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Maximal and Area Integral Characterizations of Hardy-Sobolev Spaces in the Unit Ball of $\mathbb{C}^n$

Patrick Ahern and Joaquim Bruna
Dedicated to the memory of Professor José Luis Rubio de Francia

Abstract

In this paper we deal with several characterizations of the Hardy-Sobolev spaces in the unit ball of $\mathbb{C}^n$, that is, spaces of holomorphic functions in the ball whose derivatives up to a certain order belong to the classical Hardy spaces. Some of our characterizations are in terms of maximal functions, area functions or Littlewood-Paley functions involving only complex-tangential derivatives. A special case of our results is a characterization of H^p itself involving only complex-tangential derivatives.

1. Introduction. Statement of Results

Let B^n denote the unit ball of $\mathbb{C}^n$. For a holomorphic function f on B^n with homogeneous expansion $f(z) = \sum_k f_k(z)$, the *radial fractional derivative* of order $s > 0$ is defined

$$R^s f(z) = \sum_k (k+1)^s f_k(z)$$

(so for $s = 1$, $Rf(z) = f(z) + Nf(z)$, where $N = \sum z_j(\partial/\partial z_j)$ is the normal field). The *Hardy-Sobolev* space H^p_s is defined

$$H^p_s = \left\{f: \sup_r \int_S |R^s f(r\zeta)|^p \, d\sigma(\zeta) = \|f\|^p_{p,s} < +\infty\right\}.$$

Here S denotes the boundary of B^n and $d\sigma$ its normalized Lebesgue measure.

It is a well-known general principle that holomorphic functions in B^n behave twice as well in the complex-tangential directions. Our first goal is to make precise how this principle applies to the Hardy-Sobolev spaces H^p_s. To state our results we need to introduce some definitions. A vector field

$$T = \sum_{j=1}^n a_j(z) \frac{\partial}{\partial z_j}, \qquad a_j \in C^\infty(\bar{B}^n)$$

is called *complex-tangential* if

$$\sum_{j=1}^n a_j(z)\bar{z}_j = 0.$$

If T is a complex-tangential vector field and f is holomorphic then Tf is generally no longer holomorphic. For this reason it is more natural to deal with real-variable characterizations of H^p itself. These are in terms of the following quantities, defined for a smooth function f:

(*a*) The *radial maximal function*

$$f^+(\eta) = \sup\{|f(r\eta)| : 0 \leqslant r < 1\}, \qquad \eta \in S.$$

(*b*) The *admissible maximal function*

$$f^*_\alpha(\eta) = \sup\{|f(z)| : z \in D_\alpha(\eta)\}.$$

Here $D_\alpha(\eta)$ denotes the *admissible approach region*

$$D_\alpha(\eta) = \left\{z \in B^n : |1 - \bar{z}\zeta| < \frac{\alpha}{2}(1 - |z|^2)\right\},$$

with $\alpha > 1$.

(*c*) The *admissible area function*

$$A_\alpha f(\eta) = \left\{\int_{D_\alpha(\eta)} |Rf(z)|^2 (1 - |z|)^{1-n} \, dm(z)\right\}^{1/2}.$$

(*d*) The *Littlewood-Paley g-function*

$$g(f)(\eta) = \left\{\int_0^1 |Rf(r\eta)|^2 (1 - r) \, dr\right\}^{1/2}.$$

It will be convenient to introduce the notations, for $t \in \mathbb{R}^+$ and $\gamma > 0$

$$A^t_{\alpha,\gamma} f(\eta) = \left\{ \int_{D_\alpha(\eta)} |R^t f(z)|^2 (1 - |z|)^{2\gamma - n - 1} \, dm(z) \right\}^{1/2}$$

$$g^t_\gamma f(\eta) = \left\{ \int_0^1 |R^t f(r\eta)|^2 (1 - r)^{2\gamma - 1} \, dr \right\}^{1/2}$$

(so that $A_\alpha = A^1_{\alpha,1}$ and $g = g^1_1$).

If $\varphi(f)$ denotes any of the functions in (a)-(d), then $f \in H^p$ if and only if $\varphi(f) \in L^p(S)$, (see [8]). It then follows by standard techniques (see [7, pp. 214-216] and Lemma 3.6 below) that the area function

$$A^t_{\alpha,t-s} f(\eta) = \left\{ \int_{D_\alpha(\eta)} |R^t f(z)|^2 (1 - |z|)^{2t - 2s - n - 1} \, dm(z) \right\}^{1/2}$$

with any $t > s$, characterizes H^p_s, *i.e.*, $f \in H^p_s$ if and only if $A^t_{\alpha,t-s} f \in L^p(S)$. In a similar way, the g-function

$$g^t_{t-s} f(\eta) = \left\{ \int_0^1 |R^t f(r\eta)|^2 (1 - r)^{2t - 2s - 1} \, dr \right\}^{1/2}$$

characterizes H^p_s if $t > s$.

Our first result states that if $t > s$, if k is a positive integer with $k \leqslant 2t$ and $T_1, \ldots, T_k$ are complex-tangential vector fields, then $f \in H^p_s$ implies

$$\left\{ \int_{D_\alpha(\eta)} |T_1 \cdots T_k R^{t-k/2} f(z)|^2 (1 - |z|)^{2t - 2s - n - 1} \, dm(z) \right\}^{1/2} \in L^p(S) \tag{1}$$

$$\left\{ \int_0^1 |T_1 \cdots T_k R^{t-k/2} f(r\eta)|^2 (1 - r)^{2t - 2s - 1} \, dr \right\}^{1/2} \in L^p(S). \tag{2}$$

Also, if $f \in H^p_s$ and $k \leqslant 2s$, the radial and admissible maximal functions of $T_1 \cdots T_k R^{s-k/2} f$ are in $L^p(S)$. That is, one can formally replace each $R^{1/2}$ by one complex-tangential vector field and still get a function in $L^p(S)$.

In case of the area function, this result again follows by the standard techniques in [7, pp. 214-216] and [6, Th. 12], but for the g-function and the two maximal functions it depends on some special properties of complex-tangential derivatives. We comment briefly on these. Each complex-tangential vector field is a linear combination of the T_{ij}

$$T_{ij} = \bar{z}_i \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial z_i}, \qquad i, j = 1, \ldots, n, \qquad i \neq j.$$

If f is holomorphic, then $T_{ij} f$ is no longer holomorphic, but on every complex line is annihilated by $\bar{\partial}^2$. In the same way, if the $T_1, \ldots, T_k$ are chosen among the T_{ij} (which we may assume without loss of generality), then $T_1 \cdots T_k f$ is annihilated by $\bar{\partial}^{k+1}$ on every complex line. We show that such functions share many properties of holomorphic functions, such as mean-value inequalities, Cauchy inequalities for derivatives etc., which are on the basis for the transition from the area function to the others.

Our second result states that the above property gives in fact a characterization of H_s^p. Namely, with k fixed, if (1) or (2) holds for all possible choices of $T_1 \cdots T_k$ among the T_{ij} then $f \in H_s^p$. In the same way, if $(T_1 \cdots T_k R^{s-k/2} f)^+ \in L^p$ for all $T_1 \cdots T_k$ (k fixed, $k \leqslant 2s$), then $f \in H_s^p$. Choosing k such that $k > 2s$ and $t = k/2$ we thus obtain characterizations of H_s^p in terms of area functions or g-functions involving only complex-tangential derivatives. For instance, as a particular case we obtain that $f \in H^p$ if and only if for all T_{ij}

$$\left\{\int_0^1 |T_{ij} f(r\eta)|^2 \, dr\right\}^{1/2} \in L^p(S).$$

Also, when $s = k/2$, $f \in H_s^p$ if and only if $(T_1 \cdots T_k f)^+ f \in L^p(S)$. We point out that these characterizations do not involve the $\bar{T}_{ij}$. The same result but allowing as well T_{ij} and $\bar{T}_{ij}$ would be much easier to prove because the normal derivative appears as the Lie bracket of T_{ij} and $\bar{T}_{ij}$'s.

We want to point out a basic difference that occurs, as in the real variable theory of Hardy spaces, between the area functions and the others. Indeed, for the area functions there are pointwise estimates, in the sense that the integral appearing in (1) and the $A_{\alpha, t-s}^t$ are pointwise equivalent, (up to replacement of the aperture α), but there are no such estimates for the others, *e.g.* it is not possible to bound $(T_{ij}^2 f)^+$ by $(Rf)^+$ or $(Rf)^*$, pointwise.

For example, with $n = 2$, let $f(z, w) = w^2 \log(1 - z)$ then Rf is bounded in B^2 but $T^2 f$ is not (here $T = \bar{z}(\partial/\partial w) - \bar{w}(\partial/\partial z)$). For this reason it is easier to deal with the area functions.

We also obtain analogous results for the Bergman-Sobolev spaces

$$B_{s,\gamma}^p = \left\{f \text{ holomorphic: } \int_{B^n} |R^s f(z)|^p (1 - |z|)^\gamma \, dm(z) < +\infty\right\},$$

$p > 0, \quad \gamma > -1$. Namely, one can replace $R^{k/2}$ by k complex-tangential vector fields to define these spaces. The paper [1] deals with one aspect of this.

This paper is organized as follows. In the second section we collect all properties that will be needed of functions annihilated by some power of $\bar{\partial}$ on every complex line. In the third section we prove our results for the case of area functions and in the fourth section we deal with the g-functions and maximal functions. The fifth section is more technical and deals with the following question, in case $s = k/2$: if $R^{k/2} \in H^p$ then it has a distribution boundary value which belongs to the space H_{at}^p of [4]. Is this also true for $T_1 \cdots T_k f$? We show in Section 5 that this is the case, in spite of the fact that $T_1 \cdots T_k f$ is not holomorphic. Finally in section 6 we prove our result on Bergman-Sobolev spaces.

In the remaining of this section we introduce some more notations. First, as it is easily checked, the operators T_{ij} commute with the ordinary Laplacian

Δ, so that the functions $T_1 \cdots T_k f$ are harmonic. It will be then convenient to consider R^s as defined in the class of harmonic functions. If u is harmonic and $u = \sum_k P_k$ is its expansion in spherical harmonics, then we define as well $R^s u = \sum_k (k+1)^s P_k$.

