = C(\mathcal{B})$ such that for all $f \in \mathcal{B}$, all $z \in \mathbb{R}^n$, and $t > 0$,

$$|\langle K, f^{z,t}\rangle| \leqslant Ct^{n-m}. \tag{1.25}$$

In such a case we will write $T \in \mathrm{WBP}(m)$ or, simply, $T \in \mathrm{WBP}$ if $m = 0$.

We conclude this section with some remarks that are not hard to check. The reader is referred to [T] for the details.

(*a*) Using translation, the set $\mathcal{B}$ may be replaced by more general ones whose elements have all their derivatives uniformly bounded and are supported in cubes centered on the diagonal with fixed side length (but not with fixed center).

(*b*) If, in addition, the kernel of T satifies the size condition

$$|K(x,y)| \leqslant C|x-y|^{-n-m} \quad \text{for} \quad (x,y) \in \Omega,$$

then the cubes mentioned in (*a*) do not need to be centered on the diagonal.

(*c*) If $T \in \mathrm{WBP}(m)$, then $T_{\beta,\gamma} \in \mathrm{WBP}(m - |\beta| + |\gamma|)$ for all $|\beta| \leqslant m$ and for all $|\gamma| \geqslant 0$ (compare with Lemma 2.9 in [FW]).

2. L^∞-Estimates and Pointwise Definitions

In this section we collect some technical lemmas that will be used later. Some of the results that we are going to present are already known for the case $m = 0$ and $l = 0, 1$. However, even in these cases, the proofs that we give here are somewhat different from the ones found in the references that we shall give.

The following two lemmas are generalizations of what is sometimes known as «Meyer's lemma». For $m = 0$, Y. Meyer obtained these results in [M1]; see also [MM].

Lemma 2.1. *Let $T \in \mathrm{WBP}(m)$, and assume that its kernel satisfies*

$$|K(x,y)| \leqslant C|x-y|^{-n-m}$$

for (x,y) in Ω. If $f \in \mathfrak{D}(\mathbb{R}^n \times \mathbb{R}^n)$ and $D_1^\beta D_2^\gamma f(x,x) = 0$ for all $|\beta| + |\gamma| \leqslant m$, then

$$\langle K, f \rangle = \iint_{x \neq y} K(x,y) f(x,y)\, dx\, dy,$$

where the integral on the right is absolutely convergent.

PROOF. Let $\varphi \in \mathfrak{D}(\mathbb{R}^n)$ be such that $\operatorname{supp} \varphi \subseteq B_2(0)$ and $\varphi = 1$ on $B_1(0)$. Then, for $\epsilon > 0$ we may write

$$\langle K, f \rangle = \left\langle K, \left[1 - \varphi\left(\frac{x-y}{\epsilon}\right)\right] f \right\rangle + \left\langle K, \varphi\left(\frac{x-y}{\epsilon}\right) f \right\rangle. \tag{2.2}$$

Since K is locally integrable on Ω, the first term on the right hand side of (2.2) is

$$\iint K(x,y)\left(1 - \varphi\left(\frac{x-y}{\epsilon}\right)\right) f(x,y)\, dx\, dy,$$

and, by the Lebesgue dominated convergence theorem, tends to

$$\iint K(x,y)f(x,y)\,dx\,dy$$

as $\epsilon \to 0$. Hence, we must show that the second term tends to zero.

Choose a function $\psi \in \mathfrak{D}(\mathbb{R}^n)$ with $\operatorname{supp}\psi \subseteq B_2(0)$ and so that

$$\sum_{k\in\mathbb{Z}^n} \psi(x-k) = 1.$$

Then, for $1 \geqslant \epsilon > 0$

$$f(x,y)\varphi\left(\frac{x-y}{\epsilon}\right) = f(x,y)\varphi\left(\frac{x-y}{\epsilon}\right)\sum_{k\in\mathbb{Z}^n}\psi\left(\frac{x}{\epsilon}-k\right).$$

Let $r>0$ be such that $|x|<r$ for $(x,y)\in\operatorname{supp} f$. Then

$$f(x,y)\varphi\left(\frac{x-y}{\epsilon}\right) = f(x,y)\varphi\left(\frac{x-y}{\epsilon}\right)\sum_{|k|<R}\psi\left(\frac{x}{\epsilon}-k\right),$$

where

$$R = 2 + \frac{r}{\epsilon}.$$

If we let

$$F_{k,\epsilon}(x,y) = \epsilon^{-1-m} f(\epsilon x + k, \epsilon y + k)\varphi(x-y)\psi\left(x + \frac{k}{\epsilon} - k\right),$$

then the family $\{F_{k,\epsilon}: k\in\mathbb{Z}^n \text{ and } 1\geqslant\epsilon>0\}$ is one of the sets described in remark (*a*) at the end of Section 1, and since $T\in \mathrm{WBP}\,(m)$, there exists a constant C such that

$$|\langle K, F_{k,\epsilon}^{z,t}\rangle| \leqslant Ct^{n-m}$$

for all $k\in\mathbb{Z}^n$, $1\geqslant\epsilon>0$, $z\in\mathbb{R}^n$ and $t>0$. We have used the hypothesis $D_1^\beta D_2^\gamma f(x,x)=0$ to obtain the appropriate cancellation of the powers of ϵ. In particular,

$$|\langle K, F_{k,\epsilon}^{k,\epsilon}\rangle| \leqslant C\epsilon^{n-m},$$

and, hence,

$$\begin{aligned}\left|\left\langle K, f(x,y)\varphi\left(\frac{x-y}{\epsilon}\right)\right\rangle\right| &\leqslant \sum_{|k|<R}\left|\left\langle K, f(x,y)\varphi\left(\frac{x-y}{\epsilon}\right)\psi\left(\frac{x}{\epsilon}-k\right)\right\rangle\right| \\ &\leqslant \epsilon^{1+m}\sum_{|k|<R}|\langle K, F_{k,\epsilon}^{k,\epsilon}\rangle| \\ &\leqslant C\epsilon^{1+m}R^n\epsilon^{n-m} \to 0 \quad \text{as} \quad \epsilon\to 0. \quad \square\end{aligned}$$

The next result gives a crucial L^∞-estimate, showing that T behaves well on smooth bump functions.

Lemma 2.3. *Let $T \in \mathrm{CZO}(m, \epsilon) \cap \mathrm{WBP}(m)$ and assume that $T_\beta 1 = 0$ for all $|\beta| \leqslant m$. Then T maps $\mathfrak{D}(\mathbb{R}^n)$ into L^∞. Moreover, there exists a constant C, depending only on T, such that for any $w \in \mathbb{R}^n$, $t > 0$ and $\varphi \in \mathfrak{D}$ with* supp $\varphi \subseteq B_t(w)$ *we have*

$$\|T\varphi\|_\infty \leqslant C \sum_{|\gamma| \leq 1+m} t^{|\gamma|-m} \|D^\gamma \varphi\|_\infty . \tag{2.4}$$

PROOF. We will proceed by induction on m. Assume first $m = 0$. Let $\varphi \in \mathfrak{D}$, supp $\varphi \subseteq B_t(w)$.

(*a*) For $x \notin B_{2t}(w)$,

$$T\varphi(x) = \int_{B_t(w)} K(x,y)\varphi(y)\,dy,$$

and, hence,

$$|T\varphi(x)| \leqslant \int_{B_t(w)} C|x-y|^{-n}\|\varphi\|_\infty\,dy \leqslant C\|\varphi\|_\infty . \tag{2.5}$$

(*b*) Let $\chi \in \mathfrak{D}$, $\chi(x) = 1$ on $B_4(0)$ and $\chi(x) = 0$ for $x \notin B_6(0)$, and write

$$\varphi(y) = [\varphi(y) - \varphi(x)]\chi^{w,t}(y) + \varphi(x)\chi^{w,t}(y) = f_1(x,y) + f_2(x,y).$$

Then, for $\psi \in \mathfrak{D}$ satisfying supp $\psi \subseteq B_{3t}(w)$, we have

$$\begin{aligned} |\langle T\varphi, \psi\rangle| &= |\langle K, \psi \otimes \varphi\rangle| \\ &\leqslant |\langle K, \psi(x) f_1(x,y)\rangle| + |\langle K, \psi(x) f_2(x,y)\rangle| \\ &= I + II. \end{aligned}$$

By Lemma 2.1,

$$\begin{aligned} I &= \left| \iint K(x,y)\psi(x) f_1(x,y)\,dx\,dy \right| \\ &\leqslant \int_{B_{3t}(w)} \int_{B_{6t}(w)} C|x-y|^{-n} \sum_{|\gamma|=1} \|D^\gamma \varphi\|_\infty |x-y|\,dy\,|\psi(x)|\,dx \\ &\leqslant Ct \sum_{|\gamma|=1} \|D^\gamma\varphi\|_\infty \|\psi\|_1 . \end{aligned} \tag{2.6}$$

In order to estimate II, it is sufficient to estimate

$$|\langle K, \psi \otimes \chi^{w,t}\rangle| = |\langle T\chi^{w,t}, \psi\rangle|.$$

