Revista Matemática Iberoamericana
Vol. 4, n.° 2, 1988

Fatou Theorems for Some Nonlinear Elliptic Equations

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Introduction

A classical Fatou theorem states:

For any given nonnegative harmonic function $u(x)$ defined in the unit ball B of $\mathbb{R}^n$ there exists a set of boundary points E_u, with surface measure equal to the surface measure of the entire boundary, such that for each point $P \in E_u$,

$$\lim_{\substack{x \to P \\ x \in B \cap \Gamma_P}} u(x) \text{ exists,}$$

where Γ_P is any finite cone with vertex P and interior contained in B. This behavior of u is often described by saying «u has a nontangential limit at almost every (with respect to surface measure) boundary point».

The above Fatou theorem has been generalized in many directions. It has been extended to nonnegative harmonic functions defined in nonsmooth domains ([6], [7]) and to nonnegative solutions of general second order elliptic equations with nonsmooth coefficients ([1], [2]). The statement of the Fatou theorem in these situations remains the same except that surface measure must be replaced by the harmonic measure associated with the governing partial differential operator.

In this work we wish to extend the Fatou theory to several classes of second order nonlinear elliptic operators. The primary motivating equation for us is the p-Laplacian,

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \qquad (1 < p < \infty).$$

Our Fatou theorem in this case takes the following form:

Let D be a bounded Lipschitz domain in $\mathbb{R}^n$ and suppose u is a nonnegative weak solution of $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$ in D. The set of boundary points E_u at which u has a nontangential limit has Hausdorf dimension $\geqslant \beta$, a positive number depending only on p, n, and the Lipschitz character of D. Recent examples of T. Wolff [15] and J. Lewis [11] show that even when the domain D is smooth the set E_u could have surface measure zero. The positive result at least guarantees that E_u is somewhat far from being empty.

The above Fatou theorem for the p-Laplacian in smooth domains of $\mathbb{R}^n$ and with $1 < p < 3 + (2/(n-2))$ was first proved by Manfredi and Weitsman in [12]. Several ideas in their paper were very useful and we would like to thank the authors for sharing them with us early on in their work. Also, while we were preparing the present paper Manfredi and Weitsman communicated to us a proof, different than the one presented here, of the above theorem in smooth domains of $\mathbb{R}^n$ and for any $1 < p < \infty$.

The same type of Fatou theorem remains valid for nonnegative solutions of other classes of nonlinear elliptic equations. We illustrate this in the case of a generalization of the p-Laplacian in the form

$$\operatorname{div}\left(\frac{f'(|\nabla u|)}{|\nabla u|}\nabla u\right) = 0$$

and in the case of solutions of a fully nonlinear equation of the type

$$F(D^2u, x) = 0.$$

In both situations the Hausdorf dimension of the set of nontangential limits is estimated from below by a positive number depending only on the structure constants (properties of f and F), dimension n, and the Lipschitz character of D. (See Sections II.2 and II.3.) Since the methods we employ for the general classes of equations are minor modifications of those used for the p-Laplacian, $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$, we have decided to give a detailed proof of the Fatou theorem only in this special case (Section II.1). The extension of the result to the other classes is discussed briefly in the final two sections of the paper.

Our technique for establishing a Fatou theory in the nonlinear setting is itself linear in nature. It frequently occurs that results for linear equations if achieved in sufficient generality can be successfully applied to nonlinear situations. The present work is one more example of this phenomenum. In order to find a Fatou theorem in the nonlinear setting we first develop a potential theory for linear uniformly elliptic equations in nondivergence form and for the corresponding adjoint equations. The coefficients of the linear operator are assumed to be smooth, but, more importantly for applications, all the constants involved in the basic estimates do *not* depend in any quantitative manner on the smoothness of the coefficients. These constants depend only on dimension, ellipticity, L^∞-bounds of the coefficients, and the Lipschitz character of the domain. In this respect our work is a continuation and refinement of Bauman's paper [1].

The primary results in the linear theory are what we refer to as Comparison Theorems for nonnegative solutions of either a nondivergence form elliptic equation or of the associated adjoint equation. In brief terms these theorems state that two nonnegative solutions of the equation (or adjoint equation) which vanish on an open portion of the boundary must vanish there at the same rate. For a precise statement of these results see Theorems I.2.2 and I.3.7.

The paper is divided into two parts. Part I is devoted to the potential theory for second order linear nonvariational elliptic operators and their adjoints, always aiming towards establishing the above mentioned Comparison Theorems. Part II applies the linear theory to prove the previously mentioned Fatou theorems for the p-Laplacian, its generalization, and for some fully nonlinear elliptic equations. Part II may be read somewhat independently of Part I, at least from the point of view of understanding how the linear theory enters in establishing the positive Hausdorf dimension of the set of boundary points at which the solution of the nonlinear problem has a nontangential limit.

I. POTENTIAL THEORY FOR NONDIVERGENCE FORM OPERATORS AND THEIR ADJOINTS

Before we begin the main body of this paper we would like to recall the basic definitions and introduce the primary notation which will be extensively used throughout the work.

Definition A. *A bounded domain D of $\mathbb{R}^n$ is called a Lipschitz domain if*

(i) *for each $Q \in \partial D$ there exists a coordinate system $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, a number r_0 and a function $\varphi: \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying $\|\nabla\varphi\|_{L^\infty(\mathbb{R}^{n-1})} \leqslant m$ such that*

(ii) $$B_{r_0}(Q) \cap D = \{(x', x_n): x_n > \varphi(x')\} \cap B_{r_0}(Q),$$

($B_{r_0}(Q)$ is the ball in $\mathbb{R}^n$ with center Q and radius r_0) and

$$B_{r_0}(Q) \cap \partial D = \{(x', \varphi(x'))\} \cap B_{r_0}(Q).$$

We may assume the numbers m and r_0 are the same for each $Q \in \partial D$ and we will say that these numbers determine the Lipschitz character of D.

Given the coordinate system about $Q \equiv (Q', Q_n) \in \partial D$, *we define for* $r \leqslant r_0$,

$$T_r(Q) = \{(x', x_n): |x' - Q'| < r, |x_n - Q_n| < mr\}$$
$$\Delta_r(Q) = T_r(Q) \cap \partial D, \qquad \psi_r(Q) = T_r(Q) \cap D,$$

and $A_r(Q) = (Q', Q_n + mr)$. *When the point* Q *is understood and unimportant in the discussion we will use the notation* T_r, Δ_r, ψ_r *and* A_r.

In Section 1 we consider elliptic operators of the form

$$Lu(x) = \sum_{i,j=1}^{n} a_{ij}(x) D^2_{x_i x_j} u(x) \qquad (x \in \mathbb{R}^n).$$

We assume the coefficients are smooth and the matrix $a(x) = (a_{ij}(x))$ is bounded, symmetric, and positive definite, uniformly in x, *i.e.* there exist positive numbers λ and Λ such that

$$\lambda |\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \leqslant \Lambda |\xi|^2$$

for all x and ξ in $\mathbb{R}^n$. We wish again to emphasize that the assumption of smoothness of $a_{ij}(x)$ is only a qualitative one. In our estimates the dependence of the constants on the coefficients will only be in terms of the ellipticity parameters λ and Λ and the dimension n.

