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Interpolation of Banach Spaces, Differential Geometry and Differential Equations

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In recent years the study of interpolation of Banach spaces has seen some unexpected interactions with other fields. Some of these interactions originated from work of Coifman, Cwickel, Rochberg, Sagher, and Weiss [CRSW] on the extension of the complex method of Calderón to infinite families of Banach spaces. This extension is important because some relationships with the rest of mathematics are too trivial in the classical two-space case to be seen. Examples of this include the extremal problems studied by Szegö and the theorem of Masni and Weiner on factorization of matrix-valued functions. Both of these can be viewed as special cases of the interpolation method of [CRSW].

Since then, R. Rochberg has initiated the study of interpolation from the viewpoint of curvature on vector bundles. In particular, he characterized the interpolation families in [CRSW] in terms of a vanishing curvature condition.

Also, Z. Slodkowski [S1] has shown the interpolation families of [CRSW] to be examples of analytic multifunctions, and he applied [CRSW] to problems concerning polynomial hulls in $\mathbb{C}^n$.

In this paper I shall discuss some more interactions of interpolation theory with the rest of mathematics, beginning with some joint work with Coifman

[CS]. Our basic idea was to look for the methods of interpolation that had interesting PDE's arising as examples. The Hilbert space case of the interpolation method of [CRSW], for example, gives solutions for the Dirichlet problem on the disk for the equation

$$\bar{\partial}(\Omega^{-1}\partial\Omega) = 0, \tag{0.1}$$

where $\Omega(z)$ is a positive-definite matrix-valued function.

In the course of our work we found that it is very natural to view interpolation theory as the study of the geometry of the space $\mathfrak{N}(V)$ of all norms on a vector space V. It later became clear how to make some of these formal analogies precise and thereby make sense of differential geometry on $\mathfrak{N}(V)$. This has led to some new methods of intepolation, which are defined and analyzed via differential equations.

(Let me emphasize that the reader is not assumed to have any expertise in differential geometry. The situations in this paper in which differential geometry is discussed are quite concrete, and the relevant facts are reviewed.)

These topics are discused in more detail in the sections below. At the end I shall indicate other ways that interpolation theory relates to PDE, in particular to equations of Monge-Ampére type. I would like to conclude this introductory section with some philosophical comments.

One of the main conceptual points I wish to make in this paper is that there are many ways of looking at interpolation of Banach spaces. In particular, we should not tie ourselves too tightly to interpolation estimates for operators.

An example of a natural problem in the open-minded theory of interpolation of Banach spaces is the following. Suppose we are given two convex bodies B_1 and B_2 in $\mathbb{R}^n$. What is an optimal way of connecting B_1 to B_2 by a curve of convex bodies in $\mathbb{R}^n$? What should «optimal» mean?

In the case of norming bodies —*i.e.*, convex bodies that are symmetric about the origin— we could use an interpolation method to give us a curve of norms, and hence a curve of bodies. Are there natural notions of optimality in this context?

We can also ignore classical interpolation theory altogether in our search for natural notions of optimality. We could approach this physically: what happens when we try to deform B_1 into B_2 in a way that minimizes the work done, or the total strain, or something like that? We can think of slowly squeezing one body in a maximally efficient way until we get the other body.

There are undoubtedly many natural ways to make sense of «optimal» here.

We must be more adventurous in the theory of interpolation of Banach spaces.

1. Interpolation Via Perron Processes

This section is devoted to the interpolation method in $\mathbb{C}^d$ defined in [CS].

Z. Slodkowski [S2, 3] has also considered the problem of extending the complex method of interpolation to several complex variables. His point of view is quite different than ours, and comes from his earlier work on analytic multifunctions. He works in more generality than we do, but in the case we work in, our definition is equivalent to his.

Let V be a complex vector space. We require that V be finite-dimensional to avoid technicalities, but of course we want estimates that do not depend on the dimension. Let D be a domain in $\mathbb{C}^d$. Suppose we are given some norms $\| \ \|_\zeta$ on V for each $\zeta \in \partial D$. We want to extend $\| \ \|_\zeta$ to a norm-valued function $\| \ \|_z$ in $\bar{D}$ that satisfies nice properties. For example, we may want this extension to satisfy interpolation estimates for operators.

By an interpolation method we mean, roughly speaking, some method of assigning to $\| \ \|_\zeta$ an extension $\| \ \|_z$. We would like to find interpolation methods that are related to interesting PDE's.

This motivation provides one of the basic ideas of our approach. If we want to have a PDE floating around, then we want our interpolation method to be «local». In particular, the reiteration theorem should hold: if we start with $\| \ \|_\zeta$ on ∂D, apply the interpolation method to get $\| \ \|_z$ on $\bar{D}$, restrict $\| \ \|_z$ to the boundary of a subdomain D_0 of D, and then repeat the interpolation method on D_0 again, we should get $\| \ \|_z$ back again.

This localness does not work out if we extend the ideas of Calderón and [CRSW] naively to the higher-dimensional case; their approach relies too heavily on miracles of one complex variable in the unit disk. There are problems already for multiply-connected domains in the plane.

To get the desired localness feature we must therefore take a different tack.

Let $N_z(\cdot)$ be a norm function defined in some region in $\mathbb{C}^d$. By a «norm function» we mean a function that takes values in the set of all norms on V. We say that N_z is subharmonic if $N_z(f(z))$ is a subharmonic function of z for all holomorphic V-valued functions f. (It suffices to consider only affine holomorphic functions.) This property played an important role in [CRSW]. It is clearly local.

We define our interpolation method using a Perron process based on this notion of subharmonicity. Let $\| \ \|_\zeta$, $\zeta \in \partial D$, be a given family of norms on ∂D. Set

(1.1) $\mathfrak{A} = \{N_z: N_z$ is a subharmonic norm function defined on D such that

$$\limsup_{z\to\zeta} N_z(v) \leqslant \|v\|_\zeta \quad \text{for all} \quad \zeta \in \partial D,\ v \in V\Big\} \cdot$$

We define $\| \ \|_z$ in D by

$$\|v\|_z = \sup_{N_z \in \mathfrak{A}} N_z(v). \tag{1.2}$$

There are two main issues to address for this interpolation procedure: does it have nice abstract properties, and what are the examples? I discuss first some of the abstract properties.

If D is regular for the Dirichlet problem for the usual Laplacian, then it is regular for the construction above, in the sense that if $\| \ \|_\zeta$ is continuous on ∂D, then $\| \ \|_z$ is continuous on D, and coincides with $\| \ \|_\zeta$ on ∂D. (A norm function N_z is called continuous if $N_z(v)$ is continuous in z for each $v \in V$.) One fact used in the proof of this is that if $h(z)$ is subharmonic on D and if $P(v)$ is some norm on V, then $N_z = e^{h(z)}P$ is a subharmonic norm function on D. This allows us to build barriers for the above Perron process.