Assume, first, that $\int \psi(x)\,dx = 0$. Then, since $T(1) = 0$, we have

$$0 = \langle T1, \psi \rangle = \langle T\chi^{w,t}, \psi \rangle + \langle T^*\psi, (1 - \chi^{w,t}) \rangle$$

and, hence

$$\begin{aligned}
(2.7)\quad |\langle T\chi^{w,t}, \psi \rangle| &= |\langle T^*\psi, (1 - \chi^{w,t}) \rangle| \\
&= \left| \iint K(x,y)(1 - \chi^{w,t})(y)\psi(x)\,dx\,dy \right| \\
&\leqslant \int_{B_{3t}(w)} \left| (1 - \chi^{w,t})(y) \int_{^cB_{6t}(w)} [K(x,y) - K(w,y)]\psi(x)\,dx \right| dy \\
&\leqslant C \int_{B_{3t}(w)} \int_{^cB_{6t}(w)} t^{\epsilon} |w - y|^{-n-\epsilon} |\psi(x)|\,dy\,dx \\
&\leqslant C t^{\epsilon} \|\psi\|_1 \int_{^cB_{6t}(0)} |y|^{-n-\epsilon}\,dy \\
&\leqslant C \|\psi\|_1 .
\end{aligned}$$

Now consider a general $\psi \in \mathfrak{D}$, $\operatorname{supp}\psi \subseteq B_{3t}(w)$ and $\int \psi(x)\,dx = b$. Fix $\eta \in \mathfrak{D}$, $\operatorname{supp}\eta \subseteq B_1(0)$, $\eta \geqslant 0$ and $\int \eta(x)\,dx = 1$. Then $\psi - bt^{-n}\eta^{w,t}$ has support in $B_{3t}(w)$ and has integral zero. Hence, using (2.7) and that $T \in \mathrm{WBP}\,(0)$

$$\begin{aligned}
(2.8)\qquad |\langle T\chi^{w,t}, \psi \rangle| &\leqslant |\langle T\chi^{w,t}, \psi - bt^{-n}\eta^{w,t} \rangle| + |b| t^{-n} |\langle T\chi^{w,t}, \eta^{w,t} \rangle| \\
&\leqslant C \|\psi - bt^{-n}\eta^{w,t}\|_1 + \|\psi\|_1 t^{-n} |\langle T\chi^{w,t}, \eta^{w,t} \rangle| \\
&\leqslant C \|\psi\|_1 + C \|\psi\|_1 t^{-n} t^n \\
&\leqslant C \|\psi\|_1 .
\end{aligned}$$

From (2.6) and (2.8) we see that

$$\begin{aligned}
(2.9)\qquad |\langle T\varphi, \psi \rangle| &\leqslant Ct \sum_{|\gamma| = 1} \|D^{\gamma}\varphi\|_{\infty} \|\psi\|_1 + C \|\varphi\|_{\infty} \|\psi\|_1 \\
&\leqslant C \sum_{|\gamma| \leq 1} t^{|\gamma|} \|D^{\gamma}\varphi\|_{\infty} \|\psi\|_1 .
\end{aligned}$$

Thus, (2.5) and (2.9) show that $T\varphi$ extends as an element of $(L^1(\mathbb{R}^n))^* = L^{\infty}$ with a norm bounded by the right hand side of (2.4). This ends the proof for $m = 0$.

Assume that the result is true whenever $T \in \mathrm{CZO}\,(m-1, \epsilon) \cap \mathrm{WBP}\,(m-1)$ and $T_{\beta}1 = 0$ for all $|\beta| \leqslant m - 1$, and suppose $T \in \mathrm{CZO}\,(m, \epsilon) \cap \mathrm{WBP}\,(m)$ and $T_{\beta}1 = 0$ for all $|\beta| \leqslant m$. Fix $\varphi \in \mathfrak{D}$, satisfying $\operatorname{supp}\varphi \subseteq B_t(w)$.

$(a)'$ For $x \notin B_{2t}(w)$

$$(2.10)\qquad |T\varphi(x)| \leqslant \int_{B_t(w)} C|x - y|^{-n-m} \|\varphi\|_{\infty}\,dx \leqslant Ct^{-m} \|\varphi\|_{\infty},$$

which is an estimate of the right order.

$(b)'$ Let χ be as in (b) and write

$$\varphi(y) = \left[\varphi(y) - \sum_{|\gamma| \le m} \frac{D^\gamma\varphi(x)}{\gamma!}(y-x)^\gamma\right]\chi^{w,t}(y)$$
$$+ \left[\varphi(x)\chi^{w,t}(y) + \sum_{1 \le |\gamma| \le m} \frac{D^\gamma\varphi(x)}{\gamma!}(y-x)^\gamma\right]\chi^{w,t}(y)$$
$$= f_1(x,y) + f_2(x,y) + f_3(x,y).$$

Then for $\psi \in \mathfrak{D}$, $\operatorname{supp}\psi \subseteq B_{3t}(w)$,

$$|\langle T\varphi, \psi\rangle| \leqslant \sum_{i=1}^{3} |\langle K, \psi(x) f_i(x,y)\rangle| = I + II + III.$$

As in (b), by Lemma 2.1

$$(2.11)\quad I = \left|\iint K(x,y)\psi(x) f_1(x,y)\,dx\,dy\right|$$
$$\leqslant \int_{B_{3t}(w)}\int_{B_{6t}(w)} C|x-y|^{-n-m} \sum_{|\gamma| = m+1} \|D^\gamma\varphi\|_\infty |x-y|^{m+1}\,dy\,|\psi(x)|\,dx$$
$$\leqslant Ct \sum_{|\gamma| = m+1} \|D^\gamma\varphi\|_\infty \|\psi\|_1.$$

The same argument, this time using that $T \in \mathrm{WBP}\,(m)$, yields

$$(2.12)\qquad II \leqslant Ct^{-m}\|\varphi\|_\infty\|\psi\|_1.$$

Finally, to estimate III we can apply the induction hypothesis to the operators T_γ for $1 \leqslant |\gamma| \leqslant m$ to obtain

$$(2.13)\quad |\langle K, \psi(x) f_3(x,y)\rangle| \leqslant \sum_{1 \le |\gamma| \le m} |\langle D^\gamma\varphi T_\gamma \chi^{w,t}, \psi\rangle| \frac{1}{\gamma!}$$
$$\leqslant C \sum_{1 \le |\gamma| \le m} \|D^\gamma\varphi\|_\infty \sum_{0 \le |\beta| \le m - |\gamma| + 1} t^{-m+|\gamma|+|\beta|}\|D^\beta\chi^{w,t}\|_\infty\|\psi\|_1$$
$$\leqslant C \sum_{1 \le |\gamma| \le m} \|D^\gamma\varphi\|_\infty \sum_{0 \le |\beta| \le m - |\gamma| + 1} t^{-m+|\gamma|}\|D^\beta\chi\|_\infty\|\psi\|_1$$
$$\leqslant C \sum_{1 \le |\gamma| \le m} t^{-m+|\gamma|}\|D^\gamma\varphi\|_\infty\|\psi\|_1.$$

Hence, (2.10), (2.11), (2.12) and (2.13) give (2.4). □

Corollary 2.14. *Let $T \in \mathrm{CZO}\,(l+\epsilon) \cap \mathrm{WBP}$ and suppose that $T(y^\gamma) = 0$ for all $|\gamma| \leqslant l$. Then T maps $\mathfrak{D}$ into L^∞. Moreover, there exists a constant C*

depending only on T such that, if $\varphi \in \mathfrak{D}$ *and* $\operatorname{supp} \varphi \subseteq B_t(w)$, *then*

$$\|D^\gamma T\varphi\|_\infty \leqslant C \sum_{|\nu| \le |\gamma|+1} t^{|\nu|-|\gamma|} \|D^\nu \varphi\|_\infty \quad \text{for every} \quad |\gamma| \leqslant l. \tag{2.15}$$

PROOF. Since $T \in \mathrm{CZO}\,(l+\epsilon) \cap \mathrm{WBP}$ we have

$$D^\gamma T \in \mathrm{CZO}\,(|\gamma|, l-|\gamma|+\epsilon) \cap \mathrm{WBP}\,(|\gamma|) \subseteq \mathrm{CZO}\,(|\gamma|, \epsilon) \cap \mathrm{WBP}\,(|\gamma|),$$

since $|\gamma| \leqslant l$. Hence in order to apply Lemma 2.3 we only need to check that $T_{\beta,\gamma}1 = 0$ for every $|\beta| \leqslant |\gamma|$.