Corresponding to the operator L we have the adjoint operator L^* defined by

$$L^* v(y) = \sum_{i,j=1}^{n} D^2_{y_i y_j}(a_{ij}(y) v(y)).$$

For a given Lipschitz domain D we let $g_D(x, y) = g_{D,L}(x, y)$ be the Green's function corresponding to the operator L and domain D. In particular

$$L(g_D(\cdot, y))(x) = 0 \quad \text{for} \quad x \in D \setminus \{y\},$$
$$L^*(g_D(x, \cdot))(y) = 0 \quad \text{for} \quad y \in D \setminus \{x\}, \quad \text{and}$$
$$g(Q, y) = 0 = g(x, Q) \quad \text{for} \quad Q \in \partial D, \quad x \in D \quad \text{and} \quad y \in D.$$

I.1. The Notion of a Normalized Adjoint Solution and the Boundary Harnack Principle

Definition B. *For a ball* $\mathcal{B}$ *let* $t\mathcal{B}$ *denote the ball concentric with* $\mathcal{B}$ *and radius equal to* t *times the radius of* $\mathcal{B}$. *Assume* $\frac{1}{4}\mathcal{B} \supset \bar{D}$ *and fix a point* $P \in \frac{3}{4}\mathcal{B} \setminus \frac{1}{2}\mathcal{B}$. *A normalized adjoint solution for* L^* *and* D (*briefly* n.a.s.) *is any function* $\tilde{v}$ *of the form*

$$\tilde{v}(y) = \frac{v(y)}{g_{\mathcal{B}}(P, y)}$$

where v is a solution of the adjoint equation $L^*v = 0$ in D. (We shall see that for our purposes there is no quantitative dependence of $\tilde{v}$ on $\mathcal{B}$ and $g_{\mathcal{B}}(P, y)$. Therefore we do not highlight them in the name.)

The notion of a normalized adjoint solution was used extensively in [1]. The main purpose of Part I is to refine the results there by dropping the dependence on the modulus of continuity of the coefficients and to extend the results to include a Comparison Theorem for two nonnegative solutions of the adjoint equation vanishing on a portion of the boundary. (See Theorem I.3.7.) We begin by recalling some basic facts from [1] concerning these functions.

Theorem I.1.1. (Interior Harnack Principle for n.a.s.) *Suppose $\tilde{v}$ is a nonnegative* n.a.s. *in a ball B_{2R} of radius $2R$. There exists a constant C depending only on n, λ and Λ such that for all $s \leqslant R$*

$$\sup\{\tilde{v}(y): y \in B_s\} \leqslant C \inf\{\tilde{v}(y): y \in B_s\}.$$

(B_s and B_{2R} are assumed concentric.)

Theorem I.1.2. (The Dirichlet Problem for n.a.s.) *Given $f \in C(\partial D)$ there exists a unique normalized adjoint solution, $\tilde{v}$, such that $\tilde{v} \in C(\bar{D})$ and $\tilde{v} = f$ on ∂D. Moreover the maximum and minimum values of $\tilde{v}$ occur on ∂D.*

Definition C. *Under the conditions of Theorem* I.1.2. *for fixed $y \in D$, the map $f \mapsto \tilde{v}(y)$ is a positive continuous linear functional on $C(\partial D)$. So by the Riesz Representation Theorem*

$$v(y) = \int_{\partial D} f(Q)\tilde{\omega}^y(dQ)$$

where $\tilde{\omega}^y$ is a regular Borel measure on ∂D. We will call $\tilde{\omega}^y$ the normalized adjoint measure at y (corresponding to L^* and D).

Notice that when D is a smooth domain

$$\tilde{\omega}^y(dQ) = \frac{\partial}{\partial v_Q} g(Q, y) g_{\mathcal{B}}(P, Q)\sigma(dQ)/g_{\mathcal{B}}(P, y)$$

where g denotes the Green's function for L and D, v_Q is the inward conormal to ∂D at Q. ($v_Q = a(Q)(N_Q)$ with $a = (a_{ij})$ and N_Q is the unit inward normal to ∂D at Q),

$$\frac{\partial}{\partial v_Q} g(Q, y) = \nabla_x g(x, y)|_{x=Q} \cdot v_Q,$$

and $\sigma(dQ)$ is surface measure. The normalized adjoint measure $\tilde{\omega}^y$ will play an important role in the study of normalized adjoint solutions.

Lemma I.1.3. *Let $\tilde{\omega}^y_{2r}$ denote the normalized adjoint measure corresponding to L^* and a ball B_{2r}. For $Q \in \partial B_{2r}$ and $0 < \delta < 1$ set $\Delta_{\delta r} = \partial B_{2r} \cap B_{\delta r}(Q)$. There exists a positive constant c depending only on the ellipticity parameters, n, and δ such that*

$$\inf_{y \in B_r} \tilde{\omega}^y_{2r}(\Delta_{\delta r}) \geqslant c$$

(B_r is assumed concentric with B_{2r}.)

PROOF. By a translation we may assume B_{2r} is centered at the origin and by a dilation we may assume $r = 1$,

$$\tilde{\omega}^y_2(\Delta_\delta) = \int_{\Delta_\delta} \frac{\partial}{\partial v_Q} g_2(Q, y) g_{\mathcal{B}}(P, Q)\sigma(dQ)/g_{\mathcal{B}}(P, y)$$

where $g_2(x, y)$ is the Green's function corresponding to B_2 and elliptic operator L. Fix a point $A \in \partial B_{3/2}$. All the succeeding constants in this lemma will depend at most on the dimension n, the parameters of ellipticity, λ and Λ, for L, and δ.

From Hopf's Lemma and Harnack's Inequality [4], [9] there exists $c_1 > 0$ such that for all $y \in B_1$, and $Q \in \partial B_2$,

$$\frac{\partial}{\partial v_Q} g_2(Q, y) \geqslant c_1 g_2(A, y).$$

(See [1, Lemma 4.3, p. 166].) Hence, for $y \in B_1$

$$\tilde{\omega}^y_2(\Delta_\delta) \geqslant c_1 g_2(A, y) \int_{\Delta_\delta} g_{\mathcal{B}}(P, Q)\sigma(dQ)/g_{\mathcal{B}}(P, y).$$

Let $g_3(x, y)$ denote the Green's function for B_3 and L. Then Harnack's property for normalized adjoint solutions (Theorem I.1.1.) implies

$$\frac{g_{\mathcal{B}}(P, Q)}{g_3(A, Q)} \geqslant c_2 \frac{g_{\mathcal{B}}(P, y)}{g_3(A, y)}$$

for all $Q \in \partial B_2$ and $y \in B_1$. Hence

$$\tilde{\omega}^y_2(\Delta_\delta) \geqslant c_1 c_2 \frac{g_2(A, y)}{g_3(A, y)} \int_{\Delta_\delta} g_3(A, Q)\sigma(dQ).$$

Once again using Harnack's inequality (for n.a.s.) we have for $y \in B_1$.

$$\frac{g_2(A,y)}{g_3(A,y)} \geqslant c_3 \frac{\int_{B_1} g_2(A,z)\,dz}{\int_{B_1} g_3(A,z)\,dz} \geqslant c_4 > 0.$$

We are now left to show

Lemma I.1.4. *Let $g_3(x,y)$ denote the Green's function for L and B_3. For $Q \in \partial B_2$ and $0 < \delta < 1$, set $\alpha_\delta = \partial B_2 \cap B_\delta(Q)$. There exists a positive constant c depending only on n, λ, Λ, and δ such that for all $A \in \partial B_{3/2}$*

$$\int_{\alpha_\delta} g_3(A,Q)\sigma(dQ) \geqslant c > 0.$$

PROOF. Let $D \subset B_2 \backslash B_{7/4}$ be a smooth domain such that $\alpha_\delta \subset \partial D$ and D contains a ball B' whose radius depends only on δ. Now let $D' \subset B_3$ be another smooth domain containing D such that $\alpha_\delta \subset D'$ and $\partial D' \cap \partial D = \partial D \backslash \alpha_\delta$. We pick a point $A' \in D' \backslash \bar{D}$ so that the positive constants in the following chain of inequalities depend only on the parameters of ellipticity, λ and Λ, the dimension n, and δ:

$$\int_{\alpha_\delta} g_3(A,Q)\sigma(dQ) \geqslant c_1 \int_{\alpha_\delta} g_3(A',Q)\sigma(dQ) \geqslant c_2 \int_{\alpha_\delta} g_{D'}(A',Q)\sigma(dQ).$$

In the final inequality above $g_{D'}$ denotes the Green's function for D' and L and the inequality results from the fact $g_{D'} \leqslant g_3$ since $D' \subset B_3$.

