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Inversion in Some Algebras of Singular Integral Operators

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To the memory of José Luis Rubio de Francia

Let g be a finite dimensional nilpotent Lie algebra. Suppose that g decomposes as a direct sum $g = \oplus_{\alpha \in (0,\infty)} g_\alpha$ where the Lie brackets of the summands satisfy $[g_\alpha, g_\beta] \subset g_{\alpha+\beta}$ for all $\alpha, \beta \in (0, \infty)$. Let G denote the associated connected, simply connected nilpotent Lie group. Identify elements of g with left-invariant vector fields on G, so that the exponential map $\exp : g \mapsto G$ identifies G with $\mathbb{R}^N$ for some N. Define dilations $\{\delta_r : r > 0\}$ on G by $\delta_r \exp\left(\sum X_\alpha\right) = \exp\left(\sum r^\alpha X_\alpha\right)$ for all $X_\alpha \in g_\alpha$. These are group automorphisms. Denote the homogeneous dimension of G by $d = \sum \alpha \cdot \text{dimension}\,(g_\alpha)$; of course all but finitely many of the g_α have dimension zero. Denote by $\mathcal{S}$ the Schwartz class of functions and by S the unit sphere in G, identified with $\mathbb{R}^N$. Any $x \in G \setminus \{0\}$ may be expressed uniquely as $\delta_r \theta$ for some $r > 0$, $\theta \in S$. Define $|x| = r$, and of course $|0| = 0$. In the exponential coordinates 0 is the group identity element, and $x^{-1} = -x$ for all $x \in G$. All integration over G will be with respect to Haar measure, which is unimodular and agrees with Lebesgue measure in exponential coordinates. Let I denote the identity operator.

A tempered distribution $K \in \mathcal{S}'$ is said to be homogeneous of degree $-d$ if

$$\langle K, f^r \rangle = \langle K, f \rangle$$

for all $f \in \mathcal{S}$, $r > 0$, where $f^r(x)$ denotes $f(\delta_r x)$. To any distribution ψ acting on test functions on S and annihilating constants we may assign a distribution $K \in \mathcal{S}'(G)$ as follows: For $r > 0$ and $f \in \mathcal{S}(G)$ write $f_r(\theta) = f(\delta_r \theta)$. Then define

$$\langle K, f \rangle = \int_0^\infty \langle \psi, f_r \rangle \frac{dr}{r} \cdot$$

The integral converges absolutely because $f \in \mathcal{S}$ and $\langle \psi, 1 \rangle = 0$. We denote by PV the class of all such distributions on G; they are homogeneous of degree $-d$. In fact any $K \in \mathcal{S}'$ homogeneous of degree $-d$ is necessarily of the form $K = a\delta + K'$ for some $K' \in PV$ and $a \in \mathbb{C}$, where δ denotes the Dirac mass at $0 \in G$. See [C2, Lemma 2.4] for a proof. The decomposition is unique.

For $q \in (1, \infty)$ let A_q denote the set of all operators $T: \mathcal{S}(G) \mapsto \mathcal{S}'(G)$ of the form $Tf = f * K = af + f * K'$ where $K' \in PV$ and K' restricted to S belongs to L^q. Define $|T|_q = |a| + \|K'\|_{L^q(S)}$.

Proposition 1. *For all $q \in (1, \infty)$, any $T \in A_q$ extends to an operator bounded on $L^p(G)$, for all $p \in (1, \infty)$.*

Proposition 2. *A_q is a Banach algebra for all $q \in (1, \infty)$.*

By virtue of Proposition 1, $S \circ T$ is defined for all $S, T \in A_q$, as a bounded operator on L^2. What must be proved here is that

$$S \circ T \in A_q \quad \text{and} \quad |S \circ T|_q \leqslant C_q |S|_q |T|_q.$$

(Consequently there exists an equivalent norm $|\cdot|_q'$ on A_q, such that $|I|_q' = 1$ and $|S \circ T|_q' \leqslant |S|_q' |T|_q'$ for all $S, T \in A_q$, cf. [K, p. 197].)

Let $\mathcal{B}$ denote the algebra of all bounded linear operators on $L^2(G)$, and let $|\cdot|_{\mathcal{B}}$ be the operator norm. For any algebra A and element $T \in A$, let

$$\operatorname{spec}_A(T) = \{\lambda \in \mathbb{C}: T - \lambda I \text{ is not invertible in } A\}.$$

The point of this article is

Theorem 3. *For all $q \in (1, \infty)$, for all $T \in A_q$,* $\operatorname{spec}_{A_q}(T) = \operatorname{spec}_{\mathcal{B}}(T)$.