Choose $\psi \in \mathfrak{D}_0$ and let $\{\varphi_j\}_{j=1}^\infty$ be a sequence as in Lemma 1.16. Then,

$$\begin{aligned} \langle T_{\beta,\gamma}1, \psi\rangle &= \lim_{j\to\infty} \langle T_{\beta,\gamma}\varphi_j, \psi\rangle \\ &= \lim_{j\to\infty} \langle (y-x)^\beta D_1^\gamma K, \psi \otimes \varphi_j\rangle \\ &= (-1)^{|\gamma|} \lim_{j\to\infty} \langle K, D_1^\gamma[(y-x)^\beta \psi(x)]\varphi_j(y)\rangle \\ &= (-1)^{|\gamma|} \lim_{j\to\infty} \sum_{\substack{0 \le \nu \le \gamma \\ 0 \le \beta+\nu-\gamma}} C_\alpha \langle K, (y-x)^{\beta+\nu-\gamma} D^\nu \psi(x)\varphi_j(y)\rangle \end{aligned} \tag{2.16}$$

where the inequalities under the above sum refer to each component of the multi-indexes. We claim that each term in the sum is zero. In fact, $\langle K, (y-x)^{\beta+\nu-\gamma} D^\nu\psi(x)\varphi_j(y)\rangle$ can be written as a sum of terms of the form $C_\mu \langle K, y^\mu x^{\beta+\nu-\gamma-\mu} D^\nu \psi(x)\varphi_j(y)\rangle$ with $\beta+\nu-\gamma-\mu \geqslant 0$. Since $\psi \in \mathfrak{D}_0$, it follows that $D^\nu\psi \in \mathfrak{D}_{|\nu|}$ and, hence, the function $x^{\beta+\nu-\gamma-\mu} D^\nu \psi$ has at least vanishing moments of all orders not exceeding $|\mu|$. Then by Lemma 1.19, we have

$$\begin{aligned} \lim_{j\to\infty} \langle K, y^\mu x^{\beta+\nu-\gamma-\mu} D_1^\nu \psi(x)\varphi_j(y)\rangle &= \lim_{j\to\infty} \langle T\varphi_j y^\mu, x^{\beta+\nu-\gamma-\mu} D^\nu\psi\rangle \\ &= \langle Ty^\mu, x^{\beta+\nu-\gamma-\mu} D^\nu \psi\rangle = 0, \end{aligned}$$

since $|\mu| \leqslant |\beta| + |\nu| - |\gamma| \leqslant |\gamma|$. □

Let $T \in \mathrm{CZO}\,(m, \epsilon) \cap \mathrm{WBP}\,(m)$ and assume $T_\beta 1 = 0$ for $|\beta| \leqslant m$. We have proved that for every $\varphi \in \mathfrak{D}$, $T\varphi$ is an L^∞-function. We will now show, that $T\varphi(x)$ can be well defined at every point. In fact, we will show that $T\varphi$ agrees almost everywhere with a continuous function.

Let $t > 0$, $w \in \mathbb{R}^n$ and suppose $\xi \in \mathfrak{D}$ satisfies $\xi = 1$ on $B_{2t}(w)$ and $\xi(x) = 0$ for $x \notin B_{4t}(w)$. Let $\eta = 1 - \xi$. As in the proof of Lemma 2.3, we have that for all $\psi \in \mathfrak{D}_0$ with $\operatorname{supp} \psi \subseteq B_t(w)$,

$$0 = \langle T1, \psi\rangle = \langle T\xi, \psi\rangle + \langle T^*\psi, \eta\rangle;$$

thus,

$$\begin{aligned}\langle T\xi, \psi\rangle &= -\iint [K(x,y) - K(w,y)]\eta(y)\psi(x)\,dy\,dx\\ &= \iint [K(w,y) - K(x,y)]\eta(y)\,dy\,\psi(x)\,dx.\end{aligned}$$

Since both $T\xi$ and

$$\int [K(w,y) - K(x,y)]\eta(y)\,dy$$

are bounded functions on $B_t(w)$ and they agree, as distributions, on $\mathfrak{D}_0(B_t(w))$, we must have that for a.e. x in $B_t(w)$,

$$T\xi(x) = C_{w,t} + \int [K(w,y) - K(x,y)]\eta(y)\,dy, \tag{2.17}$$

where $C_{w,t}$ is some constant which may depend on w and t but not on $x \in B_t(w)$.

Since the right hand side of (2.17) is a continuous function, we can take it as the pointwise definition of $T\xi$ on $B_t(w)$. Note also that if $m \geqslant 1$ then the «faster decay of $K(x,y)$ at infinity» allows us to compute $T\xi(x)$ as

$$T\xi(x) = C'_{w,t} - \int K(x,y)\eta(y)\,dy. \tag{2.18}$$

In any case, if $x, x' \in B_t(w)$, then

$$T\xi(x) - T\xi(x') = \int [K(x',y) - K(x,y)]\eta(y)\,dy. \tag{2.19}$$

We now want to define $T\varphi(x)$ for any $\varphi \in \mathfrak{D}$. Assume again $T \in \mathrm{CZO}\,(m, \epsilon) \cap \mathrm{WBP}\,(m)$ and $T_\beta 1 = 0$ for $|\beta| \leqslant m$. Let $\varphi \in \mathfrak{D}$ and let $\psi \in \mathfrak{D}(B_t(w))$. For ξ and η as before,

$$\begin{aligned}\langle T\varphi, \psi\rangle &= \langle K, \psi \otimes \varphi\rangle\\ &= \left\langle K, \psi(x)\left[\varphi(y) - \sum_{|\gamma| \le m} D^\gamma\varphi(x)\frac{(y-x)^\gamma}{\gamma!}\right]\xi(y)\right\rangle\\ &\quad + \left\langle K, \psi(x)\sum_{|\gamma| \le m} D^\gamma\varphi(x)\frac{(y-x)^\gamma}{\gamma!}\,\xi(y)\right\rangle + \langle K, \psi(x)\varphi(y)\eta(y)\rangle\\ &= \iint K(x,y)\left[\varphi(y) - \sum_{|\gamma| \le m} D^\gamma\varphi(x)\frac{(y-x)^\gamma}{\gamma!}\right]\xi(y)\psi(x)\,dy\,dx\\ &\quad + \sum_{|\gamma| \le m}\left\langle \frac{D^\gamma\varphi}{\gamma!}T_\gamma\xi, \psi\right\rangle + \iint K(x,y)\varphi(y)\eta(y)\psi(x)\,dy\,dx\end{aligned}$$

(the first integral representation follows from Lemma 2.1 and the second one is true because $\operatorname{supp}\eta\varphi \cap \operatorname{supp}\psi = \varnothing$). Hence, for a.e. $x \in B_t(w)$,

$$(2.20)\qquad T\varphi(x) = \int K(x,y)\left[\varphi(y) - \sum_{|\gamma|\le m} D^\gamma\varphi(x)\frac{(y-x)^\gamma}{\gamma!}\right]\xi(y)\,dy + \sum_{|\gamma|\le m}\left\{\frac{D^\gamma\varphi}{\gamma!}T_\gamma\xi\right\}(x) + \int K(x,y)\varphi(y)\eta(y)\,dy.$$

Again, since the right hand side of (2.20) is continuous we can take it as the pointwise definition of $T\varphi$ on $B_t(w)$. The continuity also guarantees that this definition is independent of $B_t(w)$ and ξ.

It will be convenient for later applications to have a formula for the difference between the values of $T\varphi$ (as well as $D^\gamma T\varphi$) at two different points. This is obtained in the next lemma. Once again, for $l = 0$ the result is due to Y. Meyer and for $l = 1$ to M. Meyer. The proof follows from a careful use of the previous pointwise definitions and it is left to the reader. The details can be found in [T]. See also [FW] for an alternative formula.

Lemma 2.21. *Assume $T \in \mathrm{CZO}(l+\epsilon) \cap \mathrm{WBP}$ and $T(x^\gamma) = 0$ for all $|\gamma| \leqslant l$. Let $x \neq x'$ be two points of $\mathbb{R}^n$, $t = |x - x'|$, $\xi \in \mathfrak{D}$, $\xi = 1$ on $B_{2t}(x')$, $\xi(x) = 0$ for $x \notin B_{4t}(x')$ and let $\eta = 1 - \xi$. Then for all $\varphi \in \mathfrak{D}$ and for all $|\gamma| \leqslant l$*

$$D^\gamma T\varphi(x) - D^\gamma T\varphi(x')$$

$$= \int D_1^\gamma K(x,y)\left[\varphi(y) - \sum_{|\beta|\le|\gamma|}\frac{D^\beta\varphi(x)}{\beta!}(y-x)^\beta\right]\xi(y)\,dy$$

$$- \int D_1^\gamma K(x',y)\left[\varphi(y) - \sum_{|\beta|\le|\gamma|}\frac{D^\beta\varphi(x')}{\beta!}(y-x')^\beta\right]\xi(y)\,dy$$

$$+ \int [D_1^\gamma K(x,y) - D_1^\gamma K(x',y)]\left[\varphi(y) - \sum_{|\beta|\le|\gamma|}\frac{D^\beta\varphi(x')}{\beta!}(y-x')^\beta\right]\eta(y)\,dy$$

$$+ \sum_{|\beta|\le|\gamma|}\frac{1}{\beta!}\left[D^\beta\varphi(x) - \sum_{|\nu|\le|\gamma|-|\beta|}\frac{D^{\beta+\nu}\varphi(x')}{\nu!}(x-x')^\nu\right]T_{\beta,\gamma}\xi(x),$$

where all integrals are absolutely convergent.