If T_z is a family of linear operators on V, T_z defined and continuous on $\bar{D}$ and holomorphic in z on D, then $\log \|T_z\|_{z,z}$ is subharmonic in z, where $\| \ \|_{z,z}$ denotes the operator norm with respect to $\| \ \|_z$. Let me indicate the proof of a special case of this: if $\|T_\zeta\|_{\zeta,\zeta} \leqslant 1$ when $\zeta \in \partial D$, then $\|T_z\|_{z,z} \leqslant 1$ on D. Indeed, under these hypotheses, if $N_z \in \mathfrak{A}$, then $N_z(T_z(\cdot))$ also lies in $\mathfrak{A}$. The desired conclusions now follow easily from the definitions.

One can prove the reiteration theorem for the above interpolation method by a straightforward chasing of definitions. Thus, if we call a norm function $\| \ \|_z$ harmonic on D if it can be recovered from its boundary values as in (1.1) and (1.2), then any norm function that is harmonic on D is also harmonic on all subdomains of D. Moreover, being harmonic is a local property: $\| \ \|_z$ is harmonic on D if for each $z \in D$ there is a neighborhood U of z on which $\| \ \|_z$ is harmonic. There is in fact a reasonable notion of a superharmonic norm function. This notion is local, and a norm function is harmonic if and only if it is both subharmonic and superharmonic.

Let us consider some examples. If D is the unit disk, then this interpolation construction agrees with the one in [CRSW]. If $D \subseteq \mathbb{C}^d$ but $V = \mathbb{C}$, so that $\|v\|_\zeta = a(\zeta)|v|$ for some positive function $a(\zeta)$, then $\|v\|_z = a(z)|v|$, with $\log a(z)$ harmonic. (Although this example is rather trivial, it is a good test case. If we tried to extend the method of [CRSW] directly to several variables, we would have problems computing even this case.)

Suppose now that we take $V = \mathbb{C}^n$ and we let $\| \ \|_\zeta$ be the $\ell^{p(\zeta)}$ norm on V, where $p: \partial D \to [1, \infty]$ is given. Then $\| \ \|_z$ is the $\ell^{p(z)}$ norm, with $1/p$ harmonic in D.

For our last example we assume that each $\| \ \|_\zeta$ is a Hilbert space norm. Let us identify V with $\mathbb{C}^n$ and let $\langle \ , \ \rangle$ denote the standard inner product there. Thus $\| \ \|_\zeta$ is given by $\|v\|_\zeta^2 = \langle \omega(\zeta)v, v\rangle$ for some positive matrix-valued function $\omega(\zeta)$ on ∂D. Then $\|v\|_z^2 = \langle \Omega(z)v, v\rangle$, where $\Omega(z)$ is a positive matrix-valued function that satisfies $\Omega|_{\partial D} = \omega$, Ω is smooth in D, and

$$\sum_{j=1}^{d} \bar{\partial}_j(\Omega^{-1}\partial_j\Omega) = 0 \quad \text{on} \quad D. \tag{1.3}$$

This, of course, generalizes (0.1).

Equation (1.3) arises in complex differential geometry and Yang-Mills theory. In particular, when $d = 2$, this is essentially a reformulation of the anti-self-dual Yang-Mills equation.

Thus our interpolation procedure gives a Perron process for (1.3), as well as reasonable notions of sub- and supersolutions for it. This is remarkable because these things usually do not make sense for systems. This also gives a precise sense in which the potential theory for (1.3) is controlled by the potential theory for the Laplacian.

Solving the Dirichlet problem for (1.3) with smooth solutions is a serious issue. We can rewrite (1.3) as

$$\Delta\Omega - \sum_{j=1}^{d} (\bar{\partial}_j\Omega)\Omega^{-1}(\partial_j\Omega) = 0.$$

Thus there are lower-order terms that are quadratic in the gradient. This is the critical case: if the growth were less than quadratic, then there are general methods for dealing with regularity issues, while on the other hand, there are well-known simple examples of second-order elliptic systems, with the leading term given by the Laplacian an lower-order terms that are quadratic in the gradient, for which regularity fails.

It turns out that one can solve the Dirichlet problem for (1.3) with, say, continuous boundary data and get smooth solutions in D. What saves us are very strong maximum principles for (1.3) that come from the Perron process. One can solve (1.3) using the continuity method, and the natural function spaces to work with involve quadratic Carleson measure conditions on $\nabla\Omega$, exactly like the usual conditions on ∇u when u is a harmonic function that is bounded or has BMO boundary values.

This is another manifestation of how the potential theory for (1.3) is controlled by the potential theory for the Laplacian: the natural notions of Carleson measures are the same in both places. Carleson measures seem to be a natural tool for treating other second-order elliptic systems with quadratic terms in the gradient as well.

Let me emphasize that although (1.3) is defined for matrix-valued functions, to define the Perron process for (1.3) we had to fatten up the target space and work with norm-valued functions. This is unavoidable; the problem is that the maximum of two Hilbert space norms is no longer a Hilbert space norm, but just a norm.

The fact that our interpolation procedure leads to a Perron process for (1.3) is interesting by itself, quite apart from its other features. This is a good example of comments I made in the introduction, about how we should view interpolation theory broadly, without retricting ourselves unduly to interpolation estimates for operators or other standard viewpoints.

Much of what we have done in this section can be recast very naturally in terms of curvature on holomorphic vector bundles, following Rochberg [R]. I shall not discuss this. Instead, I shall discuss a different way of bringing geometry into the picture, which is described in [CS].

2. Geometry of the Space of all Banach Spaces

Let $\mathfrak{N}(V)$ denote the space of all norms on V. The idea is to view $\mathfrak{N}(V)$ as some sort of infinite dimensional manifold, and to recast interpolation theory into the study of the geometry of $\mathfrak{N}(V)$.

We would like to think of $\mathfrak{N}(V)$ as being like a Riemannian manifold, although it is not at all clear how to put a natural Riemannian structure on $\mathfrak{N}(V)$. However, interpolation theory gives $\mathfrak{N}(V)$ some geometric structure so that it behaves in many ways like a Riemannian manifold.

For example, given any two points $\|\cdot\|_0$ and $\|\cdot\|_1$ in $\mathfrak{N}(V)$, we can use the Calderón method to join them by a curve in $\mathfrak{N}(V)$: we set $\|\cdot\|_\theta = [\|\cdot\|_0, \|\cdot\|_1]_\theta$, $0 < \theta < 1$. We call such a curve a Calderón curve, and we decide to view the Calderón curves as being the geodesics of $\mathfrak{N}(V)$.