Since $\partial D \backslash \alpha_\delta = \partial D \cap \partial D'$, $g_{D'}(A',Q) = 0$ for $Q \in \partial D \backslash \alpha_\delta$. Therefore

$$\int_{\alpha_\delta} g_{D'}(A',Q)\sigma(dQ) = \int_{\partial D} g_{D'}(A',Q)\sigma(dQ).$$

The function $g_{D'}(A',y)$, as a function of y, satisfies

$$L^*(g_{D'}(A',\bullet))(y) = 0 \quad \text{for} \quad y \in D.$$

The function

$$W(x) = -\int_{B'} g_D(x,y)\,dy$$

satisfies $LW = \chi_{B'}$, the characteristic function of B', and $W|_{\partial D} = 0$. Hence, an integration by parts gives

$$\int_{B'} g_{D'}(A',y)\,dy = \int_D g_{D'}(A',y) LW(y)\,dy = \int_{\partial D} g_{D'}(A',Q) \frac{\partial}{\partial v_Q} W(Q)\sigma(dQ).$$

Again from [1, Lemma 4.3, p. 166] there exists $C > 0$ such that

$$\frac{\partial W}{\partial v_Q}(Q) \leqslant C \sup_D \int_{B'} g_D(x,y)\,dy \leqslant \tilde{C},$$

$\tilde{C}$ depending only on λ, Λ, n, and δ. We now have

$$\int_{\alpha_\delta} g_{D'}(A', Q)\sigma(dQ) \geqslant c \int_{B'} g_{D'}(A', y)\, dy,$$

and using a maximum principle argument the last integral is bounded below by a positive constant depending only λ, Λ, n, and δ. (See [4, Proof of Lemma 3.3].) This concludes the proof of Lemma I.1.4 and so also the proof of Lemma I.1.3.

As a consequence of Lemma I.1.3 we obtain the Hölder continuity of a nonnegative normalized adjoint solution at the open parts of the boundary where it vanishes.

Theorem I.1.5. *Let D be a Lipschitz domain in $\mathbb{R}^n$ with constants r_0 and m determining the Lipschitz character of D. Let*

$$L = \sum_{i,j=1}^{n} a_{ij}(x) D^2_{x_i x_j}$$

be a uniformly elliptic operator with parameters of ellipticity λ and Λ. Fix $Q \in \partial D$, $r \leqslant r_0/2$, and assume $\tilde{v}$ is a nonnegative normalized adjoint solution for L^* and $\psi_{2r} = \psi_{2r}(Q)$ which continuously vanishes on $\Delta_{2r} = \Delta_{2r}(Q)$. There exists a Θ, $0 < \Theta < 1$ depending only on λ, Λ, n, and m such that

$$\sup_{\psi_r} \tilde{v} \leqslant \Theta \sup_{\psi_{2r}} \tilde{v}.$$

As a consequence there exist positive constants C and α depending only on the above parameters such that for all $y \in \psi_{2r}$,

$$\tilde{v}(y) \leqslant c\left(\frac{|y - Q|}{r}\right)^{\alpha} \sup_{\psi_{2r}} \tilde{v}.$$

PROOF. We may assume $\sup_{\psi_{2r}} \tilde{v} = 1$. Let $\tilde{\omega}^y_{2r}$ denote the normalized adjoint measure for L^* and $B_{2r} = B_{2r}(Q)$. We can find a positive number δ, depending on m, and a point $\tilde{Q} \in \partial B_{2r}$ such that

$$\alpha_{\delta r} = B_{\delta r}(\tilde{Q}) \cap \partial B_{2r} \subset \partial B_{2r} \setminus \bar{D}.$$

From the maximum principle for n.a.s. (Theorem I.1.2)

$$1 - \tilde{v}(y) \geqslant \tilde{\omega}^y_{2r}(\alpha_{\delta r}).$$

Therefore, using Lemma I.1.4.

$$\inf_{\psi_r} (1 - \tilde{v}(y)) \geqslant c > 0$$

with c depending only on λ, Λ, n, and m. Then

$$\sup_{\psi_r} v \leqslant 1 - c = \Theta.$$

To conclude the Hölder continuity we observe that the last inequality above implies: for all $y \in \psi_2 - k_r(Q)$,

$$\tilde{v}(y) \leqslant \Theta^{k+1} \sup_{\psi_{2r}} \tilde{v}, \qquad k = 0, 1, 2, \ldots$$

This immediately gives the existence of positive numbers c and α depending only on λ, Λ, n, m, such that for $y \in \psi_{2r}(Q)$

$$\tilde{v}(y) \leqslant c\left(\frac{|y - Q|}{r}\right)^{\alpha} \sup_{\psi_{2r}} \tilde{v}.$$

Theorems I.1.1 and I.1.5 allow us to repeat verbatim the arguments given in [1, Lemma 2.4, p. 157], to prove a strengthened version of Theorem I.1.5.

Theorem I.1.6. (Boundary Harnack principle for n.a.s.) *Under the hypotheses of Theorem* I.1.5. *assume again $\tilde{v}$ is a nonnegative normalized adjoint solution for L^* and $\psi_{2r}(Q)$ which vanishes continuously on $\Delta_{2r}(Q)$, $Q \in \partial D$. There exists positive constants C and α depending only on λ, Λ, n, and m, such that for $y \in \psi_r(Q)$*

$$\tilde{v}(y) \leqslant C\left(\frac{|y - Q|}{r}\right)^{\alpha} \tilde{v}(A_r(Q))$$

where, recall for $Q = (Q', Q_n)$, $A_r(Q) = (Q', Q_n + mr)$.

A consequence of Theorem I.1.6 is the validity of all the results in [1] with constants depending only on the parameters of ellipticity of L, dimension, and the Lipschitz character of D. In the next section we collect some of these results.

I.2. Potential Theory for Solutions of $Lu = 0$; the Doubling Property for L-Harmonic Measure

We remind the reader that

$$L = \sum_{i,j=1}^{n} a_{ij}(x) D^2_{x_i x_j}$$

is an elliptic operator satisfying

$$\lambda|\xi|^2 \leqslant \sum_{i,j} a_{ij}(x)\xi_i\xi_j \leqslant \Lambda|\xi|^2$$

for all x and ξ in $\mathbb{R}^n$. (λ and Λ are positive constants.) Also D is a Lipschitz domain with m and r_0 describing the Lipschitz character. (See Definition A.)

Theorem I.2.1. *For $r \leqslant r_0/6$ let $g_r(x,y)$ denote the Green's function for L and $\psi_{4r}(Q)$, $Q \in \partial D$. Pick $\delta > 0$ such that $B_{2\delta r}(A_r(Q)) \subset D$. There exists a constant C depending on λ, Λ, n, and m such that for all $x \in \psi_{4r}(Q) \setminus \psi_{2r}(Q)$*

$$\int_{\psi_r(Q)} g_r(x,y)\,dy \leqslant C \int_{B_{\delta r}(A_r(Q))} g_r(x,y)\,dy.$$

Proof. Let $\mathcal{B}$ be a ball with $\frac{1}{4}\mathcal{B} \supset \bar{D}$, P a point in $\frac{3}{4}\mathcal{B} \setminus \frac{1}{4}\mathcal{B}$, and $g_{\mathcal{B}}(x,y)$ the Green's function for L and $\mathcal{B}$. Also set

$$\tilde{v}(y) = \frac{g_r(x,y)}{g_{\mathcal{B}}(x,y)} \quad \text{for} \quad y \in \psi_{Ar}(Q), \qquad y \neq x.$$

Using Theorem I.1.6, for $x \in \psi_{4r}(Q) \setminus \psi_{2r}(Q)$,

$$\int_{\psi_r(Q)} g_r(x,y)\,dy \leqslant C\tilde{v}(A_r(Q)) \int_{\psi_r(Q)} g_{\mathcal{B}}(P,y)\,dy.$$

The (interior) doubling property for nonnegative solutions of the adjoint equation [4] implies

$$\int_{\psi_r(Q)} g_{\mathcal{B}}(P,y)\,dy \leqslant C \int_{B_{\delta r}(A_r(Q))} g_{\mathcal{B}}(P,y)\,dy.$$

Harnack's inequality for $\tilde{v}$ (Theorem I.1.1) gives

$$\tilde{v}(A_r(Q)) \int_{B_{\delta r}(A_r(Q))} g_{\mathcal{B}}(P,y)\,dy \leqslant C \int_{B_{\delta r}(A_r(Q))} g_r(x,y)\,dy$$

for all $x \in \psi_{4r}(Q) \setminus \psi_{2r}(Q)$.