3. Boundedness criteria

The following three theorems extend the results of [FHJW].

Theorem 3.1. *Let $\alpha < 0$ and $0 < p, q \leqslant \infty$. Assume that $T: \mathfrak{D} \to \mathfrak{D}'$ satisfies*

(3.2) $T \in \mathrm{WBP}$,

(3.3) $T \in \mathrm{CZO}(\delta)$ *where* $1 > \delta > 0$,

(3.4) $T^* \in \mathrm{CZO}([J - n - \alpha] + \rho)$ *where* $1 > \rho > (J - \alpha)^*$,

(3.5) $T1 = 0$, *and*

(3.6) $T^*(x^\gamma) = 0$ *for* $|\gamma| \leqslant [J - n - \alpha]$,

then T extends to a bounded operator on $\dot{F}_p^{\alpha,q}$.

Theorem 3.7. *Let $\alpha \geqslant 0$ and $\min\{p, q\} \geqslant 1$. Assume that $T: \mathfrak{D} \to \mathfrak{D}'$ satisfies*

(3.8) $T \in \mathrm{WBP}$,

(3.9) $T \in \mathrm{CZO}([\alpha] + \delta)$ *where* $1 > \delta > \alpha^*$,

(3.10) $T^* \in \mathrm{CZO}(\rho)$ *where* $1 > \rho > 0$,

(3.11) $T(y^\gamma) = 0$ *for all* $|\gamma| \leqslant [\alpha]$, *and*

(3.12) $T^*1 = 0$ *if* $\alpha = 0$,

then T extends to a bounded operator on $\dot{F}_p^{\alpha,q}$.

Theorem 3.13. *Let $\alpha \geqslant 0$ and $\min\{p, q\} < 1$. Assume that $T: \mathfrak{D} \to \mathfrak{D}'$ satisfies*

(3.14) $T \in \mathrm{WBP}$,

(3.15) $T \in \mathrm{CZO}([\alpha] + \delta)$ *where* $1 > \delta > \alpha^*$ *if* $J - n - \alpha < 0$ *and* $1 > \delta > \max\{\alpha^*, J^*\}$ *if* $J - n - \alpha \geqslant 0$,

(3.16) $T^* \in \mathrm{CZO}([J - n] + \rho)$ *where* $1 > \rho > J^*$,

(3.17) $T(y^\gamma) = 0$ *for all* $|\gamma| \leqslant [\alpha]$,

(3.18) $T^*(x^\gamma) = 0$ *for all* $|\gamma| \leqslant [J - n - \alpha]$ *if* $J - n - \alpha \geqslant 0$,

(3.19) $|D_2^\beta D_1^\gamma K(x, y) - D_2^\beta D_1^\gamma K(x, z)| \leqslant C|y - z|^\rho |x - y|^{-(n + |\beta| + |\gamma| + \rho)}$ *for* $2|y - z| \leqslant |x - y|$ *if* $|\gamma| \leqslant \min\{[J] - n, [\alpha]\}$ *and* $|\beta + \gamma| = [J] - n$, *and*

(3.20) $|D_2^\beta D_1^\gamma K(x, y) - D_2^\beta D_1^\gamma K(z, y)| \leqslant C|x - z|^\rho |x - y|^{-(n + |\beta| + |\gamma| + \delta)}$ *for* $2|x - z| \leqslant |x - y|$ *if* $|\gamma| = [\alpha]$ *and* $|\beta| = ([J] - n - [\alpha])_+$,

then T extends to a bounded operator on $\dot{F}_p^{\alpha,q}$.

As we mentioned in Section 1 we will prove these theorems by showing that smooth atoms are mapped by T into smooth molecules. Actually, we will show that if $a \in \mathfrak{D}$ is a smooth atom associated with Q_0, the unit cube with lower left corner at the origin, then Ta is a smooth molecule associated with Q_0. Moreover, we will see that Ta satisfies conditions (1.4) to (1.7) with a constant C depending only on the dimension n, the constants appearing in Lemma 2.3 and Corollary 2.14, and the constants associated to T and T^* by the hypotheses of the theorems and the definitions of generalized Calderón-Zygmund operators and WBP. It is not hard to check that all these constants, as well as the hypotheses of the theorems, remain unchanged if we replace T by any of the operators $T^{z,t}$. Now using a simple translation and dilation argument, together with the particular form of conditions (1.4) to (1.7), one can easily conclude that T maps a smooth atom associated with an arbitrary cube Q to a smooth molecule associated with Q and with the same constant C obtained for Q_0.

PROOF OF THEOREM 3.1. Let $\epsilon = \rho - (J - \alpha)^*$ and $M = J + \epsilon$. Let $a \in \mathfrak{D}$ be a smooth atom for $\dot{F}_p^{\alpha,q}$ associated with Q_0. We will show that Ta is a smooth M-molecule. Observe that since $\alpha < 0$ we only need to check that conditions (1.4) and (1.5) hold (this is also the reason why we use the term «smooth M-molecule» and not «smooth (δ, M)-molecule»).

Assume first $|x| > 4\sqrt{n}$. Then, since $a(x)$ has L-vanishing moments (where $L = [J - n - \alpha]$) and $\|a\|_\infty \leqslant 1$, we have

$$\begin{aligned}|Ta(x)| &= \left| \int_{3Q_0} K(x,y)a(y)\,dy \right| \\ &= \left| \int_{3Q_0} \left[K(x,y) - \sum_{|\beta| \leq L} D_2^\beta K(x,0) \frac{y^\beta}{\beta!} \right] a(y)\,dy \right| \\ &\leqslant \int_{3Q_0} \left| K(x,y) - \sum_{|\beta| \leq L} D_2^\beta K(x,0) \frac{y^\beta}{\beta!} \right| dy \\ &\leqslant \int_{3Q_0} \left| \sum_{|\beta| = L} [D_2^\beta K(x, z(y)) - D_2^\beta K(x,0)] \frac{y^\beta}{\beta!} \right| dy\end{aligned}$$

where the point $z(y) \in Q_0$. But, then, $2|z(y)| \leqslant 4\sqrt{n} < |x|$ and since $T^* \in \mathrm{CZO}(L + \rho)$, we obtain from the last inequality that

$$|Ta(x)| \leqslant \int_{3Q_0} C \frac{|z(y)|^\rho}{|x|^{n+L+\rho}} |y|^L\,dy \leqslant C|x|^{-n-L-\rho} = C|x|^{-M+\alpha}, \tag{3.21}$$

because $n + L + \rho = n + [J - n - \alpha] + (J - \alpha)^* + \epsilon = J - \alpha + \epsilon = M - \alpha$, where the constant C depends only on T and the dimension n. Clearly (3.21) gives (1.4) for Q_0 and $|x| \geqslant 4\sqrt{n}$.

Now let $|x| \leqslant 4\sqrt{n}$. We can use (3.2), (3.3) and (3.5) to apply Lemma 2.3 and obtain

$$|Ta(x)| \leqslant C(\|a\|_\infty + 2\sqrt{n}\,\|\nabla a\|_\infty) \leqslant C, \tag{3.22}$$

where the constant C, by (1.3), is independent of $a(x)$. Hence, (3.22) gives (1.4) for $|x| \leqslant 4\sqrt{n}$.

Since (1.4) holds, $x^\gamma Ta$ is integrable for $|\gamma| \leqslant L$; hence, by (3.4) and (3.6),

$$0 = \langle T^*x^\gamma, a\rangle = \langle Ta, x^\gamma\rangle = \int Ta(x)x^\gamma\, dx,$$

which is condition (1.5). □

PROOF OF THEOREM 3.7. Let $\epsilon = \min\{\delta - \alpha^*, \rho\}$ and let $M = J + \epsilon = n + \epsilon$. We will show that if a is a smooth atom associated with Q_0, then Ta is smooth (δ, M)-molecule also associated with Q_0.

We use the same arguments of the previous proof, and, when $|x| > 4\sqrt{n}$ we use (3.10), in order to obtain

$$\begin{aligned} |Ta(x)| &= \left|\int_{3Q_0} [K(x,y) - K(x,0)]a(y)\,dy\right| \\ &\leqslant C\int_{3Q_0} |y|^\rho |x|^{-n-\rho}\,dy \\ &\leqslant C|x|^{-n-\rho} \\ &\leqslant C|x|^{-M} \\ &\leqslant C(1+|x|)^{-M}, \end{aligned}$$

which is (1.4).