There is a natural sense in which these geodesics minimize length. Given two norms N and M on V, define

$$\delta(N, M) = \log \sup_{\substack{v \in V \\ v \neq 0}} \frac{M(V)}{N(V)}$$

and

$$d(M, N) = \max(\delta(M, N), \delta(N, M)).$$

This defines a metric on $\mathfrak{N}(V)$, a variant of the Banach-Mazur distance. Given a curve N_t in $\mathfrak{N}(V)$, $0 \leqslant t \leqslant 1$, we define its length to be

$$\sup \sum d(N_{t_j}, N_{t_{j+1}}),$$

where the supremum is taken over all partitions $\{t_j\}$ of $[0, 1]$.

It can be shown that if N_t is a Calderón curve, then its length is equal to $d(N_0, N_1)$. (This is not hard, using (2.1) below.) Unlike the situation for a Riemannian manifold, however, the converse is not true: it is easy to cook up a curve of norms M_t, $0 < t < 1$, that is not a Calderón curve but whose length is $d(M_0, M_1)$.

Calderón curves also have the following nice property. If N_t and M_t are both Calderón curves, then

(2.1) $\qquad d(N_t, M_t)$ is a convex function of t.

In the Riemannian case, where $d(\cdot, \cdot)$ is replaced by the geodesic distance, (2.1) means that the manifold has nonpositive sectional curvature.

Not only is (2.1) natural geometrically, but it is also natural from the point of view of interpolation. For example, it is also true that $\delta(N_t, M_t)$ is convex, and this contains the interpolation of operators theorem for Calderón curves. For let T be a linear operator, and assume a priori that T is invertible. If N_t is a Calderón curve, then so is $M_t(\cdot) = N_t(T(\cdot))$, and the convexity of $\delta(N_t, M_t)$ is exactly the same as the convexity of the log of the operator norm of T relative to N_t.

Let me give another example of how interpolation theory can be recast into the study of the geometry of $\mathfrak{N}(V)$. In Section 18 of [CS] a new proof of Wolff's 4-space reiteration theorem is given (due to R. Rochberg and the authors) that uses only (2.1), the fact that any two points of $\mathfrak{N}(V)$ can be joined by a Calderón curve, and elementary geometry. This argument does require substantial a priori assumptions —dim $V < \infty$ is enough— but in cases of concrete function spaces this can usually be taken care of by suitable approximation arguments. (Yves Meyer's wavelet basis can be particularly useful in this regard.)

[The proof of Wolff's theorem given in [CS] gives strong evidence that there should be a version for quasibanach spaces, but nothing rigorous has been obtained for the Calderón method.]

These ideas from [CS] on viewing interpolation theory as the study of the geometry of $\mathfrak{N}(V)$ make sense independently of the Perron process business, but there is a relationship between the two.

There is an important generalization of geodesics in Riemannian manifolds, namely, harmonic mappings. A mapping from a domain in $\mathbb{R}^n$ into a Riemannian manifold is called harmonic if it is a critical point for the energy norm, which is just the L^2-norm of its first derivatives. When $n = 1$, harmonic mappings are geodesics. In general harmonic maps are solutions of a nonlinear second-order system whose leading term is given by the Laplacian and whose lower-order terms are quadratic in the gradient.

Consider the following example. Let X_k denote the set of all $k \times k$ positive-definite matrices over $\mathbb{C}$. There is a natural Riemannian metric on X_k that is invariant under the action $\Omega \mapsto T^*\Omega T$, $T \in GL(k, \mathbb{C})$. (In fact, X_k is a negatively curved symmetric space.)

If $\Omega(x)$ is defined on some region $R \subseteq \mathbb{R}^d$ and takes values in X_k, then Ω is harmonic if and only if

$$\sum_{j=1}^{d} \partial_j(\Omega^{-1}\partial_j\Omega) = 0. \tag{2.2}$$

This equation is of course very similar to (1.3). Matrix-valued functions on $R \subseteq \mathbb{R}^d$ can be identified with matrix functions on $D = R \times i\mathbb{R}^d \subseteq \mathbb{C}^d$ that are independent of the imaginary variables. Solutions of (2.2) correspond exactly to solutions of (1.3) which are independent of the imaginary variables.

This suggests that we define harmonic maps into $\mathfrak{N}(V)$ as follows. If N_x is a norm function defined on $R \subseteq \mathbb{R}^d$, we say that N_x is a harmonic map into $\mathfrak{N}(V)$ if the corresponding norm function on $D = R \times i\mathbb{R}^d \subseteq \mathbb{C}^d$ is harmonic in the sense of our Perron process.

If we identify X_k with the subset of $\mathfrak{N}(\mathbb{C}^k)$ consisting of all Hilbert space norms, then we see that these two notions of harmonic maps of $R \subseteq \mathbb{R}^d$ into X_k —the Riemannian and interpolation definitions— coincide. In particular, the geodesics of X_k are just the Calderón curves that lie in X_k. This corresponds to $d = 1$ and $R = (0, 1)$.

There is a version of (2.1) for harmonic mappings. If N_x and M_x are two harmonic maps of $R \subseteq \mathbb{R}^d$ into $\mathfrak{N}$, then

$$d(N_x, M_x) \text{ is subharmonic in } x. \tag{2.3}$$

The obvious analogue of (2.3) for Riemannian manifolds is true when the manifold has nonpositive sectional curvature and other conditions hold, *e.g.*, if the manifold is complete and simply-connected. For example, X_k has these properties for each k. Thus harmonic maps into X_k satisfy (2.3) when $d(\cdot, \cdot)$ is the Riemannian distance or when $d(\cdot, \cdot)$ is the distance on $\mathfrak{N}(V)$.

Norm functions defined in a domain in $\mathbb{C}^d$ that are harmonic in the sense of Section 1 also satisfy (2.3). Harmonic norm functions in the sense of Section 1 have a lot in common with harmonic maps into $\mathfrak{N}(V)$, and in particular, solutions of (1.3) have a lot in common with harmonic maps into X_k.

When I mentioned in Section 1 the importance of certain strong forms of the maximum principle for (1.3) for obtaining smooth solutions of the Dirichlet problem, what I had in mind was (2.3). The methods used in [CS] for treating the Dirichlet problem for (1.3) can also be used for harmonic maps into Riemannian manifolds which are, say, complete, simply connected, and have nonpositive sectional curvature. (These hypotheses can be weakened to those used in [H].) The analogue of (2.3) again plays an important role.

Incidentally, I have restricted myself in this section to harmonic maps defined on some domain in $\mathbb{R}^d$ —rather than on a general Riemannian manifold with boundary— only for convenience.

In this section I have outlined a number of formal analogies between Riemannian geometry and the geometry of $\mathfrak{N}(V)$ that comes from interpolation. In the next section I describe a way of making these formal analogies precise.

3. Differential Geometry on the Space of all Banach Spaces

There is a way to endow a manifold with geometric structure similar to that of a Riemannian manifold, but without having a Riemannian metric. This is done by specifying a connection on that manifold. The precise definition of a connection can be rather confusing, so instead to giving the definition I shall try to motivate and explain it in a more limited way that suits our context.