The operator I^s of fractional integration is defined

$$I^s f(z) = \frac{1}{\Gamma(s)} \int_0^1 \left(\log \frac{1}{t}\right)^{s-1} f(tz)\, dt.$$

Then

$$I^s P_k = (k+1)^{-s} P_k$$

if P_k is homogeneous of degree k. Thus I^s is the inverse of R^s. This simple expression of the inverse of R^s is the technical reason for which we use R instead of N. When $s = 1$ we will write simply I. It is easily checked that I commutes with the T_{ij} and the $\bar{T}_{ij}$.

We will consider differential operators X appearing as composition

$$Xf = X_1 \cdots X_k f$$

where each X_i is R, a T_{ij} or a $\bar{T}_{ij}$. For such an operator its *weight* is defined to be $\sum_1^k w(X_k)$, the weight of R being 1 and 1/2 the weight of each T_{ij} and $\bar{T}_{ij}$. The functions Xf are most easily estimated in terms of $d^k f(X_1, \ldots, X_k)$. This is the function whose value at $z \in B^n$ is

$$\frac{\partial^k f}{\partial v_1 \cdots \partial v_k}(z)$$

with $v_j = X_j(z)$. It is easily checked by induction that Xf can be written as a linear combination of $d^j f(Y_1, \ldots, Y_j)$ with $j \leqslant k$, and

$$\sum_1^j w(Y_i) \leqslant \sum_1^k w(X_i),$$

so that when proving estimates on Xf involving its weight we will instead estimate $d^k f(X_1, \ldots, X_k)$, which is simpler because the X_j are «frozen» at z.

One final remark is in order. As said before some of our results depend on the special properties that the $T_{ij} f$ have for an holomorphic f in the ball and thus do not generalize to other domains. It is probably the case that our results hold for more general domains, but this would require other methods, as maybe the «freezing coefficient» technique. The technical difficulty involved would be similar as the one encountered when trying to generalize the Fefferman-Stein paper on real H^p spaces to a general smooth domain.

2. On Functions Annihilated by Powers of $\bar\partial$

If f is holomorphic in B^n and $T_1, \ldots, T_k$ are complex-tangential vector fields then of course $F = T_1 \cdots T_k f$ is no longer holomorphic. Indeed if we fix $z_0 \in B^n$, $\zeta_0 \in \mathbb{C}^n$ then the function $u(\lambda) = F(z_0 + \lambda\zeta_0)$ need not be harmonic not even subharmonic. Nevertheless we will see that the function u does satisfy some mean value inequalities that we can exploit to obtain «non-isotropic» mean value estimates for the function F. It is clear that u does satisfy the differential equation $(\partial^k/\partial\bar\lambda^{k+1})u \equiv 0$.

Definition. *Let $U \subseteq \mathbb{C}$ be an open set; by $H_k(U)$ we mean the set of functions u defined on U so that $(\partial^k/\partial\bar\lambda^k)u(\lambda) \equiv 0$ in U. By $H_k^p(U)$ we mean those u in $H_k(U)$ so that*

$$\int_U |u|^p \, dm = \|u\|_p^k < \infty.$$

It is easily proved, by induction on k, that if $u \in H_k(U)$ and $\lambda_0 \in \mathbb{C}$, then u has a unique expression

$$u(\lambda) = \sum_{j=0}^{k-1} (\bar\lambda - \bar\lambda_0)^j f_j(\lambda)$$

where f_j is holomorphic in U.

Lemma 2.1. *Given non-negative integers l, m there is a constant $C = C(l, m)$ such that if $u \in H_k^1(D(\lambda_0, r))$, then*

$$\left| \frac{\partial^{l+m}}{\partial\lambda^l \partial\bar\lambda^m} u(\lambda_0) \right| \leqslant \frac{C}{r^{l+m+2}} \int_{D(\lambda_0, r)} |u(\lambda)| \, dm(\lambda).$$

PROOF. By a translation and dilation it is enough to prove this when $\lambda_0 = 0$ and $r = 1$. Moreover we may assume that $u \in H_k(D(0, R))$ for some $R > 1$. For a given l, m we will find a polynomial $p = p_{l,m}$ so that

$$\frac{\partial^{l+m}}{\partial\lambda^l \partial\bar\lambda^m} u(0) = \int_{|\lambda|<1} u(\lambda) p(\lambda) \, dm(\lambda).$$

The result will follow with

$$C = \sup_{|\lambda|<1} |p(\lambda)|.$$

As observed above we have

$$u(\lambda) = \sum_{j=0}^{k-1} \bar\lambda^j f_j(\lambda),$$

with f_j holomorphic. Clearly we may assume that $m \leqslant k-1$. Then

$$\frac{\partial^{l+m}}{\partial\lambda^l\,\partial\bar\lambda^m}u(0) = m!\,f_m^{(l)}(0).$$

We try

$$p(\lambda) = \lambda^m\bar\lambda^l \sum_{\nu=0}^{k-1} \alpha_\nu |\lambda|^{2\nu},$$

where the numbers α_ν are to be determined.

$$\int_{|\lambda|<1} u(\lambda)p(\lambda)\,dm(\lambda) = \sum_{j,\nu=0}^{k-1} \alpha_\nu \int \bar\lambda^j\lambda^m\bar\lambda^l|\lambda|^{2\nu}\alpha_\nu f_j(\lambda)\,dm(\lambda).$$

Each f_j has a power series expansion

$$f_j(\lambda) = \sum_{n=0}^{\infty} \hat f_j(n)\lambda^n.$$

If we insert this above we obtain, after using polar coordinates,

$$\sum_{j,\nu=0}^{k-1} \alpha_\nu \int_0^1 r^{1+j+m+l+2\nu} \int_0^{2\pi} e^{i(m-j-l)\theta} \sum \hat f_j(n) r^n e^{in\theta}\frac{d\theta}{2\pi}\,dr$$

$$= \sum_{j,\nu=0}^{k-1} \alpha_\nu \hat f_j(j+l-m) \int_0^1 r^{1+j+m+l+2\nu+j+l-m}\,dr$$

$$= \sum_{j=0}^{k-1}\left\{\sum_{\nu=0}^{k-1} \frac{\alpha_\nu}{2(1+j+l+\nu)}\right\}\hat f_j(j+l-m).$$

So, we wish to choose the numbers $\{\alpha_\nu\}$ so that

$$\sum_{\nu=0}^{k-1} \frac{\alpha_\nu}{1+j+\nu+l} = \begin{cases} 2m!\,l! & \text{when } j=m,\\ 0 & j\neq m.\end{cases}$$

For a fixed $l \geqslant 0$, the finite Hilbert matrix

$$(H_{j\nu}) = \left(\frac{1}{1+j+\nu+l}\right)_{j,\nu=0}^{k-1}$$

is always invertible so this is possible. The proof of the lemma is complete. □

As a simple consequence of the lemma we get a boundary growth estimate.

Corollary 2.2. *There is a constant $C = C(l,m)$ so that if $u \in H_k^1(U)$ and $\lambda_0 \in U$ then*

$$\left|\frac{\partial^{l+m}u}{\partial\lambda^l\,\partial\bar{\lambda}^m}(\lambda_0)\right| \leqslant \frac{C}{\delta^{k+l+2}}\|u\|_1$$

where $\delta = \operatorname{dist}(\lambda_0, \partial U)$.

PROOF. Just apply the lemma to a disc of radius δ about λ_0. $\square$

Next we prove an analogue of a well-known theorem of Hardy and Littlewood. In fact with the above lemma and its corollary we can use their proof. We include the details for completeness.

Lemma 2.3. *Suppose* $0 < p < \infty$; *then there is a constant* C_p *such that if* $u \in H_k(D(\lambda_0, r))$ *then*

$$|u(\lambda_0)|^p \leqslant \frac{C_p}{r^2}\int_{D(\lambda_0, r)} |u(\lambda)|^p\, dm(\lambda).$$

PROOF. The case $p = 1$ is just the lemma with $l = m = 0$. The case $p > 1$ follows from the case $p = 1$ by Hölder's inequality. For the case $0 < p < 1$ we may assume that

$$\lambda_0 = 0, \qquad r = 1 \quad \text{and} \quad u \in H_k(D(0, R)), \qquad R > 1.$$

Also assume that

$$\int_{|\lambda|<1} |u(\lambda)|^p\, dm(\lambda) = 1$$

and $|u(0)| > 1$. Similarly to [3] we let

$$m_p^p(r) = \int_{|\lambda|<r} |u(\lambda)|^p\, dm(\lambda)$$

and $m_\infty(r) = \sup_{|\lambda|\le r}|u(\lambda)|$. Since $0 < p < 1$,

$$\begin{aligned} m_1(r) &= \int_{|\lambda|<r} |u(\lambda)|\, dm(\lambda) \\ &\leqslant \int_{|\lambda|<r} |u(\lambda)|^p |u(\lambda)|^{1-p}\, dm(\lambda) \\ &\leqslant m_\infty^{1-p}(r) m_p^p(r) \\ &\leqslant m_\infty^{1-p}(r). \end{aligned}$$

Now if we apply Corollary 2.2 with $l = m = 0$ to $D(0, r)$ with $r < 1$ we obtain

$$m_\infty(\rho) \leqslant C\frac{m_1(r)}{(r-\rho)^2}, \qquad 0 < \rho < r < 1.$$

So we have

$$m_\infty(\rho) \leqslant C\frac{m_\infty^{1-p}(r)}{(r-\rho)^2}, \qquad 0<\rho<r<1.$$

We take $\rho = r^\alpha$ with $\alpha > 1$ to be determined, take logarithms and integrate

$$\int_{1/2}^1 \log m_\infty(r^\alpha)\frac{dr}{r} \leqslant \log C + 2\int_{1/2}^1 \log\frac{1}{(r-r^\alpha)}\frac{dr}{r} + (1-p)\int_{1/2}^1 \log m_\infty(r)\frac{dr}{r}.$$

Letting $t = r^\alpha$ in the left hand side and rearranging we get

$$\left[\frac{1}{\alpha}-(1-p)\right]\int_{1/2}^1 \log m_\infty(r)\,dr \leqslant \frac{1}{2}\log C + 2\int_{1/2}^1 \log\frac{1}{r-r^\alpha}\frac{dr}{r}.$$

Now choose α so close to 1 that the coefficient on the left hand side is positive. Then we get

$$\log m_\infty(1/2) \leqslant 4\int_{1/2}^1 \log m_\infty(r)\,dr \leqslant C = C_p. \quad \Box$$

Next we see that there is a version of Lemma 2.1 valid for all p.