As in the previous proof, if $|x| \leqslant 4\sqrt{n}$ (3.8), (3.9), (3.11) and Lemma 2.3 yield

$$|Ta(x)| \leqslant C \leqslant C(1+|x|)^{-M}.$$

Condition (1.5) needs to be checked only if $\alpha = 0$. But in that case $T^*1 = 0$ and so

$$0 = \langle T^*1, a\rangle = \langle Ta, 1\rangle = \int Ta(x)\,dx.$$

We turn now to condition (1.6). If $|\gamma| = 0$, (1.6) is just (1.4). Thus, we may assume $\alpha \geqslant |\gamma| > 0$. Again there are two different cases. If $|x| > 4\sqrt{n}$ we have the immediate estimate

$$\begin{aligned} |D^\gamma Ta(x)| &\leqslant \int_{3Q_0} |D_1^\gamma K(x,y)|\,\|a\|_\infty\,dy \\ &\leqslant C|x|^{-n-|\gamma|} \\ &\leqslant C|x|^{-n-\delta} \\ &\leqslant C(1+|x|)^{-M}, \end{aligned}$$

while if $|x| \leqslant 4\sqrt{n}$, we may use (3.8), (3.9) and (3.11) to apply, this time, Corollary 2.14 and obtain

$$|D^\gamma Ta(x)| \leqslant C \sum_{|\beta| \leq |\gamma|+1} (2\sqrt{n})^{|\beta|-|\gamma|} \|D^\beta a\|_\infty \leqslant C \leqslant C(1+|x|)^{-M},$$

where the constant C is, again by (1.3), independent of $a(x)$.

We now show that Ta satisfies condition (1.7). Let $|\gamma| = [\alpha]$ and let x and x' be two different points of $\mathbb{R}^n$. If $|x - x'| \geqslant 1$ then, by (1.6), for $|\gamma| = [\alpha]$,

$$\begin{aligned} |D^\gamma Ta(x) - D^\gamma Ta(x')| &\leqslant |D^\gamma Ta(x)| + |D^\gamma Ta(x')| \\ &\leqslant C\{(1+|x|)^{-M} + (1+|x'|)^{-M}\} \\ &\leqslant C|x-x'|^\delta\{(1+|x|)^{-M} + (1+|x'|)^{-M}\}, \end{aligned}$$

which certainly gives (1.7) for this case.

Assume now $|x - x'| < 1$. We consider the following different positibilies:

(*a*) If $|x|$, $|x'| > 6\sqrt{n}$, then for $y \in 3Q_0$, $|x-y| \geqslant 4\sqrt{n} \geqslant 2|x-x'|$, and, also, $|x-y| \geqslant |x| - |y| \geqslant |x|/2$; thus,

$$\begin{aligned} |D^\gamma Ta(x) - D^\gamma Ta(x')| &= \left| \int_{3Q_0} D^\gamma K(x,y)a(y)\,dy - \int_{3Q_0} D^\gamma K(x',y)a(y)\,dy \right| \\ &\leqslant \int_{3Q_0} |D^\gamma K(x,y) - D^\gamma K(x',y)|\,\|a\|_\infty\,dy \\ &\leqslant \int_{3Q_0} C|x-x'|^\delta |x-y|^{-n-[\alpha]-\delta}\,dy \\ &\leqslant C|x-x'|^\delta |x|^{-n-[\alpha]-\delta} \\ &\leqslant C|x-x'|^\delta (1+|x|)^{-M}. \end{aligned}$$

(*b*) If $|x| \geqslant 6\sqrt{n}$ and $|x'| < 6\sqrt{n}$ we still have that x' is «far» from Q_0, that is $|x'| \geqslant |x| - |x-x'| \geqslant 6\sqrt{n} - 1 \geqslant 5\sqrt{n}$ and the situation can be handled as in (*a*). The same is true, of course, if $|x'| \geqslant 6\sqrt{n}$ and $|x| < 6\sqrt{n}$.

(*c*) The last case to be considered is when $|x|$, $|x'| < 6\sqrt{n}$. In this case we use the representation formula of Lemma 2.21. In fact, fix $\xi \in \mathfrak{D}$ satisfying $\operatorname{supp} \xi \subseteq B_4(0)$ and $\xi \equiv 1$ on $B_2(0)$, and let

$$\xi^{x',t}(y) = \xi\left(\frac{y-x'}{t}\right)$$

where $t = |x - x'|$. Then since $T \in \mathrm{CZO}([\alpha]+\delta) \cap \mathrm{WBP}$ and $T^\gamma(x^\gamma) = 0$, for $|\gamma| \leqslant [\alpha]$, we have for $|\gamma| = [\alpha]$

$$|D^\gamma Ta(x) - D^\gamma Ta(x')|$$

$$\leqslant \left| \int D_1^\gamma K(x,y) \left[a(y) - \sum_{|\beta| \leq [\alpha]} \frac{D^\beta a(x)}{\beta!}(y-x)^\beta \right] \xi^{x',t}(y)\,dy \right|$$

$$+ \left| \int D_1^\gamma K(x',y)\left[a(y) - \sum_{|\beta| \le [\alpha]} \frac{D^\beta a(x')}{\beta!}(y-x')^\beta\right]\xi^{x',t}(y)\,dy\right|$$

$$+ \left| \int [D_1^\gamma K(x,y) - D_1^\gamma K(x',y)] \left[a(y) - \sum_{|\beta| \le [\alpha]} \frac{D^\beta a(x')}{\beta!}(y-x')^\beta\right][1-\xi^{x',t}(y)]\,dy\right|$$

$$+ \sum_{|\beta| \le [\alpha]} \frac{1}{\beta!}\left|D^\beta a(x) - \sum_{|\nu| \le [\alpha]-|\beta|} \frac{D^{\beta+\nu}a(x')}{\nu!}(x-x')^\nu\right| |T_{\beta,\gamma}\xi^{x',t}(x)|$$

$$= I + II + III + IV.$$

We have

$$I \leqslant C \int_{|x'-y|<4|x-x'|} |x-y|^{-n-[\alpha]}\left|a(y) - \sum_{|\beta| \le [\alpha]} \frac{D^\beta a(x)}{\beta!}(y-x)^\beta\right| \|\xi\|_\infty\,dy$$

$$\leqslant C \int_{|x'-y|<4|x-x'|} |x-y|^{-n-[\alpha]} \sum_{|\beta|=[\alpha]+1} \|D^\beta a\|_\infty |y-x|^{[\alpha]+1}\,dy$$

$$\leqslant C \int_{|x'-y|<4|x-x'|} |x-y|^{-n+1}\,dy$$

$$\leqslant C|x-x'|$$

$$\leqslant C|x-x'|^\delta,$$

since $|x-x'| < 1$, where the constant C depends on ξ but not on $a(x)$ since $\|D^\beta a\|_\infty \leqslant 1$ for all $|\beta| \leqslant [\alpha]+1$. The term II can be estimated in the same way.

By (3.9),

$$III \leqslant C \int_{|x'-y|>2|x-x'|} |x-x'|^\delta |x-y|^{-n-[\alpha]-\delta} \left|a(y) - \sum_{|\beta| \le [\alpha]} \frac{D^\beta a(x')}{\beta!}(y-x')^\beta\right| dy$$

$$\leqslant C|x-x'|^\delta \left\{ \int_{|x'-y|<2} |x'-y|^{-n-[\alpha]-\delta} \sum_{|\beta|=[\alpha]+1} \|D^\beta a\|_\infty |y-x'|^{[\alpha]+1}\,dy + \int_{|x'-y|\ge 2} 2|x'-y|^{-n-[\alpha]-\delta} \sum_{|\beta|=[\alpha]} \|D^\beta a\|_\infty |y-x'|^{[\alpha]}\,dy \right\}$$

$$\leqslant C|x-x'|^\delta \left\{ \int_{|x'-y|<2} |x'-y|^{-n+1-\delta}\,dy + \int_{|x'-y|\ge 2} |x'-y|^{-n-\delta}\,dy \right\}$$

$$\leqslant C|x-x'|^\delta.$$

Finally applying Lemma 2.3 to the operators $T_{\beta,\gamma}$ we obtain

$$IV \leqslant \sum_{|\beta| \le [\alpha]} \frac{1}{\beta!} \left\{ \left| D^{\beta}a(x) - \sum_{|\nu| \le [\alpha] - |\beta|} \frac{D^{\beta+\nu}a(x')}{\nu!}(x - x')^{\nu} \right| \right.$$
$$\left. C \sum_{|\mu| \le [\alpha] - |\beta| + 1} (2t)^{|\mu| - [\alpha] + [\beta]} \| D^{\mu}\xi^{x',t} \|_{\infty} \right\}$$
$$\leqslant C \sum_{|\beta| \le [\alpha]} \sum_{|\nu| = [\alpha] - [\beta] + 1} \|D^{\nu}a\|_{\infty} |x - x'|^{[\alpha] - [\beta] + 1}$$
$$\sum_{|\mu| \le [\alpha] - |\beta| + 1} t^{|\mu| - [\alpha] + [\beta]} t^{-|\mu|} \|D^{\mu}\xi\|_{\infty}$$
$$\leqslant C \sum_{|\beta| \le [\alpha]} |x - x'|^{[\alpha] - [\beta] + 1} t^{-[\alpha] + [\beta]}$$
$$\leqslant C|x - x'|$$
$$\leqslant C|x - x'|^{\delta}.$$

We have shown that $|D^{\gamma}Ta(x) - D^{\gamma}Ta(x')| \leqslant |x - x'|^{\delta}$, which implies (1.7) for $|x - x'| < 1$ and $|x|, |x'| < 6\sqrt{n}$. This concludes the proof of Theorem 3.7. □

PROOF OF THEOREM 3.13. Let $\epsilon = \min\{\delta - \alpha^*, \rho - J^*\}$ if $J - n - \alpha < 0$ and $\epsilon = \min\{\delta - \max\{\alpha^*, J^*\}, \rho - J^*\}$ if $J - n - \alpha \geqslant 0$, let $M = J + \epsilon$, and let $a(x)$ be a smooth atom for $\dot{F}_p^{\alpha,q}$ associated to Q_0. For these choices of M, the same arguments used in the proof of Theorem 3.1 imply that Ta satisfies conditions (1.4) and (1.5) for a smooth (δ, M)-molecule associated with Q_0, and so they will not be repeated here. In order to prove that Ta also satisfies conditions (1.6) and (1.7) we will use the mixed derivatives hypotheses (3.19) and (3.20).