Theorem I.2.1. is the main result used by Bauman in [1] to pove a Comparison Theorem for solutions of $Lu = 0$. Her argument [1, p. 160-161], can be repeated to obtain

Theorem I.2.2. (Comparison Theorem for solutions of $Lu = 0$.) *Let $Q \in \partial D$ and assume u_1 and u_2 are positive solutions of $Lu = 0$ in $\psi_{2r}(Q)$ which vanish on $\Delta_r(Q)$, $r \leqslant r_0$. Then*

$$\sup_{\psi_r} \frac{u_1}{u_2} \leqslant C \frac{u_1(A_r(Q))}{u_2(A_r(Q))}$$

where as usual C depends only on λ, Λ, n, *and the Lipschitz character of* D.

Definition D. *Given an elliptic operator*

$$L = \sum_{i,j} a_{ij}(x) D^2_{x_i x_j}$$

with smooth coefficients and a bounded Lipschitz domain D, *for any given* $g \in C(\partial D)$ *there exists a unique solution* u *to the Dirichlet problem*

$$Lu = 0 \quad in \quad D, \qquad u|_{\partial D} = g.$$

Fixing a point $x \in D$, *the maximum principle implies that the map* $g \mapsto u(x)$ *is a positive linear functional on* $C(\partial D)$. *Hence there exists a unique regular Borel measure* ω^x *on* ∂D *such that for every* $g \in C(\partial D)$

$$u(x) = \int_{\partial D} g(Q) \omega^x(dQ).$$

The measure $\omega^x(\partial D)$ *is called the L-harmonic measure for D evaluated at x. At times we will emphasize the dependence of* ω^x *on L and D by using the notation* $\omega^x_{L,D}$ *or* ω^x_D.

Let us say that two objects A and B (numbers or functions) are equivalent and write $A \sim B$ if there exists a positive constant C depending at most on ellipticity parameters, dimension, and the Lipschitz character of D such that

$$\frac{1}{C} A \leqslant B \leqslant CA.$$

A consequence of the Comparison Theorem for solutions I.2.2 is

Theorem I.2.3. (Doubling property for L-harmonic measure.) *Let* ω^x *denote the L-harmonic measure for D evaluted at x. Take* $r \leqslant r_0/4$ (r_0 *as in Definition A*). *Then for all* $Q \in \partial D$ *and* $x \in D \setminus \psi_{2r}(Q)$

$$\omega^x(\Delta_r(Q)) \sim \omega^x(\Delta_{2r}(Q)).$$

PROOF. By applying Theorem I.2.2 and the maximum principle to the functions $\omega^x(\Delta_{2r})$ and $\omega^x(\Delta_r)$ in the domain $D \setminus \psi_{4r}$, we obtain

$$\omega^x(\Delta_{2r}) \leqslant C \frac{\omega^{A_{4r}}(\Delta_{2r})}{\omega^{A_{4r}}(\Delta_r)} \omega^x(\Delta_r)$$

for all $x \in D \setminus \psi_{4r}$. There exists positive constants C and α, depending on λ, Λ, n, and m, such that for $x \in \psi_{r/2}$,

$$0 \leqslant 1 - \omega^x(\Delta_r) \leqslant C\left(\frac{|x-Q|}{r}\right)^\alpha$$

and

$$0 \leqslant 1 - \omega^x(\Delta_{2r}) \leqslant C\left(\frac{|x-Q|}{r}\right)^\alpha.$$

(See [1, Lemma 2.3, p. 157].) This observation and Harnack's inequality [4], [9] imply $\omega^{A_{4r}}(\Delta_{2r})$ and $\omega^{A_{4r}}(\Delta_r)$ are each equivalent to an absolute constant depending on λ, Λ, n, and m.

Other consequences of Theorem I.2.2 are concerned with the so-called kernel function. Proofs can be found in [2].

Definition E. *The kernel function $K(x, Q)$, normalized at $x_0 \in D$, is defined for $x \in D$ and $Q \in \partial D$ by*

$$K(x, Q) = \lim_{y \to Q} \frac{g_D(x, y)}{g_D(x_0, y)}$$

where g_D denotes the Green's function corresponding to L and D. This is the same as

$$K(x, Q) = \frac{d\omega^x}{d\omega^{x_0}}(Q).$$

In fact we have the following characterization of the kernel function.

Theorem I.2.4. *The kernel function is uniquely determined by the conditions*

(*a*) $L(K(\cdot, Q))(x) = 0$ *for* $x \in D$.
(*b*) $K(x_0, Q) = 1$ *for all* $Q \in \partial D$.
(*c*) *If* $Q' \in \partial D$ *and* $Q' \neq Q$, *then* $\lim_{x \to Q'} K(x, Q) = 0$.

The relationships between L-harmonic measure evaluated at x_0 and the kernel function are given in the next theorem.

Theorem I.2.5. *Let $K(x, Q)$ be the kernel function for L and D, normalized at x_0, and set $\omega^x = \omega^x_{L,D}$. Then there exists a sequence of constants $\{C_j\}$ such that C_j depends only on λ, Λ, n, m, r_0, and the* dist $(x_0, \partial D)$, $\sum C_j < \infty$, *and*

(i) $K(A_r(Q), Q) \sim \omega^{x_0}(\Delta_r(Q))^{-1} \left(Q \in \partial D, r \leqslant \frac{r_0}{2}\right)$

(ii) *for t sufficiently small and all* $\tilde{Q} \in \partial D$

$$\sup \{K(A_t(\tilde{Q}), Q): Q \in \Delta_{2^{j+1}t}(\tilde{Q}) \backslash \Delta_{2^j t}(\tilde{Q})\} \leqslant \frac{C_j}{\omega^{x_0}(\Delta_{2^j t}(\tilde{Q}))}.$$

We conclude Section I.2 by remarking that via the techniques in [8], Theorems I.2.1.-I.2.3. imply the Hölder continuity of $K(x, \cdot)$ on ∂D with x fixed in D. The Hölder exponent can be taken depending only on λ, Λ, n, and m, and the Hölder norm can be bounded by a constant depending on these numbers, r_0, and the distances of x and x_0 to ∂D. This Hölder continuity and the properties described in Theorem I.2.5 of $K(x, Q)$ are main ingredients in the proofs of the Fatou theorems for nonlinear equations described in Part II. We approach the Hölder continuity of the kernel function from a different point of view than the one taken in [8]. We will prove that for fixed x and $x_0 \in D$ the function

$$g_D(x, y)/g_D(x_0, y)$$

is Hölder continuous in y near and up to the boundary of D. This is the subject of the next section.

I.3. A Comparison Theorem for Normalized Adjoint Solutions

We begin Section I.3 by establishing the doubling property for normalized adjoint measure.