Lemma 2.4. *If m, l are non-negative integers and $0<p<\infty$ are given then there is a constant C such that if $u \in H_k^1(D(\lambda_0, r))$ then*

$$\left|\frac{\partial^{l+m}}{\partial\lambda^l\partial\bar{\lambda}^m}u(\lambda_0)\right|^p \leqslant \frac{C}{r^{2+p(l+m)}}\int_{D(\lambda_0,r)} |u(\lambda)|^p\,dm(\lambda).$$

Proof. Let Du denote $\dfrac{\partial^{l+m}}{\partial\lambda^l\partial\bar{\lambda}^m}u$ and $\nu = l+m$. By Lemma 2.1, (the case $p=1$) we have

$$|Du(\lambda_0)| \leqslant \frac{C}{(r/2)^{2+\nu}}\int_{D(\lambda_0,r/2)} |u(\lambda)|\,dm(\lambda).$$

For each $\lambda \in D(\lambda_0, r/2)$ we have by Lemma 2.3

$$|u(\lambda)|^p \leqslant \frac{C}{(r/2)^2}\int_{D(\lambda,r/2)} |u(w)|^p\,dm(w) \leqslant \frac{C}{r^2}\int_{D(\lambda_0,r)} |u(w)|^p\,dm(w).$$

If we insert this into the above estimate for $Du(\lambda_0)$ we have

$$|Du(\lambda_0)| \leqslant \frac{C}{r^{2+\nu}} \int_{D(\lambda_0, r/2)} \left(\frac{1}{r^2} \int_{D(\lambda_0, r)} |u(w)|^p \, dm(w) \right)^{1/p} dm(\lambda)$$

$$= \frac{C}{r^{2/p+\nu}} \left(\int_{D(\lambda_0, r)} |u(\lambda)|^p \, dm(\lambda) \right)^{1/p},$$

which completes the proof of the lemma. □

We want to point out that what the above arguments actually show is the following: Suppose u is defined in a domain U and there is a constant C such that whenever $D(\lambda_0, r) \subseteq U$ we have

$$|u(\lambda_0)| \leqslant \frac{C}{r^2} \int_{D(\lambda_0, r)} |u(\lambda)| \, dm(\lambda).$$

Then for each $p, 0 < p < \infty$, there is a constant C_p such that

$$|u(\lambda_0)|^p \leqslant \frac{C_p}{r^2} \int_{D(\lambda_0, r)} |u(\lambda)|^p \, dm(\lambda)$$

whenever $D(\lambda_0, r) \subseteq U$.

We let $H_k(B^n)$ be the class of C^∞ functions in B^n that are annihilated by $\bar{\partial}^k$ on every complex line. More precisely we say that $u \in H_k(B^n)$ if for every $z \in B^n$ and every $0 \neq \zeta \in \mathbb{C}^n$ we have

$$\frac{d^k}{d\bar{\lambda}^k} u(z + \lambda\zeta) \equiv 0.$$

Let u be a C^∞ function in B^n. Take $z \in B^n$, and $\zeta \in \mathbb{C}^n$. Then we may calculate that

$$\frac{d^k}{d\bar{\lambda}^k} [u(z + \lambda\zeta)]_{\lambda=0} = \sum_{|\beta|=k} \bar{D}^\beta u(z) \bar{\zeta}^\beta.$$

From this we deduce that a C^∞ function u lies in $H_k(B^n)$ if and only if $\bar{D}^\beta u = 0$ in B^n for all multi-indexes β with $|\beta| = k$. If follows that if $u \in H_k(B^n)$ then $\bar{D}^\beta D^\gamma u \in H_k(B^n)$ for any β and γ.

In the next lemma, for $z \in B^n$ and $\delta > 0$ the polydisc $P(z, \delta)$ is defined as follows. If $z = r\zeta$, pick $\zeta_2, \ldots, \zeta_n$ so that $\{\zeta, \zeta_2, \ldots, \zeta_n\}$ is an orthonormal basis for $\mathbb{C}^n$. Then

$$P(z, \delta) = \left\{ w = r\zeta + \lambda\zeta + \sum_{j=2}^{n} \lambda_j \zeta_j : |\lambda| < \delta, |\lambda_j| < \delta^{1/2}, j = 2, \ldots, n \right\}.$$

Lemma 2.5. *Suppose $0 < p < \infty$, $1 \leqslant k$ and a differential operator X of weight m are given. Then there is a constant C such that*

$$|Xu(z)|^p \leqslant \frac{C}{\delta^{n+1+mp}} \int_{P(z,\delta)} |u(w)|^p \, dm(w)$$

for all $u \in H_k(B^n)$, as long as $P(z, \delta) \subset B^n$.

PROOF. For convenience we assume $z = (r, 0, \dots, 0)$. By the discussion in the introduction it is enough to obtain the above estimate with Xu replaced by

$$\frac{\partial^N}{\partial z_1^{k_1} \partial \bar{z}_1^{l_1} \cdots \partial z_n^{k_n} \partial \bar{z}_n^{l_n}} u(\lambda, 0)$$

where

$$\sum_{j=1}^{n} (k_j + l_j) = N$$

and

$$k_1 + l_1 + \frac{1}{2} \sum_{j=2}^{n} (k_j + l_j) \leqslant m.$$

Recalling that if $u \in H_k(B^n)$ then the same is true of any partial derivative of u we may apply Lemma 2.4 successively, one variable at a time, to obtain the desired result. □

3. Characterizations of H_s^p in Terms of Area Functions

The main purpose of this section is to obtain characterizations of H_s^p in terms of area functions, some of them involving only complex-tangential derivatives. Our starting point for this and next section is the following account of different characterizations of H^p itself, already mentioned in the introduction.

Theorem 3.1 ([8]). *The following are equivalent, (with an aperture α fixed):*

(*a*) $f \in H^p$.
(*b*) $A_\alpha(f) \in L^p(S)$.
(*c*) $g(f) \in L^p(S)$.
(*d*) $f_\alpha^* \in L^p(S)$.
(*e*) $f_\alpha^+ \in L^p(S)$.

The Theorem is well-known. The equivalence of (*a*), (*d*) and (*e*) is Koranyi's result. The conditions (*b*) and (*c*) are also known (see [8]), though a detailed proof seems to be lacking in the literature. A proof can be obtained by adapting the methods of Fefferman and Stein. This requires as in [3] the use of a version of Green's theorem for regions that appear as unions of admissible aproach regions, particularly in the equivalence of (*b*) and (*d*). We point out that for the particular case of the ball an easy proof can be obtained as follows. That (*b*) implies (*c*) is a consequence of a pointwise estimate (see Lemma 4.3 below), (*c*) implies (*a*) can be obtained by slice integration of the same result in dimension 1 and (*a*) $\Rightarrow$ (*b*) can be proved using the atomic decomposition of Garnett and Latter ([4]).

To state our result we need to introduce some more notations. For an operator X as in the introduction

$$Xf = X_1 \cdots X_k f$$

and $\gamma > 0$ we define the area function

$$A^X_{\alpha,\gamma} f(\eta) = \left\{ \int_{D_\alpha(\eta)} |Xf(z)|^2 (1 - |z|)^{2\gamma - 1 - n} \, dm(z) \right\}^{1/2}.$$

We will consider in particular operators $T = T_1 \cdots T_k$ where the $T_1, \ldots, T_k$ are chosen among the T_{ij} (not the $\bar{T}_{ij}$'s). We denote by $\{T_\delta\}$, $\delta \in C_k$, the collection of such operators and define the *k-th complex-tangential gradient* as

$$\nabla^k_T f = \sum_{\delta \in C_k} |T_\delta f|.$$

In this section we will prove

Theorem 3.2. *Let $f \in H^p_s$ and let X be a differential operator as above of weight $m > s$. Then $A^X_{\alpha, m-s} f \in L^p(S)$ for any α.*

Theorem 3.3. *If $l \geqslant 0$ and $k \in Z$ are such that $l + k/2 > s$, then $f \in H^p_s$ if and only if*

$$\left\{ \int_{D_\alpha(\eta)} |\nabla^k_T R^l f(z)|^2 (1 - |z|)^{2l + k - 2s - 1 - n} \, dm(z) \right\}^{1/2} \in L^p(S).$$

We will need a version of Hardy's inequality for which we have not found a reference so we include it here.

Lemma 3.4. *If $\alpha, \beta > 0$, $1 \leqslant p < \infty$ there is a constant C so that for all $f \geqslant 0$ we have,*

$$\int_0^1 (1 - r)^{\alpha - 1} \left(\int_0^r (r - t)^{\beta - 1} f(t) \, dt \right)^p dr \leqslant C \int_0^1 (1 - r)^{\alpha - 1 + \beta p} f(r)^p \, dr.$$

PROOF. We give a sketch. After changing variables the left hand side of the above becomes

$$\int_0^1 r^{\alpha-1}\left(\int_r^1 (t-r)^{\beta-1} f(1-t)\,dt\right)^p dr.$$

Letting $t = rs$, this becomes

$$\int_0^1 r^{\alpha-1}\left(\int_1^{1/r} r^{\beta}(s-1)^{\beta-1} f(1-rs)\,ds\right)^p dr.$$

If we now use the continuous form of Minkowski's inequality and change variables again we have that the above is at most

$$\left[\int_1^{\infty} \frac{(x-1)^{\beta-1}}{x^{\beta+\alpha/p}}\,dx\right]^p \int_0^1 (1-r)^{\alpha-1+p\beta} f(r)^p\,dr.$$

The integral occurring in the first factor is convergent since $\alpha, \beta > 0$. $\square$

In the next geometric lemma, for an approach region $D_\alpha(\eta)$ and $0 < r < 1$, $S_\alpha(r, \eta)$ denotes the region

$$S_\alpha(r,\eta) = \left\{z \in D_\alpha(\eta): \frac{1}{2}(1-r^2) < 1 - |z|^2 < 2(1-r^2)\right\}.$$

