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An Extremal Problem for Certain Subharmonic Functions in the Plane

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Dedicated to the memory of José Luis Rubio de Francia

1. Introduction

Let $\delta \in (0, 1)$, and suppose that $K \subset \mathbb{R}$ is a closed set which is δ-dense in the sense that

$$|K \cap I| \geqslant \delta \tag{1.1}$$

for every interval $I \subset \mathbb{R}$ with $|I| \equiv \text{length } I = 1$. Suppose also that every point of K is regular for the Dirichlet problem on the domain $\mathbb{C}\backslash K$. Then there exists a function $u = u(\cdot, K): \mathbb{C} \to [0, \infty)$ with the following properties:

(i) u is harmonic in $\mathbb{C}\backslash K$ and continuous on $\mathbb{C}$.
(i) $u = 0$ on K.
(iii) $u(z) = u(\bar{z}), \quad z \in \mathbb{C}$.
(iv) $\displaystyle\lim_{y \to +\infty} \frac{u(x + iy)}{y} = 1, \quad x \in \mathbb{R}$.

Such functions were constructed by A. E. Schaeffer [S], who applied them to extend Bernstein's theorem about $\sup |f'(x)|$ for entire functions of exponential type. More recently, Benedicks [Be] has studied a class of functions which includes $u(\cdot, K)$, and has proved, among other things, a conjecture of Kjellberg. From results in [Be] it follows that $u(\cdot, K)$ is unique.

B. Ya. Levin [L] posed the extremal problem of finding a δ-dense K which maximizes $\sup_{\mathbb{R}} u(x, K)$. Define

$$K_\delta = \bigcup_{n=-\infty}^{\infty} \left[n - \frac{1}{2}\delta, n + \frac{1}{2}\delta \right].$$

Levin's conjecture. For each $x \in \mathbb{R}$,

$$u(x, K) \leqslant u\left(\frac{1}{2}, K_\delta\right) = \frac{1}{\pi} \log\left(\cot \frac{\pi\delta}{4}\right). \tag{1.2}$$

We show in Section 3 how to calculate this value of $u(1/2, K_\delta)$.

In this paper we shall prove Levin's conjecture in stronger form, by showing that for every convex nondecreasing function Φ on $[0, \infty)$, $y \in \mathbb{R}$, and interval $I \subset \mathbb{R}$ with $|I| = 1$,

$$\int_I \Phi(u(x+iy, K))\, dx \leqslant \int_I \Phi(u(x+iy, K_\delta))\, dx. \tag{1.3}$$

Then (1.2) follows by taking $y = 0$, $\Phi(t) = t^p$, and letting $p \to \infty$.

The functions $u(\cdot, K)$ may be regarded as limiting cases of certain harmonic measures. For $B > 0$ define

$$\Omega(B, K) = \{z \colon |\operatorname{Im} z| < B, z \notin K\}.$$

Let $u(\cdot, B, K)$ be the function which is harmonic in $\Omega(B, K)$ and takes the boundary value zero on K, one on $\operatorname{Im} z = \pm B$. In Section 3 we will show that locally uniformly in $\mathbb{C}$,

$$\lim_{B \to \infty} Bu(z, B, K) = u(z, K). \tag{1.4}$$

Thus, (1.3) is a consequence of the following theorem.

Theorem 1. *If K satisfies* (1.1), *then for every interval $I \subset \mathbb{R}$ with $|I| = 1$, every $y \in [-B, B]$, and every convex non-decreasing function Φ on* $[0, \infty)$,

$$\int_I \Phi(u(x+iy, B, K))\, dx \leqslant \int_I \Phi(u(x+iy, B, K_\delta))\, dx. \tag{1.5}$$

We can also prove a strong uniqueness theorem.

Theorem 2. *Suppose that, with the situation of Theorem* 1, *equality holds in* (1.5) *for some Φ with $\Phi(0) = 0$, some $y \in (-B, B)$ and some I. Then either K is translation of K_δ or both sides are zero. Moreover, if K is not a translation of K_δ then for every $x \in \mathbb{R}$ and $y \in (-B, B)$,*

$$u(x+iy, B, K) < u\left(\frac{1}{2}+iy, B, K_\delta\right). \tag{1.6}$$

The corresponding uniqueness theorem also holds for $u(\bullet, K)$ and $u(\bullet, K_\delta)$. It may be proved by the same argument used to prove Theorem 2.

In [L] Levin offers the uniqueness conjecture that

$$\max u(x, K) = u\left(\frac{1}{2}, K_\delta\right)$$

implies K is a translation of K_δ. Our inequality (1.6), which holds for $u(\bullet, K)$ and $u(\bullet, K_\delta)$, confirms this when K is a set of period 1. When K is a nonperiodic set, for example, if $K = K_\delta \cup [0, 1]$, then it is easy to see that the maximum need not exist, and that

$$\sup u(x, K) = u\left(\frac{1}{2}, K_\delta\right)$$

can occur. In any case, if K is not a translation of K_δ then (1.6) holds for $u(\bullet, K)$ and $u(\bullet, K_\delta)$.

Theorem 1 is a close relative of a theorem in [Ba2, p. 167] about the behavior of harmonic measure under circular symmetrization. Suppose that Ω is a subdomain of an annulus $R_1 < |\zeta| < R_2$, $0 < R_1 < R_2 < \infty$. Let Ω^* be the domain obtained from Ω by circular symmetrization, and u be the function harmonic in Ω with boundary value zero on $\partial\Omega \cap \{R_1 < |\zeta| < R_2\}$, and one on $\{|\zeta| = R_1\} \cup \{|\zeta| = R_2\}$. Let v be the corresponding function for Ω^*.

Theorem A. *For $r \in [R_1, R_2]$ and convex nondecreasing Φ on $[0, \infty)$,*

$$\int_{-\pi}^{\pi} \Phi(u(re^{i\theta}))\, d\theta \leqslant \int_{-\pi}^{\pi} \Phi(v(re^{i\theta}))\, d\theta.$$

In [Ba2] the theorem is stated for subdomains of a disk. The annulus case is proved exactly the same way. When K has period 1 Theorem 1 follows directly from Theorem A via conformal mapping. The chief novelty of Theorem 1 is that we are able to compare a nonperiodic K with the periodic K_δ. The methods used here to do this can be applied to solve various other extremal problems in which the data are essentially one-dimensional.

It would be interesting to prove results of this type involving higher dimensional periodicities. Suppose, for example, that $K \subset \mathbb{R}^2$ is δ-dense in the sense that area $(K \cap \Delta) \geqslant \delta$ for every disk Δ with area $(\Delta) = 1$. For $B > 0$ define

$$\Omega(K, B) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \colon (x_1, x_2) \notin K, |x_3| < B\}$$

and let $u(\bullet, K, B)$ be the harmonic measure, for the Laplacian in $\mathbb{R}^3$, of

$x_3 = \pm B$ with respect to $\Omega(K, B)$. Lemma 8 of [Be, p. 63] implies existence of a constant $C(\delta, B)$ such that

$$\sup_{(x_1, x_2) \in \mathbb{R}^2} u(x_1, x_2, 0, K, B) \leqslant C(\delta, B).$$

One can pose the problem of finding the extremal configuration K. Theorems about packing and covering in the plane, see, *e.g.* [FT, Chap. 4], suggest that the extremal K should be associated with the decomposition of $\mathbb{R}^2$ into regular hexagons. The conjecture of Ahlfors and Grunsky, see, *e.g.* [M], according to which the extremal case for the theorem of Bloch involves the plane punctured at the centers of these hexagons, provides another question of this type.