Proposition 1 implies one inclusion, $\operatorname{spec}_{\mathcal{B}}(T) \subset \operatorname{spec}_{A_q}(T)$.

In the Euclidean case, that is when $G = \mathbb{R}^N$ with the Euclidean group law and $\{\delta_r\}$ are the usual dilations, these results were obtained by Calderón and Zygmund [CZ1], [CZ2]. This class of operators was further studied by Duoandikoetxea and Rubio de Francia [DR]. In the nilpotent case Proposition 1 is a corollary of the main theorem of [C1] and the Calderón-Zygmund method of rotations if K is odd, and in general is a direct consequence of the method of proof in that paper. The question was asked of us by J. Duoandikoetxea. Proposition 2 is a corollary of the proof of Proposition 1.

Our main result, Theorem 3, answers a question posed by A. Carbery. In the Euclidean case A_q is a commutative Banach algebra, the Fourier transform sets up a natural identification of its maximal ideal space with the unit sphere in $\mathbb{R}^N$, and the result follows from Plancherel's theorem [CZ2]. However in the nilpotent case commutativity is lost.

We were led to this issue in the nilpotent case in studying the regularity of solutions of $\bar{\partial}_b$ [C4], a partial differential operator which arises in complex analysis in several variables. A principal step in that work was an analogue of Theorem 3, in which the condition $K' \in L^q(S)$ was replaced by the stronger condition $K' \in L^q_\beta$ in an annular neighborhood of S, where L^q_β was a certain type of Sobolev space with exponent q and order of differentiability $\beta > 0$. The set of all associated operators T is still an algebra, and we showed that the spectrum in that algebra agrees with the spectrum in $\mathcal{B}$. Thus Theorem 3 is the limiting case $\beta = 0$ of the earlier result, and our first proof of it was by the same technique. The purpose of the present article is instead to present a considerably simpler proof which emphasizes the Banach algebraic point of view. This argument could also be used to give a simpler proof in the case $\beta > 0$, but we shall not give the details here.

The proof relies on estimation of the spectral radius in A_q. L. Carleson has pointed out that it is related to Beurling's proof [B] of the Wiener Tauberian Theorem. I am indebted to A. Carbery, L. Carleson, and A. McIntosh for helpful conversations concerning this work.

To begin let $A \subset B$ be two Banach algebras with different norms. Suppose $|x|_B \leqslant C|x|_A$ for all $x \in A$, and that $|xy|_B \leqslant |x|_B |y|_B$ for all $x, y \in B$. Define the spectral radius of an element of A by

$$\rho_A(x) = \lim_{n \to \infty} |x^n|_A^{1/n}.$$

The limit always exists [K, p. 208] and is finite.

Lemma 4. *Suppose that there exist $C < \infty$ and $\theta \in (0, 1]$ such that for all $x, y \in A$,*

$$|xy|_A \leqslant C(|x|_A |y|_A{}^{1-\theta} |y|_B^\theta + |y|_A |x|_A^{1-\theta} |x|_B^\theta).$$

Then $\rho_A(x) \leqslant |x|_B$ for all $x \in A$.

PROOF. The hypothesis gives

$$|x^{2k}|_A^{1/2k} \leqslant (2C)^{1/2k}(|x^k|_A^{(2-\theta)/2k} |x^k|_B^{\theta/2k}).$$

for all $x \in A, k > 0$. Passing to the limit gives

$$\rho_A(x) \leqslant \rho_A(x)^{1-\theta/2} |x|_B^{\theta/2}.$$

This simplification due to Carleson replaces the author's slightly longer proof. Now suppose that $B = \mathcal{B}$, and denote by x^* the adjoint of $x \in \mathcal{B}$.

Lemma 5. *Suppose that $A \subset \mathcal{B}$ satisfy the hypothesis of Lemma 4 and that A is closed under adjoints. Then*

$$\mathrm{spec}_A(x) = \mathrm{spec}_{\mathcal{B}}(x)$$

for all $x \in A$.