Lemma 3.5. *For each α, β, $1 < \alpha < \beta$, there is an $\epsilon = \epsilon(\alpha, \beta)$ such that if $\eta \in S$ and $z = r\zeta \in D_\alpha(\eta)$ then the polydisc $P(z, \epsilon(1-r^2))$ (as defined before Lemma 2.5) is contained in $S_\beta(r, \eta)$.*

PROOF. Take

$$w = (r+\lambda)\zeta + \sum_{j=2}^{n} \lambda_j \zeta_j \in P(z, \delta),\ \delta = \epsilon(1-r^2);$$

then

$$|w|^2 = |r+\lambda|^2 + \sum_{j=2}^{n} |\lambda_j|^2 \leqslant r^2 + |\lambda|^2 + 2|\lambda| + (n-1)\delta \leqslant r^2 + (n+2)\delta.$$

Also

$$|w|^2 \geqslant (r - |\lambda|)^2 = r^2 - 2r|\lambda| + |\lambda|^2 \geqslant r^2 - 2|\lambda| \geqslant r^2 - 2\delta.$$

So

$$(1-r^2) - (n+2)\delta \leqslant 1 - |w|^2 \leqslant (1-r^2) + 2\delta$$

so we have

$$\frac{1}{2}(1 - r^2) \leqslant 1 - |w|^2 \leqslant 2(1 - r^2)$$

for sufficiently small ϵ. Next we calculate

$$\begin{aligned} |1 - \langle w, \eta\rangle| &= \left|1 - r\langle \zeta, \eta\rangle + \lambda\langle \zeta, \eta\rangle + \sum_{j=2}^{n} \lambda_j \langle \zeta_j, \eta\rangle\right| \\ &\leqslant \frac{\alpha}{2}(1 - r^2) + \delta + \sum_{j=2}^{n} \delta^{1/2}|\langle \zeta_j, \eta\rangle| \\ &= \frac{\alpha}{2}(1 - r^2) + \delta + \sum_{j=2}^{n} \delta^{1/2}|\langle \zeta_j, \eta - r\zeta\rangle| \\ &\leqslant \frac{\alpha}{2}(1 - r^2) + \delta + n\delta^{1/2}|\eta - r\zeta| \\ &\leqslant \frac{\alpha}{2}(1 - r^2) + \delta + n\delta^{1/2}\sqrt{2}\,|1 - r\langle \zeta, \eta\rangle|^{1/2} \\ &\leqslant \frac{\alpha}{2}(1 - r^2) + \delta + n\sqrt{2}\left(\frac{\alpha}{2}(1 - r^2)\right)^{1/2}\delta^{1/2} \\ &= \left(\frac{\alpha}{2} + \epsilon + n\alpha^{1/2}\epsilon^{1/2}\right)(1 - r^2). \end{aligned}$$

Also from the first part of the proof

$$\frac{\beta}{2}(1 - |w|^2) \geqslant \frac{\beta}{2}[1 - (n + 2)\epsilon](1 - r^2)$$

and so

$$|1 - \langle w, \eta\rangle| < \frac{\beta}{2}(1 - |w|^2)$$

for all $w \in P(z, \epsilon(1 - r^2))$ for ϵ sufficiently small. □

Lemma 3.6. *Fix* $1 < \alpha < \beta$, $q \geqslant 1$, $l \geqslant 0$, $\gamma > 0$ *and let* X *be an operator of weight* m. *Then there is a constant* C *such that for all* $f \in H_q(B^n)$ *and all* $\eta \in S$ *we have*

$$(3)\quad \int_{D_\alpha(\eta)} |I^l X f(z)|^2 (1 - |z|)^{2\gamma - n - 1}\, dm(z) \leqslant C \int_{D_\beta(\eta)} |f(z)|^2 (1 - |z|)^{2(\gamma + l - m) - n - 1}\, dm(z).$$

PROOF. As already mentioned, the proof follows the same lines as in [7, pp. 214-216]. Combining Lemma 2.5 with Lemma 3.5 we see that if $r\zeta \in D_\alpha(\eta)$ then

$$(4) \qquad |Xf(r\zeta)| \leqslant \left\{ \frac{c}{(1-r)^{n+1+2m}} \int_{S_\beta(r,\eta)} |f(w)|^2 \, dm(w) \right\}^{1/2} = J_r.$$

Assume $l > 0$. In this case,

$$|I^l Xf(r\zeta)| \leqslant \frac{c}{r} \int_0^r (r-t)^{l-1} J_t \, dt, \qquad r > 0.$$

Having obtained a bound for $I^l Xf(r\zeta)$ which just depends on r, we integrate in polar coordinates in $D_\alpha(\eta)$, using the fact that for fixed r,

$$\sigma\{\zeta \in S : r\zeta \in D_\alpha(\eta)\} \leqslant c(1-r)^n.$$

This gives that the left hand side of (3) is bounded by

$$C \int_0^1 (1-r)^{2\gamma-1} \left(\int_0^r (r-t)^{l-1} J_t \, dt \right)^2 dr$$

and, using Lemma 3.4, by

$$(5) \qquad C \int_0^1 (1-r)^{2\gamma-1+2l} J_r^2 \, dr.$$

Inserting the definition of J_r we obtain the bound

$$\int_0^1 (1-r)^{2(\gamma+l-m-1)-n} \int_{S_\beta(r,\eta)} |f(w)|^2 \, dm(w) \, dr.$$

If $w \in S_\beta(r,\eta)$, then $1-|w|$ is comparable to $1-r$ hence the above integral is dominated by

$$\int_{D_\beta(\eta)} (1-|z|)^{2(\gamma+l-m)-n-1} |f(z)|^2 \, dm(z).$$

For $l = 0$ we use the same argument to pass directly from (4) to (5). □

PROOF OF THEOREM 3.2. If X is as in Theorem 3.2, we apply Lemma 3.6 to $R^{1+s}f$ instead of f, $l = 1+s$ and $\gamma = m-s$ to obtain (recall that I^l commutes with the T_{ij})

$$A^X_{\alpha, m-s} f(\eta) \leqslant c A^{1+s}_{\beta,1} f(\eta) = c A_\beta(R^s f)(\eta).$$

By Theorem 3.1, $A_\beta(R^s f) \in L^p(S)$ and so part (*a*) is proved. □

PROOF OF THEOREM 3.3. Now, taking in Lemma 3.6 as X powers of R it is easily checked by symmetry the following

Corollary 3.7. *For $t \in \mathbb{R}$ and $\gamma > 0$, recall the notation*

$$A^t_{\alpha,\gamma} f(\eta) = \left\{ \int_{D_\alpha(\eta)} |R^t f(z)|^2 (1 - |z|)^{2\gamma - n - 1} \, dm(z) \right\}^{1/2}.$$

(where $R^t = I^{-t}$ if $t < 0$). Then if, $t, l \in \mathbb{R}$ and $\gamma, r > 0$ are such that $l - t = r - \gamma$, the area functions $A^t_{\alpha,\gamma} f$ and $A^l_{\alpha,r} f$ are pointwise equivalent, up to replacement of the apertures.

Now, by Theorem 3.1, $f \in H^p_s$ is equivalent to $A_\alpha(R^s f) = A^{1+s}_{\alpha,1}(f)$ being in $L^p(S)$. By Corollary 3.7, it follows that $f \in H^p_s$ if and only if $A^l_{\alpha,r} f \in L^p(S)$, for each α and l, τ, $\tau > 0$, with $l - \tau = s$, which proves Theorem 3.3 in case $k = 0$. To finish the proof of Theorem 3.3 we must show that in this characterization we can replace $R^{k/2}$ by $T_1 \cdots T_k$. This will be a consequence of

Theorem 3.8. *Fix $\gamma > 0$ and let E be the space of all functions such that*

$$\left\{ \int_{D_\alpha(\eta)} |f(z)|^2 (1 - |z|)^{2\gamma - n - 1} \, dm(z) \right\}^{1/2} \in L^p(S)$$

for all α. Let k be any positive integer. Then, if $f \in H_q(B^n)$ for some q, f belongs to E if and only $\nabla^k_T I^{k/2} f \in E$.

Note that we just need this lemma for an holomorphic f, but one is lead to the above statement by the non-analyticity of the $T_1 \cdots T_k I^{k/2} f$.

Choosing $X = T_1 \cdots T_k$ and $I^l = I^{k/2}$ in Lemma 3.6, we obtain that $f \in E$ implies $T_1 \cdots T_k I^{k/2} f \in E$. For the reverse implication, we need in turn several Lemmas. In the first one, $H(p, q)$ denotes the space of harmonic polynomials of bidegree (p, q).

Lemma 3.9. *$\sum \bar{T}_{ij} T_{ij} = -2p(q + n - 1) Id$ on $H(p, q)$, the sum being extended over all pairs (i, j) $1 \leqslant i, j \leqslant n$.*

PROOF. It is convenient for the proof *not* to exclude the case $i = j$, even though $T_{ii} = 0$. With this in mind we calculate that

$$\bar{T}_{ij} T_{ij} = |z_i|^2 \frac{\partial^2}{\partial z_j \partial \bar{z}_j} + |z_j|^2 \frac{\partial^2}{\partial z_i \partial \bar{z}_i} - z_i \bar{z}_j \frac{\partial^2}{\partial z_i \partial \bar{z}_j} - \bar{z}_i z_j \frac{\partial^2}{\partial \bar{z}_i \partial z_j}$$
$$- z_i \frac{\partial}{\partial z_i} - z_j \frac{\partial}{\partial z_j} + \delta_{ij} \left[z_i \frac{\partial}{\partial z_i} + z_j \frac{\partial}{\partial z_j} \right].$$

Now if we add on *all* i and j we have

$$2|z|^2 \Delta - 2 \sum_{i,j=1}^n z_i \bar{z}_j \frac{\partial^2}{\partial z_i \partial \bar{z}_j} - 2n \sum_{i=1}^n z_i \frac{\partial}{\partial z_i} + 2 \sum_{i=1}^n z_i \frac{\partial}{\partial z_i}.$$

Now if we apply this to a function in $H(p, q)$ the part involving Δ is zero so we need only check the action of the rest of the terms on an element of $H(p, q)$. Now each element of $H(p, q)$ is a linear combination of terms of the form $z^\alpha \bar{z}^\beta$ where $|\beta| = p$, $|\beta| = q$. Applying the terms not involving Δ to $z^\alpha \bar{z}^\beta$ yields

$$\left\{-2 \sum_{i,j=1}^{n} \alpha_i \beta_j - 2(n-1) \sum_{i=1}^{n} \alpha_i\right\} z^\alpha \bar{z}^\beta = -2(|\alpha|\,|\beta| + 2(n-1)|\alpha|) z^\alpha \bar{z}^\beta$$
$$= -2p(q+n-1) z^\alpha \bar{z}^\beta. \quad \square$$

In the statement of next Lemma, if $T_\delta = T_i \cdots T_1$ we define $\overline{T_\delta} = \overline{T_k} \cdots \overline{T_1}$.