The main tool used to prove Theorem A was the subharmonicity of a certain maximal function u^*. See Theorem B in Section 2. In order to adapt this proof we introduce in Section 2 certain variants u^*_λ, whose subharmonicity under appropriate hypotheses is established in Theorem 3.

Essén and Shea [ES] proved a uniqueness theorem associated with Theorem A. For annuli, it states that if equality holds for some $r \in (R_1, R_2)$ and some strictly convex Φ then Ω^* is a rotation of Ω. Our Theorem 2 has a stronger conclusion, in that «strictly» is not assumed. The method also works for sufficiently regular domains Ω, and gives a slight extension of the Essén-Shea theorem for those domains. Probably with more effort it could be made to work for all domains.

It is a pleasure to acknowledge discussions of Levin's problem I had with J. L. Fernández in Zaragoza while visiting Spain under the auspices of the Hispano-American accord.

In sadness I dedicate this paper to the memory of José Luis Rubio de Francia. I met him on only a few occasions, but his penetrating mind, cheerful demeanor, and strength of character made a vivid impression that I will carry with me always.

2. A Variant of the *-function

Suppose that $-\infty < A_1 < A_2 < \infty$ and that f is a real valued integrable function on $[A_1, A_2]$. Let

$$L = \frac{1}{2}(A_2 - A_1).$$

Define the symmetric decreasing rearrangement $\tilde{f}\colon [-L, L] \to \mathbb{R}$ and the $*$-function $f^*\colon [0, L] \to \mathbb{R}$ by

$$\tilde{f}(x) = \inf\{t: |\{y: f(y) > t\}| \leqslant 2x\}, \qquad |x| < L,$$

$$\tilde{f}(\pm L) = \lim_{x \to L-} \tilde{f}(x) = \operatorname{ess\,inf} f,$$

$$f^*(x) = \sup \int_E f(t)\,dt, \tag{2.1}$$

where the supremum is over all $E \subset [A_1, A_2]$ with $|E| = 2x$. These functions were studied in [Ba2, 3] where the following properties were proved.

(2.2) For each $x \in [0, L]$ there exists $E \subset [A_1, A_2]$ for which the supremum in (2.1) is attained.

$$f^*(x) = \int_{-x}^{x} \tilde{f}(t)\,dt, \qquad 0 \leqslant x \leqslant L. \tag{2.3}$$

(2.4) For $f, g \in L^1[A_1, A_2]$ the following two statements are equivalent:

(*a*) $$f^*(x) \leqslant g^*(x), \quad \text{for every} \quad x \in [0, L],$$

(*b*) $$\int_{A_1}^{A_2} \Phi(f(x))\,dx \leqslant \int_{A_1}^{A_2} \Phi(g(x))\,dx,$$

for any convex non-decreasing function Φ on $(-\infty, \infty)$.

These results were stated for $A_1 = -\pi$, $A_2 = \pi$ in [Ba2]. The more general versions considered here can be reduced to that one. The argument which proves (2.4) gives also a statement about strict inequality.

(2.5) For non-negative $f, g \in L^1[A_1, A_2]$, with g not the zero function, the following two statements are equivalent:

(*a*) $f^*(x) < g^*(x)$, for every $x \in (0, L]$,

(*b*) for each convex non-decreasing Φ on $(0, \infty)$ with $\Phi(0) = 0$, either

$$\int_{A_1}^{A_2} \Phi(f(x))\,dx < \int_{A_1}^{A_2} \Phi(g(x))\,dx,$$

or both integrals are zero.

Suppose next that u is a real function defined in the annulus $R_1 < |\zeta| < R_2$ for which $u(re^{i\theta}) \in L^1[-\pi, \pi]$, $R_1 < r < R_2$. Then u^* is defined in the semi-annulus $S = \{re^{i\theta}: R_1 < r < R_2, 0 \leqslant \theta \leqslant \pi\}$.

Theorem B. *Suppose that u is subharmonic in $R_1 < |\zeta| < R_2$. Then*

(*a*) $u^* \in C(S)$.

(*b*) *u^* is subharmonic in the interior of S.*

(*c*) *For each $\theta \in [0, \pi]$ the function $r \to u^*(re^{i\theta})$ is a convex function of $\log r$ on (R_1, R_2).*

For (*a*) and (*b*) see [Ba2, pp. 141, 147] and for (*c*) [Ba3, p. 85].

Let us return now to $f \in L^1[A_1, A_2]$. For $E \subset \mathbb{R}$ write $\operatorname{diam} E = \sup E - \inf E$. For $0 < \lambda \leqslant L$ define $f_\lambda^*: [0, \lambda] \to \mathbb{R}$ by

$$f_\lambda^*(x) = \sup \int_E f(t)\, dt \tag{2.6}$$

where the supremum is over all $E \subset [A_1, A_2]$ with $|E| = 2x$ and $\operatorname{diam} E \leqslant 2\lambda$. Relatives of f_λ^* appear in [Ba1]. It is easy to see that again a set $E = E(x, \lambda)$ exists for which the supremum in (3.6) is attained.

We adopt the following notation for open rectangles:

$$R(A_1, A_2, B_1, B_2) = \{x + iy: A_1 < x < A_2, B_1 < y < B_2\}.$$

The closure of such an R will be denoted by $\bar{R}$.

Suppose that u is defined in $R(A_1, A_2, B_1, B_2)$ and that $u(x + iy) \in L^1[A_1, A_2]$ for each $y \in (B_1, B_2)$. Then u_λ^* is defined in $R(0, \lambda, B_1, B_2)$. In general, subharmonicity of u does not imply subharmonicity of u_λ^*. Consider, for example, $u(x, y) = x^2 - y^2$ in $R(-A, A, -\infty, \infty)$, with $A \in (1, \infty)$. Take $\lambda = 1/2$, then for $x + iy \in R(0, 1/2, -\infty, \infty)$,

$$u_\lambda^*(x + iy) = \int_{A-2x}^{A} (t^2 - y^2)\, dt, \qquad \Delta u_\lambda^* = 12x - 8A < 0.$$

To get a positive result we need to impose a condition which prevents the maximal sets E from hitting the left or right hand boundary of R. This is the role of (2.7) in the theorem below. Recall that $L = (A_2 - A_1)/2$.