Lemma I.3.1. *Let* $Q \in \partial D$, $0 < r \leqslant r_0/2$, *and denote by* $\tilde{\omega}_r^y$ *the normalized adjoint measure corresponding to* L^* *and* $\psi_r(Q)$. *Let*

$$\alpha_r = \partial\psi_r(Q) \cap D \quad \textit{and} \quad \beta_r = B_{\delta_r/2}(A_r(Q)) \cap \partial\psi_r(Q),$$

where $\delta_r = \min\{\operatorname{dist}(A_r(Q), \partial D), r\}$. *Then for* $y \in \psi_{r/2}(Q)$,

$$\tilde{\omega}_r^y(\alpha_r) \sim \tilde{\omega}_r^y(\beta_r).$$

PROOF. Since β_r is a smooth portion of the boundary,

$$\tilde{\omega}_r^y(\beta_r) = \int_{\beta_r} \frac{\partial}{\partial v_Q} g_r(Q, y) g_{\mathcal{B}}(P, Q)\sigma(dQ)/g_{\mathcal{B}}(P, y)$$

where g_r is the Green's function corresponding to L and $\psi_r(Q)$. For $y \in \psi_{r/2}(Q)$, Hopf's Lemma [1, Lemma 4.3, p. 166], gives

(I.3.2) $$\tilde{\omega}_r^y(\beta_r) \sim \frac{g_r(A_{r/2}(A), y)}{r g_{\mathcal{B}}(P, y)} \int_{\beta_r} g_{\mathcal{B}}(P, Q)\sigma(dQ).$$

The remaining part of the proof is dedicated to showing that for $y \in \psi_{r/2}(Q)$

$$\tilde{\omega}_r^y(\alpha_r) \leqslant C \frac{g_r(A_{r/2}(Q), y)}{r g_{\mathcal{B}}(P, y)} \int_{\beta_r} g_{\mathcal{B}}(P, Q)\sigma(dQ).$$

Choose h a smooth function in $\mathbb{R}^n$ such that $h = 1$ in $T_r(Q) \setminus T_{r/3}(Q)$, $h = 0$ in $T_{r/4}(Q)$, and $0 \leqslant h \leqslant 1$ everywhere. Then

(I.3.3) $$\tilde{\omega}^y(\alpha_r) \leqslant \tilde{\omega}_r^y(h) = \tilde{\omega}_r^y(h) - h(y) \quad \text{for} \quad y \in \psi_{r/4}.$$

The function $g_{\mathcal{B}}(P, y)\tilde{\omega}_r^y(h)$ is a solution in ψ_r of the adjoint equation $L^*v = 0$ with boundary values $g_{\mathcal{B}}(P, Q)h(Q)$. Hence

(I.3.4) $$g_{\mathcal{B}}(P, y)[\tilde{\omega}_r^y(h) - h(y)] = \int_{\psi_r} L^*[g_{\mathcal{B}}(P, z)h(z)]g_r(z, y)\, dz.$$

Now

$$L^*[g_{\mathcal{B}}(P, z)h(z)] = g_{\mathcal{B}}(P, z)Lh(z) + \sum_{i,j} D_{z_i}(a_{ij}(z)g_{\mathcal{B}}(P, z))D_{z_j}h(z).$$

We may choose h so that

$$|Lh| \leqslant \frac{C}{r^2}, \qquad |\nabla h| \leqslant \frac{C}{r},$$

and ∇h is supported in $T_{r/3}(Q) \setminus T_{r/4}(Q)$. Therefore, for $y \in \psi_{r/6}(Q)$ the integral in I.3.4 can be written, after an integration by parts, as

$$2\int_{\psi_{r/3}\setminus\psi_{r/4}} g_{\mathcal{B}}(P, z)Lh(z)g_r(z, y)\, dz + 2\int_{\psi_{r/3}\setminus\psi_{r/4}} g_{\mathcal{B}}(P, z)a(z)\nabla h(z) \cdot \nabla_z g_r(z, y)\, dz \equiv I + II.$$

Applying to $g_r(\bullet, y)$ the «boundary Harnack principle for positive solutions of $Lu = 0$", [1 Lemma 2.4, p. 157], $g_r(z, y) \leqslant C g_r(A_{r/2}(Q), y)$ for all $z \in \psi_{r/3} \setminus \psi_{r/4}$ and $y \in \psi_{r/6}$. Hence

$$|I| \leqslant \frac{C}{r^2} g_r(A_{r/2}(Q), y) \int_{\psi_{r/3}\setminus\psi_{r/2}} g_{\mathcal{B}}(P, z)\, dz, \qquad y \in \psi_{r/6}.$$

Again from the interior doubling property of the measure $g_{\mathcal{B}}(P, z)\, dz$, [4],

$$\int_{\psi_{r/3}} g_{\mathcal{B}}(P, z)\, dz \leqslant C \int_{B_{\delta_r/2}(A_r(Q))} g_{\mathcal{B}}(P, z)\, dz.$$

Let $g_{\delta_r}(x, y)$ denote the Green's function for L and $B_{\delta_r}(A_r(Q))$. Then for suitably chosen $\eta > 0$ and from Theorem I.1.1.,

$$\int_{B_{\delta_r/2}(A_r(Q))} g_{\mathcal{B}}(P, z)\, dz \leqslant C \frac{g_{\mathcal{B}}(P, A_r(Q))}{g_{\delta_r}(A_{\eta r}(Q), A_r(Q))} \int_{B_{\delta_r/2}(A_r(Q))} g_{\delta_r}(A_{\eta r}(Q), z)\, dz.$$

Using a dilation with center $A_r(Q)$ we have

$$\int_{B_{\delta_r/2}(A_r(Q))} g_{\delta_r}(A_\eta^r(Q), z)\, dz \sim r^2.$$

With this same dilation and Lemma I.1.4.

$$\int_{\beta_r} g_{\delta_r}(A_{\eta r}(Q), Q)\sigma(dQ) \sim r.$$

Hence

$$|I| \leqslant \frac{C}{r} g_r(A_{r/2}(Q), y) \frac{g_{\mathcal{B}}(P, A_r(Q))}{g_{\delta_r}(A_{\eta r}(Q), A_r(Q))} \int_{\beta_r} g_{\delta_r}(A_{\eta r}(Q), Q)\sigma(dQ),$$

and from Theorem I.1.1

$$\text{(I.3.5)} \qquad |I| \leqslant \frac{C}{r} g_r(A_{r/2}(Q), y) \int_{\beta_r} g_{\mathcal{B}}(P, Q)\sigma(dQ) \qquad (y \in \psi_{r/6}).$$

We now handle

$$II \equiv 2 \int_{\psi_{r/3} \setminus \psi_{r/4}} g_{\mathcal{B}}(P, z) a(z) \nabla h(z) \cdot \nabla_z g_r(z, y)\, dz.$$

By Schwartz's inequality

$$|II| \leqslant \frac{C}{r} \left(\int_{\psi_r \setminus \psi_{r/4}} g_{\mathcal{B}}(P, z)\, dz \right)^{1/2} \left(\int_{\psi_r \setminus \psi_{r/4}} g_{\mathcal{B}}(P, z) |\nabla_z g_r(z, y)|^2\, dz \right)^{1/2}.$$

Choose $l(z) \in C_0^\infty(\mathbb{R}^n)$, $0 \leqslant l \leqslant 1$, satisfying $l = 1$ on $T_r \setminus T_{r/4}$, $l = 0$ on $T_{r/5}$, $|\nabla l| \leqslant cr^{-1}$ and $|D^2_{z_i z_j} l| \leqslant cr^{-2}$ for all i and j. For $z \in T_r \setminus T_{r/4}$ and $y \in \psi_{r/6}$,

$$\begin{aligned} |\nabla_z (L(z) g_r(z, y))|^2 &\leqslant c a(z) \nabla_z (l(z) g_r(z, y)) \cdot \nabla_z (l(z) g_r(z, y)) \\ &\leqslant c\{ L[(l(z) g_r(z, y))^2] - 2 l(z) L(l(z)) g_r^2(z, y) \\ &\qquad + g_r^2((z, y) a(z) \nabla l(z)) \cdot \nabla l(z) \\ &\qquad\quad - 2 l(z) g_r(z, y) a(z) \nabla l(z) \cdot \nabla_z (l(z) g_r(z, y)) \}. \end{aligned}$$

Since $l(\cdot) g_r(\cdot, y) = 0$ on $\partial(\psi_r \setminus \psi_{r/5})$, for $y \in \psi_{r/6}$

$$\int_{\psi_r \setminus \psi_{r/5}} g_{\mathcal{B}}(P, z) L[(l(z) g_r(z, y))^2]\, dz = 0.$$