To do this we first look more closely at how the notion of a geodesic is determined by a Riemannian metric. Let us work in local coordinates, that is, we identify a piece of a given Riemannian manifold with an open subset U of $\mathbb{R}^n$, with the Riemannian metric given by some matrix-valued function (g_{ij}) on U. A curve $\gamma(t)$ in U is a geodesic relative to g if

$$\ddot{\gamma}_k(t) + \sum_{i,j} \Gamma^k_{ij}(\gamma(t))\dot{\gamma}_i(t)\dot{\gamma}_j(t) = 0,$$

where the dots denote derivatives in t. The functions $\Gamma^k_{ij}(x)$ can be expressed explicitely in terms of g and its first derivatives, but the specific formula is not relevant for the present purpose.

On a Riemannian manifold there is a globally-defined object called the Levi-Cevita connection, which in local coordinates is given in terms of the Γ^k_{ij}'s above. This object encodes a great deal of the geometry of the Riemannian manifold without actually giving the metric. As in (3.1), the class of geodesics on the manifold is determined by the connection, and t his is true for harmonic maps into the manifold as well. The Riemann curvature tensor is also given in terms of the connection, without recourse to the metric.

We can also consider connections on a manifold that do not come from a Riemannian metric. A connection can be determined by specifying Γ^k_{ij}'s in local coordinates in a coherent way. By choosing a connection we endow the manifold with a lot of the same geometric structure as in the Riemannian case —geodesics, harmonic maps into the manifold, and curvature— without actually having a metric.

It turns out that there is a natural choice of connection on $\mathfrak{N}(V)$, whose geodesics are the Calderón curves, and such that harmonic maps from a domain in $\mathbb{R}^n$ into $\mathfrak{N}(V)$ are the same as the harmonic maps defined in Section 2. This allows us to ask new questions about the geometry of $\mathfrak{N}(V)$, *e.g.*, what is its curvature?

The task of defining this connection is simplified by the existence of a natural choice of global coordinates on $\mathfrak{N}(V)$, so that choosing a connection reduces to choosing Γ^k_{ij} in that one coordinate system. Before describing how this choice is made, let me reformulate slightly the Γ^k_{ij}'s in a way that is more convenient in infinite dimensions. Given a point x, $\Gamma^k_{ij}(x)$ defines a bilinear operator Γ_x: $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ given by $c = \Gamma_x(a, b)$,

$$c_k = \sum_{ij} \Gamma^k_{ij}(x) a_i b_j, \qquad a, b, c \in \mathbb{R}^n.$$

Thus in local-coordinates a connection is determined by a function Γ that takes values in bilinear operators.

To see how to choose our connection on $\mathfrak{N}(V)$, we must first find a differential equation for Calderón curves; then we choose the connection so that the Calderón curves are geodesics.

It is a beautiful fact due to Rochberg [R] that there is a PDE that characterizes Calderón curves. Let N_t, $0 < t < 1$, be a Calderón curve of norms on $V = \mathbb{C}^d$, and set $F(t, v) = N_t(v)^2$. Then N_t is a Calderón curve if and only if

$$\ddot{F} - \sum_{jk} F^{j\bar{k}} \dot{F}_j \dot{F}_{\bar{k}} = 0. \tag{3.2}$$

Here F_j, $F_{\bar{k}}$, and $F_{j\bar{k}}$ denote the various complex derivatives of F in $v \in \mathbb{C}^d$, *e.g.*

$$F_{j\bar{k}} = \frac{\partial^2}{\partial v_j \partial \bar{v}_k} F,$$

and $F^{j\bar{k}}$ denotes the inverse of $F_{j\bar{k}}$, so that

$$\sum_k F^{j\bar{k}} F_{l\bar{k}} = \delta^j_l.$$

The dots denote derivatives in t.

To be honest, Calderón curves may not satisfy (3.2) in the usual sense, because F may not be smooth enough, or $F_{j\bar{k}}$ may not be invertible. There are reasonable senses in which Calderón curves are generalized solutions of (3.2). I shall not discuss this here. The main point is that (3.2) tells us how to choose a connection on $\mathfrak{N}(V)$.

Let us first set some preliminaries. Let $C^2_2(V)$ denote the Banach space of real-valued functions on V that are C^2 except at the origin and also complex homogeneous of degree 2, *i.e.*, $h(\alpha v) = |\alpha|^2 h(v)$ for any $\alpha \in \mathbb{C}$, $h \in C^2_2(V)$. Let $\mathfrak{N}_0(V)$ denote the open subset of $C^2_2(V)$ of positive functions that are strictly convex. (For some purposes —such as making sense of (3.2)— strict plurisubharmonicity would suffice.) For the discussion below we work with $\mathfrak{N}_0(V)$ instead of $\mathfrak{N}(V)$ so that everything is well defined.

We view $\mathfrak{N}_0(V) \subseteq C^2_2(V)$ as playing the same role as $U \subseteq \mathbb{R}^n$ did before. In this context Γ is now a function on $\mathfrak{N}_0(V)$ that takes values in bilinear operators defined on $C^2_2 \times C^2_2$: Given $F \in \mathfrak{N}_0(V)$ and $A, B \in C^2_2(V)$, we define

$$\Gamma_F(A, B) = -Re \sum_{jk} F^{j\bar{k}} A_j B_{\bar{k}}. \tag{3.3}$$

This defines our connection on $\mathfrak{N}_0(V)$. By definition, the geodesics for Γ are precisely the solutions of (3.2), *i.e.*, Calderón curves.

Notice that $\Gamma(A, B)$ is symmetric in A and B. This property is natural and useful, and it is one of the defining properties of the Levi-Cevita connection on a Riemannian manifold. (In technical jargon, this symmetry means that the connection is torsion-free.) Notice also that the definition of Γ is forced on us once we decide that it should be symmetric and that its geodesics should be Calderón curves.

As I promised, the class of harmonic maps of a domain in $\mathbb{R}^n$ into $\mathfrak{N}_0(V)$ defined by Γ corresponds exactly to the definition given in Section 2.

Now that we have found the connection on $\mathfrak{N}_0(V)$ that corresponds to interpolation theory, let us consider its properties from the point of view of elementary differential geometry.

Let us start with a very simple question. What can we say about the initial-value problem for geodesics on $\mathfrak{N}_0(V)$? Given $F_0 \in \mathfrak{N}_0(V)$, $A_0 \in C_2^2(V)$, when can we find a solution $F(t, v)$, $0 < t < ?$, of (3.2) such that $F = F_0$ and $\dot{F} = A_0$ at $t = 0$?