Theorem 3. *Suppose that u is subharmonic in $R(A_1, A_2, B_1, B_2)$ and belongs to $L^1[A_1, A_2]$ for each $y \in (B_1, B_2)$, that $0 < \lambda < L/2$, and that, as $z \to \zeta$ from inside $R(A_1, A_2, B_1, B_2)$,*

$$\overline{\lim_{z\to\zeta}}\, u(z) < \underline{\lim_{z\to\zeta}}\, u(z + 2\lambda), \qquad \overline{\lim_{z\to\zeta}}\, u(z) < \underline{\lim_{z\to\zeta}}\, u(z - 2\lambda), \tag{2.7}$$

where in the first inequality $\zeta = A_1 + iy$ and in the second $\zeta = A_2 + iy$, $B_1 < y < B_2$. Then

(*a*) $u_\lambda^* \in C[R(0, \lambda, B_1, B_2) \cup \{\lambda + iy: B_1 < y < B_2\}]$.

(*b*) *u_λ^* is subharmonic in $R(0, \lambda, B_1, B_2)$.*

(*c*) *For each $x \in [0, \lambda]$, the function $y \to u_\lambda^*(x + iy)$ is convex on (B_1, B_2).*

For the proof we need two lemmas about sets $E \subset \mathbb{R}$. For $\epsilon \in \mathbb{R}$ write $E_\epsilon = E + \epsilon = \{x + \epsilon: x \in E\}$. Equivalence of two sets means that they differ by a set of measure zero. Also, we will write $\sup E$, $\inf E$, and $\operatorname{diam} E$ instead of $\operatorname{ess\,sup} E$, $\operatorname{ess\,inf} E$, $\operatorname{ess\,diam} E$ to denote the essential supremum, infimum and diameter of E.

Lemma 1. *Suppose that E is bounded, $|E| > 0$, and that E is not equivalent to an interval. Then for all sufficiently small $\epsilon > 0$, $|E_\epsilon \cap E_{-\epsilon}| \leqslant |E| - 4\epsilon$.*

PROOF. If E is equivalent to a pair of disjoint closed intervals the lemma follows by direct verification. Suppose that E is equivalent to neither a single interval nor a pair. Then there exist points $a_1 < b_1 < a_2 < b_2 < a_3 < b_3$ such that the a_i are density points of $\mathbb{R}\backslash E$ and the b_i are density points of E. Take $c_1 \in (b_1, a_2)$, $c_2 \in (b_2, a_3)$. Then for $\epsilon > 0$ and $\chi = \chi_E$,

$$|E_\epsilon \cap E_{-\epsilon}| = \int_{-\infty}^{\infty} \chi(t+\epsilon)\chi(t-\epsilon)\,dt = \int_{-\infty}^{c_1} + \int_{c_1}^{c_2} + \int_{c_2}^{\infty}.$$

Take $\varkappa > 0$. The proof on page 146 of [Ba2] shows that for ϵ sufficiently small

$$\int_{-\infty}^{c_1} \chi(t+\epsilon)\chi(t-\epsilon)\,dt \leqslant \int_{-\infty}^{c_1} \chi(t+\epsilon)\,dt - (2-\varkappa)\epsilon.$$

The same inequality holds when $(-\infty, c_1)$ is replaced by (c_1, c_2) and (c_2, ∞). Hence

$$|E_\epsilon \cap E_{-\epsilon}| \leqslant \int_{-\infty}^{\infty} \chi(t+\epsilon)\,dt - (6-3\varkappa)\epsilon < |E| - 4\epsilon,$$

provided $\varkappa < 2/3$.

Lemma 2. *Let E satisfy the hypotheses of Lemma 1. Then for each sufficiently small $\epsilon > 0$ there exist sets $E', E'' \subset \mathbb{R}$ such that*

(*a*) $E_\epsilon \cup E_{-\epsilon} = E' \cup E''$ *and* $E_\epsilon \cap E_{-\epsilon} = E' \cap E''$.
(*b*) $|E'| = |E| - 2\epsilon$ *and* $|E''| = |E| + 2\epsilon$.
(*c*) $\operatorname{diam} E' \leqslant \operatorname{diam} E$ *and* $\operatorname{diam} E'' \leqslant \operatorname{diam} E$.

PROOF. Let $F = (E_\epsilon \cup E_{-\epsilon})\backslash(E_\epsilon \cap E_{-\epsilon})$. Then $|F| \geqslant 4\epsilon$, by Lemma 1. Let G be a set of the form $G = F \cap [a, \infty)$ for which

$$|G| = |E| - |E_\epsilon \cap E_{-\epsilon}| - 2\epsilon.$$

Then, by Lemma 1, G exists and $|G| \geqslant 2\epsilon$. Define $E' = (E_\epsilon \cap E_{-\epsilon}) \cup G$, $E'' = (E_\epsilon \cap E_{-\epsilon}) \cup (F\backslash G)$. Then (*a*) is true. Also, $|E'| = |E| - 2\epsilon$. Now (*a*) can be restated as

$$\chi_{E_\epsilon} + \chi_{E_{-\epsilon}} = \chi_{E'} + \chi_{E''}.$$

Hence $2|E| = |E'| + |E''|$, so that $|E''| = |E| + 2\epsilon$ and (*b*) is proved.

Suppose that $\inf G < \epsilon + \inf E$. (Recall that inf means essential infimum.) Then, since $(F\backslash G) \subset [-\epsilon + \inf E, \inf G]$, we would have $|F\backslash G| < 2\epsilon$, so that

$|E''| < |E| + 2\epsilon$, a contradiction, Similarly, $\sup(F\backslash G) > -\epsilon + \sup E$ would imply $|G| < 2\epsilon$, contradicting an earlier inequality. Hence

$$\inf G \geqslant \epsilon + \inf E, \qquad \sup(F\backslash G) \leqslant -\epsilon + \sup E.$$

Thus $E' \subset [\epsilon + \inf E, \epsilon + \sup E]$, $E'' \subset [-\epsilon + \inf E, -\epsilon + \sup E]$, and (*c*) is proved.

PROOF OF THEOREM 3. Suppose first that $u \in C^2(R(A_1, A_2, B_1, B_2))$ and that $u \in C(\bar{R}(A_1, A_2, B_1, B_2))$. Then continuity of u_λ^* follows from a straightforward argument, and we omit it. Let us prove subharmonicity. Take $z_0 = x_0 + iy_0 \in R(0, \lambda, B_1, B_2)$ and $E_0 \subset [A_1, A_2]$ with $\operatorname{diam} E_0 \leqslant 2\lambda$, $|E_0| = 2x_0$ and

$$u_\lambda^*(z_0) = \int_{E_0} u(t + iy_0)\, dt. \tag{2.8}$$

From (2.7) it follows that there exists $\varkappa > 0$ such that

$$u(t + iy_0) < u(t + 2\lambda + iy_0), \qquad u(t + iy_0) < u(t - 2\lambda + iy_0),$$

when, respectively,

$$A_1 \leqslant t \leqslant A_1 + \varkappa \quad \text{or} \quad A_2 - \varkappa \leqslant t \leqslant A_2.$$