Lemma 3.10.

$$\tau_k = \sum_{\delta \in C_k} \bar{T}_\delta T_\delta = (-2)^k \frac{p!}{(p-k)!} \frac{(q+k+n-2)!}{(q+n-2)!} \quad on \quad H(p, q).$$

PROOF. We do induction on k. The case $k = 1$ is just Lemma 3.9.
Take $u \in H(p, q)$, then

$$\tau_k u = \sum_{\delta \in C_{k-1}} \bar{T}_\delta \sum_{i,j} \bar{T}_{ij} T_{ij} T_\delta u.$$

Now $T_\delta u \in H(p-k+1, q+k-1)$ and so

$$\sum_{i,j} \overline{T_{ij}} T_{ij} T_\delta u = -2(p-k+1)(q+k+n-1) T_\delta u,$$

by Lemma 3.9. We see that

$$\tau_k u = -2(p-k+1)(q+k+n-1)\tau_{k-1} u$$
$$= (-2)^{k-1}(-2) \frac{p!}{p-(k-1))!} \frac{(q+k+n-2)!}{(q+n-2)!} (p-k+1)(q+k)u$$
$$= (-2)^k \frac{p!}{(p-k)!} \frac{(q+k+n-2)!}{(q+n-2)!} u$$

as promised. $\square$

Lemma 3.11. *For f holomorphic in B^n we have*

$$\sum_{\delta \in C_k} I^{k/2} \bar{T}_\delta T_\delta I^{k/2} = \sum_{j=0}^{k} d_j I^j f$$

for some constants $d_0, \ldots, d_k$ depending only on k, $d_0 \neq 0$.

PROOF. Take an holomorphic monomial $z^\alpha \in H(p,0)$, $p = |\alpha| > k$. Then

$$\tau_k z^\alpha = (-2)^k \frac{p!}{(p-k)!} \frac{(k+n-2)!}{(n-2)!} z^\alpha = c_k \frac{p!}{(p-k)!} z^\alpha.$$

Now $p!/(p-k)!$ is a polynomial of degree k in p whose coefficients depend only on k. If we express this polynomial as a function of $p+1$ we obtain

$$\tau_k z^\alpha = C_k(p+1)^k z^\alpha + C_{k-1}(p+1)^{k-1} z^\alpha + \cdots + C_0 z^\alpha = \left(\sum_{j=0}^{k} C_j R^j\right) z^\alpha.$$

This shows that

$$\tau_k f = \sum_{j=0}^{k} c_j R^j f, \qquad c_k \neq 0$$

for an holomorphic function f. We now just apply I^k to both sides, using again the fact that the I^k commute with the T_δ. □

Lemma 3.12. *Let*

$$P(x) = \sum_{j=1}^{k} c_j x^j$$

be a polynomial and denote

$$P(I)f = \sum_{j=1}^{k} c_j I^j f.$$

Then there are c, r_0 depending on α and P, c, $r_0 < 1$ such that for f holomorphic

$$\begin{aligned}\int_{D_\alpha(\eta)} |f(z)|^2(1-|z|)^{2\gamma-n-1}\,dm(z) &\leq c\int_{D_\alpha(\eta)} |f+P(I)f(z)|^2(1-|z|)^{2\gamma-n-1}\,dm(z)\\ &\quad + c\sup\{|f(z)|^2 : |z| \leq r_0\}.\end{aligned}$$

PROOF. Note that $\{f : f + P(I)f = 0\}$ is a finite dimensional subspace, hence all norms are equivalent on it and thus it really does not matter what norm is considered in the last term. We will prove that

$$(6)\quad \int_{D_\alpha(\eta)} |If(z)|^2(1-|z|)^{2\gamma-n-1}\,dm(z) \leq c\int_{D_\alpha(\eta)} |f(z)|^2(1-|z|)^{2\gamma-n}\,dm(z).$$

This implies by iteration that associated to P there is a constant M such that

$$\int_{D_\alpha(\eta)} |P(I)f(z)|^2(1-|z|)^{2\gamma-n-1}\,dm(z) \leq M\int_{D_\alpha(\eta)} |f(z)|^2(1-|z|)^{2\gamma-n}\,dm(z)$$

Hence

$$\begin{aligned}\int_{D_\alpha(\eta)} |f(z)|^2(1-|z|)^{2\gamma-n-1}\,dm(z) \\ \leqslant c\int_{D_\alpha(\eta)} |f+P(I)f(z)|^2(1-|z|)^{2\gamma-n-1}\,dm(z) \\ + Mc\sup\{|f(z)|^2: |z|\leqslant r_0\} \\ + M(1-r_0)\int_{D_\alpha(\eta)} |f(z)|^2(1-|z|)^{2\gamma-n-1}\,dm(z).\end{aligned}$$

and the lemma will follow choosing r_0 such that $M(1-r_0)<1$.

Note that (6) is similar to (3) but the aperture of the admissible approach region is not changed in (6). To prove it, we use polar coordinates in $D_\alpha(\eta)$ and

$$|If(r\zeta)|^2 \leqslant \frac{1}{r}\int_0^r |f(t\zeta)|^2\,dt$$

to bound the left hand side of (6) by

$$\int_0^1 (1-r)^{2\gamma-n-1}r^{2n-2}\,dr\int_{A(r,\eta)} d\sigma(\zeta)\int_0^r |f(t\zeta)|^2\,dt.$$

Here $A(r,\eta)=\{\zeta\in S: r\zeta\in D_\alpha(\eta)\}$. Now we use Fubini's theorem: denoting

$$I(t,r)=\int_{A(r,\eta)} |f(t\zeta)|^2\,d\sigma(\zeta)$$

and using that $I(t,r)\leqslant I(t,t)$ for $t\leqslant r$ (because $A(r,\eta)\subset A(t,\eta)$), we obtain the bound

$$\int_0^1 (1-r)^{2\gamma-n-1}\,dr\int_0^r I(t,t)\,dt = c\int_0^1 I(t,t)(1-t)^{2\gamma-n}\,dt. \tag{7}$$

On the other hand by the same argument, the right hand side equals

$$c\int_0^1 (1-t)^{2\gamma-n}t^{2n-1}I(t,t)\,dt.$$

Thus the part of (7) corresponding to the integral for t far from 0 is right. The other part, for $t\leqslant\epsilon$, is harmless: we bound it by $\sup\{|f(z)|^2: |z|\leqslant\epsilon\}$, that is in turn bounded by the right-hand side of (6), choosing ϵ so that $D_\alpha(\eta)$ contains the ball $|z|\leqslant 2\epsilon$. □

As a consequence of the above, we note that if

$$N=\sum_j z_j\frac{\partial}{\partial z_j}$$

is the true radial derivative, then, since $Nf = Rf - f = Rf - IRF$, we see

$$\int_{D_\alpha(\eta)} |Nf(z)|^2(1-|z|)^{2\gamma-n-1}\,dm(z) \leqslant c\int_{D_\alpha(\eta)} |Rf(z)|^2(1-|z|)^{2\gamma-n-1}\,dm(z)$$
$$\leqslant c\int_{D_\alpha(\eta)} |Nf(z)|^2(1-|z|)^{2\gamma-n-1}\,dm(z) + \sup\{|f(z)|: |z| \leqslant r_0\}.$$

(The first inequality is elementary and the second is a particular case of Lemma 3.12.) Thus, it does not matter to use R or N in the definitions of the area functions.

END OF PROOF OF THEOREM 3.8 (AND THEOREM 3.3). We have to show that if $T_\delta I^{k/2} f \in E$ for all $\delta \in C_k$ then $f \in E$. Applying Lemma 3.6 to $T_\delta I^{k/2} f$ (this is why we need Lemma 3.6 for functions more general than holomorphic) $X = \bar{T}_\delta$, $l = k/2$, we obtain

$$\int_{D_\alpha(\eta)} |I^{k/2}\bar{T}_\delta T_\delta I^{k/2} f(z)|^2(1-|z|)^{2\gamma-n-1}\,dm(z)$$
$$\leqslant C\int_{D_\alpha(\eta)} |\nabla_T^k I^{k/2} f(z)|^2(1-|z|)^{2\gamma-n-1}\,dm(z).$$

Adding on $\delta \in C_k$, using Lemma 3.11 and then Lemma 3.12 the proof of Theorem 3.8 is finished. □

4. Characterizations of H_s^p in Terms of Littlewood-Paley Functions and Maximal Functions

In this section we will obtain criteria for $f \in H_s^p$ is terms of g-functions and maximal functions, some of them just depending on complex-tangential derivatives.

For an operator X and $\gamma > 0$ we consider the g-function

$$g_\gamma^X f(\eta) = \left\{\int_0^1 |Xf(r\eta)|^2(1-r)^{2\gamma-1}\,dr\right\}^{1/2}.$$

In this section we will prove results corresponding to Theorems 3.2 and 3.3.

Theorem 4.1. *If $f \in H_s^p$ and X has weight $m > s$, then $g_{m-s}^X f \in L^p(S)$.*

Theorem 4.2. *The following are equivalent*:

(*a*) $f \in H_s^p$.

(*b*) $\left\{\int_0^1 |\nabla_T^k R^l f(r\eta)|^2(1-r)^{2l+k+2s-1}\,dr\right\}^{1/2} \in L^p(S)$, *for some* l, k, $l + k/2 > s$ (*and hence for all*).