Therefore, applying again to $g_r(\cdot, y)$ the «boundary Harnack principle for positive solutions of $Lu = 0$» [1, Lemma 2.4, p. 157], it easily follows that for $y \in \psi_{r/6}$

$$\int_{\psi_r \setminus \psi_{r/5}} g_{\mathcal{B}}(P,z) |\nabla_z(l(z) g_r(z,y))|^2 \, dz \leqslant \frac{C}{r^2} g_r^2(A_{r/2}(Q), y) \int_{\psi_r \setminus \psi_{r/5}} g_{\mathcal{B}}(P,z) \, dz.$$

Finally from the arguments controlling I

$$\text{(I.3.6)} \qquad |II| \leqslant \frac{C}{r^2} g_r(A_{r/2}(Q), y) \int_{\psi_r \setminus \psi_{r/5}} g_{\mathcal{B}}(P,z) \, dz$$

$$\leqslant \frac{C}{r} g_r(A_{r/2}(Q), y) \int_{\beta_r} g_{\mathcal{B}}(P,Q) \sigma(dQ).$$

The conclusion from I.3.2-I.3.6 is: for $y \in \psi_{r/6}(Q)$,

$$\tilde{\omega}_r^y(\alpha_r) \leqslant \frac{C}{r} \frac{g_r(A_{r/2}(Q), y)}{g_{\mathcal{B}}(P,y)} \int_{\beta_r} g_{\mathcal{B}}(P,Q) \sigma(dQ) \sim \tilde{\omega}_r^y(\beta_r).$$

We are now in position to prove

Theorem I.3.7. (Comparison Theorem for adjoint solutions.) *Let D be a Lipschitz domain whose Lipschitz character is determined by the numbers r_0 and m. Assume L is a uniformly elliptic operator with parameters of ellipticity λ and Λ (and having smooth coefficients). Let v and w be two nonnegative adjoint solutions, i.e. $L^*v = L^*w = 0$, in $\psi_{2r}(Q)$, $Q \in \partial D$, continuously vanishing on $\Delta_{2r}(Q)$ $(r \leqslant r_0/2)$.*

Then

$$\frac{v(y)}{w(y)} \sim \frac{v(A_r(Q))}{w(A_r(Q))} \quad \textit{for all} \quad y \in \psi_r(Q).$$

PROOF. The functions

$$\tilde{v}(y) = \frac{v(y)}{g_{\mathcal{B}}(P,y)} \quad \text{and} \quad \tilde{w}(y) = \frac{w(y)}{g_{\mathcal{B}}(P,y)}$$

are normalized adjoint solutions satisfying the hypothesis of Theorem I.1.5. From this theorem, Lemma I.3.1, and the technique in [1, Proof of Theorem 2.1, p. 160], I.3.7 follows for v and w replaced by $\tilde{v}$ and $\tilde{w}$. Since $\tilde{v}/\tilde{w} = v/w$ the conclusion also holds for v and w.

Corollary I.3.8. *Let v and w satisfy the hypotheses of Theorem I.3.7 in $\psi_{r_0}(Q)$, $Q \in \partial D$. Then there exist positive constants C and α depending on λ,*

Λ, n, and m such that

$$\left|\frac{v(y)}{w(y)} - \frac{v(y')}{w(y')}\right| \leqslant C\frac{v(A_{r_0}(Q))}{w(A_{r_0}(Q))}\left(\frac{|y-y'|}{r_0}\right)^{\alpha}$$

for all y and y' in $\psi_{r_0/2}(Q)\cap D$.

PROOF. The proof follows the method of Moser who obtains Hölder continuity from the uniform Harnack principle [12]. In fact that argument and Theorem I.1.5 give the above result when $y \in D\cap\overline{\psi_{r_0/2}(Q)}$ and $y' \in B_{s/2}(y)$, with $s = \operatorname{dist}(y, \partial D)$. Hence it is enough to show the Hölder continuity at a boundary point $Q_0 \in \Delta_{r_0/2}(Q)$. We may also assume $v(A_{r_0}(Q)) = w(A_{r_0}(Q)) = 1$.

Set

$$M(s) = \sup\left\{\frac{v(y)}{w(y)} : y \in \psi_s(Q_0)\right\}$$

and

$$m(s) = \inf\left\{\frac{v(y)}{w(y)} : y \in \psi_s(Q_0)\right\}\cdot$$

Then for $s \leqslant r_0/4$,

$$M(s) - \frac{v}{w} = \frac{M(s)w - v}{w}$$

and

$$\frac{v}{w} - m(s) = \frac{v - m(s)w}{w}$$

are quotients of positive adjoint solutions in $\psi_s(Q_0)$ which vanish on $\Delta_s(Q_0)$. By Theorem I.3.7,

$$\sup_{\psi_{s/2}(Q_0)}\left(M(s) - \frac{v}{w}\right) = C\inf_{\psi_{s/2}(Q_0)}\left(M(s) - \frac{v}{w}\right)$$

and

$$\sup_{\psi_{s/2}(Q_0)}\left(\frac{v}{w} - m(s)\right) \leqslant C\inf_{\psi_{s/2}(Q_0)}\left(\frac{v}{w} - m(s)\right)$$

i.e.

$$M(s) - m(s/2) \leqslant C(M(s) - M(s/2))$$

and

$$M(s/2) - m(s) \leqslant C(m(s/2) - m(s)).$$

Adding the final two inequalities we obtain

$$M(s/2) - m(s/2) \leqslant \Theta(M(s) - m(s))$$

where

$$0 < \Theta = \frac{C-1}{C+1} < 1.$$

Iterating gives

$$M(s) - m(s) \leqslant \Theta^{-1}\left(\frac{s}{r_0}\right)^{\alpha}(M(r_0) - m(r_0))$$

with

$\alpha = -\log_2 \Theta$.

This concludes the proof of the Corollary.

When v and w represent the Green's function for an elliptic operator at two fixed poles we obtain the following important

Theorem I.3.9. *Assume $L = \sum a_{ij}(x)D^2_{x_i x_j}$, an elliptic operator, and D, a Lipschitz domain, satisfy the hypotheses of Theorem* I.3.7. *Let $g(x, y)$ denote the Green's function corresponding to L and D. If x and x_0 are fixed points of D there exist positive constants C and α, α depending only on λ, Λ, n, m, r_0 but C depending in addition on $\delta = \min\{\operatorname{dist}(x, \partial D), \operatorname{dist}(x_0, \partial D)\}$, such that*

$$\left|\frac{g(x,y)}{g(x_0,y)} - \frac{g(x,y')}{g(x_0,y')}\right| \leqslant C|y' - y|^{\alpha}$$

for all y and y' belonging to $\{z \in D\colon \operatorname{dist}(z, \partial D) \leqslant \delta/2\}$. As a consequence, the kernel function $K(x, Q)$ corresponding to L and D and normalized at x_0 satisfies

$$|K(x, Q) - K(x, Q')| \leqslant C|Q - Q'|^{\alpha}$$

for all Q and Q' on ∂D; C and α as above.

II. FATOU THEOREMS FOR SOLUTIONS OF NONLINEAR EQUATIONS

In Part II, for the sake of simplicity but without any loss of generality, we consider Lipschitz domains D starshaped with respect to the origin.

II.1. The p-Laplacian

The first application we give of the potential theory developed in Part I is to the boundary behavior of nonnegative p-harmonic functions.