I claim that E_0 is essentially contained in $[A_1 + \varkappa, A_2 - \varkappa]$. Let

$$b = \sup E_0 = (\operatorname{ess\,sup} E_0), \qquad a = \inf E_0, \quad \text{and} \quad F = E_0 \cap [A_2 - \varkappa, A_2].$$

If $|F| > 0$ then $b \in (A_2 - \varkappa, A_2]$ and hence $a \geqslant b - 2\lambda > A_2 - \varkappa - 2\lambda$. Let

$$E_1 = (E_0\backslash F) \cup (-2\lambda + F).$$

Then $|E_1| = |E_0|$, $\operatorname{diam} E_1 \leqslant 2\lambda$, and the integral in (2.8) strictly increases when E_0 is replaced by E_1, contradicting the definition of u_λ^*. Similarly, $|E_0 \cap [A_1, A_1 + \varkappa]| = 0$. Thus, after possible deletion of a null set,

$$E_0 \subset [A_1 + \varkappa, A_2 - \varkappa]. \tag{2.9}$$

For $0 < \rho < \varkappa$, $-\pi/2 \leqslant \phi \leqslant \pi/2$, define a function Q by

$$Q(\rho, \phi) = \int_{E_0} [u(t + \rho\cos\phi + i(y_0 + \rho\sin\phi)) + u(t - \rho\cos\phi + i(y_0 + \rho\sin\phi))]\, dt.$$

Using (2.8) and subharmonicity of u, we have

$$2\pi u_\lambda^*(z_0) \leqslant \int_{E_0} dt \int_{-\pi}^{\pi} u(t + iy_0 + \rho e^{i\phi})\, d\phi = \int_{-\pi/2}^{\pi/2} Q(\rho, \phi)\, d\phi. \tag{2.10}$$

Fix ρ, ϕ and write $y = y_0 + \rho \sin \phi$, $\epsilon = \rho \cos \phi$. Then

$$Q(\rho, \phi) = \int_{(E_0)_\epsilon} u(t + iy)\, dt + \int_{(E_0)_{-\epsilon}} u(t + iy)\, dt.$$

If E_0 is not equivalent to an interval, then, for small enough ρ, we can find sets E', E'' for E_0 satisfying the conclusions of Lemma 2, and (2.9) guarantees that these sets are contained in $[A_1, A_2]$. From Lemma 2(a) it follows that

$$Q(\rho, \phi) = \int_{E'} u(t + iy)\, dt + \int_{E''} u(t + iy)\, dt,$$

and from Lemma 2(b), (c) we obtain

$$Q(\rho, \phi) \leqslant u_\lambda^*(z_0 - \rho e^{i\phi}) + u_\lambda^*(z_0 + \rho e^{i\phi}). \tag{2.11}$$

If $E_0 = [a, b]$, then $b - a = 2x_0 < 2\lambda$ and we achieve (2.11) by taking $E' = [a - \epsilon, b - \epsilon]$, $E'' = [a + \epsilon, b + \epsilon]$ (assumming $\rho < \lambda - x_0$). Inserting (2.11) in (2.10), we arrive at

$$2\pi u_\lambda^*(z_0) \leqslant \int_{-\pi}^{\pi} u_\lambda^*(z_0 + \rho e^{i\phi})\, d\phi, \tag{2.12}$$

so that u_λ^* is subharmonic in $R(0, \lambda, B_1, B_2)$.

Turning now to the proof of convexity in the C^2 case, let $u_\epsilon(z) = u(z) + \epsilon y^2$. It will suffice to prove that $(u_\epsilon)_\lambda^*(x + iy)$ is convex in y for each $\epsilon > 0$.

Take $z_0 = x_0 + iy_0$ with $0 < x_0 \leqslant 2\lambda$, $y_0 \in (B_1, B_2)$, and let E_0 be a maximal set for u_λ^* as in (2.8). Then (2.9) holds. Since u_λ^* and $(u_\epsilon)_\lambda^*$ have the same maximal sets, (2.8) holds also with u replaced by u_ϵ. Define

$$J_1(x) = \int_{E_0} u_\epsilon(x + t + iy_0)\, dt, \qquad J_2(y) = \int_{E_0} u_\epsilon(t + iy)\, dt.$$

Then J_1 is C^2 in a neighborhood of 0, J_2 is C^2 in a neighborhood of y_0, and J_1 has a local maximum at $x = 0$. Hence

$$\begin{aligned} 0 \geqslant J_1''(0) &= \int_{E_0} (u_\epsilon)_{xx}(t + iy_0)\, dt \\ &= \int_{E_0} [(\Delta u_\epsilon)(t + iy_0) - (u_\epsilon)_{yy}(t + iy_0)]\, dt \\ &\geqslant 2\epsilon |E_0| - J_2''(y_0). \end{aligned}$$

Thus $J_2''(y_0) > 0$, and J_2 is convex in a neighborhood of y_0. Hence, for small $\tau > 0$,

$$\begin{aligned} (u_\epsilon)_\lambda^*(z_0) = J_2(y_0) &\leqslant \frac{1}{2} [J_2(y_0 + \tau) + J_2(y_0 - \tau)] \\ &\leqslant \frac{1}{2} [(u_\epsilon)_\lambda^*(y_0 + \tau) + (u_\epsilon)_\lambda^*(y_0 - \tau)], \end{aligned}$$

which proves (*c*) in the C^2 case.

It remains to prove Theorem 3 in the nonsmooth case. Take B_3, B_4 with $B_1 < B_3 < B_4 < B_2$. By (2.7), there exists $A_3 \in (A_1, A_1 + \lambda)$, $A_4 \in (A_2 - \lambda, A_2)$ such that $u(z) < u(z + 2\lambda)$, $u(z) < (z - 2\lambda)$ when $z = x + iy$ with $y \in [B_3, B_4]$ and, respectively, $x \in (A_1, A_3)$ or $x \in (A_4, A_2)$. As in the C^2 case any maximal set for $u_\lambda^*(x + iy)$ will be essentially contained in $[A_3, A_4]$ when $y \in (B_3, B_4)$.

Let $K \geqslant 0$ be a radial C^2 function on $\mathbb{C}$ such that $K(|z|)$ decreases as $|z|$ increases, $K(z) = 0$ for $|z| \geqslant 1/2$, and $\int_{\mathbb{C}} K(z)\,dx\,dy = 1$. Let u_ϵ be the $\mathbb{R}^2$-convolution of u with $\epsilon^{-2}K(z\epsilon^{-1})$. Then u_ϵ is defined on a slightly smaller rectangle than u, and satisfies a boundary condition like (2.7) when ϵ is small. For each $z \in R\ (A_1, A_2, B_1, B_2)$ we have $u_\epsilon(z) \downarrow u(z)$. Moreover, using continuity of the mean value

$$y \to \int_{A_1+\tau}^{A_2-\tau} u(x + iy)\,dx, \qquad \tau > 0,$$

as in the proof of (11) on page 145 of [Ba2], one can show that for each $\tau > 0$,

$$\lim_{\epsilon \to 0} \sup_{y \in [B_3, B_4]} \int_{A_1+\tau}^{A_2-\tau} |u_\epsilon(x + iy) - u(x + iy)|\,dx = 0.$$

Using this, it is not hard to prove that $(u_\epsilon)_\lambda^* \to u_\lambda^*$ uniformly on $\bar{R}[0, \lambda, B_3, B_4]$. Then the general case of Theorem 3 follows from the C^2 case.