(*c*) $(\nabla_T^k I^{k/2} R^s f)^+ \in L^p(S)$, $k \leqslant 2s$.
(*d*) $(\nabla_T^k I^{k/2} R^s f)^* \in L^p(S)$, $k \leqslant 2s$.
(*e*) $\int_S |\nabla_T^k I^{k/2} R^s f(r\eta)|\, d\sigma(\eta) \leqslant C, \quad 0 \leqslant r \leqslant 1$.

The proof of Theorem 4.1 (and part (*a*) $\Rightarrow$ (*b*) of Theorem 4.2) is a consequence of the next Lemma and Theorem 3.2.

Lemma 4.3. *Let $f \in H_q(B^n)$ for some q, let $\gamma > 0$, and let α be an aperture. Then*

$$\int_0^1 |f(r\eta)|^2 (1-r)^{2\gamma-1}\, dr \leqslant c \int_{D_\alpha(\eta)} |f(z)|^2 (1-|z|)^{2\gamma-1-n}\, dm(z).$$

PROOF. We combine Lemma 2.5 (with $p = 2$) with Lemma 3.5 to obtain

$$|f(r\eta)|^2 \leqslant \frac{c}{(1-r)^{n+1}} \int_{S_\alpha(r,\eta)} |f(z)|^2\, dm(z) = c(1-r)^{-n-1} J_r.$$

Recall that

$$S_\alpha(r,\eta) = \left\{ z \in D_\alpha(\eta): \frac{1}{2}(1-r^2) < 1-|z|^2 < 2(1-r^2) \right\}.$$

Hence

$$\int_0^1 |f(r\eta)|^2 (1-r)^{\gamma+n}\, dr \leqslant c \int_0^1 (1-r)^{\gamma-1} J_r\, dr.$$

and Fubini's theorem finishes the proof. □

That (*c*) implies (*d*) follows from the next Lemma applied to each $T_\delta I^{k/2} R^s f$, $\delta \in C_k$.

Lemma 4.4. *Let $u \in H_q(B^n)$ for some q. Then $\|u_\alpha^*\|_p \leqslant c \|u^+\|_p$.*

PROOF. The proof is very much alike the corresponding real variable result in [3], and depends on the Hardy-Littlewood result, Lemma 2.5. For $z \in D_\alpha(\eta)$ let P_z be a polydisc as in Lemma 3.5, with any $\beta > \alpha$. By Lemma 2.5,

$$|u(z)|^{p/2} \leqslant c(1-r)^{-n-1} \int_{P_z} |u(w)|^{p/2}\, dm(w).$$

If we use now polar coordinates, $w = r\zeta$, and bound $|u(w)|$ by $u^+(\zeta)$ we obtain

$$|u(z)|^{p/2} \leqslant c(1-r)^{-n-1} \int_{\{r\zeta \in P_z\}} |u^+(\zeta)|^{p/2}\, dr\, d\sigma(\zeta).$$

If $K(\eta, r)$ denotes the Koranyi ball $K(\eta, r) = \{\zeta \in S: |1 - \bar{\zeta}\eta| < r\}$, the projection of P_z on S is contained in $K(\eta, c(1-r))$, because $z \in D_\alpha(\eta)$, for some constant c. Hence

$$|u(z)|^{p/2} \leqslant c(1-r)^{-n} \int_{K(\eta, c(1-r))} |u^+(\zeta)|^{p/2} \, d\sigma(\zeta).$$

Since $\sigma(K(\eta, c(1-r))) \simeq (1-r)^n$, this shows that

$$|u(z)|^{p/2} \leqslant cM[(u^+)^{p/2}](\eta).$$

i.e. $(u^*)^{p/2} \leqslant cM[(u^+)^{p/2}]$, where M denotes the Hardy-Littlewood maximal operator (defined with Korányi balls). The fact that M is bounded in L^2 then finishes the proof. □

For the remaining implications we need to recall some facts from the real variable theory of Hardy spaces, which we state as follows for the reader's convenience.

Lemma 4.5. *Let u be harmonic in B^n and for $\ell \in \mathbb{R}$, $\gamma > 0$ recall the notation*

$$g_\gamma^\ell u(\eta) = \left\{ \int_0^1 |R^\ell u(r\eta)|^2 (1-r)^{2\gamma - 1} \, dr \right\}^{1/2}.$$

Then, if $\ell, m \in \mathbb{R}$, $\gamma, r > 0$ and $m - \tau = \ell - \gamma$, the conditions $g_\gamma^\ell u \in L^p(S)$ and $g_\tau^m u \in L^p(S)$ are equivalent. In case $\ell = \gamma$, they are equivalent to $u^+ \in L^p(S)$.

PROOF. The proof depends on a result like Corollary 3.7 for the area functions in the real variable theory

$$S_{\alpha,\gamma}^\ell u(\eta) = \left\{ \int_{\Gamma_\alpha(\eta)} |R^\ell u(z)|^2 (1-|z|)^{2(\gamma - n)} \, dm(z) \right\}^{1/2}, \qquad \gamma > 0.$$

Here $\Gamma_\alpha(\eta)$ is the cone of aperture α and vertex η. Namely, for $\gamma, r > 0$ and $m - r = \ell - \gamma$, $S_{\alpha,\gamma}^\ell$ and $S_{\alpha,\tau}^m$ are pointwise equivalent up to replacement of the apertures. This is proved as in [7, pp. 214-216] (the proof of Lemma 3.6 was precisely an adaptation of this).

Now, there is a pointwise estimate

$$g_\gamma^\ell u(\eta) \leqslant c_\alpha S_{\alpha,\gamma}^\ell u(\eta)$$

proved as in Lemma 4.3. On the other hand, the same argument as in [3, Cor. 3 p. 171], but usig the Hilbert space $L^2(0, 1)$ with the measure $(1-r)^\gamma \, dr$, shows that if $g_\gamma^\ell u \in L^p(S)$ then $S_{\alpha,\gamma+1}^{\ell+1} u \in L^p(S)$, which proves the first part of the Lemma. (We remark that in this case there are no pointwise estimates among the g_γ^ℓ.)

In case $\ell = \gamma$, one can assume $\ell = \gamma = 1$ and then we have the situation of the real Hardy spaces described in [3]. □

PROOF OF $(b) \Rightarrow (c)$. According to Lemma 4.5, condition (b) is equivalent to (recall that $T_1 \cdots T_k R^\ell f$ are harmonic)

$$\left\{\int_0^1 |R^q \nabla_T^k R^\ell f(r\eta)|^2 (1-r)\, dr\right\}^{1/2} \in L^p(S)$$

with $2q - 1 = 1 + 2s - 2\ell - k$, *i.e.* $q = 1 + s - \ell - k/2$. Writing

$$R^q \nabla_T^k R^\ell = R I^{1-q} \nabla_T^k R^\ell = R \nabla_T^k I^{k/2} R^s$$

(note that $q - 1 < 0$ and recall that I^ℓ commutes with the T_δ) and applying last part of Lemma 4.5 we conclude that (c) holds.

For the implication $(d) \Rightarrow (a)$ we need

Lemma 4.6. *Let u be an harmonic function in the ball and assume that $u_\alpha^* \in L^p(S)$. Then, if $\beta < \alpha$, and*

$$A_\beta u(\eta) = \left\{\int_{D_\beta(\eta)} |Ru(z)|^2 (1 - |z|)^{1-n}\, dm(z)\right\}^{1/2}$$

then $A_\beta u \in L^p(S)$.

PROOF. This is proved exactly as in the first part of [3, Thm. 8], but replacing the regions appearing as unions of cones by regions which are unions of admissible approach ones, and using Green's theorem (also as in [7]). □

We notice that the lemma has somewhat an unnatural statement, because the notion of harmonicity, a real variable one, is mixed with complex variable notions such as the one of admissible approach region. It simply says that among harmonic functions, maximal functions control area functions. The converse would not be true for a general harmonic function. The lemma is most easily understood when $p = 2$. In this case it does no matter which kind of area function, the non-tangential Su or the admissible Au is used, because as it is easily checked their L^2-norms are comparable to the L^2-norm of u if u is harmonic, and their finiteness is thus equivalent to the non-tangential maximal function being in L^2. So it is clear that $u_\alpha^* \in L^2$ is a strictly stronger condition.

PROOF OF $(d) \Rightarrow (a)$. By the lemma just proved, from (d) we conclude that

$$\left\{\int_{D_\beta(\eta)} |R \nabla_T^k I^{k/2} R^s f(z)|^2 (1 - |z|)^{1-n}\, dm(z)\right\}^{1/2}$$

belongs to $L^p(S)$. For $\delta \in C_k$, $RT_\delta I^{k/2}R^s = T_\delta R^{1+s-k/2}$ and thus we get that the condition in Theorem 3.3 holds with $\ell = 1 + s - k/2$, and hence $f \in H^p_s$.

It remains to prove the equivalence of (*c*) and (*e*). Obviously (*c*) implies (*e*). We prove the converse for $k = 1$ showing that if g is holomorphic and $T_{ij}g(r\eta)$, $0 \leqslant r \leqslant 1$ have bounded L^p norms, then their supremum is in L^p, once again as if $T_{ij}g$ were holomorphic. If $1 < p$ this is clear because $T_{ij}g$ is harmonic and for $p \leqslant 1$ it is shown as follows. We may assume that g is holomorphic in a neighbourhood of the closed ball. Let $h(\eta) = \sup\{|T_{ij}g(r\eta)| : 0 \leqslant r \leqslant 1\}$. Then

$$\int_0^{2\pi} |h(e^{i\theta}\eta)|^p \, d\theta \leqslant \int_0^{2\pi} \sup_r \left| \bar{\eta}_i \frac{\partial g}{\partial \eta_j}(e^{i\theta}r\eta) - \bar{\eta}_j \frac{\partial g}{\partial \eta_i}(e^{i\theta}r\eta) \right|^p d\theta.$$

By the one variable result for H^p applied to the function in the last integral, which is holomorphic in $re^{i\theta}$, this is less than some constant times the same integral but without the supremum and with $r = 1$. Now the result follows integrating in η on the sphere. The proof for $k > 1$ is similar. □

We also point out here that if $s = k/2$ where k is an integer then we have proved

$$\sup_{0<r<1} \int_S |\nabla_T^k f(r\zeta)|^p \, d\sigma(\zeta) \leqslant C \int_S |R^{k/2} f(\zeta)|^p \, d\sigma(\zeta).$$

This will be used in Section 6.