PROOF. What must be demonstrated is that if $x \in A$ is invertible in $\mathcal{B}$, then it is also invertible in A. It suffices to invert both x^*x and xx^* in A, for then x will have both a left and a right inverse. If $\epsilon > 0$ is chosen sufficiently small, then $|I - \epsilon xx^*|_{\mathcal{B}} < 1$ because x is invertible in $\mathcal{B}$. Set $y = I - \epsilon xx^* \in A$. Then $\rho_A(y) \leqslant |y|_{\mathcal{B}} < 1$ by Lemma 4, so $I - y$ may be inverted in A by summing the series $I + y + y^2 + \cdots$. But $I - y = \epsilon xx^*$. The reasoning for x^*x is the same. This argument was pointed out to us by J. L. Journé.

Introduce an auxiliary function $\zeta \in C_0^\infty(G)$, with $\zeta(x)$ a function of $|x|$ alone, identically one in a neighborhood of 0 and supported in a small neighborhood of 0. Denote by $\|\cdot\|_{q,q}$ the operator norm on $L^q(G)$.

Lemma 6. *Suppose that $T \in \mathcal{B}$ is of the form $Tf = f * K$ where $K = a\delta + K'$, $K' \in PV$. Then $|a| \leqslant C\|T\|_{\mathcal{B}}$.*

PROOF. Fix $\phi \in C_0^\infty(G)$, identically one in a neighborhood of 0. If the support of ζ is chosen to be sufficiently small then $\phi * (\zeta K') \equiv 0$ in some neighborhood of 0, since $\zeta K'$ annihilates constants by the definition of PV and the fact that ζ is itself radial. But $(1 - \zeta)K' = (1 - \zeta)K$ and the L^2 operator norm of $f \mapsto f * (1 - \zeta)K$ is majorized by $C_\zeta \|T\|_{\mathcal{B}}$ [C2, Lemma 2.10]. In a neighborhood of 0, $T\phi = a\phi + \phi * ((1 - \zeta)K)$. So $\|T(\phi)\|_{L^2} \geqslant C_1|a| - C_2|T|_{\mathcal{B}}$.

Assume for now the validity of Proposition 1. To complete the proof of Theorem 3 we show that $A = A_q$, $B = \mathcal{B}$ satisfy the hypothesis of Lemma 4. Suppose $q \neq 2$. Let $S, T \in A_q$ be given by convolution on the right with $K = a\delta + K'$ and by $L = b\delta + L'$ respectively, where $a, b \in \mathbb{C}$ and $K', L' \in PV$. Set $K_\infty = (1 - \zeta)K = (1 - \zeta)K'$ and similarly define L_∞. Since $S \circ T$ is well-defined as a bounded operator on $L^2(G)$, $L * K$ is a well-defined element of $\mathcal{S}'$, manifestly homogeneous of degree $-d$. Furthermore

$$L * K = (\zeta L) * (\zeta K) + [(1 - \zeta)L] * K + L * [(1 - \zeta)K] + [(1 - \zeta)L] * [(1 - \zeta)K]. \tag{1}$$

Since $L * K$ is homogeneous of degree $-d$, its restriction to S will be in L^q if and only if its restriction to $\{x \in G: |x| \geqslant 1\}$ is in L^q with respect to Haar measure on G. If we choose ζ to be supported sufficiently near 0, then $(\zeta L) * (\zeta K)$ will be supported in $\{|x| < 1\}$, so may be neglected.

For the second term of (1), $\|(1-\zeta)L\|_{L^q(G)} \leqslant C|T|_q$ so that

$$\begin{aligned}\|[(1-\zeta)L]*K\|_{L^q} &\leqslant \|(1-\zeta)L\|_q \|S\|_{q,q} \\ &\leqslant C|T|_q \|S\|_{q,q} \\ &\leqslant C|T|_q \|S\|_{r,r}^{1-\theta} \|S\|_{2,2}^{\theta} \\ &\leqslant C|T|_q |S|_q^{1-\theta} |S|_{\mathcal{B}}^{\theta}\end{aligned}$$

where $r \in (1,\infty)$ is chosen so that q lies in the open interval with endpoints $r, 2$. The second-to-last inequality then follows from the Riesz-Thörin interpolation theorem, and the last from Proposition 1.

The fourth term in (1) may be estimated in the same way since again by Lemma 2.10 of [C2], the L^q operator norm of $f \mapsto f*[(1-\zeta)K]$ is majorized by $C_\zeta \|S\|_{q,q}$. As for the third, it will be no loss of generality to assume that K' is real, and ζ may be chosen to be real also. Then $L*[(1-\zeta)K](-x)$ is the complex conjugate of $T^*(h)(x)$ where $h(y) = (1-\zeta(-y))K'(-y)$ and T^* denotes the adjoint of T. Inversion about 0 leaves the $L^q(G)$ norm invariant, so the reasoning of the last paragraph applies.