3. Other Preliminaries

There are also versions of u_λ^* and of Theorem 3 in the periodic case. Suppose that f is an integrable real valued function on the unit circle T and that $0 < \lambda \leqslant \pi$. Define $f_\lambda^*\colon [0, \lambda] \to \mathbb{R}$ by

$$f_\lambda^*(\theta) = \sup \int_E f(e^{it})\,dt, \tag{3.1}$$

where the supremum is over all $E \subset T$ with $|E| = 2\theta$ and E contained in some arc I of T with $|I| = 2\lambda$. This f_λ^* differs from the one defined in Section 2 for the function $\theta \mapsto f(e^{i\theta})$ on $[-\pi, \pi]$, since in (3.1) sets like $E = \{e^{i\theta}\colon \pi - \epsilon < \theta < \pi + \epsilon\}$ are permitted in the competition, when $2\epsilon \leqslant 2\lambda$.

Suppose that u is defined in the annulus $R_1 < |z| < R_2$ and is integrable on each r circle, $R_1 < r < R_2$. Then u_λ^* is defined in the annular sector

$$S = \{re^{i\theta}\colon R_1 < r < R_2, 0 \leqslant \theta \leqslant \lambda\}.$$

Theorem 4. *Suppose that u is subharmonic in $R_1 < |z| < R_2$. Then*

(*a*) $u_\lambda^* \in C(S)$.

(*b*) *u_λ^* is subharmonic in the interior of S.*
(*c*) *For each $\theta \in [0, \lambda]$, $r \mapsto u^*(re^{i\theta})$ is a convex function of* $\log r$ *on* (R_1, R_2).

When $\lambda = \pi$ this theorem reduces to Theorem B. Note that in the periodic case we do not need any extra conditions such as (2.7). The proof of Theorem 4 follows the same pattern as that of Theorem 3, with a few simplifications. We will not use Theorem 4 in this paper, except for the previously known case $\lambda = \pi$. It is recorded here for possible use in other contexts.

Next, we examine the extremal function $u(z, B, K_\delta)$. For fixed $\delta \in (0, 1)$ and $0 < B < \infty$ define

$$v(z) = u(z, B, K_\delta). \tag{3.2}$$

Then

$$v(z) = v(z+1) \quad \text{for} \quad z \in R\ (-\infty, \infty, -B, B).$$

Thus there is a well defined function V in the annulus $e^{-2\pi B} \leqslant |\zeta| \leqslant e^{2\pi B}$ given by

$$v(z) = V(-e^{2\pi i z}).$$

This function is harmonic in the annulus with the slit $\{e^{i\phi}: \pi - \pi\delta \leqslant \phi \leqslant \pi + \pi\delta\}$ deleted, takes the value zero on the slit and the value one when $|\zeta| = e^{\pm 2\pi B}$. The argument in Proposition 5 of [Ba2, p. 153] is applicable, and one finds that $V(re^{i\phi})$ is a symmetric decreasing function of ϕ, for $|\log r| \leqslant 2\pi B$. Equivalently, for $z = re^{i\theta}$ in the upper half of the annulus,

$$V^*(re^{i\theta}) = \int_{-\theta}^{\theta} V(re^{i\phi})\, d\phi$$

where V^* is the usual periodic *-function (with $\lambda = \pi$). Moreover, V^* is continuous in the closed half-annulus, harmonic in the open half-annulus with $\{e^{i\theta}: \pi - \pi\delta \leqslant \phi \leqslant \pi\}$ deleted and, since $V^*(re^{i\pi})$ is the mean value over the circle $|\zeta| = r$, it is a linear function of $\log r$ for $1 < r < e^{2\pi B}$.

Returning to v, define, for $z \in \bar{R}(0, 1/2, -B, B)$,

$$v^*(z) = \int_{-x}^{x} v\left(\frac{1}{2} + t + iy\right) dt. \tag{3.3}$$

Define also

$$\alpha = \frac{1}{2}(1 - \delta). \tag{3.4}$$

Then $v^*(z) = V^*(e^{2\pi i z})$ and the properties of V^* translate to the following properties of v^*.

Proposition 1. *For v defined by* (3.2),

(*a*) $v^* \in C(\bar{R}(0, 1/2, -B, B))$,
(*b*) v^* *is harmonic in* $R(0, 1/2, -B, B)\setminus[\alpha, 1/2)$,
(*c*) $y \to v^*(1/2 + iy)$ *is linear on* $[0, B]$.
(*d*) $\sup_{\mathbb{R}} v(x) = v\left(\frac{1}{2}\right)$.

Next, we shall show that

$$\lim_{B\to\infty} Bu\left(\frac{1}{2}, B, K_\delta\right) = \frac{1}{\pi}\log\left(\cot\frac{\pi\delta}{4}\right). \tag{3.5}$$

Let

$$F_1(z) = \sin\pi z, \qquad F_2(w) = \frac{1}{2}\left(\sin\frac{\pi\delta}{2}\right)(w + w^{-1}).$$

Then F_1 conformally maps the half strip $R(-1/2, 1/2, 0, \infty)$ onto the upper half plane, and F_2 conformally maps the exterior of the unit disk onto

$$\mathbb{C}\Big\backslash\left[-\sin\frac{\pi\delta}{2}, \sin\frac{\pi\delta}{2}\right].$$

Let

$$\rho = \frac{1}{4}\left(\csc\frac{\pi\delta}{2}\right)e^{\pi B}.$$

Then, for large B, $F_1^{-1}\circ F_2$ maps the half disk $1 < |w| < \rho$, $\operatorname{Im} w > 0$, into $R(-1/2, 1/2, 0, B)$. Let $h_B(w) = u(F_1^{-1}(F_2(w)), B, K_\delta)$. Then h_B is harmonic in the half disk, $h_B = 0$ on $|w| = 1$, $h_B \leqslant 1$ on $|w| = \rho$ and $\partial h_B/\partial n = 0$ on $\operatorname{Im} w = 0$, $1 < |\operatorname{Im} w| < \rho$, except perhaps at the points $F_2^{-1}\circ F_1(\pm 1/2)$, where h_B remains bounded. Extend h_B to the annulus $1 < |w| < e^{\pi B}$ by $h_B(\bar{w}) = h_B(w)$. Then h_B is harmonic in the annulus and

$$h_B(w) \leqslant \frac{\log|w|}{\log\rho}, \qquad 1 < |w| < \rho.$$

Now

$$F_2^{-1}\circ F_1\left(\frac{1}{2}\right) = \cot\frac{\pi\delta}{4}.$$

Hence, as $B\to\infty$,

$$u\left(\frac{1}{2}, B, K_\delta\right) \leqslant \frac{\log\left(\cot\frac{\pi\delta}{4}\right)}{\pi B + O(1)}. \tag{3.6}$$

Choosing instead

$$\rho = 4\left(\csc \frac{\pi\delta}{2}\right)e^{\pi B},$$

then $F_1^{-1} \circ F_2$ maps the half disk onto a super-domain of $R(-1/2, 1/2, 0, B)$. One obtains the inequality opposite to (3.6), and (3.5) follows.