If F_0 and A_0 are real-analytic except at $v = 0$, then we can solve the initial value problem for short time using the Cauchy-Kowalevsky theorem. I don't think one can do much better. I can prove the following: Given $F_0 \in \mathfrak{N}_0(V)$, if we can solve the initial value problem whenever $A_0(v)$ *is a quadratic form, and with reasonable bounds, then* F must be in the Gevrey class of order 2, *i.e.*, F is C^∞ (except at $v = 0$), and its n^{th} derivative grows in n no worse that $C^n(n!)^2$.

Let us now address a more interesting issue: what can we say about the curvature? The curvature associated to Γ can be expressed in terms of Γ and its first derivatives exactly as in the Riemannian case. I will not give the formula for it, but it can be found in most texts on differential geometry.

Let me describe three examples of how we can learn something from looking at the curvature.

The curvature of Γ is rather a mess. One reason is that the complex conjugations and the «Re» make the algebra more complicated. This suggests that we first consider a model situation without these problems.

Let $W = \mathbb{R}^d$, and let $\mathfrak{N}^r(W)$ denote the set of all norms on W. Let $C_2^2(W)$ and $\mathfrak{N}_0^r(W)$ be as before, but with real homogeneity replacing complex homogeneity. Define $\bar{\Gamma}$ for $\mathfrak{N}_0^r(W)$ by

$$\bar{\Gamma}_F(A, B) = -\sum_{j,k} F^{jk} A_j B_k, \qquad F \in \mathfrak{N}_0^r(W), \quad A, B \in C_2^2(W). \tag{3.4}$$

Thus $\bar{\Gamma}$ is exactly like Γ, except that we have removed all vestiges of the complex numbers.

The curvature of $\bar{\Gamma}$ is much simpler that that of Γ: it vanishes identically. General nonsense says that there is a change of variables under which $\bar{\Gamma}$ transforms to 0. (The way $\bar{\Gamma}$ transforms under a change of variables is a little complicated.) A good guess and some calculation show that the desired change of variables is given by $\delta(\|\bullet\|) = \|\bullet\|^*$, *i.e.*, the map that takes a norm to its dual norm. (We view $\|\bullet\|^*$ as a norm on W, via the standard pairing on $W = \mathbb{R}^d$.)

More calculation allows one to show that Γ is preserved by this same change of variables. This contains in particular the duality theorem for the Calderón method. Still more calculation of the same type allows one to prove a duality theorem for the interpolation method described in Section 1. This result was obtained first by Slodkowski [S2, part I], and in fact he proved a much more general theorem.

A better real-variable model for Γ on $\mathfrak{N}_0^r(W)$ is Γ', defined by

$$(3.5) \qquad \Gamma_F'(A, B) = -\frac{1}{2}\sum F^{jk}A_jB_k, \qquad F \in \mathfrak{N}_0^r(W), \qquad A, B \in C_2^2(W).$$

The same calculations that show that $\tilde{\Gamma}$ is trivialized by δ also give that Γ' is invariant under δ. In particular, the class of geodesics associated to Γ' is invariant under δ, just like for the Calderón method.

Let me briefly describe a computation that arises when computing how Γ, $\tilde{\Gamma}$, and Γ' transform under δ and which is interesting in its own right. To compute these transformations, we must first compute the differential of δ. Because we are working in a somewhat unusual situation, let me spell out what I mean by this. Fix $F \in \mathfrak{N}_0^r(W)$. Given $A \in C_2^2(W)$, define

$$(3.10) \qquad d\delta_F(A) = \lim_{t \to 0} \frac{\delta(F + tA) - \delta(F)}{t},$$

i.e., the directional derivative of δ at F in the direction of A.

It turns out that there is a nice formula for $d\delta_F(A)$. Let J be the duality map associated to F, that is, if N is the norm such that $N^2 = F$, then $N^*(J(v)) = N(v)$ and $\langle v, J(v)\rangle = N(v) \cdot N^*(J(v))$. Here $\langle \bullet, \bullet \rangle$ denotes the standard pairing on $\mathbb{R}^d$, which is used to define the dual norm. Our formula for $d\delta$ is

$$(3.11) \qquad d\delta_F(A) = -A \circ J^{-1}.$$

The proof of (3.11) is not infinitely difficult. First we recall the well-known fact that

$$J(V) = \frac{1}{2}\nabla F(v).$$

Indeed, by definition the linear functional $w \mapsto \langle w, J(v) \rangle$ attains its maximum on the F-sphere $S = \{w: F(w) = F(v)\}$ at v, so that the hyperplane $\{w: \langle w, J(v) \rangle = \langle v, J(v) \rangle\}$ must be tangent to S at v, which implies that $J(v)$ is proportional to ∇F at v. The constant $1/2$ is determined by the relation $F(v) = F^*(J(v))$.

Next, we differentiate the relation

$$F = F^* \circ J, \tag{3.12}$$

viewing J as a function of F, and apply the chain rule. When we differentiate both sides of (3.12) in the direction of A, just as in the right side of (3.10), we get

$$A = (d\delta_F(A)) \circ J + \sum_l \left(\left(\frac{\partial}{\partial v_l} F^* \right) \circ J \right) \cdot (dJ_F(A))_l. \tag{3.13}$$

Let me explain the right side of (3.13). By definitions, $d\delta_F(A)$ is a function on W (homogeneous of degree 2), and when we compose that function with J, we get the first term on the right. The second term on the right comes from the chain rule. Because J is a map from W to W for each F, so is $dJ_F(A)$, and $(dJ_F(A))_l$ denotes the l^{th} component of this vector-valued function.

We can simplify this considerably. Because

$$J = \frac{1}{2} \nabla F, \qquad dJ_F(A) = \frac{1}{2} \nabla A.$$

Also,

$$K = \frac{1}{2} \nabla F^*$$

is just the duality map for F^*, so that $K = J^{-1}$, and thus

$$\frac{1}{2} (\nabla F^*) \circ J$$

is the identity. If we write out the last term in (3.13) as a function of v using these remarks, we get

$$\sum_l 2v_l \cdot \frac{1}{2} \frac{\partial}{\partial v_l} A(v).$$

Because A is homogeneous of degree 2, one can show that this equals $2A(v)$. Plugging this into (3.13) gives (3.11).

Once we have computed $d\delta$, it is not difficult to compute the effect of the change of variables δ on $\tilde{\Gamma}$ and Γ'. The case of Γ is more complicated, but in the same spirit.

This concludes the discussion of the first example of how we can learn something from computing the curvature of these connections. We have seen that it has led us to important formulae, which helps us compute the action of duality on Γ. It also led to an interesting real-variable model for Γ, to wit, Γ'.

The second example deals with the following question: Is there some natural sense in which $\mathfrak{N}_0(V)$ is negatively curved? Usually «negatively curved» is defined to mean that the sectional curvatures are negative, and the definition of sectional curvature involves both the curvature tensor and the Riemannian metric. In the case of $\mathfrak{N}_0(V)$, it is not clear how «negatively curved» should be interpreted.