For any $S, T \in A_q$, $S \circ T$ is given by convolution with a distribution homogeneous of degree $-d$, which must have the form $a\delta + K'$ for some $a \in \mathbb{C}$, $K' \in PV$. The argument just completed shows that $K' \in L^q(S)$ with bound $O(|S|_q|T|_q)$, so Proposition 2 is also proved.

To prove Proposition 1 suppose that $K \in PV$ restricts to an L^q function on S. Introduce an auxiliary function $\eta \in C_0^\infty(G)$, depending only on $|x|$ and vanishing identically in some neighborhood of 0. Thus ηK annihilates constants. Choose it so that

$$\int_0^\infty \eta(\delta_s(x))\, ds/s \equiv 1$$

on $G\setminus\{0\}$. Set $K_t(x) = \eta(\delta_t^{-1}x)K(x)$. Then $\int_0^\infty K_t\, dt/t$ converges in $\mathcal{S}'$ and equals K. Set $T_t f = f * K_t$ for all $f \in \mathcal{S}$.

Lemma 7. *There exist $C < \infty$ and $\epsilon > 0$ such that for all $s, t > 0$,*

$$\|T_t T_s^*\|_{2,2} + \|T_t^* T_s\|_{2,2} \leqslant C \min(s/t, t/s)^\epsilon.$$

L^2 boundedness of $f \mapsto f*K$ then follows from the Cotlar-Stein almost-orthogonality lemma.

SKETCH OF PROOF. Consider $T_s T_t^*$ in the case when $t \leqslant s$; the case $s < t$ as well as $T_s^* T_t$ may be handled by the same method. Rescaling via the dilation δ_s then reduces matters to the case $s = 1$. Since $K_t \in L^1$ uniformly in t, the issue is the factor of t^ϵ as $t \to 0$. Let $\tilde{K}_s(x)$ be the complex conjugate of $K_s(-x)$. As in [C1] it suffices to prove that

$$\|\tilde{K}_t * (K_1 * \tilde{K}_1 * K_1 * \tilde{K}_1 * \cdots)\|_{L^q} \leqslant Ct^\epsilon \tag{2}$$

where the convolution product has a total of N factors of K_1, $\tilde{K}_1$. This holds with $\epsilon = 0$ when $q = 1$, so by interpolation it will suffice to prove it for some $\epsilon > 0$ when $q = \infty$.

Consider N points $\theta_1, \ldots, \theta_N \in S$, and regard S as a subset of the Lie algebra $\mathfrak{g}$. Write $s = (s_1, \ldots, s_N) \in \mathbb{R}^N$ and $\theta = (\theta_1, \ldots, \theta_N) \in S^N$. Consider the map from $\mathbb{R}^N \times S^N$ to G given by $F(s, \theta) = \exp(s_1\theta_1)\exp(s_2\theta_2)\cdots\exp(s_N\theta_N)$, where the product is taken with respect to the group structure on G. Denote by $J(s, \theta)$ the determinant of the $N \times N$ matrix of first partial derivatives of F with respect to s, a real analytic function of both variables. Whenever $\theta_1, \ldots, \theta_N$ are linearly independent elements of $\mathfrak{g}$, $J(0, \theta) \neq 0$. Therefore J is real analytic and not identically zero. Now $K_1 * \tilde{K}_1 * \cdots * (x)\,dx$ is the push-forward by F of the measure $\prod_{j=1}^N (\Omega_j(\theta_j))\phi(s_j)\,d\theta_j\,ds_j$, for a certain function $\phi \in C_0^\infty(\mathbb{R}^+)$, where Ω_j denotes the restriction of either K_1 or $\tilde{K}_1$ to S, with the choice alternating in j. Since $\tilde{K}_t$ annihilates constants and is supported on $\{x \in G: |x| \leqslant Ct\}$, (2) now follows directly from a combination of the proofs of [C1, Lemma 5.4] and [C3, Lemma 2.2]. See also [RS] and [C5] for arguments of this type.

Inequality (2) remains valid, with the same proof, if $\tilde{K}_t$ is replaced by any measure supported on $\{x \in G: |x| \leqslant Ct\}$ which annihilates constants; the constant on the right-hand side of (2) depends only on the total mass of the measure. Therefore the boundedness of any element of A_q on L^p, for any $p, q \in (1, \infty)$, follows from the methods of Duoandikoetxea and Rubio de Francia [DR] and the author [C1].

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