PROOF OF (1.4). This can easily be derived by the methods of [Be], but is particularly simple if one uses Theorem 1 and (3.5). These results imply existence of $C = C(\delta)$ such that $Bu(x, B, K) \leqslant C$ for all $B > 0$, $x \in \mathbb{R}$. Hence, by the maximum principle, for $0 \leqslant y \leqslant B$, and $z = x + iy$,

$$y \leqslant Bu(z, B, K) \leqslant y + C. \tag{3.7}$$

Thus, the symmetric functions $Bu(\bullet, B, K)$ possess a subsquence which converges locally uniformly on $\mathbb{C}\backslash K$ to a harmonic function u, which satisfies $|y| \leqslant u(z) \leqslant |y| + C$. So, for $x \in \mathbb{R}$,

$$\lim_{y \to \infty} \frac{u(x + iy)}{y} = 1.$$

It remains to show that for each $\zeta \in K$,

$$\lim u(z) = 0, \tag{3.8}$$

when $z \to \zeta$, $z \in K^c$.

Fix $\zeta \in K$ and let $S = \{z: A_1 < \operatorname{Re} z < A_2\}$ be a strip with $\zeta \in S$. Let H be the solution of the Dirichlet problem in $S\backslash K$ with boundary data $H = 0$ on $K \cap S$, $H(z) = C + |y|$ on ∂S. Since the boundary function is integrable with respect to the harmonic measure on $S\backslash K$ and each point of K is Dirichlet regular, potential theory shows that such an H exists, and that $\lim_{z \to \zeta} H(z) = 0$. From (3.7) we deduce $u(z) \leqslant H(z)$ in $S\backslash K$, and (3.8) follows.

For the proof of Theorems 1 and 2 it will be convenient to have a special form of the maximum principle. Suppose that $0 < \alpha < a$ and $b > 0$. Write $R = R(0, a, -b, b)$, and let w be a function defined on $\bar{R}$.

Proposition 2. *Suppose that*

(3.9) *w is subharmonic in $R\backslash[\alpha, a)$ and continuous on $\bar{R}$,*

(3.10) *$y \mapsto w(a + iy)$ is convex on $[0, b]$,*

(3.11) $w(z) = w(\bar{z})$, $z \in \bar{R}$,

(3.12) $w(iy) = 0$ *for* $|y| \leqslant b$, *and* $w(x + ib) \leqslant 0$ *for* $0 \leqslant x \leqslant a$, (3.13) *$w(x) = w(\alpha)$ for $\alpha \leqslant x \leqslant a$, and*

$$\lim_{x \to \alpha -} \frac{w(\alpha) - w(x)}{\alpha - x} = 0.$$

Then

(3.14) *$w \leqslant 0$ in $\bar{R}$.*

Moreover, if either

(3.15) *w is not harmonic in $R \setminus [\alpha, a)$*

or

(3.16) *$w(a + iy) < 0$ for $0 < y < b$,*

then $w(z) < 0$ for $z \in R \cup \{a + iy: |y| < b\}$.

PROOF. Let $M = \sup_R w$. Then $M \geqslant 0$, by (3.12). The hypotheses imply that $M = w(z_0)$ for some z_0 with $\operatorname{Re} z_0 = 0$, $\operatorname{Im} z_0 = b$ or $z_0 = \alpha$. In the first two cases we have $M = 0$, which is (3.14). If $M > 0$ then $z_0 = \alpha$, w is non-constant, and hence $w < M$ everywhere in $R(0, \alpha, -b, b)$. From Hopf's lemma (see, *e.g.* [GT, p. 34]), it follows that $\partial w(\alpha)/\partial x > 0$ (derivative from the left). This contradicts (3.13), and therefore (3.14) holds.

If either (3.15) or (3.16) holds, then $w < 0$ in $R \setminus [\alpha, a)$. Hopf's lemma shows again that $w(\alpha) < 0$, and thus $w(x) < 0$ for $x \in [\alpha, a]$, by (3.13). Thus $w(a) < 0$, while $w(a + ib) \leqslant 0$, by (3.12). Hence $w(a + iy) < 0$ for $|y| < b$, by (3.10), and the proof is complete.

4. Proof of Theorem 1

We may assume that K contains no interval (a, ∞) or $(-\infty, a)$, since otherwise we can replace K by a closed subset with this property which still satisfies the δ-density condition. The corresponding $u(\cdot, B, K)$ pointwise dominates the original one.

Choose sequences $A'_n \downarrow -\infty$, $A''_n \uparrow +\infty$ such that $1 + A'_n \notin K$, $-1 + A''_n \notin K$. Let u_n be the function harmonic in $R(A'_n, A''_n, -B, B) \setminus K$ with boundary values zero on $K \cup (\operatorname{Re} z = A'_n$ or $A''_n)$, one on $\operatorname{Im} z = \pm B$. Then $u_n \uparrow u(\cdot, B, K)$ locally uniformly in $|y| \leqslant B$, so it suffices to prove Theorem 1 with u_n in place of $u(\cdot, B, K)$, and $I \subset [A'_n, A''_n]$.

When $A''_n - A'_n > 1$, Theorem 3, with $\lambda = 1/2$, is applicable to u_n. Moreover, it is easy to see that $(u_n)^*_X \in C[\bar{R}(0, 1/2, -B, B)]$. Let $w = (u_n)^*_X - v^*$, where v^*

is defined by (3.3) and v by (3.2). Take a set E_0 with $|E_0| = 2\alpha = 1 - \delta$ for which

$$(u_n)^*_X(\alpha) = \int_{E_0} u_n(t)\,dt$$

and $E_0 \subset I$, where I is an interval of length 1. Define

$$u^*_I(x) = \sup \int_E u_n(t)\,dt$$

where the supremum is over $E \subset I$ with $|E| = 2x$. By the δ-density condition and (2.3),

$$\lim_{x \to \alpha -} \frac{u^*_I(\alpha) - u^*_I(x)}{\alpha - x} = 0. \tag{4.1}$$

Now $u^*_I(\alpha) = (u_n)^*_X(\alpha)$, and $u^*_I(x) \leqslant (u_n)^*_X(x)$ for $0 \leqslant x < \alpha$. Thus (4.1) holds with $(u_n)^*_X$ in place of u^*_I, and by (3.3) it holds for v. Hence w satisfies (3.13). From Theorem 3 and Proposition 1 it follows that w satisfies the other hypotheses of Proposition 2 in $R(0, 1/2, -B, B)$, so that $w \leqslant 0$ there. By (2.4), this proves Theorem 1.