Let $|x| > 4\sqrt{n}$. Assume, first, that $0 < |\gamma| \leqslant \min\{[J] - n, [\alpha]\}$. Then by (3.19)

$$(3.23)\quad |D^{\gamma}Ta(x)| \leqslant \int_{3Q_0} \left| D_1^{\gamma}K(x,y) - \sum_{|\beta| \le [J] - n - |\gamma|} D_2^{\beta}D_1^{\gamma}K(x,0)\frac{y^{\beta}}{\beta!} \right| \|a\|_{\infty}\, dy$$
$$\leqslant \int_{3Q_0} \sum_{|\beta| = [J] - n - |\gamma|} |D_2^{\beta}D_1^{\gamma}K(x,z(y)) - D_2^{\beta}D_1^{\gamma}K(x,0)| \left| \frac{y^{\beta}}{\beta!} \right| dy$$
$$\leqslant \int_{3Q_0} C|z(y)|^{\rho}|x|^{-n-([J]-n-|\gamma|)-|\gamma|-\rho}|y|^{[J]-n-|\gamma|}\, dy$$
$$\leqslant C|x|^{-[J]-\rho}$$
$$= C|x|^{-J-(\rho-J^*)}$$
$$\leqslant C|x|^{-J-\epsilon} = C|x|^{-M}.$$

While if $[J] - n + 1 \leqslant |\gamma| \leqslant [\alpha]$,

$$
\begin{aligned}
(3.24) \qquad |D^\gamma Ta(x)| &\leqslant \int_{3Q_0} |D_1^\gamma K(x,y)| \, \|a\|_\infty \, dy \\
&\leqslant \int_{3Q_0} C|x-y|^{-n-|\gamma|} \, dy \\
&\leqslant C|x|^{-n-|\gamma|} \\
&\leqslant C|x|^{-n-([J]-n+1)} \\
&\leqslant C|x|^{-M}.
\end{aligned}
$$

The inequalities (3.23) and (3.24) imply (1.6) for $|x| > 4\sqrt{n}$. On the other hand, if $|x| \leqslant 4\sqrt{n}$, we use (3.14), (3.15) and (3.17) in order to apply, as in the previous theorems, Corollary 2.14 to obtain

$$|D^\gamma Ta(x)| \leqslant C,$$

which again gives (1.6) for $|x| \leqslant 4\sqrt{n}$.

We now want to check condition (1.7) for $|\gamma| = [\alpha]$. Going back to the proof of Theorem 3.7, we see that the only case that needs a different argument is when $|x - x'| < 1$ and $|x|$, $|x'| > 6\sqrt{n}$. We consider three different possibilities:

(*a*) If $J - n - \alpha < 0$, then

$$
\begin{aligned}
|D^\gamma Ta(x) - D^\gamma Ta(x')| &\leqslant \int_{3Q_0} |D_1^\gamma K(x,y) - D_1^\gamma K(x',y)| \, \|a\|_\infty \, dy \\
&\leqslant C|x-x'|^\delta |x|^{-n-[\alpha]-\delta} \\
&\leqslant C|x-x'|^\delta |x|^{-M},
\end{aligned}
$$

since

$$n + [\alpha] + \delta = n + \alpha - \alpha^* + \delta \geqslant n + \alpha + \epsilon \geqslant J + \epsilon = M.$$

(*b*) If $J - n - \alpha \geqslant 0$ and $[J] - n - [\alpha] = 0$, then again

$$
\begin{aligned}
|D^\gamma Ta(x) - D^\gamma Ta(x')| &\leqslant \int_{3Q_0} |D_1^\gamma K(x,y) - D_1^\gamma K(x',y)| \, \|a\|_\infty \, dy \\
&\leqslant C|x-x'|^\delta |x|^{-n-[\alpha]-\delta} \\
&\leqslant C|x-x'|^\delta |x|^{-M},
\end{aligned}
$$

because, in this case, $\epsilon = \min\{\delta - \max\{\alpha^*, J^*\}, \rho - J^*\}$ and, thus,

$$n + [\alpha] + \delta = [J] + \delta = J - J^* + \delta \geqslant J + (\delta - \max\{\alpha^*, J^*\}) \geqslant J + \epsilon = M.$$

(*c*) If $J - n - \alpha \geqslant 0$ and $k = [J] - n - [\alpha] > 0$, then, using the vanishing moments of $a(x)$, the integral form of the remainder and (3.20), we get

$$
\begin{aligned}
&|D^\gamma Ta(x) - D^\gamma Ta(x')| \\
&\leqslant \left| \int_{3Q_0} \left[D_1^\gamma K(x,y) - \sum_{|\beta|<k} D_2^\beta D_1^\gamma K(x,0) \frac{y^\beta}{\beta!} \right] a(y)\, dy \right. \\
&\qquad \left. - \int_{3Q_0} \left[D_1^\gamma K(x',y) - \sum_{|\beta|<k} D_2^\beta D_1^\gamma K(x',0) \frac{y^\beta}{\beta!} \right] a(y)\, dy \right| \\
&\leqslant \int_{3Q_0} \int_0^1 \frac{(1-s)^{k-1}}{(k-1)!} \left| \sum_{|\beta|=k} [D_2^\beta D_1^\gamma K(x,sy) - D_2^\beta D_1^\gamma K(x',sy)] \frac{y^\beta}{\beta!} \right| ds\, dy \\
&\leqslant C \int_{3Q_0} \int_0^1 |x-x'|^\delta |x-sy|^{-n-|\gamma|-k-\delta} |y|^k\, ds\, dy \\
&\leqslant C|x-x'|^\delta |x|^{-n-|\gamma|-k-\delta} \\
&\leqslant C|x-x'|^\delta |x|^{-M},
\end{aligned}
$$

since

$$n + |\gamma| + k + \delta = n + [\alpha] + [J] - n - [\alpha] + \delta = [J] + \delta = J + \delta - J^* \geqslant M.$$

In all of the above three cases we obtain (1.7) because $|x - x'| < 1$ and $|x|, |x'| > 6\sqrt{n}$. This ends the proof of Theorem 3.13. □

We want to make some remarks regarding the above theorems, duality and interpolation for the case $1 \leqslant p, q < \infty$. In this case, $(\dot{F}_p^{\alpha,q})^* = \dot{F}_{p'}^{-\alpha,q'}$ for any $\alpha \in \mathbb{R}$, where $p' = p/(p-1)$ and $q' = q/(q-1)$. Thus, for $\alpha > 0$ and $1 < p, q \leqslant \infty$, Theorem 3.7 can be obtained from Theorem 3.1 under the additional assumption $T^*1 = 0$ (which we know is actually unnecessary). In fact, in such a case, T^* is, by Theorem 3.1, bounded on $\dot{F}_{p'}^{-\alpha,q'}$ and, hence, T is bounded on $\dot{F}_p^{\alpha,q}$. This, together with an interpolation argument, gives an alternative proof of the L^2 boundedness result of David and Journé [DJ] for the case $T1 = T^*1 = 0$. As was already observed in [FHJW], the result of David and Journé can also be obtained from Theorem 3.7 since $L^2 = \dot{F}_2^{0,2}$. Thus, Theorem 3.7 can also be used to prove directly the boundedness of Calderón-Zygmund operators on $L^p = \dot{F}_p^{0,2}$, for $1 < p < \infty$, and $H^1 = \dot{F}_1^{0,2}$, avoiding the usual duality and interpolation argument (see, for example, [CM]).

For $\alpha > 0$, $\dot{F}_2^{\alpha,2} = \dot{L}_2^\alpha$, where $\dot{L}_2^\alpha$ is the homogeneous (L^2) Sobolev or Beppo Levi space of orther α. In [MM] a criterion for the boundedness of generalized Calderón-Zygmund operators on $\dot{L}_2^\alpha$ for $0 < \alpha < 2$ is obtained, without the assumption (3.10) of our Theorem 3.7 *i.e.* without smoothness in the variable y). The reason for our assumption is that we are proving a little more about T, namely that it maps smooth atoms into smooth molecules. We also recover and extend (again under the additional smoothness in y), the result of P. Lemarié [L], for Besov-Lipschitz spaces. In fact, for $\alpha > 0$ and $\min\{p, q\} \geqslant 1$, we can use the real interpolation result:

$$\dot{B}_p^{\alpha,q} = (\dot{B}_{p_0}^{\alpha_0,p_0}, \dot{B}_{p_1}^{\alpha_1,p_1})_{\theta,q},$$

where $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$ and $1/p = (1-\theta)(1/p_0) + \theta(1/p_1)$ with $0 < \theta < 1$, and the fact that $\dot{B}_p^{\alpha,p} = \dot{F}_p^{\alpha,p}$, to get from Theorem 3.7 a criterion for these spaces.