Let me give two pieces of evidence that suggest that $\mathfrak{N}_0(V)$ is negatively curved in some reasonable sense. First, we have seen that if M_t, N_t are two Calderón curves, then $d(N_t, M_t)$ is convex. In the Riemannian case this means that the sectional curvature is nonpositive.

Second, $\mathfrak{N}_0(V)$ contains the space of all Hilbert space norms on V as a submanifold, which we identify with the manifold of all $d \times d$ positive-definite matrices. This manifold has a natural Riemannian structure with nonpositive sectional curvature, and the corresponding connection is the same as what you get by restricting Γ to it.

There are in fact natural senses in which Γ and Γ' have nonpositive curvature. Understanding how this works leads to a better knowledge of the maximum principles for geodesics and harmonic maps of Γ and Γ', and the relationship between the curvature and these maximum principles.

For the third example we start with the following question: is there a Riemannian structure on $\mathfrak{N}_0(V)$ or $\mathfrak{N}_0^r(W)$ compatible with Γ or Γ'?

Let me begin by explaining how this relates to curvature. I shall restrict the discussion to the specific example of $\mathfrak{N}_0^r(W)$, equipped with Γ'.

A Riemannian metric on $\mathfrak{N}_0^r(W)$ is a bilinear-form valued function g on $\mathfrak{N}_0^r(W)$; more precisely, for each $F \in \mathfrak{N}_0^r(W)$, $g_F(\cdot, \cdot)$ should be a positive bilinear form on $C_2^2(W)$. (I am also willing to consider g's that are only defined on subclasses of C_2^2 with more smoothness, even real-analyticity.)

There are a number of ways of thinking of the curvature tensor R associated to Γ'. I prefer to think of it as follows. For each $F \in \mathfrak{N}_0^r(W)$ and every $A, B \in C_2^2(W)$, $R_F(A, B)$ defines a linear operator on homogeneous functions of degree 2 on W. Thus for each $C \in C_2^2(W)$, $R_F(A, B)C$ is a homogeneous function of degree 2 on W. Also, $R_F(A, B)$ is linear in A and B, and $R_F(A, B) = -R_F(B, A)$.

In the case of Γ', the formula for the curvature is not so bad:

$$R_F(A, B)C = \frac{1}{4} \sum_{ijkl} \{F^{ki} B_{ij} F^{jl} A_k C_l - F^{ki} A_{ij} F^{jl} B_k C_l\}. \tag{3.14}$$

As usual, subscripts denote derivatives, and F^{ki} denotes the inverse of F_{lm}. (The curvature for Γ is more complicated, but in the same spirit.)

If Γ' is the canonical connection associated to a Riemannian metric g, then the curvature tensor must satisfy some extra conditions. For example, if $F \in \mathfrak{N}_0^r(W)$ is given, and $A, B, C, D \in C_2^2(W)$ are arbitrary, then we would have to have

$$g_F(R_F(A, B)C, D) = -g_F(C, R_F(A, B)D). \tag{3.15}$$

In other words, $R_F(A, B)$ should define an antisymmetric operator relative to $g_F(\bullet, \bullet)$.

Let us choose F to be the square of the ordinary Euclidean norm, and let us denote by g_0 and R_0 the values of g_F and R_F at this particular point. In this case (3.14) reduces to

$$R_0(A, B)C = \frac{1}{16} \sum_{jk} \{B_{jk} A_j C_k - A_{jk} B_j C_k\}. \tag{3.16}$$

Suppose now that dim $W = 2$. We can identify homogeneous functions of degree 2 with functions on the circle. One can easily show that every tangent vector field X on the circle arises as the scum of two operators of the form $R_0(A, B)$. Hence the compatibility condition (3.15) tells us that such X must be antisymmetric with respect to $g_0(\bullet, \bullet)$. It is not hard to show that this implies that

$$g_0(A, B) = a \int_0^{2\pi} A(t) B'(t)\, dt$$

for some $a \in \mathbb{R}$, where I have now identified functions on the circle with periodic functions on $[0, 2\pi]$. Because this bilinear form is never symmetric and positive-definite, we conclude that Γ' is not compatible with any Riemannian metric when dim $W = 2$.

It does not seem reasonable to expect that in higher dimensions the linear span of the operators $R_0(A, B)$ should give all tangent vector fields on the sphere. It seems much more reasonable to hope that the Lie algebra generated by these operators would give all such tangent vector fields. For the purposes above, it would suffice if the Lie algebra generated by $R_0(A, B)$ and covariant derivatives of the curvature operator gave everything. In any case, it is very likely still true in higher dimensions that Γ' is never compatible with a Riemannian metric, and that one can prove this using (3.15) and its variants.

The preceeding discussion can and should be phrased in terms of the holonomy group of the connection Γ'. Let me briefly indicate some of the main points.

The holonomy group of Γ' can be naturally realized as a subgroup of the group of all orientation-preserving diffeomorphisms of the unit sphere of

$W = \mathbb{R}^d$, no matter what dim W is. Calculations like those above show that when dim $W = 2$ the holonomy group is in some moral sense the group of all the orientation-preserving diffeomorphisms on the circle. To be specific, the Lie algebra of the holonomy group contains all real-analytic vector fields on the circle. I do not know whether the analogue of this last fact is true in higher-dimensions.

Although there is more to say about the behavior of Γ and Γ' from the viewpoint of differential geometry, and on the relationship of this to interpolation theory, it is probable better to move on to the next topic.

4. Interpolation Theory Via Differential Equations

Since Γ' seems to be a good real-variable model for Γ, it is natural to ask whether the geodesics for Γ' —*i.e.*, solutions $F(t, v)$ of

$$\ddot{F} - \frac{1}{2}\sum_{jk} F^{jk}\dot{F}_j\dot{F}_k = 0 \tag{4.1}$$

— share many properties with Calderón curves. For example, does (4.1) define a good interpolation method?

I think that this is a fun question. We are starting with nothing but the equation, and we want to analyze its interpolation-theoretic properties directly from the PDE.

One of the first questions is whether we can solve the boundary value problem for (4.1). We will not be able to get smooth solutions in general, but we can look for generalized solutions via a Perron process.

Let me warn the reader from the outset that in this discussion I intend to ignore all issues related to making sense of generalized solutions, and instead do the computations as though everything is smooth. This will help me to convey the essential ideas without getting bogged down in less interesting technicalities.

To define a Perron process we first need to have a good notion of suharmonicity. It can be shown that $F(t, v)$ is a subsolution of (4.1) (*i.e.*, the left side is $\geqslant 0$) if and only if for each affine W-valued function $f(t)$, we have

$$F(t, f(t))^{\cdot\cdot} \geqslant -\sum_{j,k} F_{jk}(t, f(t))\dot{f}_j\dot{f}_k. \tag{4.2}$$

This should be compared with the notion of subharmonicity discussed in Section 1. [Incidentally, subsolutions for the geodesic equation for $\tilde{\Gamma}$ (which amounts to replacing the 1/2 in (4.1) with a 1) are characterized by the property that $F(t, f(t))^{\cdot\cdot} \geqslant 0$ for all affine functions $f(t)$. Also, $F(t, v)$ is a geodesic for $\tilde{\Gamma}$ if and only if F^* is affine. This is because $\tilde{\Gamma}$ is trivialized by the change of variables $\delta(\|\ \ \|) = \|\ \ \|^*$.]