5. Proof of Theorem 2. Periodic case

Assume in this section that there exists a closed set $K' \subsetneq [0, 1]$ such that

$$K = \bigcup_{n=-\infty}^{\infty} (n + K').$$

Fix $B > 0$ and write $u(z) = u(z, B, K)$. Then $u(z + 1) = u(z)$.

Define U in the annulus $e^{-2\pi B} \leqslant |\zeta| \leqslant e^{2\pi B}$ by the formula $U(e^{2\pi i z}) = u(z)$. Then u is the function harmonic in the annulus with the set $e^{2\pi i K'}$ deleted, which takes the values one on $|\zeta| = e^{\pm 2\pi B}$ and zero on $e^{2\pi i K'}$. Define

$$U^*(re^{i\theta}) = \sup_E \int_E U(re^{it})\,dt, \qquad u^*(z) = \sup_E \int_E u(t + iy)\,dt, \tag{5.1}$$

where in the first case the supremum is over subsets of the unit circle with $|E| = 2\theta$ and in the second over $E \subset \mathbb{R}$ with $|E| = 2x$ and $\operatorname{diam} E \leqslant 1$. Then $u^*(z) = U^*(e^{2\pi i z})$. By Theorem B, u^* is subharmonic in the rectangle

$$R = R(0, 1/2, -B, B),$$

continuous on its closure, and the function $y \to u^*(1/2 + iy)$ is convex on $[-B, B]$.

Let $w = u^* - v^*$, where v is defined in (3.2) and v^* by (3.3). Then $w \leqslant 0$ in R, by Theorem 1 and (2.4). By (2.5), the strict Φ-inequality statement of Theorem 2 is equivalent to $w(z) < 0$ for $z \in R \cup \{1/2 + iy: |y| < B\}$. By propositions 1 and 2, it is sufficient to show that w fails to be harmonic in $R \setminus [\alpha, 1/2)$, unless $e^{2\pi i K'}$ is a single arc on the unit circle with length exactly $2\pi\delta$.

Let $\alpha' = (1 - |K'|)/2$. Then $0 < \alpha' \leqslant \alpha = (1 - \delta)/2$. Let $H = e^{2\pi i K'}$. If H is a single arc then $\alpha' < \alpha$, and consideration of the function U shows that u^* is harmonic in $R \setminus [\alpha', 1/2)$ but not in $R \setminus [\alpha, 1/2)$. Hence w fails to be harmonic in $R \setminus [\alpha, 1/2)$, and we are done.

Assume, then that H is not a single arc. We are going to show that for x_0 slightly smaller than α' and for all small enough ρ,

$$u^*(x_0) < \frac{1}{2\pi} \int_{-\pi}^{\pi} u^*(x_0 + \rho e^{i\phi})\, d\phi. \tag{5.2}$$

Since v^* is harmonic at x_0 this will show that w is not harmonic in $R \setminus [\alpha, 1/2)$, as required.

Suppose that f is a continuous real function on the unit circle for which $|\{\phi \in [-\pi, \pi]: f(e^{i\phi}) = t\}| = 0$ for each $t > 0$. Let $E(t) = \{\phi: f(e^{i\phi}) > t\}$. Then the function $t \to |E(t)|$ is strictly decreasing and continuous for $0 < t < \sup f$, and for each $\theta \in (0, |E(0)|/2)$ there is an essentially unique set E for which the supremum

$$\sup_{|E| = 2\theta} \int_E f(e^{i\phi})\, d\phi, \qquad E \subset |\zeta| = 1,$$

is attained. This set is $E = E(t)$ with t chosen so that $|E(t)| = 2\theta$.

The function U is harmonic in $e^{-2\pi B} < |\zeta| < e^{2\pi B}$ except on H and is not constant on $|\zeta| = 1$. It follows that for $t > 0$ the sets $E(t) = \{\phi: U(e^{i\phi}) > t\}$ have only finitely many components. Thus, when $r = 1$ and $0 < \theta < 2\pi\alpha'$ each maximal set E in (5.1) can be taken to consist of finitely many open arcs, or of finitely many disjoint closed arcs. We work with the latter, and note that U is constant on ∂E. Since H is not a single arc, the equation $U(e^{i\phi}) = t$ will have at least four solutions when t is close to zero, and hence the closed maximal set E will consist of at least two disjoint closed arcs, when θ is close to $2\pi\alpha'$.

Return now to the function u. For $x_0 \in (0, \alpha')$ and sufficiently close to α' there is, by the above discussion, a set

$$E_0 = \bigcup_{i=1}^{N} [a_i, b_i]$$

with

$$a_i < b_i < a_{i+1}, \quad i = 1, \ldots, N,$$

$b_N - a_1 < 1$ (note the strict inequality), $|E_0| = 2x_0$, and $2 \leqslant N < \infty$, for which

$$u^*(x) = \int_{E_0} u(t)\,dt.$$

Moreover, $u(a_i) = u(b_i) = t_0$ for some t_0, $u(t) > t_0$ for $t \in E_0$, and $u(t) < t_0$ for t slightly to the left of an a_i or to the right of a b_i.

For $\epsilon > 0$ define sets $E' = E'(\epsilon)$ and $E'' = E''(\epsilon)$ by

$$E' = [a_1 + \epsilon, b_1 - \epsilon] \cup \left(\bigcup_{i=2}^{N} [a_i + \epsilon, b_i + \epsilon] \right),$$

$$E'' = [a_1 - \epsilon, b_1 + \epsilon] \cup \left(\bigcup_{i=2}^{N} [a_i - \epsilon, b_i - \epsilon] \right).$$

When ϵ is small enough these sets satisfy the conclusion of Lemma 2 in Section 2, with $\operatorname{diam} E' \leqslant 1$, $\operatorname{diam} E'' \leqslant 1$. Furthermore,

$$\int_{E''} u(t)\,dt < u^*(x_0 + \epsilon),$$

since the integral strictly increases when E'' is replaced by

$$[a_1 - \epsilon, b_1 + \epsilon] \cup \left(\bigcup_{i=2}^{N} [a_i, b_i] \right).$$

Let $Q(\rho, \phi)$ be defined as in (2.10). The analysis which led to (2.11) shows that, for small enough ρ, $Q(\rho, \phi)$ will be strictly smaller than

$$u^*(x_0 - \rho e^{i\phi}) + u^*(x_0 + \rho e^{i\phi}) \quad \text{for} \quad \phi = 0.$$

By continuity this is true in a neighborhood of $\phi = 0$, and the argument used to establish (2.12) proves (5.2). The proof of the Φ inequality in Theorem 2 is complete.

To prove (1.6), suppose again that K is not translation of K_δ. Fix $x_0 \in \mathbb{R}$ and define for $z \in R(0, 1/2, -B, B) = R$,

$$u_1(z) = \int_{-x}^{x} u(x_0 + t + iy)\,dt, \qquad z = x + iy.$$

Then u_1 is subharmonic in r, and we proved above that $u_1(z) \leqslant u^*(z) < v^*(z)$ there. Applying Hopf's lemma, as in the proof of Proposition 2, Section 3, in $R(0, \alpha, -B, B)$ to $w_1 = u_1 - v^*$ at $z = iy$, we obtain for $|y| < B$,

$$2\left(u(x_0 + iy) - v\left(\frac{1}{2} + iy \right) \right) = \lim_{x \to 0+} \frac{w_1(x + iy)}{x} < 0,$$

which gives (1.6).