5. The Boundary Distribution of Complex Tangential Derivatives

One consequence of the results of the previous sections is the following: if k is a positive integer and if f is holomorphic in B^n with $R^{k/2}f \in H^p(B^n)$ then $T_\delta f$, $\delta \in C_k$ has admissible maximal function in $L^p(S)$. For $0 < p \leqslant 1$ we may inquire more closely about the boundary behaviour of $T_\delta f$, $\delta \in C_k$. If $0 < p \leqslant 1$, since $T_\delta f$ is harmonic it follows that $T_\delta f$ has a boundary distribution which lies in the real Hardy space, $\operatorname{Re} H^p(S)$ and hence has an atomic decomposition as a sum of $\operatorname{Re} H^p$ atoms. We have seen however that the functions $T_\delta f$ share many properties with holomorphic functions (non-isotropic mean value inequalities for example). We see here that the boundary distribution of $T_\delta f$ also has a «holomorphic nature» by showing that it also lies in the non-isotropic Hardy space H^p_{at} of Garnett and Latter [4]. We do this by showing that the mapping $K = T_\delta I^{k/2}$ which leads from $R^{k/2}f$ to $T_\delta f$ can be realized as a non-isotropic singular integral operator. Then we apply the method of Coifman and Weiss [2] to show that it maps the non-isotropic Hardy space H^p_{at} to itself. Then we just have to check that the boundary distribution of the harmonic function $T_\delta f$ is the same as the distribution $KR^{k/2}f$ determined by the singular integral theory.

We will prove the following

Theorem 5.1. *There is a singular integral operator K which maps H^p_{at} to H^p_{at} for all $0 < p$ such that if $f \in H^p(B^n)$, then $Kf = T_\delta I^{k/2} f$ in the following sense: we regard f as an element of H^p_{at} and apply K to obtain a distribution Kf in H^p_{at}; on the other hand the harmonic function $T_\delta I^{k/2} f$ also has a boundary distribution; these distributions are the same.*

PROOF. First we show that $\nabla^k_T I^{k/2}$ can be realized as a singular integral. To see this let $T_1, \ldots, T_k$ be complex-tangential operators and take $F \in H^1(B^n)$. If we use the definition of the operator $I^{k/2}$ and the Cauchy integral formula for F we see that

$$(I^{k/2}F)(z) = \frac{1}{\Gamma(k/2)} \int_S F(\zeta) \int_0^1 \left(\log\frac{1}{t}\right)^{\beta-1} (1 - t\langle z, \zeta\rangle)^{-n}\, dt\, d\sigma(\zeta).$$

If we now apply $T_1 \cdots T_k$ and differentiate under the integral sign we find that

$$(KF)(z) = \int F(\zeta) K(z, \zeta)\, d\sigma(\zeta),$$

where

$$K(z, \zeta) = C\psi_1(z, \zeta) \cdots \psi_k(z, \zeta) \int_0^1 \left(\log\frac{1}{t}\right)^{\frac{k}{2}-1} (1 - t\langle z, \zeta\rangle)^{-n-k} t^k\, dt.$$

Here C is a constant and each of the factors $\psi_1, \ldots, \psi_k$ is of the form $\overline{z_k}\overline{\zeta_l} - \overline{z_l}\overline{\zeta_k}$ for some $l \neq k$. In particular we have $\psi_j(\zeta, \zeta) \equiv 0$. Now it is not hard to see that the integral factor in the expression for $K(z, \zeta)$ behaves essentially like $(1 - \langle z, \zeta\rangle)^{-n-k/2}$. However, rather than getting an asymptotic development of this integral and keeping track of the error terms it seems easier to deal with it directly.

We state now some elementary lemmas.

Lemma 5.2. *There are constants C, A such that if $|1 - \langle \zeta, \zeta_0\rangle| < \delta$ and $|1 - \langle \zeta_1, \zeta_0\rangle| \geqslant C\delta$, then for $0 < r < 1$, we have*

(i) $|1 - \langle r\zeta_1, \zeta\rangle| \geqslant \delta$, *and*
(ii) $1/A\,|1 - \langle r\zeta_1, \zeta\rangle| \leqslant |1 - \langle r\zeta_1, \zeta_0\rangle| \leqslant A|1 - \langle r\zeta_1, \zeta\rangle|$.

Lemma 5.3. *Suppose $\alpha, \beta > 0$ and $\beta < \alpha$. Then there is C such that for all $|\lambda| < 1$ we have*

$$\int_0^1 \left(\log\frac{1}{t}\right)^{\beta-1} |1 - t\lambda|^{-\alpha}\, dt \leqslant C|1 - \lambda|^{\beta-\alpha}.$$

Lemma 5.4. *If $\zeta, \zeta_0 \in S$ and $z \in \bar{B}^n$ then*

$$|\langle z, \zeta - \zeta_0\rangle| \leqslant 2|1 - \langle \zeta, \zeta_0\rangle|^{1/2}\{|1 - \langle \zeta, \zeta_0\rangle|^{1/2} + |1 - \langle z, \zeta_0\rangle|^{1/2}\}.$$

Lemma 5.5. *If $z, w \in \bar{B}^n$, then*

$$|z - w| \leqslant \sqrt{2}\,|1 - \langle z, w\rangle|^{1/2}.$$

At first we will assume that $n/(n + \tfrac{1}{2}) < p \leqslant 1$.

In order to apply the results of Coifman and Weiss we need to show that:

(i) $f \mapsto \int K(z, \zeta) f(\zeta)\, d\sigma(\zeta)$ is bounded on $L^2(S)$, and

(ii) There is a constant C such that if $|1 - \langle \zeta, \zeta_0\rangle| < \delta$ and $|1 - \langle \zeta_1, \zeta_0\rangle| \geqslant C\delta$ then

$$|K(\zeta_1, \zeta) - K(\zeta_1, \zeta_0)| \leqslant C\frac{|1 - \langle \zeta, \zeta_0\rangle|^{1/2}}{|1 - \langle \zeta_1, \zeta_0\rangle|^{n+1/2}}.$$

Actually, our strategy will be to let $K_r(\zeta, \eta) = K(r\zeta, \eta)$ and show that K_r satisfies (i) and (ii) with bounds that are independent of r and then take a limit.

First we discuss (i). Fix $0 < r < 1$. Note that the kernel $\overline{K_r(\eta, \zeta)}$ is holomorphic in ζ and hence we have, for $f \in L^2$, $K_r f = K_r F$, where F is the orthogonal projection of f onto $H^2(B)$. From the previous discussion we know that $(K_r F)(\eta) = (T_1 \cdots T_k I^{k/2} F)(r\eta)$. Or, $K_r F = (T_1 \cdots T_k I^{k/2} F)_r$. It follows from the results of Section 3 that $\|(T_1 \cdots T_k I^{k/2} F)^+\|_{L^2(S)} \leqslant C\|F\|_{H^2} \leqslant C\|f\|_{L^2}$, and in particular $\int |K_r F(\eta)|^2\, d\sigma(\eta) \leqslant C\|f\|_{L^2}$, where the constant C is independent of r and of f.

We now show that (ii) holds for K_r with constants independent of r. We write

$$K(z, \zeta) = \psi(z, \zeta) L(z, \zeta) \quad \text{where} \quad \psi(z, \zeta) = C \prod_{j=1}^{k} \psi_j(z, \zeta)$$

and $L(z, \zeta)$ is the integral. We have

$$\begin{aligned} K_r(\zeta_1, \zeta) - K_r(\zeta_1, \zeta_0) &= [\psi(r\zeta_1, \zeta) - \psi(r\zeta_1, \zeta_0)] L(r\zeta_1, \zeta) \\ &\quad + \psi(r\zeta_1, \zeta_0)[L(r\zeta_1, \zeta) - L(r\zeta_1, \zeta_0)] \\ &= I + II. \end{aligned}$$

Note that

$$\begin{aligned} |L(r\zeta_1, \zeta)| &\leqslant C \int_0^1 \left(\log\frac{1}{t}\right)^{\frac{k}{2}-1} |1 - tr\langle \zeta_1, \zeta\rangle|^{-n-k}\, dt \\ &\leqslant C|1 - r\langle \zeta_1, \zeta\rangle|^{-n-\frac{k}{2}} \\ &\leqslant C|1 - \langle \zeta_1, \zeta_0\rangle|^{-n-\frac{k}{2}} \end{aligned}$$

because of the assumptions in (ii) concerning ζ, ζ_0, ζ_1 and Lemmas 5.2 and 5.3.

Next note that

$$\begin{aligned} &|\psi(r\zeta_1, \zeta) - \psi(r\zeta_1, \zeta_0)| \\ &\quad\leqslant C \sum_{l=0}^{k-1} \left| \prod_{j=1}^{l} \psi_j(r\zeta_1, \zeta) \prod_{i=l+2}^{k} \psi_i(r\zeta_1, \zeta_0)[\psi_{l+1}(r\zeta_1, \zeta) - \psi_{l+1}(r\zeta_1, \zeta_0)] \right| \\ &\quad\leqslant C|1 - r\langle \zeta_1, \zeta\rangle|^{l/2}|1 - \langle r\zeta_1, \zeta_0\rangle|^{(k-l-1)/2}|1 - \langle \zeta, \zeta_0\rangle|^{1/2} \\ &\quad\leqslant C|1 - r\langle \zeta_1, \zeta_0\rangle|^{(k-1)/2}|1 - \langle \zeta, \zeta_0\rangle|^{1/2}. \end{aligned}$$

It follows that

$$|I| \leqslant C \frac{|1 - \langle \zeta, \zeta_0\rangle|^{1/2}}{|1 - \langle \zeta_1, \zeta_0\rangle|^{n+1/2}}.$$

Now we turn to II. Note that

$$\begin{aligned} &|L(r\zeta_1, \zeta) - L(r\zeta_1, \zeta_0)| \\ &\quad\leqslant \int_0^1 \left(\log\frac{1}{t}\right)^{k/2-1} t^n \left| \frac{1}{(1 - rt\langle \zeta_1, \zeta\rangle)^{n+k}} - \frac{1}{(1 - rt\langle \zeta_1, \zeta_0\rangle)^{n+k}} \right| dt. \end{aligned}$$