Notice that the subharmonicity condition (4.2) is equivalent to requiring that $F(t, f(s+t))$ be a subharmonic function of (s, t) for all affine functions f. This helps make clear certain basic properties of this notion of a subharmonic norm function, for instance, that the maximum of two such things is also subharmonic.

Using this notion of subharmonicity we can define a Perron process that gives us generalized solutions of the boundary value problem for (4.1).

Just as in the case of Calderón curves, if $F(t, v)$ and $G(t, v)$ are two solutions of (4.1), then $\delta(F, G)$ and $d(F, G)$ are convex functions of t, where δ and d are as defined in Section 2. One can derive from this an interpolation of operators theorem: if $\| \ \|_t$ is a curve of norms such that $F(t, v) = \|v\|_t^2$ satisfies (4.1), and if $T: W \to W$ is linear, then $\log \|T\|_{t,t}$ is convex, where $\| \ \|_{t,t}$ denotes the operator norm of T relative to $\| \ \|_t$.

Thus (4.1) does define an interpolation method. There is a duality theorem for this method because Γ' is invariant under the duality mapping on $\mathfrak{N}^r(W)$, as we saw in Section 3.

Let us now address the issue of computing examples for this interpolation method. The Hilbert space case works out in essentially the same way as for the Calderón method. The only difference is that now the Hilbert spaces are real, and one computes with real symmetric positive-definite matrices instead of their complex cousins.

The situation is quite different for L^p spaces. If $\| \ \|_t$ is taken to be the $\ell^{p(t)}$ norm on $W = \mathbb{R}^d$, then $F(t, v) = \|v\|_t^2$ gives a solution of (4.1) if and only if $p(t)$ is constant.

This turns out to be not as bad as it may seem. If $1/p(t)$ is affine, then F gives a subsolution of (4.1) when $p(t) \geqslant 2$, and it gives a supersolution when $p(t) \leqslant 2$.

More generally, given any function $p(t)$, $1 < p(t) < \infty$, with $1/p$ affine, we can define another interpolation method using the equation

$$\ddot{F} - \frac{p-1}{p} \sum F^{jk} \dot{F}_j \dot{F}_k + \left(\frac{\dot{p}}{p}\right)^2 F = 0. \tag{4.3}$$

Here F corresponds to a norm function $\| \ \|_t$ by $F(t, v) = \|v\|_t^{p(t)}$. As with (4.1), we can find generalized solutions for the boundary-value problem for (4.3) using a Perron process, based on a notion of subharmonicity that is similar to (4.2), but more complicated. There is an interpolation of operators theorem for (4.3), as well as a duality theorem: the dual of a solution of (4.3) is again a solution of (4.3), but with p replaced by its conjugate exponent.

If $1/q(t)$ is affine, then the $\ell^{q(t)}$ norm yields a solution of (4.3) if $q(t) = p(t)$, it gives a subsolution when $q(t) \geqslant p(t)$, and it gives a supersolution when $q(t) \leqslant p(t)$.

Using these interpolation methods we recover the theorem of Marcel Riesz, which states that if $1/p(t)$ and $1/q(t)$ are affine, $p(t) \leqslant q(t)$, and if T is a linear operator, then the log of the L^p-L^q operator norm of T is convex. Here L^p denotes the L^p-space of real functions. I should emphasize that for real L^p-spaces this theorem is false if we do not assume that $p \leqslant q$.

These new interpolation methods seem to be natural versions of the complex method for real Banach spaces. Although the method of analysis of (4.1) and (4.3) is based on differential equations, and hence is quite different from the usual approaches for the complex method, a very similar analysis can be used for the complex method.

These new methods hold open the alluring prospect of being able to build interpolation methods to suit your needs, by writting down a differential equation.

Let me mention one other property of the interpolation methods defined by (4.3), concerning the relationship between these methods and the classical real method. Suppose we are given two norms A and B on W, and let $[A, B]_{\theta,q}$ denote the norms obtained from A and B by the real method. Let $p(t)$ be a function such that $1/p(t)$ is affine. Then given any $\epsilon > 0$ there is a constant $C < \infty$ and a norm function N_t defined for $\epsilon \leqslant t \leqslant 1 - \epsilon$, such that $F(t, v) = N_t(v)^{p(t)}$ is a generalized solution of (4.3), and N_t is equivalent to $[A, B]_{t,p(t)}$ on $\epsilon \leqslant t \leqslant 1 - \epsilon$, with constant C. (Of course, C does not depend on dim W.)

In other words, the classical real method generates interpolation families for (4.3).

Let me sketch the proof of this. Let $K(t, v)$ denote the K-functional associated to the couple (A, B). Then the $[A, B]_{t,p(t)}$ norm is equivalent

$$H(t, v) = \Big(\sum_j (2^{-jt}K(2^j, V))^{p(t)}\Big)^{1/p(t)}.$$

I claim that $H(t, v)^{p(t)}$ is a subsolution of (4.3). Indeed, for any sequence of norms $\| \ \|_j$,

$$G(t, v) = \sum (2^{-jt}\| v\|_j)^{p(t)} \tag{4.4}$$

is a subsolution of (4.3). This is because a sum of subsolutions is a subsolution, and one can simply compute that each of the individual terms is a subsolution.

Thus the K-functional definition of $[A, B]_{t,p(t)}$ gives a subsolution for (4.3). One can show that the J-functional definition gives a supersolution in much the same way (or you could use duality). From here it is not difficult to get N_t.

From the point of view of nonlinear PDE, this is quite remarkable: we are getting some sort of superposition principle for (4.3).

Note that although $G(t, v)$ as in (4.4) always gives a subsolution, in general it need not be anywhere near a supersolution. In this particular case, where all the $\| \ \|_j$'s come from the K-functional as above, what we get is also equivalent to a supersolution.

Before leaving this section I should mention that we can also consider higher-dimensional versions of the interpolation methods defined by (4.1) and (4.3), just as in Section 1.

5. A Few Last Comments

It seems natural at this stage to reiterate my opinion (indicated in the introduction) that we should look for new ways of viewing interpolation theory and new connections with other fields. To give an example of this I shall discuss some observations of Coifman and myself.