Theorem 3.13 can be used, in particular, to obtain a boundedness criterion for real Hardy spaces, since $\dot{F}_p^{0,2} = H^p$ for $p \leqslant 1$. The statement of this theorem is clearly more technical than the previous two due to the presence of the mixed derivative conditions. However, if T is a convolution singular integral operator, and, thus $K(x, y) = K(x-y)$, then the hypotheses of Theorem 3.13 are much easier to state. In fact, T is (essentially) its own transpose, and for $\alpha = 0$, $q = 2$ and $0 < p \leqslant 1$ the hypotheses of the theorem are reduced to:

$$T \in \mathrm{WBP},$$
$$|D^\gamma K(x)| \leqslant C|x|^{-n-|\gamma|} \quad \text{for} \quad |\gamma| \leqslant [n(1/p-1)],$$
$$|D^\gamma K(x) - D^\gamma K(x')| \leqslant C|x-x'|^\epsilon |x|^{-n-|\gamma|-\epsilon} \quad \text{for} \quad |\gamma| = [n(1/p-1)],$$

$2|x-x'| < |x|$ and $1 > \epsilon > (n/p)^*$, and

$$T(y^\gamma) = 0 \quad \text{for} \quad |\gamma| \leqslant [n(1/p-1)].$$

Moreover, the last of these conditions is really superfluous since it is guaranteed by the translation invariance of T, as the reader may check (cf. [T]). Thus we recover the classical result about the boundedness of singular integrals on H^p-spaces (see [FS, p. 191]).

Conditions (3.11) and (3.17) in the above theorems are to some extent necessary as the following theorem shows.

Theorem 3.25. *Suppose that* $\alpha > 0$, $0 < p, q \leqslant \infty$, *and*

$$T \in \mathrm{CZO}([\alpha] + \epsilon) \cap \mathrm{WBP}$$

has the norm boundedness property

$$\|T\varphi\|_{\dot{F}_p^{\alpha,q}} \leqslant C\|\varphi\|_{\dot{F}_p^{\alpha,q}} \tag{3.26}$$

for all $\varphi \in \mathfrak{D} \cap \dot{F}_p^{\alpha,q}$. *Then,*

$$T(x^\gamma) = 0 \quad \textit{whenever} \quad |\gamma| < \alpha - \frac{n}{p}. \tag{3.27}$$

Let us delay the proof of this theorem for a moment and make some comments and describe some of its consequences.

The norm boundedness property of the statement of this theorem does not, a priori, allow us to extend T to a continuous operator on $\dot{F}_p^{\alpha,q}$ for the following reason. $\dot{F}_p^{\alpha,q}$ can be regarded as a space of tempered distributions modulo polynomials of degree less than or equal to $[\alpha - n/p]$ (see [FJ2]). Hence, it does not make sense to regard T as defined on $\dot{F}_p^{\alpha,q}$, unless $T(y^\gamma) = 0$ for $|\gamma| \leqslant [\alpha - n/p]$. Nevertheless, it does make sense to consider (3.26) for $\varphi \in \mathfrak{D} \cap \dot{F}_p^{\alpha,q}$. Except for the gap in the case where $\alpha - n/p$ is integer, Theorem 3.25 shows that the norm property can not hold unless T annihilates polynomials of appropriate degree, so that T, in fact, can be extended to a bounded operator on $\dot{F}_p^{\alpha,q}$.

Moreover, there is in general, a gap between this theorem and conditions (3.11) or (3.17). However, if $p = \infty$ and $\alpha > 0$ is not an integer, we obtain from Theorems 3.7 and 3.25 the following

Corollary 3.28. *Let $\alpha > 0$, $\alpha \notin \mathbb{Z}$ and $1 < q < \infty$. Assume that $T: \mathfrak{D} \to \mathfrak{D}'$ satisfies*

$$T \in \mathrm{WBP},$$
$$T \in \mathrm{CZO}\,([\alpha] + \delta) \qquad 1 > \delta > \alpha^*,$$
$$T^* \in \mathrm{CZO}\,(\rho) \qquad 1 > \rho > 0.$$

Then T extends to a bounded operator on $\dot{F}_\infty^{\alpha,q}$ if and only if $T(y^\gamma) = 0$ for all $|\gamma| \leqslant [\alpha]$.

If, in addition, $q = \infty$ we have $\dot{F}_\infty^{\alpha,\infty} = \dot{\Lambda}^\alpha$, the (homogeneous) Lipschitz space of order α, and, in this particular case, a careful look at the proof of Theorem 3.7 shows that, if we do not require that T is to map smooth atoms into smooth molecules, we can get rid of the smoothness in the y variable and still obtain the boundedness on $\dot{\Lambda}^\alpha$. Thus, we recover the result of Y. Meyer for the boundednes of generalized Calderón-Zygmund opertors on Lipschitz spaces [M2].

The gap between conditions (3.11) and (3.27) disappears if we consider conditions yielding boundedness for all $\alpha > 0$. In fact, fix p and q with $1 \leqslant p, q \leqslant \infty$. From Theorems 3.7 and 3.25 we immediately obtain

Corollary 3.29. *Let $T: \mathfrak{D} \to \mathfrak{D}'$ be a linear and continuous operator with kernel K whose restriction to Ω is a C^∞-function of x. If, in addition, we have*

$$T \in \mathrm{WBP}, \tag{3.30}$$

$$|D_1^\gamma K(x, y)| \leqslant C_\gamma |x - y|^{-n-|\gamma|} \quad \textit{for all} \quad \gamma, \tag{3.31}$$

and

$$|K(x,y) - K(x,z)| \leqslant C|y-z|^{\epsilon}|x-y|^{-n-\epsilon} \tag{3.32}$$

for some $\epsilon > 0$ *whenever* $2|y-z| < |x-y|$, *then* T *extends to a bounded operator on* $\dot{F}_p^{\alpha,q}$ *for all* $\alpha > 0$ *if and only if* $T(P) = 0$ *for all polynomials* P.

Observe that in this case we also have that T is bounded if and only if T maps smooth atoms into smooth molecules for all $\alpha > 0$.

For the inhomogeneous spaces $F_p^{\alpha,q}$ (defined in [Tr], [FJ2]) we have the following

Corollary 3.33. *Suppose* $1 \leqslant p, q < \infty$. *Let* $T: \mathfrak{D} \to \mathfrak{D}'$ *be a linear and continuous operator satisfying* (3.31) *and* (3.32). *Suppose, also, that* T *is bounded on* L^p *and that* $T(P) = 0$ *for all polynomials* P. *Then* T *extends to a bounded operator on* $F_p^{\alpha,q}$ *for all* $\alpha > 0$.

This result follows from Theorem 3.7 (boundedness in L^p implies WBP) and the standard fact that for $\alpha > 0$ and $0 < p, q \leqslant \infty$, we have

$$\|f\|_{F_p^{\alpha,q}} \cong \|f\|_{\dot{F}_p^{\alpha,q}} + \|f\|_{L^p} \quad \text{for} \quad f \in \mathcal{S}'.$$

Recall that for $\alpha > 0$ and $1 < p < \infty$, $F_p^{\alpha,q} = L_p^{\alpha}$, the ($L^p$) Sobolev space of order α. Thus, the above corollary gives, in particular, conditions for T to be simultaneously bounded on all Sobolev spaces for a fixed p.

PROOF OF THEOREM 3.25. Let $0 < \alpha$ and $0 < \min\{p, q\} < \infty$. We may assume $\alpha > n/p$, otherwise (3.27) is void. We will need to use the following standard facts about Triebel-Lizorkin spaces

Fact 1 (see, for example, [FJ2] or [FW]). Let $\psi \in \mathfrak{D}_l$ for some $l \geqslant [\alpha - n/p]$. Then there exists a constant $C_\psi > 0$ such that for all $f \in \dot{F}_p^{\alpha,q}$,

$$|\langle f, \psi\rangle| \leqslant C_\psi \|f\|_{\dot{F}_p^{\alpha,q}}.$$

Fact 2 (see [Tr, p. 239]). Let $f \in \mathfrak{D} \cap \dot{F}_p^{\alpha,q}$, then $\|f^t\|_{\dot{F}_p^{\alpha,q}} \leqslant C t^{(n/p)-\alpha}\|f\|_{\dot{F}_p^{\alpha,q}}$.