6. Proof of Theorem 2. Nonperiodic case

Suppose that $u(z) = u(z, B, K)$ is not a 1-periodic function. For $z \in R(-\infty, \infty, -B, B)$ define, for $z = x + iy$,

$$h(z) = \int_0^1 u(x + t + iy)\, dt, \qquad H(z) = \int_0^1 v(x + t + iy)\, dt,$$

where v is defined by (3.2).

Then H is actually a function of the form $a|y| + b$, since v is 1-periodic, while h is subharmonic in $R(-\infty, \infty, -B, B)$. By Theorem 1 we have $h \leqslant H$ in $R = R(-\infty, \infty, -B, B)$. Thus, either $h < H$ in R or $h \equiv H$. If $h \equiv H$ in R then $h(x) = H(x)$ is constant for $x \in \mathbb{R}$ and hence

$$0 = h'(x) = u(x + 1) - u(x).$$

This violates our non-periodicity assumption. Hence, for all

$$z \in R(-\infty, \infty, 0, B),\ z = x + iy,$$

$$\int_0^1 u(x + t + iy)\, dt < \int_0^1 v(x + t + iy)\, dt. \tag{6.1}$$

Let us suppose that the Φ-inequality statement in Theorem 2 is false. Then from Theorem 1 and (2.5) it follows that there exists $z_0 \in R(0, 1/2, -B, B) \cup \{1/2 + iy\colon -B < y < B\}$ and $E_0 \subset \mathbb{R}$ with $\operatorname{diam} E_0 \leqslant 1$, $|E_0| = 2x_0$, such that

$$\int_{E_0} u(t + iy_0)\, dt = v^*(z_0), \qquad z_0 = x_0 + iy_0. \tag{6.2}$$

The rest of the proof will consist of showing that no such E_0 can exist. Assuming that it does, let $I \subset \mathbb{R}$ be an interval with $E_0 \subset I$ and $|I| = 1$. Define

$$f(t) = u(t + 1 + iy_0) - u(t + iy_0).$$

I claim that $f(t) < 0$ for some $t \in E_0$. Suppose not. Then

$$\int_{E_0} u(t + iy_0 + 1)\, dt \geqslant \int_{E_0} u(t + iy_0)\, dt = v^*(z_0),$$

with strict inequality unless $f \equiv 0$ on E_0. The strict inequality is ruled out by (2.4) and Theorem 1. Hence

$$u(t + iy_0 + 1) = u(t + iy_0) \quad \text{for all} \quad t \in E_0.$$

Now u is harmonic in $(|\operatorname{Im} z| < B)\backslash K$, constant on $y = \pm B$, and satisfies $u(z) = u(\bar{z})$. Moreover, E_0 has positive measure, and it cannot be contained in K if (6.2) holds. One can easily show that under these conditions we must have in fact $u(z) = u(z + 1)$ for every z with $|\operatorname{Im} z| \leqslant B$. Since this contradicts non-periodicity our claim is established, and thus there exists $t_1 \in E_0 \subset I$ with $u(t_1 + iy_0 + 1) < u(t_1 + iy_0)$.

Similarly, there exists $t_2 \in E_0$ with $u(t_2 + iy_0 - 1) < u(t_2 + iy_0)$. For $\epsilon > 0$ let $R = R(t_2 - 1, t_1 + 1, -\epsilon, \epsilon)$. By continuity, when ϵ is small enough the function $u(z + iy_0)$ satisfies the hypotheses of Theorem 3 in R with $\lambda = 1/2$. Write $u^* = u^*_{1/2}$ for the corresponding $*$-function, let v, v^* be as in Sections 4, 5, and define

$$w(z) = u^*(z + iy_0) - v^*(z + iy_0).$$

Suppose first that $y_0 \neq 0$. We may assume that $y_0 > 0$, $\epsilon < y_0$, and then, by Theorem 3 and Proposition 1, w is subharmonic in $R_1 = R(0, 1/2, -\epsilon, \epsilon)$ and continuous on the closure. It follows from Theorem 1 that $w \leqslant 0$ in R_1 and, by (6.1), $w(1/2 + iy) < 0$ for $|y| < \epsilon$. Hence $w < 0$ in R. But since $E_0 \subset I \subset [t_2 - 1, t_1 + 1]$, (6.2) implies $w(x_0) = 0$, and (6.1) implies $x_0 < 1/2$, so that $x_0 \in R$. These contradictions show that $y_0 \neq 0$ is impossible.

If $y_0 = 0$ then w is subharmonic in $R \setminus [\alpha, 1/2)$, where α is defined by (3.4). The argument of the preceding paragraph rules out the possibility that $0 < x_0 < \alpha$. Suppose that $x_0 = \alpha$, so that $w(\alpha) = 0$. For our interval I define

$$u^*_I(x) = \sup \int_E u(t)\, dt$$

where the supremum is over E with $|E| = 2x$ and $E \subset I$. Then

$$u^*_I(\alpha) = u^*(\alpha) = v^*(\alpha).$$

Hence

$$0 \leqslant u^*(\alpha) - u^*(x) \leqslant u^*_I(\alpha) - u^*_I(x).$$

Using (2.3) with $I = [A_1, A_2]$ and the δ-density condition (1.1), we deduce that

$$\lim_{x \to \alpha -} \frac{u^*(\alpha) - u^*(x)}{\alpha - x} = 0.$$

It follows that w satisfies all the conditions of Proposition 2 in $R(0, 1/2, -\epsilon, \epsilon)$. By (6.1) we have $w(1/2 + iy) < 0$ for $0 < y < \epsilon$, and Proposition 2 implies that $w(\alpha) < 0$, a contradiction.

The last remaining possibility is that $\alpha < x_0 \leqslant 1/2$, $y_0 = 0$. But the δ-density condition and (2.3) imply that if (6.2) holds for such a z_0 and E_0 then it also holds for some E_1 with $|E_1| = 2\alpha$ and $y_0 = 0$. By the previous paragraph this is impossible. Thus, there are no circumstances under which (6.2) can hold. The Φ inequality for Theorem 2 is now established. The inequality $u(x + iy) < v(1/2 + iy)$ is proved exactly as in the periodic case.

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Note

After this paper was typeset I learned that Theorem 1 had already been obtained by A. E. Fryntov, (Dokl. Akad. Nauk USSR, Tom 300 (1988), No. 4, English Translation in Soviet Math. Dokl. **37**(1988), 754-755). His proof is similar to the one given here. Fryntov's article makes no mention of uniqueness questions. I learned also about a result related to Levin's conjecture which had been proved by E. V. Gleizer and A. A. Gol'dberg (Analysis Matematica **11**(1985), 23-28). I thank Professor Havin for telling me about the Gleizer-Gol'dberg result, and Professor Gol'dberg for telling me about Fryntov's.