Much of what we did in Section 3 still makes sense if we work with spaces of convex functions instead of norms. For example, if we let $H(W)$ denote the space of strictly convex C^2 functions on $W = \mathbb{R}^d$, then we can still define connections $\tilde{\Gamma}$ and Γ' on $H(W)$ exactly as before, and we can define geodesics associated to these connections, just as before. Similarly, we can also define Γ on $H(V)$, $V = \mathbb{C}^d$.

The equation for geodesics for $\tilde{\Gamma}$ is

$$\ddot{F} - \sum F^{jk}\dot{F}_j\dot{F}_k = 0, \qquad F = F(t, v), \qquad v \in W. \tag{5.1}$$

A little computation shows that (5.1) is equivalent to the real Monge-Ampére equation, *i.e.*, $\det(\operatorname{Hess} F) = 0$, where $\operatorname{Hess} F$ denotes the Hessian of F, that is, the matrix of second derivatives of F in all variables (t and v).

In the context of the Monge-Ampére equation it is natural not to require our functions to be defined on all of W, and even to allow their domains to change with t. Although this is a bit awkward, the formalism with connections and such still makes sense, at least locally.

Even in this larger setting, the curvature of $\tilde{\Gamma}$ vanishes. There is also a nice change of variables that trivializes $\tilde{\Gamma}$, analogous to duality in the context of norms. Given a convex function F defined on some domain in W, its conjugate F^* is defined by

$$F^*(w) = \sup_{v \in W} \{ \langle v, w \rangle - F(v) \}. \tag{5.2}$$

Here $\langle \ , \ \rangle$ is the standard pairing on $W = \mathbb{R}^d$. To be precise, the domain of F^* is defined to be those $w \in W$ such that the above supremum is finite. The change of variables $F \mapsto F^*$ can be shown to trivialize $\tilde{\Gamma}$, just as the duality function did in the norm case.

The preceeding contains in particular the following well-known fact. If u is a solution of Monge-Ampére in a domain in $\mathbb{R}^n$, with Rank (Hess u) $= n - 1$, then u determines a foliation of its domain by lines, with Du constant on each line. Note that this is a fact about individual solutions of Monge-Ampére, while the above stuff about trivializing $\tilde{\Gamma}$ is a property of the Monge-Ampére operator.

[To be honest, to get this business about foliations by lines in full generality, we have to have a version of the conjugate function F^* when F is not convex but satisfies det (Hess F) $\neq 0$. This can be done, but only locally, which is good enough for the foliation story.]

Calderón curves, and more generally interpolation families in the sense of [CRSW], give rise to solutions of the complex Monge-Ampére equation. If $\|\cdot\|_z$ is a family of norms on the unit disk, and $F(z, v) = \|v\|_z^2$, then $\|\cdot\|_z$ is an interpolation family in the sense of [CRSW] if and only if

$$F_{z\bar{z}} - \sum F^{j\bar{k}} F_{jz} F_{\bar{k}\bar{z}} = 0. \tag{5.3}$$

Here the z and $\bar{z}$ subscripts denote $\partial/\partial z$ and $\partial/\partial \bar{z}$ derivatives.

This characterization is due to Rochberg [R]. (He does not write it in quite this form, however.) Equation (3.2) is a special case of this, corresponding to the case where $F(z, v)$ is defined on the strip, instead of the disk, and where $F(z, v)$ does not depend on Im (y).

A little calculation shows that (5.3) is equivalent to requiring that $F(z, v)$ satisfies the complex Monge-Ampére equation, as a function of both z and v. The complex Monge-Ampére equation is defined by

$$\det\left(\frac{\partial^2}{\partial \zeta_i \partial \bar{\zeta}_j} u\right) = 0, \tag{5.4}$$

for $u(\zeta)$ defined in some domain in $\mathbb{C}^n$.

The relationship between complex Monge-Ampére and the interpolation construction of [CRSW] is more fundamental than one sees at first glance. Let me give some examples of this. If u satisfies (5.4) and also the rank of the complex Hessian $\partial_i \bar{\partial}_j u$ is $n - 1$, then it is well known that the domain of u is foliated by a family of Riemann surfaces, such that u restricts to a harmonic function on each of these surfaces. This should be compared with the foliation by lines in the real case. In the case where u is given by $\|v\|_z^2$, these Riemann surfaces are just the graphs of the extremal functions given in [CRSW]. (Recall that these extremal fuctions are the holomorphic V-valued functions $f(z)$ such that $\|f(z)\|_z$ in constant.)

L. Lempert [L1, 2] has solved certain boundary-value problems for the homogeneous complex Monge-Ampére equation on domains in $\mathbb{C}^n$ using a set-up that is strikingly similar to the interpolation method in [CRSW]. Lempert builds his solutions by first solving extremal problems for holomorphic functions defined on the unit disc and taking values in the given domain. These extremal problems are much like those in [CRSW]. The way that the solution of Monge-Ampére is built out of the solutions of these extremal problems is also very similar to the interpolation construction in [CRSW]. Convexity assumptions on the given domain (in $\mathbb{C}^n$) play an important role in [L1, 2], just as convexity is important in [CRSW] (in obtaining the reiteration theorem, for example).

It is not hard to see why extremal problems like those in [CRSW] and [L1, 2] should be related to finding solutions of the homogeneous complex Monge-Ampére equation. As I have mentioned, such a solution often has associated to it a foliation of Riemann surfaces on which the solution is harmonic. If you can find these Riemann surfaces, and if their boundaries lie inside the boundary of the domain on which you are working, then you can obtain the solution from its boundary values by solving the Dirichlet problem for harmonic functions on these Riemann surfaces. In [L1, 2] and [CRSW], these Riemann surfaces are analytic disks obtained by solving an extremal problem.

These Riemann surfaces will always be associated to extremal problems. If u is a plurisubharmonic solution to (5.4), then its restriction to any Riemann surface is subharmonic, and the ones in the foliation are exactly those for which the restriction is harmonic. If these good Riemann surfaces have their boundaries contained in the boundary of the domain you are working on, then the preceeding remark can be used to write down an extremal problem in terms of the boundary values of u for which the good Riemann surfaces are the solutions.

I should emphasize that the details of the analysis in [L] and [CRSW] are quite different, despite the similarities in the set-up. The techniques Lempert introduces are very interesting and should prove useful in interpolation theory.

The preceeding discussion gives another example of how interpolation theory interacts with PDE in nontraditional ways.

Many of the questions that we asked at the end of the introduction in the case of norms also make sense in the more general context indicated at the beginning of the section. For example, is there a reasonable sense in which the geodesics for $\Gamma, \tilde{\Gamma}$, or Γ' give «optimal» curves of convex functions joining a given pair of convex functions? What are natural notions of «optimality» in this context, not necessarily related to these connections? Are there other natural connections on the space of convex functions?

In the norm case we saw that Γ' has more in common with Γ than $\tilde{\Gamma}$ does. In what ways is this manifested in this more general setting? Perhaps the geodesic equation for Γ' is interesting in the Monge-Ampére scheme of things.

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