Let $|\gamma| = l < \alpha - n/p$. We have to show that $Ty^\gamma = 0$ in $\mathfrak{D}_l'$. Now, although Ty^γ is initially only defined on $\mathfrak{D}_l'$, using the Hahn-Banach theorem we can extend it to a distribution S in $\mathfrak{D}'$ (two such extensions will, of course, differ by a polynomial of degree less than or equal to l). Let β be a multi-index such that $|\beta| > [\alpha - n/p]$. Then, $d^\beta\psi \in \mathfrak{D}_{|\beta|-1} \subseteq \mathfrak{D}_l$ for all $\psi \in \mathfrak{D}$ and we have

$$\langle D^\beta S, \psi\rangle = (-1)^{|\beta|}\langle S, D^\beta\psi\rangle = (-1)^\beta\langle Ty^\gamma, D^\beta\psi\rangle.$$

Choose $\varphi \in \mathfrak{D}$ such that $\varphi(0) \neq 0$ and $y^\gamma \varphi(y) \in \mathfrak{D}_L$, where as before, $L = [J - n - \alpha]$. It is easy to check that, up to a multiplicative constant, $y^\gamma \varphi$ is a smooth atom for $\dot{F}_p^{\alpha, q}$, and, therefore, $y^\gamma \varphi \in \mathfrak{D} \cap \dot{F}_p^{\alpha, q}$. Since

$$T \in \mathrm{CZO}([\alpha] + \epsilon) \subseteq \mathrm{CZO}([\alpha - n/p] + \epsilon),$$

we have, from Lemma 1.22 and the two facts stated at the beginning of this proof, that

$$\begin{aligned}
|\varphi(0)\langle D^\beta S, \psi\rangle| &= |\varphi(0)\langle T(y^\gamma), D^\beta \psi\rangle| \\
&= \lim_{t \to \infty} |\langle T(y^\gamma \varphi^t), D^\beta \psi\rangle| \\
&= \lim_{t \to \infty} t^{|\gamma|} |\langle T[(y^\gamma \varphi)^t], D^\beta \psi\rangle| \\
&\leqslant \lim_{t \to \infty} C_\psi t^{|\gamma|} \| T[(y^\gamma \varphi)^t] \|_{\dot{F}_p^{\alpha, q}} \\
&\leqslant C_{\psi, T} \lim_{t \to \infty} t^{|\gamma|} \| (y^\gamma \varphi)^t \|_{\dot{F}_p^{\alpha, q}} \\
&\leqslant C_{\psi, T} \lim_{t \to \infty} t^{|\gamma| + n/p - \alpha} \| y^\gamma \varphi \|_{\dot{F}_p^{\alpha, q}} \\
&= 0
\end{aligned}$$

since $|\gamma| < \alpha - n/p$. Thus, since $\varphi(0) \neq 0$, we see that $D^\beta S = 0$ in $\mathfrak{D}'$ for all $|\beta| > [\alpha - n/p]$. But this implies that S coincides with a polynomial of degree at most $[\alpha - n/p]$ on $\mathfrak{D}$, and, a fortiori, Ty^γ coincides on $\mathfrak{D}_l$ with a polynomial of degree at most $[\alpha - n/p]$.

Put $P = T(y^\gamma)$. We claim that the weak boundedness property implies that the degree of $P(x)$ is at most l. Since a polynomial of degree at most l acts as the zero linear fuctional on $\mathfrak{D}_l$, this will imply that $T(y^\gamma) = 0$ in $\mathfrak{D}_l'$.

Suppose that

$$P(x) = \sum_{|\nu| \leq d} c_\nu x^\nu = \sum_{|\nu| < d} c_\nu x^\nu + \sum_{|\nu| = d} c_\nu x^\nu \equiv Q(x) + R(x),$$

where $d > l$ and $\sum_{|\nu| = d} |c_\nu| \neq 0$.

Let $\psi \in \mathfrak{D}_l$ be such that $\int \psi(x) R(x)\, dx \neq 0$, and let $r > 0$ be such that $\operatorname{supp} \psi \subseteq B_r(0)$. Let $\chi \in \mathfrak{D}$ satisfy $\chi \equiv 1$ on $B_{2r}(0)$ and $\|\chi\|_\infty = 1$. For all $t > 1$,

$$\begin{aligned}
\langle P, \psi^t\rangle &= \langle T(y^\gamma), \psi^t\rangle \\
&= \langle T(\chi^t y^\gamma), \psi^t\rangle + \langle T[(1 - \chi^t) y^\gamma], \psi^t\rangle \\
&= \langle T(\chi^t y^\gamma), \psi^t\rangle + \langle T^* \psi^t, (1 - \chi^t) y^\gamma\rangle \\
&\equiv I + II.
\end{aligned}$$

By the WBP,

$$|I| \leqslant t^{|\gamma|}|T(\chi y^\gamma)^t, \psi^t\rangle| \leqslant C_{\psi,\chi} t^{n+l},$$

and, from (1.21) we have that

$$\begin{aligned}
|II| &= |\langle T^*\psi^t, (1-\chi^t)y^\gamma\rangle| \\
&= \left|\iint K(x,y)\psi^t(x)(1-\chi^t(y))y^\gamma\,dx\,dy\right| \\
&\leqslant \int_{|y|>2rt} 2|y|^{|\gamma|}\left|\int K(x,y)\psi^t(x)\,dx\right|dy \\
&\leqslant C(rt)^{n+l+\epsilon}\|\psi\|_\infty \int_{|y|>2rt} |y|^{-n-l-\epsilon+l}dy \\
&\leqslant C_\psi t^{n+l}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\langle P, \psi^t\rangle &= \int P(x)\psi^t(x)\,dx \\
&= t^n \int P(tx)\psi(x)\,dx \\
&= t^n \int Q(tx)\psi(x)\,dx + t^n \int R(tx)\psi(x)\,dx \\
&= \sum_{|\nu|<d} c_\nu t^{n+|\nu|} \int x^\nu \psi(x)\,dx + t^{n+d}\int R(x)\psi(x)\,dx,
\end{aligned}$$

and, since $\int \psi(x)R(x)\,dx \neq 0$ and $d > l$, if we let t tend to infinity, we obtain a contradiction. Therefore, the degree of the polynomial $P(x)$ is at most l. This completes the proof of the theorem. □

References

[CZ] Calderón A. and Zygmund, A. On the Existence of Certain Singular Integrals, *Acta Math.*, **88**(1952), 85-139.

[CM] Coifman, R. and Meyer, Y. Au Delá des Operateurs Pseudo-Différentiels, *Astérisque,* **57**(1982).

[CW] Coifman, R. and Weiss, G. Extensions of Hardy Spaces and Their Use in Analysis, *Bull. Amer. Math. Soc.,* **83**(1977), 569-645.

[DJ] David, G. and Journé, J.-L. A Boundedness Criterion for Generalized Calderón-Zygmund operators, *Ann. of Math.,* **120**(1984), 371-397.

[FS] Fefferman, C. and Stein, E. H^p Spaces of Several Variables, *Acta Math.,* **129**(1972), 137-193.

[FHJW] Frazier, M., Han, Y. S., Jawerth, B. and Weiss, G. The $T1$ Theorem for Triebel-Lizorkin Spaces, to appear in *Proceedings of the Harmonic Analysis.* El Escorial, June 1987; to be published by Springer Verlag (Mathematical Notes series).

[FJ1] Frazier, M. and Jawerth, B. The φ-Transform and Applications to Distribution Spaces, *Lecture Notes in Math.* **1302**(1988), 223-246.

[FJ2] Frazier, M. and Jawerth, B. A Discrete Transform and Decomposition of Distribution Spaces, preprint.

[FW] Frazier, M. and Weiss, G. Calderón-Zygmund Operators on Triebel-Lizorkin Spaces, MSRI, Berkeley, 02221-88 (1988).

[GC-RF] García-Cuerva, J. and Rubio de Francia, J. L. *Weighted Norm Inequalities and related Topics*, North Holland Mathematics Studies, Amsterdam, 1985.

[L] Lemarié, P. Continuité sur les Espaces de Besov des Operateurs Définis par des Integrales Singuliérs, *Ann. Inst. Fourier, Grenoble,* **35**(1985), 175-187.

[MM] Meyer, M. Continuité Besov de Certains Opérateurs Integraux Singulières, Thèse de 3e Cycle, Orsay 1985.

[M1] Meyer, Y. Les Nouveaux Opérateurs de Calderón-Zygmund, *Astèrisque,* **131**(1985), 237-254.

[M2] Meyer, Y. Continuité sur les Espaces de Hölder et the Sobolev des Opérateurs Définis par des Intégrales Singulières, *Recent progress in Fourier Analysis*, I. Peral and J. L. Rubio de Francia ed., North Holland, (1985), 145-172.

[S] Stein, E. *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press (1970).

[TW] Taibleson, M. and Weiss, G. The Molecular Characterization of Certain Hardy Spaces, *Astérisque*, **77**(1980), 67-149.

[T] Torres, R. Washington University, Ph. D. thesis.

[Tr] Triebel, H. *Theory of Function Spaces, Monographs in Mathematics*, Birkhaus, 1983.

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* Supported in part by NSF Postdoctoral Fellowship DMS 8705935.
** Supported in part by NSF Grant MCS 8200884.