Now, $|(1 - \lambda)^{-(n+k)} - (1 - \lambda_0)^{-(n+k)}| \leqslant C|\lambda - \lambda_0|M$ where M is the maximum of $|1 - z|^{-(n+k+1)}$ where z lies on the line segment joining λ to λ_0. A point on the segment joining $rt\langle \zeta_1, \zeta_0\rangle$ to $rt\langle \zeta_1, \zeta\rangle$ is of the form $rt\langle \zeta_1, \zeta'\rangle$ where $|1 - \langle \zeta', \zeta_0\rangle| < \delta$. It follows from Lemma 5.2 that $|1 - rt\langle z, \zeta'\rangle|$ is essentially equal to $|1 - rt\langle \zeta_1, \zeta_0\rangle|$. It follows that

$$\begin{aligned} &|L(r\zeta_1, \zeta) - L(r\zeta_1, \zeta_0)| \\ &\quad\leqslant C|\langle \zeta_1, \zeta - \zeta_0\rangle| \int_0^1 \left(\log\frac{1}{t}\right)^{\frac{k}{2}-1} |1 - rt\langle \zeta_1, \zeta_0\rangle|^{-(n+k+1)}\, dt \\ &\quad\leqslant C \frac{|1 - \langle \zeta, \zeta_0\rangle|^{1/2}}{|1 - r\langle \zeta_1, \zeta_0\rangle|^{n+k/2+1/2}}. \end{aligned}$$

We also have $|\psi(r\zeta_1, \zeta_0)| \leqslant C|1 - r\langle \zeta_1, \zeta_0\rangle|^{k/2}$ and hence

$$II \leqslant \frac{C|1 - \langle \zeta, \zeta_0\rangle|^{1/2}}{|1 - \langle \zeta_1, \zeta_0\rangle|^{n+1/2}},$$

as required.

Now we may proceed to apply the method of Coifman and Weiss [2] and the atomic decomposition of Garnett and Latter [4]. We refer to these papers for the relevant definitions. Since

$$p > \frac{n}{n + \frac{1}{2}} = \frac{2}{2 + \frac{1}{n}}$$

we may find ϵ, $0 < \epsilon < 1/n$ so that $p > 2/(2 + \epsilon)$. Now if we let

$$\gamma = \frac{2 - p}{p(2 + \epsilon) - 2}$$

then it follows from the above estimates for K_r that

$$\left\{\int |K_r a(\zeta)|^2\, d\sigma(\zeta)\right\}\left\{\int |K_r a(\zeta)|^2 |1 - \langle \zeta, \zeta_0\rangle|^{n(1+\epsilon)}\, d\sigma(\zeta)\right\}^{\gamma} \leqslant C$$

whenever a is a p-atom centered at ζ_0. Here the constant C is independent of both r and a. Now by a straightforward modification of the proof of Theorem C on page 594 of [2] we see that $\|K_r a\|_{H^p_{at}} \leqslant C$, where C is independent of r and a.

If follows that we may define $K_r(f) = \sum \lambda_k K_r a_k$ whenever $f = \sum \lambda_k a_k \in H^p_{at}$ and we have $\|K_r f\|_{H^p_{at}} \leqslant C\|f\|_{H^p_{at}}$. Now if $f = \sum \lambda_k a_k \in H^p_{at}$ and $r, s < 1$ we have

$$\|K_r f - K_s f\|^p_{H^p_{at}} \leqslant \sum_{k \leq N} |\lambda_k|^p \|K_r a_k - K_s a_k\|^p_{H^p_{at}} + C \sum_{k > N} |\lambda_k|^p.$$

Now for an atom a

$$\|K_r a - K_s a\|^p_{H^p_{at}} \leqslant \|K_r a - K_s a\|_1 \to 0$$

as $r, s \to 1$. So we see that for each $f \in H^p_{at}$, $Kf = \lim_{r\to 1} K_r f$ exists in H^p_{at}.

Now suppose $f \in H^p(B^n)$, and let u be the harmonic function

$$u = T_1 \cdots T_K I^{k/2} f$$

defined in B^n. We know that $u^+ \in L^p$ from Section 2. Hence the functions u_r, where $u_r(\zeta) = u(r\zeta)$ have a distribution limit U as $r \to 1$. We will identify U with the distribution Kf. First note that if f is holomorphic in a neighborhood of $\bar{B}^n$ then

$$Kf = T_1 \cdots T_k I^{k/2} f = U \quad \text{on} \quad S.$$

Also if $f \in H^p(B^n)$ then $f_r \to f$ in H^p_{at} and so $Kf_r \to Kf$ in H^p_{at}, and hence in the sense of distributions. But $Kf_r = u_r$ and this shows that

$$U = Kf$$

as desired.

We will only say a few words about the case $p \leqslant n/(n + \frac{1}{2})$. In this case the definition of H^p atoms involves some cancellation. In this case the estimate (ii) above must be replaced by an estimate of the type

$$|K_r(\eta, \zeta) - P(\eta, \zeta_0)| \leqslant C \frac{|1 - \langle \zeta, \zeta_0 \rangle|^a}{|1 - \langle \eta, \zeta_0 \rangle|^{n+a}}$$

where a is some positive number depending on p and $P(\eta, \zeta_0)$ is a certain non-isotropic Taylor polynomial expansion of K_r about ζ_0.

6. Bergman-Sobolev Spaces

In this section we give an application of the results and techniques of the previous sections to the Bergman-Sobolev spaces $B^p_{s,\gamma}$ defined in the introduction. In some ways these spaces are much easier to deal with than the spaces $H^p_s(B^n)$. The reason is that this family of spaces $(B^p_{s,\gamma})$ is stable under differentiation, a property not shared by the Hardy-Sobolev spaces. More precisely, we have $B^p_{s,\gamma} = B^p_{t,\beta}$ as long as $sp - \gamma = tp - \beta$, (and $\gamma, \beta > -1$, of course). This is immediate in the case of the unit ball B^n since the case $n = 1$ is a well-know theorem of Hardy and Littlewood and the general case follows by slice integration.

The fact that each radial derivative controls two tangential derivatives in the context of the spaces $B^p_{s,\gamma}$ has been proved in [1], even for general domains. In the theorem below we show how this follows immediately from the results of Sections 3-4. We also show that the converse holds. This does not seem to follow directly from the statements of our earlier results but it is a simple consequence of the techniques developed in the previous sections.

Theorem 6.1. *Suppose that $k/2 \leqslant s$, k is an integer; then $f \in B^p_{s,\gamma}$ if and only if*

$$\int_{B^n} |\nabla^k_T R^{s-k/2} f(z)|^p (1 - |z|)^\gamma \, dm(z) < \infty.$$

PROOF. We may assume that $k/2 = s$. One direction follows from the inequality, (see Section 4)

$$\int |\nabla_T^k f(r\zeta)|^p \, d\sigma(\zeta) \leqslant C \int |R^{k/2} f(r\zeta)|^p \, d\sigma(\zeta),$$

upon multiplying by $(1-r)^\gamma$ and integrating on r. For the other direction, by the remarks above it suffices to prove that

$$\int |R^k f(z)|^p (1-|z|)^{\gamma+kp/2} \, dm(z) \leqslant \int |\nabla_T^k f(z)|^p (1-|z|)^\gamma \, dm(z).$$

It was proved in Lemma 3.11 that

$$\sum_{j=0}^{k} d_j R^{k-j} f = \sum_{\delta \in C_k} \bar{T}_\delta T_\delta f.$$

Fix $\delta \in C_k$, then from Lemma 2.5 we have

$$|\bar{T}_\delta T_\delta f(r\zeta)|^p \leqslant \frac{C}{(1-r)^{n+1+kp/2}} \int_{P(r\zeta, \epsilon(1-r^2))} |T_\delta f(z)|^p \, dm(z)$$

where ϵ is chosen as in Lemma 3.5 so that $P(r\zeta, \epsilon(1-r^2)) \subset S_\alpha(r,\zeta)$ for some $\alpha > 1$. So we have

$$|\bar{T}_\delta T_\delta f(\zeta)|^p \leqslant C(1-r)^{-n-1-kp/2} \int |T_\delta f(z)|^2 \chi(\zeta, z) \, dm(z)$$

where χ is the characteristic function of $S_\alpha(r,\zeta)$. Now we integrate this inequality over S and use Fubini's theorem to obtain

$$\int_S |\bar{T}_\delta T_\delta f(r\zeta)|^p \, d\sigma(\zeta) \leqslant C(1-r)^{-n-1-kp/2} \int |T_\delta f(z)|^2 \int_S \chi(\zeta, z) \, d\sigma(\zeta) \, dm(z).$$

Now $\int \chi(\zeta, z) \, d\sigma(\zeta) = 0$ unless $z \in L_r = \{z : (1-r)/2 < 1-|z| < 2(1-r)\}$ and in any case $\int \chi \, d\sigma \leqslant C(1-r)^n$ so we obtain

$$\int_S |\bar{T}_\delta T_\delta f(r\zeta)| \, d\sigma(\zeta) \leqslant C(1-r)^{-1-kp/2} \int_{L_r} |T_\delta f(z)|^2 \, dm(z).$$

Now we multiply this inequality by $(1-r)^{\gamma+kp/2}$, integrate on r and use Fubini's theorem once more. If we now add on $\delta \in C_k$ we obtain

$$\int \left| \sum d_j R^{k-j} f(z) \right|^p (1-|z|)^{\gamma+kp/2} \, dm(z) \leqslant C \int |\nabla_T^k f(z)|^p (1-|z|)^\gamma \, dm(z).$$

To replace the function in the integral on the left hand side by $R^k f$ above we note that for $j > 0$ we have

$$\int |R^{k-j} f(z)|^p (1-|z|)^{\gamma+kp/2} \, dm(z) \leqslant C \int |R^k f(z)|^p (1-|z|)^{\gamma+kp/2+jp} \, dm(z)$$

and proceed as in the proof of Lemma 3.12. □

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Patrick Ahern*
Department of Mathematics
University of Wisconsin
Madison WI 53706, USA

Joaquim Bruna**
Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra, SPAIN

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