Definition F. *Fix $1 < p < \infty$. A function $u(x)$, $x \in D$, is a solution of the equation*

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \text{ in } D \tag{II.1.1}$$

if $u \in W^{1,p}_{\text{loc}}(D) = \{u \in L^p_{\text{loc}}(D)\colon \nabla u \in L^p_{\text{loc}}(D)\}$ and

$$\int_D |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \, dx = 0$$

for all $\varphi \in C_0^\infty(D)$. A function u which is a solution of II.1.1. *is called a p-harmonic function and Δ_p is called the p-Laplacian.*

Despite the degenerate character of II.1.1. solutions belong to $C^{1,\alpha}_{\text{loc}}(D)$ with $\alpha = \alpha(p, n) > 0$. (See [3] and [10].) Concerning boundary behavior we have

Theorem II.1.2. *Let u be a nonnegative p-harmonic function in D. Then $\{Q \in \partial D\colon u$ has nontangential limit at $Q\}$ has Hausdorf dimension $\geqslant \beta > 0$ where β depends only on p, n, and the Lipschitz character of D.*

PROOF. Since D is starshaped with respect to the origin, for $0 < r < 1$, we set $u_r(x) = u(rx)$ and obtain a p-harmonic function u_r in D which is $C^{1,\alpha}$ in $\bar{D}$, the closure of D. Now consider the regularized p-harmonic operator

$$\Delta_p^\epsilon v = \operatorname{div}((|\nabla v|^2 + \epsilon)^{(p-2)/2}\nabla v), \qquad \epsilon > 0, \tag{II.1.3}$$

and denote by $v_{\epsilon,r}$ the unique solution of the Dirichlet problem

$$\Delta_p^\epsilon v = 0 \quad \text{in} \quad D, \qquad v|_{\partial D} = u_r|_{\partial D}.$$

Because of the nondegeneracy of Δ_p^ϵ, $v_{\epsilon,r}$ belongs to $C^\infty(D)$. Therefore we can perform the differentiation in II.1.3 and, after dividing by $(\epsilon + |\nabla v_{\epsilon,r}|^2)^{(p-2)/2}$, we see that $v_{\epsilon,r}$ is a solution of the linear equation

$$L^u_{\epsilon,r} w = \sum_{i,j=1}^n \left(\delta_{ij} + (p-2)\frac{D_{x_i}v_{\epsilon,r}D_{x_j}v_{\epsilon,r}}{\epsilon + |\nabla v_{\epsilon,r}|^2}\right) D^2_{x_i x_j} w = 0.$$

(We have added the superscript u to the operator to keep in mind the dependence of the coefficients on the p-harmonic function u.) The operator $L^u_{\epsilon,r}$ has parameters of ellipticity λ, Λ depending only on p; namely, $\lambda = \min\{1, p-1\}$ and $\Lambda = \min\{1, p-1\}$, and the coefficients belong to $C^\infty(D)$.

A key property of $v_{\epsilon,r}$ is that $v_{\epsilon,r} \to u_r$ in $W^{1,p}(D)$ as $\epsilon \to 0$, [10]. Also as $\epsilon \to 0$ $v_{\epsilon,r} \to u_r$ uniformly on compact subsets of D.

We make one more dilation: the function $v_{\epsilon,r}(rx)$ satisfies an elliptic equation $\tilde{L}^u_{\epsilon,r}w = 0$ of the same form as that defined by $L^u_{\epsilon,r}$. The parameters of ellipticity can be taken dependent only on p and the coefficients now belong to $C^\infty(\bar{D})$. For $v_{\epsilon,r}(rx)$ we have the following representation formula:

$$\text{(II.1.4)} \qquad v_{\epsilon,r}(rx) = \int_{\partial D} v_{\epsilon,r}(rQ)\tilde{K}^u_{\epsilon,r}(x, Q)\tilde{\omega}^u_{\epsilon,r}(dQ)$$

where $\tilde{\omega}^u_{\epsilon,r}$ denotes the $\tilde{L}^u_{\epsilon,r}$-harmonic measure for D evaluated at the origin and $\tilde{K}^u_{\epsilon,r}(x, Q)$ denotes the kernel function associated with $\tilde{L}^u_{\epsilon,r}$ and D, normalized at the origin.

Since

$$\int_{\partial D} \tilde{\omega}^u_{\epsilon,r}(dQ) = 1,$$

we can select a sequence $\epsilon_j \to 0$ such that $\tilde{\omega}^u_{\epsilon_j,r}(dQ)$ converges weakly (as $j \to \infty$) to a regular Borel measure $\tilde{\omega}^u_r(dQ)$. Also, from the local uniform convergence of $v_{\epsilon,r}$ to u_r in D,

$$\lim_{j\to\infty} v_{\epsilon_j,r}(rQ) = u(r^2Q)$$

uniformly on ∂D. Finally recall that each kernel function $\tilde{K}^u_{\epsilon,r}(x,Q)$ is a solution in x of a second order elliptic equation with parameters of ellipticity depending only on p. Since $\tilde{K}^u_{\epsilon,r}(0, Q) = 1$. $\tilde{K}^u_{\epsilon,r}(x, Q)$ is locally Hölder continuous in X with local Hölder exponent and norm independent of Q, ϵ, r, and u. Also from Theorem I.3.9 when x varies over a fixed compact subset of D. $\tilde{K}^u_{\epsilon,r}(x, Q)$ is Hölder continuous on ∂D, with Hölder exponent depending only on p, n, and the Lipschitz character of D, and Hölder norm bounded independently of ϵ and r. Hence we may assume the sequence $\{K^u_{\epsilon_j,r}(x, Q)\}$ converges uniformly on ∂D for each fixed $x \in D$. The function

$$\tilde{K}^u_r(x, Q) = \lim_{j\to\infty} \tilde{K}^u_{\epsilon_j,r}(x, Q)$$

satisfies the same Hölder continuity just described for each kernel function of the sequence. We can now allow ϵ_j tend to zero in II.1.4 and obtain the representation

$$\text{(II.1.5)} \qquad u(r^2x) = \int_{\partial D} u(r^2Q)\tilde{K}^u_r(x, Q)\tilde{\omega}^u_r(dQ).$$

We repeat once more the arguments of the previous paragraph:

$$u(0) = \int_{\partial D} u(r^2Q)\tilde{\omega}^u_r(dQ) \quad \text{and} \quad 1 = \int_{\partial D} \tilde{\omega}^u_r(dQ)$$

imply there exists a sequence $r_j \nearrow 1$ and regular Borel measures $\mu^u(dQ)$ and $\tilde{\omega}^u(dQ)$ such that as $r_j \nearrow 1$, $u(r_j^2 Q)\tilde{\omega}^u_{r_j}(dQ)$ converges weakly to $\mu^u(dQ)$ while $\tilde{\omega}^u_{r_j}(dQ)$ converges weakly to $\tilde{\omega}^u(dQ)$. As already noted in the previous paragraph $\tilde{K}^u_r(x, Q)$ is Hölder continuous in x and Q for $Q \in \partial D$ and x restricted to a compact subset of D. The Hölder norm can be bounded and the Hölder exponent can be written independently of r. Hence as $r_j \nearrow 1$ we may assume

$$\tilde{K}^u_{r_j}(x, Q) \to \tilde{K}^u(x, Q)$$

uniformly on ∂D for each $x \in D$. Letting $r_j \nearrow 1$ we obtain the final representation

$$u(x) = \int_{\partial D} \tilde{K}^u(x,Q)\mu^u(dQ). \tag{II.1.6}$$

The measures $\tilde{\omega}^u_{\epsilon,r}(dQ)$ enjoy the doubling condition of Theorem I.2.3 (for $x = 0$) with a doubling constant depending only on p, n, and the Lipschitz character of D. The same property holds for the weak limits $\tilde{\omega}^u_r$ and $\tilde{\omega}^u$. On the other hand, the relationships between $\tilde{K}^u_{\epsilon,r}$ and $\tilde{\omega}^u_{\epsilon,r}$ expressed in Theorem I.2.5 carry over to $\tilde{K}^u$ and $\tilde{\omega}^u$. These relationships and the doubling property of $\tilde{\omega}^u$ allow us to apply the procedure in [2] obtaining the existence of nontangential limits of u at all points $Q \in \partial D$ except for a set of $\tilde{\omega}^u$ measure zero.

In particular $\tilde{\omega}^u(\{Q \in \partial D\colon u$ has nontangential limit at $Q\}) = \tilde{\omega}^u(\partial D) = 1$. But the doubling property of $\tilde{\omega}^u$; namely.

$$\tilde{\omega}^u(\Delta_{2r}) \leqslant C\tilde{\omega}^u(\Delta_r) \qquad (r \leqslant r_0)$$

with C depending only on p, n, and the Lipschitz character of D implies there exist positive constants C and β also depending only on these parameters such that

$$\tilde{\omega}^u(\Delta_r) \leqslant Cr^\beta \qquad (r \leqslant r_0).$$

In particular the Hausdorf dimension of the $\{Q \in \partial D$: u has nontangential limit at $Q\}$ is $\geqslant \beta$.

In the next two sections we discuss two other classes of equations for which one can obtain a Fatou theorem of the type of Theorem II.1.2. Since the proofs follow so closely the one for the p-Laplacian, we will only briefly indicate the arguments.

II.2. The Equation $\operatorname{div}(f'(|\nabla u|)\nabla u/|\nabla u|) = 0$

We begin with a function $f(t)$ defined for $t > 0$, in $C^2(0, \infty)$, and which is positive, increasing and convex in $(0, \infty)$ see [5]. We also assume there exists

$C > 0$ such that (II.2.1)

(i) $\frac{1}{C}(t^p - 1) \leqslant tf'(t) \leqslant C(t^p + 1)$, $t > 0$, p fixed $1 < p < \infty$.

(ii) $\frac{1}{C} \leqslant t\frac{f''(t)}{f'(t)} \leqslant C, \qquad t > 0.$

We note that II.2.1 (ii) implies $\lim_{t \to 0+} f'(t) = 0$.

Definition G. *We say that $u \in W^{1,p}(D)$ is a weak solution in D of*

$$\operatorname{div}\left(\frac{f'(|\nabla u|)}{|\nabla u|}\nabla u\right) = 0 \tag{II.2.2}$$

if

$$\int_D \frac{f'(|\nabla u(x)|)}{|\nabla u(x)|}\nabla u(x) \cdot \nabla \varphi(x) = 0$$

for all $\varphi \in C_0^\infty(D)$. (The integrand in the integral is intepreted to be 0 *at each x where $\nabla u(x) = 0$.)*

Theorem II.2.3. *Assume u is a nonnegative weak solution of* II.2.2 *in a (starshaped) Lipschitz domain D. There exists $\beta > 0$ depending only on the constant C in* (II.2.1) (i) *and* (ii), *dimension, and the Lipschitz character of D, such that the Hausdorf dimension of $\{Q \in \partial D$: u has nontangential limit at $Q\}$ is $\geqslant \beta$.*

PROOF. We proceed in the manner of Theorem II.1.2. For $0 < r < 1$ set $u_r(x) = u(rx)/r$. This function is a solution of (II.2.2) with data $u(rQ)/r$. For $\epsilon > 0$ let $u_{\epsilon,r}$ be the solution to the Dirichlet problem

$$\operatorname{div}\left(\frac{f'[(|\nabla v|^2 + \epsilon)^{1/2}]}{(|\nabla v|^2 + \epsilon)^{1/2}}\nabla v\right) = 0,$$

$$v|_{\partial D} = u_r|_{\partial D}.$$

Again the nondegenerate nature of this problem implies $u_{\epsilon,r}$ is smooth in D. ($u_{\epsilon,r}$ at least belongs to $W^{2,p}(D)$ for all finite p.) $u_{\epsilon,r}$ is itself a solution of the nonvariational equation

$$0 = L_{\epsilon,r}(w)(x) = \sum_{i,j=1}^n a^{\epsilon,r}(x) D^2_{x_i x_j} w(x),$$

where for $t = (|\nabla u_{\epsilon,r}(x)|^2 + \epsilon)^{1/2}$

$$a_{ij}^{\epsilon,r}(x) = \delta_{ij} + \frac{(f''(t)t - f'(t))}{f'(t)} \frac{D_{x_i}u_{\epsilon,r}(x)D_{x_j}u_{\epsilon,r}(x)}{(|\nabla u_{\epsilon,r}(x)|^2 + \epsilon)}.$$

Condition (II.2.1)-(ii) implies $L_{\epsilon,r}$ is uniformly elliptic with parameters of ellipticity depending on the constant C in (II.2.1)-(ii).

The function $u_{\epsilon,r} \to u_r$ as $\epsilon \to 0$ uniformly on compact subsets of D and the function $u_{\epsilon,r}(x) = u_{\epsilon,r}(rx)/r$ satisfies the same elliptic equation as $u_{\epsilon,r}$ with $\nabla u_{\epsilon,r}(x)$ in the coefficients replaced by $\nabla u_{\epsilon,r}(rx)$. We now repeat verbatim the remaining arguments in the case of the p-Laplacian to conclude the proof of Theorem II.2.3.

II.3. Fully Nonlinear Equations

As indicated in the introduction, a Fatou theorem holds for nonnegative solutions of fully nonlinear equations. Specifically we consider equations of the type

$$\text{(II.3.1)} \qquad F(D^2u(x), x) = 0 \quad \text{in} \quad D,$$

where $F(M, x)$ is smooth with respect to the $n \times n$ matrix variables M and n-dimensional variable x in D. We further assume a uniformly ellipticity condition

$$\lambda|\xi|^2 \leqslant \sum_{i,j=1}^{n} \frac{\partial F}{\partial M_{ij}}(M, x)\xi_i \xi_j \leqslant \Lambda|\xi|^2$$

for all $M \in \mathbb{R}^{n^2}$, $x \in D$, $\xi \in \mathbb{R}^n$.

Theorem II.3.2. *Assume $u \in C^2(D)$ is a nonnegative solution of* II.3.1 *in D, a Lipschitz domain starshaped with respect to* 0. *There exists a positive number β depending only on λ, Λ, n, and the Lipschitz character of D such that if $F(0, x) \in L^n(D)$, then the Hausdorf dimension of $\{Q \in \partial D$: u has non-tangential limit at $Q\}$ is $\geqslant \beta$.*

PROOF. Again the proof follows the techniques used for p-harmonic functions. In fact, u satisfies the linear equation

$$Lv = \sum_{i,j} a_{ij}(x) D^2_{x_i x_j} v(x) = f(x)$$

where

$$a_{ij}(x) = \int_0^1 \frac{\partial F}{\partial M_{ij}}(tD^2u(x), x)\, dt \quad \text{and} \quad f(x) = -F(0, x).$$

The function

$$u_r(x) = u(rx), \qquad 0 < r < 1,$$

satisfies

$$L_r u_r = \sum_{i,j} a_{ij}(rx) D^2_{x_i x_j} u_r(x) = f(rx) r^2$$

and has the representation

$$u(rx) = \int_{\partial D} u(rQ) K^u_r(x, Q)\omega^u_r(dQ) - \int_D r^2 g^u_r(x, y) f(ry)\, dy$$

where K^u_r, ω^u_r, and g^u_r denote respectively the kernel function (normalized at 0), the L_r-harmonic measure (evaluated at 0), and the Green's function corresponding to L_r and D. Therefore the conclusions of Theorem II.3.2 follow if for every $Q \in \partial D$

$$\text{(II.3.3)} \qquad \lim_{x \to Q} \int_D g^u_r(x, y) f(ry)\, dy = 0$$

uniformly for $Q \in \partial D$ and r near 1. This follows easily from the observations

$$\sup_x \int_D g^u_r(x, y)^{n/(n-1)}\, dy \leqslant C(\lambda, \Lambda, n, \operatorname{diam} D),$$

[14] and for $x \in \psi_{s/2}(Q)$,

$$\left| \int_{D \setminus \psi_s(Q)} g^u_r(x, y) f(ry)\, dy \right| \leqslant C\left(\frac{|x - Q|}{s}\right)^{\alpha} \int_{D \setminus \psi_s(Q)} g^u_r(A_{s/2}(Q), y) |f(ry)|\, dy,$$

with $0 < C$ and $0 < \alpha$ depending only on λ, Λ, n, and the Lipschitz character of D. The last observation results from an application to $g^u_r(\bullet, y)$ of the «boundary Harnack principle» and Hölder continuity for nonnegative solutions of $L_r w = 0$ ([1, Lemmas 2.3 and 2.4, p. 157]).

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[1,2] Partially supported by NSF/DMS-8421377-03.